TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 287, Number 2, February 1985

# INEQUALITIES FOR SOME MAXIMAL FUNCTIONS. I

BY

MICHAEL COWLING AND GIANCARLO MAUCERI

ABSTRACT. This paper presents a new approach to maximal functions on $\mathbf{R}^n$. Our method is based on Fourier analysis, but is slightly sharper than the techniques based on square functions. In this paper, we reprove a theorem of E. M. Stein [**16**] on spherical maximal functions and improve marginally work of N. E. Aguilera [**1**] on the spherical maximal function in $L^2(\mathbf{R}^2)$. We prove results on the maximal function relative to rectangles of arbitrary direction and fixed eccentricity; as far as we know, these have not appeared in print for the case where $n \geqslant 3$, though they were certainly known to the experts. Finally, we obtain a best possible theorem on the pointwise convergence of singular integrals, answering a question of A. P. Calderón and A. Zygmund [**3, 4**] to which N. E. Aguilera and E. O. Harboure [**2**] had provided a partial response.

We prove, in a unified manner, the boundedness of various maximal functions. The basis of our approach is the use of the Mellin transformation, and development in spherical harmonics, to reduce "difficult" maximal functions to "easy" maximal functions and expressions involving singular integrals. We are able to treat various maximal functions which arise in differentiation theory and in the study of singular integrals; unfortunately maximal functions based on oscillatory integrals (such as those involved in the pointwise convergence of Fourier series) require more sophisticated ideas.

Our fundamental inequality is the following. If

$$k(x, y) = \int_{\mathbf{R}} du \sum_{m \in \mathbf{N}} \sum_{j \in D_m} a_{j,m,u}(x) k_{j,m,u}(y)$$

($a_{j,m,u}$ and $k_{j,m,u}$ are described below), then

$$\left| \int_{\mathbf{R}^n} dy\, k(x, y) f(x-y) \right| \leqslant \int_{\mathbf{R}} du \sum_{m \in \mathbf{N}} \sum_{j \in D_m} \left| a_{j,m,u}(x) k_{j,m,u} * f(x) \right|.$$

In some cases, it is easy to verify that, for all $x$ in $\mathbf{R}^n$,

$$\left[ \sum_{j \in D_m} \left| a_{j,m,u}(x) \right|^2 \right]^{1/2} \leqslant b_m(u), \qquad m \in \mathbf{N}, u \in \mathbf{R},$$

Received by the editors November 4, 1983.

1980 *Mathematics Subject Classification*. Primary 42B25; Secondary 42B20.

*Key words and phrases*. Maximal functions, Mellin transform, spherical harmonics.

and consequently

$$\left|\int_{\mathbf{R}^n} dy\, k(x,y) f(x-y)\right| \leqslant \int_{\mathbf{R}} du \sum_{m\in\mathbf{N}} b_m(u)\left[\sum_{j\in D_m} \left|k_{j,m,u} * f(x)\right|^2\right]^{1/2},$$

and so we obtain a bound, which is pointwise, in terms of convolutions. If $k^\alpha$ are different kernels such that the corresponding $d_m$-vectors $\mathbf{a}_{m,u}$ are dominated by the same $b_m(u)$, we obtain the maximal function estimate

$$\sup_\alpha \left|\int_{\mathbf{R}^n} dy\, k^\alpha(x,y) f(x-y)\right| \leqslant \int_{\mathbf{R}} du \sum_{m\in\mathbf{N}} b_m(u) |\mathbf{k}_{m,u} * f(u)|,$$

and hence

$$\left[\int_{\mathbf{R}^n} dx \sup_\alpha \left|\int_{\mathbf{R}^n} dy\, k^\alpha(x,y) f(x-y)\right|^p\right]^{1/p}$$
$$\leqslant \int_{\mathbf{R}} du \sum_{m\in\mathbf{N}} b_m(u)\left[\int_{\mathbf{R}^n} dx |\mathbf{k}_{m,u} * f(x)|^p\right]^{1/p}$$
$$\leqslant \left(\int_{\mathbf{R}} du \sum_{m\in\mathbf{N}} b_m(u) \||\mathbf{k}_{m,u} * \||_p\right)\|f\|_p,$$

where $\||g * \||_p$ denotes the operator norm on $L^p(\mathbf{R}^n)$ of convolution by $g$. These and similar inequalities are the backbone of our work.

The kernels $k_{j,m,u}$ are homogeneous functions, smooth on $\mathbf{R}^n \setminus \{0\}$. Precisely, if $Y_{1,m}, Y_{2,m}, \dots, Y_{d_m,m}$ is an orthonormal basis for the space $\mathscr{H}_m$ of spherical harmonics of degree $m$ on the unit sphere $S^{n-1}$ in $\mathbf{R}^n$ (often written just $S$), equipped with normalised Lebesgue measure (written $\omega_n^{-1}\, dy'$, or $\omega^{-1}\, dy'$), then

$$k_{j,m,u}(y) = Y_{j,m}(y')|y|^{-iu-n}, \qquad y \in \mathbf{R}^n \setminus \{0\},$$

where $y = y'|y|$. The functions $a_{j,m,u}(x)$ are defined by the rules

$$a_{j,m,u}(x) = \omega^{-1}\int_S dy'\, k_u(x,y')\overline{Y}_{j,m}(y'),$$

$$k_u(x,y') = (2\pi)^{-1}\int_{\mathbf{R}^+} dt\, k(x,ty')t^{n-1+iu}.$$

It is clear that, at least formally, if $D_m = \{1,2,\dots,d_m\}$, then

$$\int_{\mathbf{R}} du \sum_{m\in\mathbf{N}} \sum_{j\in D_m} a_{j,m,u}(x) k_{j,m,u}(y)$$
$$= \int_{\mathbf{R}} du\, k_u(x,y')|y|^{-iu-n} = k(x,y),$$

by the properties of spherical harmonics and the Mellin transform. We also observe that

$$a_{j,m,u}(x) = (2\pi\omega)^{-1}\int_{\mathbf{R}^n} dy\, k(x,y)\overline{Y}_{j,m}(y')|y|^{iu}.$$

It will sometimes save effort to write $\mathbf{a}_{m,u}(x)$ for the $d_m$-vector and $|\mathbf{a}_{m,u}(x)|$ for its norm.

The first part of this paper will be devoted to a few preliminary results. We derive the mapping properties of convolution with the vector-valued kernel $\mathbf{k}_{m,u}$ from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n; \mathbf{C}^{d_m})$, and recall the definition and elementary properties of the Handy-Littlewood maximal function. This latter is used to control

$$\int_J du\, a_{0,u}(x) k_{0,u} * f(x),$$

where $J$ is the interval $[-1, 1]$, when $a_{0,u}(x)$, considered as a function of $u$, is the restriction to $J$ of a Fourier-Stieltjes transform.

The second part of the paper treats some radial maximal functions. We give two general results, and derive from these easy proofs and generalizations of some results of E. M. Stein [**16**] and N. E. Aguilera [**1**]. These results are used in §4.

In §3, certain maximal functions arising in differentiation theory are studied. Again, we prove a few general results and use these to reprove results of A. Córdoba [**6**] and J.-O. Strömberg [**19**], and to sharpen up some results of N. E. Aguilera and E. O. Harboure [**2**].

In §4, we apply our method to the theory of singular integrals. We improve results of N. E. Aguilera and E. O. Harboure [**2**] on a question raised by A. P. Calderón and A. Zygmund [**3**, **4**].

In this paper we systematically denote by $C$ a constant which may depend on any or all of the parameters in play, and which may vary from line to line, while by $C_{p,q}$ we mean a constant which depends only on the parameters $p$, $q$, and possibly $n$, the dimension of the space. We ignore the dependence of constants on $n$. By $J$ we denote the interval $[-1, 1]$; $J_m$ means $J_m$ if $m = 0$ and $\varnothing$ (the empty set) if $m \in \mathbf{N}^*$ (the nonzero natural numbers).

