FINITE SUBGROUPS OF FORMAL A -MODULES OVER $\mathfrak{p}$ -ADIC INTEGER RINGS

BY
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ABSTRACT. Let $B \supset A$ be $\mathfrak{p}$ -adic integer rings such that $A/\mathbb{Z}_p$ is finite and B/A is unramified. Generalizing a result of Fontaine on finite commutative p -group schemes, we show that galois homomorphisms of finite subgroups of one-dimensional formal A -modules over B are given by power series.

Introduction. Let K be a finite extension of the rational p -adic number field $\mathbb{Q}_p$, and A the integer ring of K . Let L be a complete unramified extension of K , B the ring of integers of L , and $\mathfrak{p}$ the maximal ideal of B . We write $\bar{\mathfrak{p}}$ for the maximal ideal of the integer ring of the algebraic closure $\bar{L}$ of L . Let F denote an n -dimensional formal A -module defined over B of finite A -height. Then F induces an A -module structure on $\bar{\mathfrak{p}}^n$, which we denote by $F(\bar{\mathfrak{p}})$; it is an $A[\mathfrak{G}]$ -module, where $\mathfrak{G} = \text{Gal}(\bar{L}/L)$. Let P be a finite sub- $A[\mathfrak{G}]$ -module of $F(\bar{\mathfrak{p}})$ (henceforth, simply of F). In this paper, we attach to P a couple $ML(P)$ of modules over a noncommutative power series ring. Let G be another formal A -module over B of finite A -height and Q be a finite sub- $A[\mathfrak{G}]$ -module of G . Then we describe the $A[\mathfrak{G}]$ -homomorphisms from P to Q by morphisms from $ML(Q)$ to $ML(P)$ (Theorem 1). If $A = \mathbb{Z}_p$ (the p -adic integer ring), this result follows from Fontaine [4], but our proof depends rather on Tate modules of formal groups. Furthermore, if F and G are one-dimensional, we can show that every $A[\mathfrak{G}]$ -homomorphism from P to Q is of the form $g^{-1} \circ cf$ for some $c \in B$, where f and g are the logarithms of F and G , respectively (Theorem 3). In [8], Lubin has obtained a rather weaker version of this result.

In the following, let $K, A, L, B, \mathfrak{p}, \bar{\mathfrak{p}}$ and $\mathfrak{G}$ be as above. We write π for a fixed prime element of A and q for the cardinality of the residue field of A . Let σ denote the Frobenius automorphism of L/K . We write $E = B_\sigma[[T]]$ for the ring of noncommutative power series ring over B in a variable T with respect to the multiplication rule: $Tb = b^\sigma T$ for all $b \in B$. Call $F^A(B)$ the category of finite-dimensional formal A -modules over B of finite A -height.

I would like to thank the referee for calling my attention to Lubin [8].

1. Homomorphisms of finite subgroups of formal A -modules. We write $T(F)$ for the Tate module of a formal A -module F . $T(F)$ is an $A[\mathfrak{G}]$ -module, A -free of rank h ($= A$ -height of F). Let DH' be the category defined in Decauwert [2]. Let $M(F)$ and $L(F)$ be as in [2]; $M(F)$ is an E -module, B -free of rank h and

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$L(F)$ is a sub- B -module of $M(F)$. The functor $ML(F) = (M(F), L(F))$ induces an antiequivalence between $F^A(B)$ and DH' [2, Théorème 2].

Let $\alpha: F \rightarrow G$ be a morphism in $F^A(B)$. We also write α for the homomorphism $T(F) \rightarrow T(G)$ induced by α . We write $\tilde{\alpha}$ for the morphism $ML(G) \rightarrow ML(F)$ induced by α .

LEMMA 1. *Let F, G and H be objects of $F^A(B)$. Let $\alpha: F \rightarrow H$ and $\beta: H \rightarrow G$ be homomorphisms over B . Then $0 \rightarrow T(F) \xrightarrow{\alpha} T(H) \xrightarrow{\beta} T(G) \rightarrow 0$ is exact if and only if $0 \rightarrow ML(G) \xrightarrow{\tilde{\beta}} ML(H) \xrightarrow{\tilde{\alpha}} ML(F) \rightarrow 0$ is exact.*

SKETCH OF PROOF. For a morphism s in DH' , we see that $\text{Ker } s$ and $\text{Im } s$ are in DH' . The “if” part follows easily from this. By Fontaine [5, Chapter V, §2] we can express $ML(F)$ by means of special elements. Choosing an appropriate special element of H , we can prove the “only if” part (cf. also Honda [6]).