We recall that $d_m$, the dimension of $\mathscr{H}_m$, is given by the formula (see E. M. Stein and G. Weiss [**18**, p. 147]).

$$d_m = \frac{(n + 2m - 2)\Gamma(n + m - 2)}{\Gamma(m + 1)\Gamma(n + 1)};$$

we shall use the notation

$$\gamma_{m,iu} = i^m \pi^{n/2 + iu} \frac{\Gamma(m/2 - iu/2).}{\Gamma(n/2 + m/2 - iu/2)}.$$

The Fourier transformation is defined as follows:

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} dx\, f(x) e^{-2\pi i \xi \cdot x}$$

for nice $f$, and then extended.

These results were announced in [**7**]; related work appears in [**8**]. A sequel to this paper is in preparation. We are grateful to G. Weiss for his encouragement during the researching and writing of this paper.

**1. Some preliminaries.** In this section, we obtain estimates for the operator from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n; \mathbf{C}^{d_m})$ of convolution with $\mathbf{k}_{m,u}$, for $p$ between 1 and 2. These are proved by establishing $L^2 - L^2$ and $H^1 - L^1$ estimates, and then interpolating. Later, we recall the definition of the Hardy-Littlewood maximal function, and use this to control

$$\int_J du\, a_{0,u}(x) k_{0,u} * f(x)$$

for nice $a_{0,u}$.

LEMMA 1.1. *With the notation established in the introduction,*

$$\| |\mathbf{k}_{m,u} * f| \|_2 = |\gamma_{m,-iu}| d_m^{1/2} \|f\|_2.$$

PROOF. We denote by $\mathscr{F}$ the Fourier transformation. Clearly

$$\begin{aligned} \| |\mathbf{k}_{m,u} * f| \|_2 &= \| |\mathscr{F}(\mathbf{k}_{m,u} * f)| \|_2 = \| |\mathscr{F}\mathbf{k}_{m,u}| \mathscr{F}f \|_2 \\ &\leqslant \| |\mathscr{F}\mathbf{k}_{m,u}| \|_\infty \|\mathscr{F}f\|_2 = \| |\mathscr{F}\mathbf{k}_{m,u}| \|_\infty \|f\|_2, \end{aligned}$$

so it suffices to show that

$$|\mathscr{F}k_{m,u}(\xi)| = |\gamma_{m_1-iu}| d_m^{1/2}, \qquad \xi \in \mathbf{R}^n \setminus \{0\}.$$

This is straightforward. The Fourier transform of $k_{j,m,u}$ is $\gamma_{m,-iu}$ times the function $Y_{j,m}(\xi')|\xi|^{iu}$ (see, e.g., p. 78 of Stein [**15**]), and

$$\left[ \sum_{j \in D_m} \left| Y_{j,m}(\xi') \right|^2 \right]^{1/2} = C, \qquad \xi' \in S^{n-1};$$

clearly

$$C^2 = \omega^{-1} \int_S d\xi' \sum_{j \in D_m} \left| Y_{j,m}(\xi') \right|^2 = d_m. \quad \square$$

We note explicitly that $|\gamma_{m,-iu}|$ explodes when $m = 0$ and $u$ is small. This is why we have to treat this case separately.

LEMMA 1.2. *With the notation established in the introduction,*

$$\| |\mathbf{k}_{m,u} * f| \|_1 \leqslant C_{m,u} \|f\|_{H^1},$$

*where*

$$C_{m,u} \leqslant C d_m \log(2 + m + |u|), \qquad m \in \mathbf{N}, u \in \mathbf{R} \setminus J_m,$$

*and*

$$C_{m,u} \leqslant C|u|^{-1}, \qquad m = 0, u \in J.$$

PROOF. We refer the reader to R. R. Coifman and G. Weiss [**5**] for an excellent discussion of $H^1(\mathbf{R}^n)$. From their work and Lemma 1.1, it is clearly enough to estimate $I$, where

$$I = \int_{|x| > 2|y|} dx |\mathbf{k}_{m,u}(x - y) - \mathbf{k}_{m,u}(x)|.$$

This we do by considering separately first $I_1$, where

$$\begin{aligned} I_1 &= \int_{(2+m+|u|)|y|>|u|>2|y|} dx|\mathbf{k}_{m,n}(x-y)-\mathbf{k}_{m,u}(x)| \\ &\leqslant 2\int_{(3+m+|u|)|y|>|x|>|y|} dx|\mathbf{k}_{m,u}(x)| \\ &= 2\log(3+m+|u|)\int_S dx'|\mathbf{k}_{m,u}(x')| \\ &\leqslant 2\log(3+m+|u|)\,\omega^{1/2}\left[\int_S dx'|\mathbf{k}_{m,u}(x')|^2\right]^{1/2} \\ &= 2\log(3+m+|u|)\,\omega d_m^{1/2} \end{aligned}$$

and then $I_2$, where

$$\begin{aligned} I_2 &= \int_{|x|>(2+m+|u|)|y|} dx|\mathbf{k}_{m,u}(x-y)-\mathbf{k}_{m,u}(x)| \\ &= \int_{|x|>(2+m+|u|)|y|} dx\left|\int_0^1 ds\, d/ds\,\mathbf{k}_{m,u}(x-sy)\right| \\ &\leqslant \int_{|x|>(1+m+|u|)|y|} dx|\nabla\mathbf{k}_{m,u}(x)|\,|y| \\ &\leqslant \int_{|x|>(1+m+|u|)|y|} dx\left(|iu+n|\,|\mathbf{k}_{m,u}(x')|\,|x|^{-n-1}+|\nabla_t\mathbf{k}_{m,u}(x)|\right)|y|, \end{aligned}$$

where $\nabla_t$ is the tangential component of the gradient. Continuing by applying Hölder's inequality and integrating by parts, we obtain

$$\begin{aligned} I_2 &\leqslant (1+m+|u|)^{-1}|iu+n|\omega^{1/2}\left[\int_S dx'|\mathbf{k}_{m,u}(x')|^2\right]^{1/2} \\ &\quad+(1+m+|u|)^{-1}\omega^{1/2}\left[-\int_S dx'\,\Delta_t\mathbf{k}_{m,u}(x')\cdot\mathbf{k}_{m,u}(x')\right]^{1/2} \\ &= (1+m+|u|)^{-1}\omega d_m^{1/2}\left[|iu+n|+[m(m+n-2)]^{1/2}\right]. \end{aligned}$$

Combining the estimates for $I_1$ and for $I_2$, we conclude that

$$I \leqslant Cd_m^{1/2}\log(2+m+|u|).$$

From this estimate and Lemma 1.1 we obtain the desired result. □

**PROPOSITION 1.3.** *Suppose that* $1<p\leqslant 2$. *Then the following estimates hold*:

$$\begin{aligned} \|\mathbf{k}_{m,u}*f\|_p &\leqslant C_p(m+1)^{n/2-1}(m+|u|)^{-n/p'}\left[\log(2+m+|u|)\right]^{2/p-1}\|f\|_p, \\ &\qquad\qquad m\in\mathbf{N},\ u\in\mathbf{R}\setminus J_m, \\ &\leqslant C_p|u|^{-1}\|f\|_p, \qquad m=0,\ u\in J. \end{aligned}$$

**PROOF.** This is a corollary of Lemmata 1.1 and 1.2 and the interpolation theorem of C. Fefferman and E. M. Stein **[10]**. We also use easy estimates for Γ-functions. □

Since the operator norm of convolution with $k_{0,u}$ blows up as $u$ tends to 0, we need to exercise particular care in dealing with expressions of the form

$$\int_J du\, a_{0,u}(x) k_{0,u} * f(x).$$

The Hardy-Littlewood maximal function $Mf$ of a locally integrable function $f$ is a valuable tool in our investigations.

The Hardy-Littlewood maximal function $Mf$ is defined as follows:

$$Mf(x) = \sup_{r \in \mathbf{R}^+} |B(x,r)|^{-1} \int_{B(x,r)} dy |f(y)|,$$

where $B(x, r)$ is the ball centred at $x$ of radius $r$, and $|B(x, r)|$ is its measure. The essential facts about $Mf$ may be found on p. 5 of Stein [**15**]. We recall only that, if $1 < p \leqslant \infty$, there is a constant $A_p$ such that

$$\|Mf\|_p \leqslant A_p \|f\|_p, \qquad f \in L^p(\mathbf{R}^n).$$

LEMMA 1.4. *Suppose that $F$: $\mathbf{R}^+ \to \mathbf{R}^+$ is decreasing, and that*

$$\int_{\mathbf{R}^n} dy\, F(|y|) < \infty.$$

*Then for any $\lambda$ in $\mathbf{R}^+$ and $f$ in $L^p(\mathbf{R}^n)$,*

$$\int_{\mathbf{R}^n} dy\, \lambda^n F(\lambda|y|) |f(x-y)| \leqslant \left[\int_{\mathbf{R}^n} dy\, F(|y|)\right] Mf(x).$$

PROOF. The function $y \to \lambda^n F(\lambda|y|)$ satisfies the same hypotheses as $y \to F(|y|)$, so without loss of generality we may take $\lambda$ to be 1. Since

$$F(|y|) = -\int_{|y|}^{\infty} dF(t) = -\int_{\mathbf{R}^+} dF(t) \chi_{B(0,t)}(y)$$

(where $B$ is the closed ball),

$$\begin{aligned}
\int_{\mathbf{R}^n} dy\, F(|y|) |f(x-y)| &= -\int_{\mathbf{R}^+} dF(t) \int_{\mathbf{R}^n} dy\, \chi_{B(0,t)}(y) |f(x-y)| \\
&\leqslant -\int_{\mathbf{R}^+} dF(t) |B(0,t)| Mf(x) \\
&= -\int_{\mathbf{R}+} dF(t) \int_{\mathbf{R}^n} dy\, \chi_{B(0,t)}(y) Mf(x) \\
&= \int_{\mathbf{R}^n} dy\, F(|y|) Mf(x). \quad \square
\end{aligned}$$

PROPOSITION 1.5. *Suppose that for some measure $\mu$ on $\mathbf{R}$ of bounded variation,*

$$a_{0,u}(x) = \int_{\mathbf{R}} d\mu_x(v) e^{-2\pi i u v}, \qquad u \in J,$$

*where $\int_{\mathbf{R}} |d\mu_x(v)| \leqslant A$. Then*

$$\left| \int_J du\, a_{0,u}(x) k_{0,u} * f(x) \right| \leqslant 4\pi\omega A Mf(x) + A \int_J du |u|\, |k_{0,u} * f(x)|.$$

PROOF. We observe first that

$$\begin{aligned}\left|\int_J du\, a_{0,u}(x) k_{0,u} * f(x)\right| &\leqslant \int_J du|u|\, |a_{0,u}(x)|\, |k_{0,u} * f(x)| \\ &\quad + \left|\int_J du(1-|u|)\int_{\mathbf{R}} d\mu_x(v) e^{-2\pi i u v} k_{0,u} * f(x)\right| \\ &\leqslant A\int_J du|u|\, |k_{0,u} * f(x)| + \int_{\mathbf{R}} |d\mu_x(v)| \left|\int_J du(1-|u|) e^{-2\pi i u v} k_{0,u} * f(x)\right|.\end{aligned}$$

Next, let $H$: $\mathbf{R} \to \mathbf{R}$ be the function $H(w) = \sin^2(w/2)(w/2)^2$, and let $G$: $\mathbf{R} \to \mathbf{R}$ be the majorant $G(w) = 8/4 + w^2$. Then

$$\begin{aligned}\int_J du(1-|u|) e^{-2\pi i u v} |y|^{-iu-n} &= \int_J du(1-|u|)\exp\left(-iu\left[2\pi v + \log|y|\right]\right)|y|^{-n} \\ &= H\left(\log\left(e^{2\pi v}|v|\right)\right)|y|^{-n},\end{aligned}$$

and so, by Lemma 1.4,

$$\begin{aligned}&\int_{\mathbf{R}} |d\mu_x(v)| \left|\int_J du(1-|u|) e^{-2\pi i u v} k_{0,u} * f(x)\right| \\ &\quad = \int_{\mathbf{R}} |d\mu_x(v)| \left|\int_{\mathbf{R}^n} dy\, H\left(\log\left(e^{2\pi v}|y|\right)\right)|y|^{-n} f(x-y)\right| \\ &\quad \leqslant \int_{\mathbf{R}} |d\mu_x(v)| \int_{\mathbf{R}^n} dy\, G\left(\log\left(e^{2\pi v}|y|\right)\right)|y|^{-n}|f(x-y)| \\ &\quad \leqslant \int_{\mathbf{R}} |d\mu_x(v)| \int_{\mathbf{R}^n} dy\, G\left(\log\left(|y|\right)\right)|y|^{-n} Mf(x) \\ &\quad \leqslant A\int_S dy' \int_{\mathbf{R}^+} dt\, t^{-1} G(\log(t)) Mf(x) \\ &\quad = 4\pi\omega A Mf(x). \quad \square\end{aligned}$$

We remark that if $a_{0,u}(x)$ is Hölder continuous with exponent more than $1/2$, as a function of $u$, then it is certainly equal to a Fourier-Stieltjes transform in $J$. More to the point of what follows, if

$$a_{0,u}(x) = \int_{\mathbf{R}} d\mu_x(v) e^{-2\pi i u v}, \qquad u \in J,$$

then, for any $s$ in $\mathbf{R}^+$,

$$s^{iu} a_{0,u}(x) = \int_{\mathbf{R}} d\mu_x(v + \log(s)/2\pi) e^{-2\pi i u v}, \qquad u \in J,$$

and of course

$$\int_{\mathbf{R}} |d\mu_x(v + \log(s)/2\pi)| = \int_{\mathbf{R}} |d\mu_x(v)|.$$

We also point out that the above proof goes through verbatim for vector-valued functions $f$, provided that convolution be appropriately defined. See the discussion of the point at the end of the proof of Theorem 2.1.

**2. Radial maximal functions.** In this section we prove two general theorems, and show that these contain the results of Stein [16] and Aguilera [1] as particular cases. We shall treat a vector-valued version of one of these results, since it will be useful later in this form. Naturally the other theorem also could be given a vectorial formulation.

THEOREM 2.1. *Suppose that $p \in (1,2]$, that $A \in \mathbf{R}^+$, and that $b\colon \mathbf{R} \to \mathbf{R}^+$ be such that*

$$B = \int_{\mathbf{R}\setminus J} du\, b(u)|u|^{-n/p'} \log(2 + |u|)^{2/p-1} < \infty.$$

*Let $\Phi$ be the set of all radial distributions $\phi$ on $\mathbf{R}^n$, integrable near 0 and at $\infty$, for which the function $a$, given by the formula*

$$a_{0,u} = (2\pi\omega)^{-1} \int_{\mathbf{R}^n} d\phi(y)|y|^{iu}, \qquad u \in \mathbf{R},$$

*is such that*

$$a_{0,u} = \int_{\mathbf{R}} d\mu(v) e^{-2\pi i u v}, \qquad u \in J,$$

*with $\int_{\mathbf{R}} |d\mu(v)| \leqslant A$, and $|a_{0,u}| \leqslant b(u)$, $u \in \mathbf{R} \setminus J$.*

*Then for any $d$ in $\mathbf{N}^*$, and for all $f$ in $L^p(\mathbf{R}^n; \mathbf{C}^d)$,*

$$M_\Phi f = \sup\{|\phi * f| \colon \phi \in \Phi\}$$

*exists in $L^p(\mathbf{R}^n)$; more precisely,*

$$M_\Phi f \leqslant 4\pi\omega A M|f| + A\int_J du|u|\,|k_{0,u} * f| + \int_{\mathbf{R}\setminus J} du\, b(u)|k_{0,u} * f|,$$

*and so*

$$\|M_\Phi f\|_p \leqslant C_p(A + B)\|f\|_p.$$

PROOF. Since $\phi$ is radial, we obtain, by inverting the Mellin transformation, that, distributionally

$$\phi = \int_{\mathbf{R}} du\, a_{0,u} k_{0,u}.$$

(Integrability of $\phi$ near 0 avoids the difficulties one encounters trying to synthesize the Laplace operator in 0 in this manner; integrability at $\infty$ appears in the hypothesis of the theorem purely to assure an easy definition of $a_{0,u}$, and may be relaxed). Thus

$$\begin{aligned}\phi * f &= \int_{\mathbf{R}} du\, a_{0,u} k_{0,u} * f \\ &= \int_J du\, a_{0,u} k_{0,u} * f + \int_{\mathbf{R}\setminus J} du\, a_{0,u} k_{0,u} * f.\end{aligned}$$