Now let $F \in F^A(B)$, and let P be a finite sub- $A[\mathfrak{G}]$ -module of F . Denote by S the superlattice of $T(F)$ in $T(F) \otimes_A K$ such that $S/T(F) \cong P$. Then by Waterhouse [10, Theorem 1.3] there exists an isogeny $\nu: F \rightarrow F'$ defined over B such that $S = \nu^{-1}T(F')$. As S is an $A[\mathfrak{G}]$ -module, we see that $F' \in F^A(B)$. Define $ML(P) = (M(P), L(P))$, where $M(P) = M(F)/\tilde{\nu}M(F')$ and $L(P) = L(F)/\tilde{\nu}L(F')$. Then $M(P)$ is an E -module and $L(P)$ is a sub- B -module of $M(P)$. Let M, M' be left E -modules and N, N' be sub- B -modules of M and M' , respectively. By $\text{Hom}_E((M, N), (M', N'))$ we denote the set of E -linear maps $\delta: M \rightarrow M'$ such that $\delta(N) \subset N'$. Then clearly P determines $ML(P)$ up to an E -isomorphism.

THEOREM 1. *Let $F, G \in F^A(B)$. Let P and Q be finite sub- $A[\mathfrak{G}]$ -modules of F and G , respectively. Then $\text{Hom}_{A[\mathfrak{G}]}(P, Q)$ is A -isomorphic to*

$$\text{Hom}_E(ML(Q), ML(P)).$$

SKETCH OF PROOF. We refer to the method used in Oort [9]. Let $\alpha: F \rightarrow F'$ and $\beta: G \rightarrow G'$ be isogenies over B such that $\text{Ker } \alpha = P$ and $\text{ker } \beta = Q$. Write $T_1 = T(F)$, $T_2 = T(G)$, $M_1 = ML(F)$ and $M_2 = ML(G)$; let T'_1, T'_2, M'_1, M'_2 be similarly defined for F' and G' . We note that $P \cong T'_1/\alpha(T_1)$ and $Q \cong T'_2/\beta(T_2)$. Let $\eta \in \text{Hom}_{A[\mathfrak{G}]}(P, Q)$ and $\Gamma(\eta)$ be the superlattice of $\alpha(T_1) \times \beta(T_2)$ in $T'_1 \times T'_2$ such that $\Gamma(\eta)/\alpha(T_1) \times \beta(T_2)$ is the graph of η . We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \rightarrow T_2 & \xrightarrow{i} & T_1 \times T_2 & \xrightarrow{j} & T_1 & \rightarrow & 0 \\ & \parallel & \downarrow \varepsilon & & \downarrow \alpha & & \\ 0 \rightarrow T_2 & \rightarrow & \Gamma(\eta) & \rightarrow & T'_1 & \rightarrow & 0 \\ & \downarrow \beta & \downarrow & & \parallel & & \\ 0 \rightarrow T'_2 & \xrightarrow{i'} & T'_1 \times T'_2 & \xrightarrow{j'} & T'_1 & \rightarrow & 0 \end{array}$$

where i, i' are the canonical injections, j, j' the canonical projections and ε the composite map $T_1 \times T_2 \xrightarrow{\alpha \times \beta} \alpha(T_1) \times \beta(T_2) \hookrightarrow \Gamma$. Then the functor ML gives the

following commutative diagram, whose rows are exact by Lemma 1:

$$\begin{array}{ccccccccc}
 0 & \rightarrow & M'_1 & \rightarrow & M'_1 \times M'_2 & \rightarrow & M'_2 & \rightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow \tilde{\beta} & & \\
 0 & \rightarrow & M'_1 & \rightarrow & ML(H) & \rightarrow & M_2 & \rightarrow & 0 \\
 & & \downarrow \tilde{\alpha} & & \downarrow & & \parallel & & \\
 0 & \rightarrow & M_1 & \rightarrow & M_1 \times M_2 & \rightarrow & M_2 & \rightarrow & 0
 \end{array}$$

where $H \in F^A(B)$ is such that $T(H) \cong \Gamma(\eta)$ (cf. [10]). By the above diagram we have a morphism $ML(Q) = M_2/\beta M'_2 \rightarrow ML(P) = M_1/\tilde{\alpha} M'_1$, which does not depend on the choice of H ; we denote it by $\theta(\eta)$. By construction we see easily that $\theta: \text{Hom}_{A[\mathfrak{G}]}(P, Q) \rightarrow \text{Hom}_E(ML(Q), ML(P))$ is a bijection. Let $\eta_1, \eta_2 \in \text{Hom}_{A[\mathfrak{G}]}(P, Q)$. As the exact sequence $0 \rightarrow T_2 \rightarrow \Gamma(\eta_1 + \eta_2) \rightarrow T'_1 \rightarrow 0$ is the Baer sum of the extensions $0 \rightarrow T_2 \rightarrow \Gamma(\eta_i) - T'_1 \rightarrow 0$ ($i = 1, 2$), we see that θ is a homomorphism using the functor ML . Clearly θ is an A -isomorphism.