By Proposition 1.5 and the hypotheses,

$$\begin{aligned}|\phi * f| &\leqslant \left|\int_J du\, a_{0,u}k_{0,u} * f\right| + \int_{\mathbf{R}\setminus J} du|a_{0,u}|\,|k_{0,u} * f| \\ &\leqslant 4\pi\omega AM|f| + A\int_J du|u|\,|k_{0,u} * f| \\ &\quad + \int_{\mathbf{R}\setminus J} du\, b(u)|k_{0,u} * f|.\end{aligned}$$

The right-hand side is independent of $\phi$ in $\Phi$, and so

$$M_\Phi f \leqslant 4\pi\omega AM|f| + A\int_J du|u|\,|k_{0,u} * f| + \int_{\mathbf{R}\setminus J} du\, b(u)|k_{0,u} * f|.$$

(The "supremum" is defined in the separable Banach lattice $L^p(\mathbf{R}^n)$; this avoids measurability questions. See N. Dunford and J. T. Schwartz [**9**, Chapter 3] for the general theory necessary). We conclude by noting that, from Proposition 1.3 and the properties of $M|f|$,

$$\|M_\Phi f\|_p \leqslant 4\pi\omega A\|M|f|\,\|_p + A\int_J du|u|\,\|\,|k_{0,u} * f|\,\|_p + \int_{\mathbf{R}\setminus J} du\, b(u)\|\,|k_{0,u} * f|\,\|_p.$$

For vector-valued $f$, we have been guilty of omitting the definition of $k_{0,u} * f$; we mean, of course, the operator, perhaps more correctly written $k_{0,n} \otimes I * f$ (but is pedantry correct?) which acts in each component of the vector by convolution with $k_{0,u}$. It is a well-known result of J. Marcinkiewicz and A. Zygmund [**13**] that the norm of this operator on $L^p(\mathbf{R}^n; \mathbf{C}^d)$ is that of convolution by $k_{0,u}$ on $L^p(\mathbf{R}^n)$.

Consequently, $\|M_\Phi f\|_p \leqslant C_p(A + B)\|f\|_p$. □

The next result is a variant on the same theme.

THEOREM 2.2. *Suppose that $p \in (1,2]$, that $r \in [1, p]$, that $\alpha \in (-\infty, n/p' - 1/r)$, and that $A, B \in \mathbf{R}^+$. Let $\Phi$ be the set of all radial distributions $\phi$ on $\mathbf{R}^+$, integrable near 0 and at $\infty$, for which the function a, given by the formula*

$$a_{0,u} = (2\pi\omega)^{-1}\int_{\mathbf{R}^n} d\phi(y)|y|^{iu}, \qquad u \in \mathbf{R},$$

*is such that,*

$$a_{0,u} = \int_{\mathbf{R}} d\mu(v)e^{-2\pi iuv}, \qquad u \in J,$$

*with $\int_{\mathbf{R}}|d\mu(v)| \leqslant A$, and also such that*

$$\left[\int_{\mathbf{R}\setminus J} du\left[|a_{0,u}|\,|u|^{-\alpha}\right]^{r'}\right]^{1/r'} \leqslant B$$

*(with the obvious modification if $r' = \infty$). Then for all f in $L^p(\mathbf{R}^n)$,*

$$M_\Phi f = \sup\{|\phi * f| : \phi \in \Phi\}$$

*exists in $L^p(\mathbf{R}^n)$. More precisely*

$$M_\Phi f \leqslant 4\pi\omega AMf + A\int_J du|u|\,|k_{0,u} * f| + B\left(\int_{\mathbf{R}\setminus J} du|u|^{\alpha r}|k_{0,u} * f|^r\right)^{1/r},$$

*whence $\|M_\Phi f\|_p \leqslant C_{p,r,\alpha}(A + B)\|f\|_p$.*

PROOF. As in the proof of Theorem 2.1, we have that

$$|\phi * f| \leqslant \left|\int_J du\, a_{0,u} k_{0,u} * f\right| + \int_{\mathbf{R}\setminus J} du |a_{0,u}|\, |k_{0,u} * f|.$$

By Proposition 1.5, Hölder's inequality, and the hypothesis,

$$\begin{aligned}|\phi * f| &\leqslant 4\pi\omega A Mf + A\int_J du|u|\,|k_{0,u} * f|\\ &\quad + \left(\int_{\mathbf{R}\setminus J} du\left[|a_{0,u}|\,|u|^{-\alpha}\right]^{r'}\right)^{1/r'}\left(\int_{\mathbf{R}\setminus J} du|u|^{\alpha r}|k_{0,u} * f|^r\right)^{1/r}\\ &\leqslant 4\pi\omega A Mf + A\int_J du|u|\,|k_{0,u} * f| + B\left(\int_{\mathbf{R}\setminus J} du|u|^{\alpha r}|k_{0,u} * f|^r\right)^{1/r}.\end{aligned}$$

As before, it follows that the right-hand side of this inequality dominates $M_\Phi f$. We conclude by observing that Minkowski's inequality finishes off the proof:

$$\begin{aligned}\left\|\left(\int_{\mathbf{R}\setminus J} du|u|^{\alpha r}|k_{0,u} * f|^r\right)^{1/r}\right\|_p &\leqslant \left(\int_{\mathbf{R}\setminus J} du|u|^{\alpha r}\|k_{0,u} * f\|_p^r\right)^{1/r}\\ &\leqslant \left(\int_{\mathbf{R}\setminus J} du|u|^{\alpha r} C_p|u|^{-n/p'}\log(2+|u|)^{2/p-1}\|f\|_p r\right)^{1/r}\\ &= C_{p,r,\alpha}\|f\|_p,\end{aligned}$$

from Proposition 1.3 and the initial restrictions on $p$ and $r$. □

Before we pass to a few applications of these results, we remark that by duality, the estimates for $|||k_{0,u} *|||_p$ are valid for $|||k_{0,u} *|||_{p'}$, and so Theorems 2.1 and 2.2 have easy extensions to $L^p(\mathbf{R}^n)$ with $p$ in $(2, \infty)$. However, for those $p$, a better strategy for treating maximal functions seems to be interpolation between $L^2$-results (of the type just proved) and $L^\infty$-results (obtained trivially if $\Phi$ contains only measures of uniformly bounded variation). So we omit these extensions.

Some applications of Theorems 2.1 and 2.2 will now be given. First, we discuss the results of Stein [**16**].

Put $m^z(x) = b^z([1-|x|^2]_+)^{z-1}$, $x \in \mathbf{R}^n$, where

$$b^z = \left[\int_{\mathbf{R}^n} dx\left(\left[1-|x|^2\right]_+\right)^{z-1}\right]^{-1} = [(\omega/2)B(n/2, z)]^{-1}.$$

Then $m^z$, defined initially for $\mathrm{Re}(z) > 0$ as an integrable function, extends meromorphically to $z$ in $\mathbf{C}$ as a distribution with singular support $S^{n-1}$. Further, $m^z = \int_{\mathbf{R}} du\, a_u^z k_{0,u}$ where

$$\begin{aligned}a^z &= (2\pi\omega)^{-1}\int_{\mathbf{R}^n} dy\, m^z(y)|y|^{iu}\\ &= (2\pi)^{-1}b^z\int_{\mathbf{R}^+} dt([1-t^2])^{z-1}t^{iu+n-1}\\ &= (2\pi\omega)^{-1}B(n/2, z)^{-1}B((m+iu)/2, z).\\ &= (2\pi\omega)^{-1}\frac{\Gamma((n+iu)/2)}{\Gamma(n/2)}\,\frac{\Gamma(n/2+z)}{\Gamma((n+iu)/2+z)}.\end{aligned}$$

This is analytic in $\mathrm{Re}(z) > -n/2$. As a function of $u$ it is clearly real-analytic, and so certainly admits an expansion as a Fourier-Stieltjes transform in $J$.

From the well-known asymptotic formulae for the $\Gamma$-function, it is clear that, if $1 \geqslant \mathrm{Re}(z) > -n/2$, then

$$|a_u^z| \leqslant C^z |u|^{-\mathrm{Re}(z)}, \qquad u \in \mathbf{R} \setminus J.$$

We therefore obtain the following corollary of Theorem 2.1.

COROLLARY 2.3 (STEIN [**16**]). *If* $1 < p \leqslant 2$, *and* $\mathrm{Re}(z) > 1 - n/p'$, *then* $\|M_z f\|_p \leqslant C^z \|f\|_p$ *for all* $f$ *in* $L^p(\mathbf{R}^n)$, *where*

$$M_z f = \sup\left\{\left|\int_{\mathbf{R}^n} dy\, t^{-n} m^z(y/t) f(x-y)\right| : t \in \mathbf{R}^+\right\}.$$

PROOF. We take some representation of $a_u^z$ as a Fourier-Stieltjes transform in $J$,

$$a_u^z = \int_{\mathbf{R}} d\mu(v) e^{-2\pi i u v}, \qquad u \in J,$$

and let $b\colon \mathbf{R} \to \mathbf{R}^+$ be the function given by the rule

$$b(u) = (2\pi\omega)^{-1} \left|B(n/2, z)^{-1} B((n+iu)/2, z)\right|, \qquad u \in \mathbf{R}.$$

Since $m^z(y) = \int_{\mathbf{R}} du\, a_u^z k_{0,u}(y)$ then

$$t^{-n} m^z(y/t) = \int_{\mathbf{R}} du\, a_u^z t^{-n} k_{0,u}(y/t) = \int_{\mathbf{R}} du\, t^{iu} a_u^z k_{0,u}(y).$$

Alternatively

$$(2\pi\omega)^{-1} \int_{\mathbf{R}^n} dy\, t^{-n} m^z(y/t) |y|^{iu} = t^{iu} (2\pi\omega)^{-1} \int_{\mathbf{R}^n} dy\, m^z(y) |y|^{iu}$$
$$= t^{iu} a_u^z.$$

Now as remarked at the end of §1,

$$t^{iu} a_u^z = \int_{\mathbf{R}} d\mu(v + \log(t)/2\pi) e^{-2\pi i u v}, \qquad u \in J,$$

and

$$\int_{\mathbf{R}} |d\mu(v) + \log(t)(2\pi)| = \int_{\mathbf{R}} |d\mu(v)| = A,$$

say. Further,

$$|t^{iu} a_u^z| = b(u), \qquad u \in \mathbf{R}.$$

So with $\Phi$ as in the enunciation of Theorem 2.1, $M_z f \leqslant M_\Phi f$, and the estimate for $M_z f$ follows from that for $M_\Phi f$. □

Now we consider some work of N. Aguilera [**1**]. If $\phi \in L^1(\mathbf{R}^n)$, and $\phi$ is radial (let us write $\phi(x) = F(|x|)$), then

(1)
$$a_{0,u} = (2\pi\omega)^{-1} \int_{\mathbf{R}^n} dx\, \phi(x) |x|^{iu}$$
$$= (2\pi)^{-1} \int_{\mathbf{R}^+} dt\, F(t) t^{n-1+iu}$$
$$= (2\pi)^{-1} \int_{\mathbf{R}} dv\, G(v) e^{iuv},$$

where $G(v) = e^{nv}F(e^v)$. Since $\phi \in L^1(\mathbf{R}^n)$,

$$\int_{\mathbf{R}} dv|G(v)| = \int_{\mathbf{R}^n} dx|\phi(x)| < \infty,$$

and so $a_{0,u}$ is automatically a Fourier-Stieltjes transform.

By using Theorem 2.2, we recover Aguilera's results.

COROLLARY 2.4. *If $n = 2$, $1 < q < 2$, and $\Phi$ is the set of all $\phi$ in $L^1(\mathbf{R}^n)$ such that $\|\phi\|_1 \leqslant A$ and*

$$\left(\int_{\mathbf{R}^n} dx|x|^{2q-2}|\phi(x)|^q\right)^{1/q} \leqslant B,$$

*then $M_\Phi f(\sup_{\phi\in\Phi}|\phi * f|)$ exists in $L^2(\mathbf{R}^2)$, and*

$$\|M_\Phi f\|_2 \leqslant C_q(A + B)\|f\|_2, \qquad f \in L^2(\mathbf{R}^2).$$

PROOF. We apply Theorem 2.2 with $r$ equal to 2, and $\alpha = 1/2q$. For $\phi$ in $\Phi$, $a_{0,u}$ is equal to a Fourier-Stieltjes transform, as remarked above (after formula (1)). Let $G(v) = e^{2v}F(e^v)$, as before. Then $a_{0,u} = (2\pi)^{-1}\hat{G}(-(2\pi)^{-1}u)$.

It is sufficient to show that, if $\phi \in \Phi$, then

$$\left[\int_{\mathbf{R}\setminus J} du\left[|a_{0,u}|\,|u|^{-1/2q}\right]^2\right]^{1/2} \leqslant B',$$

where $B'$ may depend on $B$, $A$, $q$, etc. but not on $\phi$. We estimate this integral as follows:

$$\begin{aligned}
&\left[\int_{\mathbf{R}\setminus J} du\left[|a_{0,u}|\,|u|^{-1/2q}\right]^2\right]^{1/2}\\
&\qquad \leqslant \left[\int_{\mathbf{R}\setminus J} du|a_{0,u}|^{q'}\right]^{1/q'}\left[\int_{\mathbf{R}\setminus J} du|u|^{-q'/q(q'-2)}\right]^{(q'-2)/2q'}\\
&\qquad = \left[\int_{\mathbf{R}\setminus J} du\left|(2\pi)^{-1}\hat{G}\left(-(2\pi)^{-1}u\right)\right|^{q'}\right]^{1/q'}[2(q'-2)]^{(q'-2)/2q'}\\
&\qquad \leqslant (2\pi)^{-1/q}\left[\int_{\mathbf{R}} dv|G(v)|^q\right]^{1/q}[2(q'-2)]^{(q'-2)/2q'}\\
&\qquad = C_q\left[\int_{\mathbf{R}^+} dt\, t^{-1}t^{2q}|F(t)|^q\right]^{1/q}\\
&\qquad = C_q\left[\int_{\mathbf{R}^2} dx|x|^{2q-2}|\phi(x)|^q\right]^{1/q}\\
&\qquad \leqslant C_q B,
\end{aligned}$$

as required, by Hölder's inequality and the Hausdorff-Young theorem. □

On the other hand, by applying Theorem 2.1, we may obtain a slightly sharper theorem for a different class $\Phi$. Let $b\colon \mathbf{R} \to \mathbf{R}^+$ be such that

$$B = \int_{\mathbf{R}\setminus J} du\, b(u)|u|^{-1} < \infty.$$

In particular, if we take $\phi\colon \mathbf{R}^2 \to \mathbf{C}$ such that $G$, defined above (after formula (1)), is in $h^1(\mathbf{R})$, as defined by D. Goldberg [**11**], then we may set $b(u) = |a_{0,u}|$.

COROLLARY 2.5. *Suppose that $A \in \mathbf{R}^+$ and that $b$ is as above, and let $\Phi$ be the set of integrable functions $\phi$ such that $\|\phi\|_1 \leqslant A$ and $|a_{0,u}| \leqslant b(u)$, $u \in \mathbf{R}\setminus \mathcal{Y}$. Then $\mathcal{M}_\Phi f$ exists in $L^1(\mathbf{R}^2)$ for any $f$ in $L^2(\mathbf{R}^2)$, and*

$$\|\mathcal{M}_\Phi f\|_2 \leqslant C_{A,b}\|f\|_2, \qquad f \in L^2(\mathbf{R}^2).$$

PROOF. Omitted, since it is essentially like those of Corollaries 2.3 and 2.4.