2. One-dimensional formal A -modules. Let v be the valuation of $\bar{L}$ which is normalized so that $v(\pi) = 1$. Here we assume that all formal A -modules are one-dimensional. Let $u = \pi + \sum_{\nu=1}^{\infty} b_{\nu} T^{\nu}$ be a special element in E . We write $(u^{-1}\pi)^*(x)$ for the element $f(x)$ of $L[[x]]$ such that $f(0) = 0$ and $\pi x = \pi f(x) + \sum_{\nu=1}^{\infty} b_{\nu} f^{\sigma^{\nu}}(x^{q^{\nu}})$. Then $F(x, y) = f^{-1}(f(x) + f(y))$ is a formal A -module over B . This shows that the strong isomorphism classes of formal A -modules over B , of A -height h , correspond bijectively to the special elements of the form $\pi + \sum_{\nu=1}^h b_{\nu} T^{\nu}$, where $b_1, \dots, b_{h-1} \in \mathfrak{p}$ but b_h is a unit of B (cf. Cox [1]). Let F be a formal A -module of A -height h defined over B . We write $\Lambda_{F,m} = \text{Ker}[\pi^m]_F = \{x \in \bar{\mathfrak{p}} \mid [\pi^m]_F(x) = 0\}$ for $m \geq 0$, which is a finite subgroup of order q^{hm} in $F(\bar{\mathfrak{p}})$.

THEOREM 2. Let $u_1 = \pi + \sum_{i=1}^h b_i T^i$ and $u_2 = \pi + \sum_{i=1}^h c_i T^i$ be special elements of E such that $b_i, c_i \in \mathfrak{p}$ ($1 \leq i \leq h-1$) and b_h, c_h are units of B . Let $f_1(x) = (u_1^{-1}\pi)^*(x) = \sum_{n=0}^{\infty} a_n x^{q^n}$, $f_2(x) = (u_2^{-1}\pi)^*(x) = \sum_{n=0}^{\infty} a'_n x^{q^n}$ and $\psi = f_2^{-1} \circ f_1$. Let m be an integer such that $u_1 \equiv u_2 \pmod{\mathfrak{p}^m}$ but $u_1 \not\equiv u_2 \pmod{\mathfrak{p}^{m+1}}$. Put $w_i = (b_i - c_i)/\pi^m$ for $1 \leq i \leq h$ and let e ($1 \leq e \leq h$) be such that $w_i \in \mathfrak{p}$ for $1 \leq i \leq e-1$ and w_e is a unit. Then the convergence domain of ψ contains $\{x \in \bar{\mathfrak{p}} \mid v(x) > q^{-e} r^{-m+1} (r-1)^{-1}\}$, where $r = q^h$.

For the proof of Theorem 2 we need the following

LEMMA 2. Assume the same hypothesis as in Theorem 2, and put $A_n = a_n - a'_n$. Then we have $v(A_n) \geq (m-1) - [(n-e)/h]$ for $n \geq 0$, where $[\alpha]$ denotes the largest integer not exceeding α .

PROOF. We proceed by induction on n . First we note that $v(a'_i) \geq -[t/h]$ by [1, Proposition 4.1.1]. By the definition of f_1 and f_2 we can show that

$$A_n = - \sum_{i=1}^h \pi^{-1} b_i A_n^{\sigma^i} - \pi^{m-1} \sum_{i=1}^h w_i a_n^{\sigma^i}.$$

Then it is clear that $v(A_n) \geq m$ for $0 \leq n \leq e$. Hence we may assume that the assertion of our lemma holds for n' with $n' < n = h(j-1) + e + k$, where $0 \leq k < h$

and $j \geq 1$. We have $v(\pi^{-1}b_i A_{n-i}) \geq m-1 - [(n-i-e)/h] \geq m-j$ for $1 \leq i \leq h-1$ and $v(\pi^{-1}b_h A_{n-h}) \geq m-j$. Noting $e+k < 2h$, we have

$$v(\pi^{m-1}w_i a_{n-i}^{\prime\sigma^i}) \geq m-1 - [(n-h)/h] = m-j - [(e+k-h)/h] \geq m-j.$$

Therefore $v(A_n) \geq m-j = (m-1) - [(n-e)/h]$. This completes our proof by induction.