We note explicitly that this corollary contains the following result: if $\phi$ is such that $G$ is in $h^1(\mathbf{R})$, then, putting $\phi_{,t}(x) = t^{-2}\phi(x/t)$,

$$\Big\|\sup_{t>0} |\phi_{,t} * f|\Big\|_2 \leqslant C_\phi\|f\|_2, \qquad f \in L^2(\mathbf{R}^2).$$

We were unable to prove this using Aguilera's technique.

**3. Nonradial maximal functions.** As in §2, we shall first prove some general results on maximal functions, and then apply them.

THEOREM 3.1. *Suppose that $1 < p \leqslant 2$, and that $b\colon \mathbf{N} \times \mathbf{R} \to \mathbf{R}^+$ is such that*

$$\sum_{m\in\mathbf{N}} \int_{\mathbf{R}\setminus J_m} du\, b(m,u)(1+m)^{m/2-1}(m+|u|)^{-n/p'}\log(2+m+|u|)^{2/p-1}$$

*is finite. Suppose that $A \in \mathbf{R}^+$. Let $\Phi$ be the set of all distributions $\phi$ on $\mathbf{R}^n$, integrable near $0$ and at $\infty$ such that, if*

$$a_{j,m,u} = (2\pi\omega)^{-1}\int_{\mathbf{R}^n} d\varphi(y)|y|^{iu}\overline{Y}_{j,m}(y'),$$

*then*

$$\Big(\sum_{j\in D_m} |a_{j,m,u}|^2\Big)^{1/2} \leqslant b(m,u), \qquad u \in \mathbf{R}\setminus J_m,$$

*and also*

$$a_{0,u} = \int_{\mathbf{R}} d\mu(v) e^{-2\pi i u v}, \qquad u \in J,$$

*with $\int_{\mathbf{R}} |d\mu(v)| \leqslant A$. Then for all $f$ in $L^p(\mathbf{R}^n)$,*

$$M_\Phi f = \sup\{|\phi * f|\colon \phi \in \Phi\}$$

*exists in $L^p(\mathbf{R}^n)$. More precisely,*

$$M_\Phi f \leqslant 4\pi\omega A M f + A\int_J du|u|\,|k_{0,u} * f|$$

$$+ \sum_{m\in\mathbf{N}} \int_{\mathbf{R}\setminus J_m} du\, b(m,u)|\mathbf{k}_{m,u} * f|,$$

*whence $\|M_\Phi f\|_p \leqslant C_{p,A,b}\|f\|_p$.*

PROOF. This is just a routine generalization of the proof of Theorem 2.1, and we omit it.

THEOREM 3.2. *Suppose* $p \in (1,2]$, $r \in [1, p]$, $A, B \in \mathbf{R}^+$, $\alpha, \beta, \gamma \in \mathbf{R}$, *and*

(i) $\alpha + \gamma > 1/r - n/p'$;

(ii) $\alpha + \beta + \gamma > 2/r - n/p' + (n/2 - 1)$;

(ii) $\beta + \gamma > 1/r - n/p' + (n/2 - 1)$.

*Let* $\Phi$ *be the set of all distributions* $\phi$ *on* $\mathbf{R}^n$, *integrable near* 0 *and at* $\infty$, *for which the function* $a$, *given by the formula*

$$a_{j,m,u} = (2\pi\omega)^{-1}\int_{\mathbf{R}^n} d\phi(y)|y|^{iu}\overline{Y}_{j,m}(y')$$

*is such that*

$$a_{0,u} = \int_{\mathbf{R}} d\mu(v)e^{-2\pi iuv}, \qquad u \in J,$$

*with* $\int_{\mathbf{R}} |d\mu(v)| \leqslant A$, *and also such that*

$$\left[\sum_{m\in\mathbf{N}} \int_{\mathbf{R}\setminus J_m} du|\mathbf{a}_{m,u}|^{r'}(1+|u|)^{r'\alpha}(1+m)^{r'\beta}(m+|u|)^{r'\gamma}\right]^{1/r'} \leqslant B$$

(*with the obvious modifications if* $r' = \infty$). *Then for all* $f$ *in* $L^p(\mathbf{R}^n)$,

$$M_\Phi f = \sup\{|\phi * f| : \phi \in \Phi\}$$

*exists in* $L^p(\mathbf{R}^n)$. *More precisely*,

$$M_\Phi f \leqslant 4\pi\omega AMf + A\int_J du|u|\,|k_{0,u} * f|$$
$$+B\left[\sum_{m\in\mathbf{N}} \int_{\mathbf{R}\setminus J_m} du(1+|u|)^{-r\alpha}(1+m)^{-r\beta}(m+|u|)^{-r\gamma}|\mathbf{k}_{m,u} * f|^r\right]^{1/r},$$

*and so*

$$\|M_\Phi f\|_p \leqslant C_{p,r,\alpha,\beta,\gamma}(A+B)\|f\|_p.$$

PROOF. As before, it is straightforward to see that

$$|\phi * f| \leqslant 4\pi\omega AMf + A\int_J du|u|\,|k_{0,u} * f|$$
$$+ \sum_{m\in\mathbf{N}} \int_{\mathbf{R}\setminus J_m} du|a_{m,u}|\,|k_{m,u} * f|.$$

It will suffice to consider the last term, which we estimate using Hölder's inequality:

$$\sum_{m\in\mathbf{N}} \int_{\mathbf{R}\setminus J_m} du|\mathbf{a}_{m,u}|\,|\mathbf{k}_{m,u} * f|$$
$$\leqslant \left[\sum_{m\in\mathbf{N}} \int_{\mathbf{R}\setminus J_m} du|\mathbf{a}_{m,u}|^{r'}(1+|u|)^{r'\alpha}(1+m)^{r'\beta}(m+|u|)^{r'\gamma}\right]^{1/r'}$$
$$\cdot\left[\sum_{m\in\mathbf{N}} \int_{\mathbf{R}\setminus J_m} du(1+|u|)^{-r\alpha}(1+m)^{-r\beta}(m+|u|)^{-r\gamma}|\mathbf{k}_{m,u} * f|^r\right]^{1/r}.$$

By hypothesis, the first term of this product is bounded by $B$, while, from Minkowski's inequality,

$$\left\| \left[ \sum_{m \in \mathbf{N}} \int_{\mathbf{R} \setminus J_m} du (1 + |u|)^{-r\alpha} (1 + m)^{-r\beta} (m + |u|)^{-r\gamma} |\mathbf{k}_{m,u} * f|^r \right]^{1/r} \right\|_p$$

$$\leqslant \left[ \sum_{m \in \mathbf{N}} \int_{\mathbf{R} \setminus J_m} du (1 + |u|)^{-r\alpha} (1 + m)^{-r\beta} (m + |u|)^{-r\gamma} \|\mathbf{k}_{m,n} * f\|_p^r \right]^{1/r}$$

$$\leqslant C_p \left[ \sum_{m \in \mathbf{N}} \int_{\mathbf{R} \setminus J_m} du (1 + |u|)^{-r\alpha} (1 + m)^{-r\beta + rn/2 - r} \cdot (m + |u|)^{-r\gamma - rn/p'} \log(2 + m + |u|)^{2r/p - r} \right]^{1/r} \|f\|_p.$$

The last expression in square brackets may be estimated easily by considering separately $|u| \leqslant m + 1$ and $|u| \geqslant m + 1$, and using hypotheses (i), (ii) and (iii). We leave this to the reader. □

Our applications regard functions (or distributions) on $\mathbf{R}^n$ which may be readily expressed in polar coordinates. It will be useful to remind the reader of a few familiar results on spherical harmonic expansions, so that the angular part of the functions may be treated.

LEMMA 3.3. *Suppose that $q \in [1, 2]$, and that $\beta_0 = (2 - n)(1/q - 1/2)$. If $f \in L^q(S^{n-1})$, and*

$$b_{j,m} = \omega^{-1} \int_S dx' f(x') \overline{Y}_{j,m}(x'),$$

*then if $\beta = \beta_0$ and $r' \geqslant q'$, or if $\beta < \beta_0$ and $1/r' < 1/q' + \beta_0 - \beta$,*

$$\left( \sum_{m \in \mathbf{N}} (1 + m)^{r'\beta} |\mathbf{b}_m|^{r'} \right)^{1/r'} \leqslant C_{q,r,\beta} \|f\|_q.$$

PROOF. If $f \in L^2(S^{n-1})$, we have the Plancherel formula

$$\left( \sum_{m \in \mathbf{N}} |\mathbf{b}_m|^2 \right)^{1/2} = \left( \omega^{-1} \int_S dx' |f(x')|^2 \right)^{1/2}.$$

Next, by an obvious estimation,

$$|\mathbf{b}_m| \leqslant \omega^{-1} \int_S dx' |f(x')| \left( \sum_{j \in D_m} |Y_{j,m}(x')|^2 \right)^{1/2}$$

$$= \omega^{-1} \int_S dx' |f(x')| d_m^{1/2}.$$

By interpolation we obtain the Hausdorff-Young inequality

$$\left( \sum_{m \in \mathbf{N}} d_m \left( d_m^{1/2} |\mathbf{b}_m| \right)^{q'} \right)^{1/q'} \leqslant \|f\|_q$$

(where the norm is relative to normalised Lebesgue measure in $S^{n-1}$). Since $d_m$ grows like $(1+m)^{n-2}$, we obtain

$$\left(\sum_{m\in\mathbf{N}}(1+m)^{q'\beta_0}|\mathbf{b}_m|^{q'}\right)^{1/q'} \leqslant C_q\|f\|_q.$$

Evidently, if $r' \geqslant q'$,

$$\left(\sum_{m\in\mathbf{N}}(1+m)^{r'\beta_0}|\mathbf{b}_m|^{r'}\right)^{1/r'} \leqslant C_q\|f\|_q,$$

while if $\beta < \beta_0$ and $1/r' < 1/q' + \beta_0 - \beta$, then Hölder's inequality, dextrously applied, proves the rest of the lemma. □

LEMMA 3.4. *Suppose that $f \in L^1(S^{n-1})$, and that*

$$b_{j,m} = \omega^{-1}\int_S dx'\, f(x')\overline{Y}_{j,m}(x').$$

*If $n = 2$ and $f \in L\log^+ L$, then*

$$\sum_{m\in\mathbf{N}}(1+m)^{-1}|\mathbf{b}_m| \leqslant C\int_S dx'|f(x')|\left[\log^+|f(x')|+1\right].$$

*If $n \geqslant 3$, $q = 2(1-1/n)$, and $f \in L^{q,1}(S^{n-1})$ (the usual Lorentz space, as defined and studied in* **[18]**, *e.g.) then*

$$\sum_{m\in\mathbf{N}}(1+m)|\mathbf{b}_m| \leqslant C_m\|f\|_{q,1}.$$

PROOF. In the case where $n = 2$, the result is well known. It follows readily from Hardy's inequality and the fact that $L\log^+ L(S^1) \subseteq H^1(S^1)$ (à propos, see E. M. Stein **[14]**).

To prove the second result, we use the Lorentz spaces $l^{p,q}(\mathbf{N}; d)$ on $\mathbf{N}$ relative to the weight $d$. Let $\mathbf{c}_m$ be a $d_m$-vector such that $|\mathbf{c}_m| = (1+m)^{-1}$, for $m$ in $\mathbf{N}$. Then the generalized Hausdorff-Young inequality implies that

$$\begin{aligned}\left|\sum_{m\in\mathbf{N}}\mathbf{b}_m\cdot\mathbf{c}_m\right| &\leqslant \sum_{m\in\mathbf{N}} d_m\left(d_m^{-1/2}|\mathbf{b}_m|\right)\left(d_m^{-1/2}|\mathbf{c}_m|\right)\\ &\leqslant \|d^{-1/2}|\mathbf{b}|\,\|_{q',1}\|d^{-1/2}|\mathbf{c}|\,\|_{q,\infty} \leqslant C\|f\|_{q,1},\end{aligned}$$

whence $\sum_{m\in\mathbf{N}}(1+m)^{-1}|\mathbf{b}_m| \leqslant C\|f\|_{q,1}$. □

To state our applications, we need a little notation. If $u \in C_c^\infty(\mathbf{R}^n)$, $\sigma \in SO(n)$, and $t \in \mathbf{R}^+$, we denote by $u^{\sigma,t}$ the $C_c^\infty(\mathbf{R}^n)$-function defined thus

$$u^{\sigma,t}(x) = u(\sigma t x), \qquad x \in \mathbf{R}^n.$$

For a distribution $\phi$ on $\mathbf{R}^n$, we denote by $\phi_{\sigma,t}$ the distribution given by the rule

$$\phi_{\sigma,t}(u) = \phi(u^{\sigma,t}), \qquad u \in C_c^\infty(\mathbf{R}^n).$$

For obvious reasons, we call $\phi_{\sigma,t}$ a "normalised dilate and rotate of $\phi$".

Suppose that $E$ be an open subset of $\mathbf{R}^n$, of finite measure $|E|$, starlike about 0. Then there exists a (measurable) function $R$: $S^{n-1} \to \mathbf{R}^+$ such that

$$E = \left\{rx' : x' \in S^{n-1}, r \in \left[0, R(x')\right)\right\}.$$

Let $\phi$ be $|E|^{-1}1_E$ ($1_E$ is the characteristic function of $E$).

We shall be interested in the maximal functions $M_\Phi f$:

$$M_\Phi f = \sup\{|\phi * f| : \phi \in \Phi\}$$

for various families of distributions $\Phi$. See M. de Guzmán's book **[12]** for the relevance of these maximal functions in differentiation theory.

COROLLARY 3.5. *Suppose that $R^2 \in L\log^+ L(S^1)$ if $n = 2$, or that $R^n \in L^{q,1}(S^{n-1})$, where $q = 2 - 2/n$ if $n \geqslant 3$. Then if $\phi$ is as above, and $\Phi$ is the set of all normalized dilates and rotates of $\phi$, we have the inequalities following*:

$$\|M_\Phi f\|_2 \leqslant C\left[\|R^2\|_{L\log^+ L} + \|R^2\|_1\right]\|R^2\|_1^{-1}\|f\|_2.$$

*for $n = 2$ and all $f$ in $L^2(\mathbf{R}^2)$; if $n \geqslant 3$, then for all $f$ in $L^2(\mathbf{R}^n)$,*

$$\|M_\Phi f\|_2 \leqslant C\|R^n\|_{q,1}\|R^n\|_2^{-1}\|f\|_2.$$

PROOF. Let $\psi$ be the distribution associated to the function given by the formula

$$\psi(x) = |E|^{-1}\exp(-|x|/R(x')), \qquad x \in \mathbf{R}^n\setminus\{0\},$$

and let $\Psi$ be the family of all its mentioned dilates and rotates. Since

$$|\phi_{\sigma,s} * f| \leqslant \phi_{\sigma,s} * |f| e\psi_{\sigma,s} * |f|,$$

it suffices to show that, for any $f$ in $L^2(\mathbf{R}^n)$,

$$\|M_\Psi f\|_2 \leqslant C_E\|f\|_2.$$

Let $b$: $\mathbf{N} \times \mathbf{R} \to \mathbf{R}^+$ be defined by the rule

$$b(m,u) = (2\pi\omega)^{-1}\left(\sum_{j\in D_m}\left|\int_{\mathbf{R}^n} d\psi(y)|y|^{iu}\overline{Y}_{j,m}(y')\right|^2\right)^{1/2}.$$

We observe that, for any $\sigma$ and $s$ in $SO(n)$ and $\mathbf{R}^+$,

$$(2\pi\omega)^{-1}\left(\sum_{j\in D_m}\left|\int_{\mathbf{R}^n} d\psi_{\sigma,s}(y)|y|^{iu}\overline{Y}_{j,m}(y')\right|^2\right)^{1/2} = b(m,u).$$

Theorem 3.1 may be applied as long as the function

$$u \to (2\pi\omega)^{-1}\int_{\mathbf{R}^n} d\psi(y)|y|^{iu}$$

may be written as a Fourier-Stieltjes transform in $J$, and provided

$$\sum_{m\in\mathbf{N}}\int_{\mathbf{R}\setminus J_m} du\, b(m,u)(1+m)^{n/2-1}(m+|u|)^{-n/2} < \infty.$$