PROOF OF THEOREM 2. We can write $\psi(x) = \sum_{n=0}^{\infty} \alpha_n x^{n(q-1)+1}$ with $\alpha_n \in L$ by Lubin [7, p. 475]. Let ξ be an element of $\bar{\mathfrak{p}}$ such that $\xi^{q^e r^{m-1}(r-1)} = \pi$. Put $\beta_n = \alpha_n \xi^{n(q-1)+1}$. By induction on n we shall show that $v(\beta_n) \geq 1/(r-1)$. Let R be the set whose points are sequences $\mathbf{n} = (n_0, n_1, n_2, \dots)$, where n_i are nonnegative integers for all i and $n_i = 0$ for almost all i . For $\mathbf{n} \in R$, define $|\mathbf{n}| = \sum_{k=0}^{\infty} n_k$, $\mathbf{n}^* = \sum_{k=0}^{\infty} k n_k$ and $C(\mathbf{n}) = |\mathbf{n}|! / (\prod_{k=0}^{\infty} (n_k!))$. They are rational integers. We define an element $\alpha^{\mathbf{n}}$ of L to be $\prod_{k=0}^{\infty} \alpha_k^{n_k}$. Put $Q_s = (q^s - 1)/(q - 1)$. Let t be an integer such that $Q_t < N+1 \leq Q_{t+1}$. On comparing the coefficients of $x^{(N+1)(q-1)+1}$, we get by the equation $f_2(\psi(x)) = f_1(x)$ that

$$(*) \quad \alpha_{N+1} + \sum_{k=1}^t a'_k \left(\sum_{\mathbf{n}} C(\mathbf{n}) \alpha^{\mathbf{n}} \right) = \begin{cases} 0 & \text{if } N+1 < Q_{t+1}, \\ A_{t+1} & \text{if } N+1 = Q_{t+1}, \end{cases}$$

where the sum $\sum_{\mathbf{n}}$ is taken over all $\mathbf{n} \in R$ such that $|\mathbf{n}| = q^k$ and $\mathbf{n}^* = N+1 - Q_k$. We have easily by Lemma 2 that $v(A_1) \geq m-1$ if $e=1$ and $v(A_1) \geq m$ if $e > 1$. Then

$$v(\beta_1) = v(\xi^q \alpha_1) \geq q^{1-e} r^{1-m} (r-1)^{-1} + v(A_1) \geq 1/(r-1).$$

Therefore by induction hypothesis we assume that $v(\beta_n) \geq 1/(r-1)$ for $1 \leq n \leq N$. For $\mathbf{n} = (n_0, n_1, n_2, \dots) \in R$ with $|\mathbf{n}| = q^k$ ($k \geq 1$) and $\mathbf{n}^* = N+1 - Q_k$, let

$$D_{k,\mathbf{n}}^{N+1} = \xi^{(N+1)(q-1)+1} a'_k C(\mathbf{n}) \alpha^{\mathbf{n}} = a'_k C(\mathbf{n}) \xi^{n_0} \prod_{k=1}^{\infty} \beta_k^{n_k}.$$

Now let $g(x) = r^x (r-1)^{-1} - x$; it is clear that $g(n) \geq 1/(r-1)$ for all integers n and $g(n) = 1/(r-1)$ if and only if $n=0$ or $n=1$. Now if $n_0=0$, then

$$v(D_{k,\mathbf{n}}^{N+1}) \geq v(a'_k) + v(C(\mathbf{n})) + q^k/(r-1) \geq g([k/h]) \geq 1/(r-1).$$

If $n_0 \neq 0$, then $0 < n_0 < q^k$. Writing $q = p^j$, $n_0 = q^s d$ with $q \nmid d$ and $d = p^{j'} d_1$ with $(p, d_1) = 1$, we easily get

$$\text{ord}_p({}_q C_{n_0}) = j(k-s) - j' \geq k-s.$$

Clearly ${}_q C_{n_0}$ is a divisor of $C(\mathbf{n})$ and $q^s \leq q^k - n_0$. Therefore

$$\begin{aligned} v(D_{k,\mathbf{n}}^{N+1}) &\geq v(a'_k) + v(C(\mathbf{n})) + nq^{-e} r^{1-m} (r-1)^{-1} + (q^k - n_0)(r-1)^{-1} \\ &> -[k/h] + (k-s) + q^s (r-1)^{-1} \geq g([s/h]) \geq (r-1)^{-1}. \end{aligned}$$

Let us now assume $N+1 = Q_{t+1}$. Then, by Lemma 2, we have

$$v(\xi^{(N+1)(q-1)+1} A_{t+1}) \geq g(-(m-1) + [(t+1-e)/h]) \geq (r-1)^{-1}.$$

In view of (*), we have thus established that $v(\beta_{N+1}) \geq 1/(r-1)$; therefore $v(\beta_n) \geq 1/(r-1)$ for all $n \geq 1$. As $\psi(x) = \sum_{n=0}^{\infty} \beta_n (x/\xi)^{n(q-1)+1}$, the proof is completed.

REMARK. By further computations we can show that $v(\beta_{Q_{h+s}}) = 1/(r-1)$ for $s \geq m$. Therefore the convergence domain of ψ is $\{x \in \bar{\mathfrak{p}} | v(x) > q^{-e} r^{1-m} (r-1)^{-1}\}$.

COROLLARY. *Assumptions and notation being as in Theorem 2, let $F_i(x, y) = f_i^{-1}(f_i(x) + f_i(y))$ ($i = 1, 2$). Then ψ defines an $A[\mathfrak{G}]$ -isomorphism $\Lambda_{F_1, m} \rightarrow \Lambda_{F_2, m}$.*

As $v(x) \geq r^{1-m}(r-1)^{-1}$ for $x \in A_{F_1, m}$, this is clear.

THEOREM 3. *Let F and G be one-dimensional formal A -modules of the same A -height h defined over B and f, g be the logarithms of F, G , respectively. Then every element of $\text{Hom}_{A[\mathfrak{G}]}(\Lambda_{F, m}, \Lambda_{G, m})$ is of the form $g^{-1} \circ cf$ for some $c \in B$. If f and g are of type $u_1 = \pi + \sum_{i=1}^h b_i T^i$ and $u_2 = \pi + \sum_{i=1}^h c_i T^i$, respectively (cf. [1, p. 295]), then $g^{-1} \circ cf \in \text{Hom}_{A[\mathfrak{G}]}(\Lambda_{F, m}, \Lambda_{G, m})$ for $c \in B$ if and only if $u_2 c \equiv cu_1 \pmod{\mathfrak{p}^m}$.*

PROOF. As $M(F) \cong E/Eu_1$ and $M(\Lambda_{F, m}) \cong E/(Eu_1 + E\pi^m)$, we get easily by Theorem 1 that

$$\text{Hom}_{A[\mathfrak{G}]}(\Lambda_{F, m}, \Lambda_{G, m}) \cong \{c \in B \mid u_2 c \equiv cu_1 \pmod{\mathfrak{p}^m}\} / \mathfrak{p}^m.$$

Let $c \in B$ be such that $u_2 c \equiv cu_1 \pmod{\mathfrak{p}^m}$. We assume $v(c) = s \leq m$ and write $c = b\pi^s$ with a unit b in B . Let $u' = bu_1 b^{-1}$. Then u' is special and $u' \equiv u_2 \pmod{\mathfrak{p}^{m-s}}$. Let $f_1(x) = (u'^{-1}\pi)^*(x)$ and $F_1(x, y) = f_1^{-1}(f_1(x) + f_1(y))$. Then $g^{-1} \circ cf = (g^{-1} \circ f_1)[\pi^s]_{F_1} \circ (f_1^{-1} \circ bf)$, where $f_1^{-1} \circ bf: F \rightarrow F_1$ is an isomorphism. By the Corollary above, $g^{-1} \circ cf$ defines an element $\eta(c)$ of $\text{Hom}_{A[\mathfrak{G}]}(\Lambda_{F, m}, \Lambda_{G, m})$; clearly $\eta(c) = \eta(c')$ if and only if $c \equiv c' \pmod{\mathfrak{p}^m}$. Our assertion is now obvious.

REMARK. For a formal A -module F over B of finite A -height h , we have the results which are completely analogous to those in Fontaine [3]. Let $\rho: \mathfrak{G} \rightarrow \text{Aut}_A(T(F)) (\cong \text{GL}_h(A))$ be the π -adic representation attached to F . Then by [3] we have

- (1) $\rho(\mathfrak{G}) \supset A^\times$. Therefore $\mathfrak{G}$ -endomorphisms of $\Lambda_{F, m}$ are $A[\mathfrak{G}]$ -endomorphisms.
- (2) For $h = 1$ or 2 , applying our Theorem 3 we can determine the closed subgroup $\rho(\mathfrak{G})$ of $\text{GL}_h(A)$ (up to an isomorphism) by the special element of F .

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