Since

$$(2\pi\omega)^{-1}\int_{\mathbf{R}^n} d\psi(y)|y|^{iu} = (2\pi\omega|E|)^{-1}\int_S dx'\int_{\mathbf{R}^+} dt\, t^{n-1}\exp(-t/R(x'))t^{iu}$$

$$= (2\pi\omega|E|)^{-1}\Gamma(n+iu)\int_S dx'\, R(x')^{n+iu},$$

which is differentiable qua function of $u$ provided that

$$\int_S dx' \, R(x')^n |\log(R(x'))| < \infty$$

(true by hypothesis), while

$$\begin{aligned}
&\sum_{m \in \mathbf{N}} \int_{\mathbf{R} \setminus J_m} du \, b(m,u)(1+m)^{n/2-1}(m+|u|)^{-n/2} \\
&\quad \leqslant 2^{n/2} \sum_{m \in \mathbf{N}} \int_{\mathbf{R}} du \, b(m,u)(1+m)^{-1} \\
&\quad \leqslant C|E|^{-1} \sum_{m \in \mathbf{N}} \int_{\mathbf{R}} du |\Gamma(n+iu)| \left( \sum_{j \in D_n} \left| \int_S dy' \, R(y')^{n+iu} \overline{Y}_{j,m}(y') \right|^2 \right)^{1/2} \\
&\quad \leqslant C\|R\|_1^{-1} \left[ \|R^2\|_{L \log^+ L} + \|R^2\|_1 \right] \quad \text{if } n = 2 \\
&\quad \leqslant C\|R^n\|_1^{-1} \|R^n\|_{q,1} \quad \text{if } n \geqslant 3
\end{aligned}$$

by Lemma 3.4; the corollary is proved. □

COROLLARY 3.6. *Suppose that $2n/n+1 < p \leqslant 2$, and that $p'(1-1/n) < q \leqslant 2$. Let $\Phi$ be the set of all functions of the form $|E|^{-1} 1_E$, where $E = \{x \in \mathbf{R}^n \colon 0 \leqslant |x| < R(x')\}$ and $\|R^n\|_q \leqslant C\|R^n\|_1 < \infty$.*

*Then*

$$\|M_\Phi f\|_p \leqslant C_{p,q} \|f\|_p, \qquad f \in L^p(\mathbf{R}^n).$$

PROOF. This result follows from Theorem 3.2 and Lemma 3.3 much as Corollary 3.5 followed from Theorem 3.1 and Lemma 3.4. We omit the details.

We obtain similar results when we consider distributions $\phi$ on $\mathbf{R}^n$ of the form

$$\phi(f) = \int_0^1 dt \, t^{n-1} \int_S dx' \, \theta(x') f(tx'), \qquad f \in C(\mathbf{R}^n), \tag{2}$$

where $\theta \in L^1(S^{n-1})$, and let $\Phi$ be the set of all normalized dilates and rotates of $\phi$.

COROLLARY 3.7. *With the notation just described, $M_\Phi$ is bounded on $L^2(\mathbf{R}^n)$ if $\theta \in H^1(S^1)$ (when $n = 2$), or if $\theta \in L^{q,1}(S^{n-1})$, where $q = 2 - 2/n$ (when $n \geqslant 3$).*

COROLLARY 3.8. *If $2n/(n+1) < p \leqslant 2$ and $(n-1)p'/n < q \leqslant 2$, and if $\Phi$ is the set of all normalized dilates and rotates of distributions $\phi$ described above, where $\|\theta\|_q \leqslant 1$, then $M_\Phi$ is bounded on $L^p(\mathbf{R}^n)$.*

PROOFS. We omit these: Corollary 3.7 is like Corollary 3.5, and Corollary 3.8 is like Corollary 3.6.

By interpolation with easily proved results for $L^1(\mathbf{R}^n)$ and $L^\infty(\mathbf{R}^n)$, we may extend the above results. For instance, we have the following improvement of Corollary 3.8.

COROLLARY 3.9. *If $1 < p \leqslant 2$ and $(n-1)p'/n < q$, and if $\Phi$ is the set of all normalized dilates and rotates of distributions $\phi$ described above* (2), *where $\|\theta\|_q \leqslant 1$, then $M_\Phi$ is bounded on $L^p(\mathbf{R}^n)$.*

PROOF. If $p = \infty$, then $M_\Phi$ is bounded when $q = 1$; if $p = 1$, then $M_\Phi$ is of weak type (1, 1) if $q = \infty$. Consequently, if $q = \infty$, $M_\Phi$ is bounded on $L^{1+\varepsilon}(\mathbf{R}^n)$ for any $\varepsilon$ in $\mathbf{R}^+$. Interpolating between this result and that of Corollary 3.8 for $p = 2$, we obtain the claim.

Of course, we could also obtain information for $p$ in $(2, \infty)$ by interpolation (it would suffice to have $q' < p(n-1)/(n-2)$), but this is perhaps not best possible, at least if $n \geqslant 3$.

It may be of interest that the results of Corollary 3.8 are as sharp as those of Corollary 3.9, for those $p$ for which Corollary 3.8 holds. It seems to be impossible to modify the argument leading to Corollary 3.8 to obtain the whole range $(1, 2]$ for $p$. The essential difficulty is that we cannot prove estimates for the decrease of the coefficients in the spherical harmonic expansion of $f$ in $L^q(S^{n-1})$ for $q$ greater than 2 which are better than those for $q$ equal to 2.

The results of Corollaries 3.8 and 3.9 are essentially best possible if $1 < p \leqslant 2$ (see N. E. Aguilera and E. O. Harboure [**2**]). From Corollary 3.5, we may obtain estimates for maximal functions involving rectangles of fixed eccentricity and arbitrary directions, essentially equivalent to the results of A. Córdoba [**6**] and J.-O. Strömberg [**19**]. If we take $E$ to be the rectangle $[-l, l] \times [-1/l, 1/l]$ in $\mathbf{R}^2$ (with $l > 1$), then parametrising $x'$ in $S^1$ by the argument $\theta$, we obtain

$$R(\theta) = \min\left(l|\cos(\theta)|^{-1}, l^{-1}|\sin(\theta)|^{-1}\right).$$

For norm calculations, we might as well use

$$\begin{aligned} R'(\theta) &= \min\left(l, l^{-1}|\theta|^{-1}\right), \qquad |\theta| \leqslant \pi/2, \\ &= \min\left(l, l^{-1}|\theta - \pi|^{-1}\right), \qquad |\theta - \pi| \leqslant \pi/2. \end{aligned}$$

It follows that the norm of the maximal operator on $L^2(\mathbf{R}^1)$ is of the order of $\log(l)$. If we take $E$ to be the rectangle $[-l^{n-1}, l^{n-1}] \times [-l, l]^{n-1}$ in $\mathbf{R}^n$ ($n \geqslant 3$), then similar calculations lead to the estimate that the norm of the maximal operator on $L^2(\mathbf{R}^n)$ is of the order of $l^{n(n-2)/2}$. This is essentially best possible. However, as E. M. Stein [**17**] points out, the interesting case is when $p > 2$, and we have no significant contribution to the case $p = n$.

**4. Convergence of singular integrals.** In this section, we apply our techniques to the pointwise convergence of singular integrals. Our results complement work of A. P. Calderón and A. Zygmund [**3, 4**] and of N. E. Aguilera and E. O. Harboure [**2**].

A singular integral kernel is a measurable function $k$: $\mathbf{R}^n \times \mathbf{R}^n \setminus \{0\} \to \mathbf{C}$ such that

$$k(x, \lambda y) = \lambda^{-n} k(x, y), \qquad \lambda \in \mathbf{R}, x \in \mathbf{R}^n, y \in \mathbf{R}^n \setminus \{0\},$$

and

$$\int_S k(x, y')\, dy' = 0, \qquad x \in \mathbf{R}^n.$$

Given a singular integral kernel, we attempt to associate to it an operator $K$, by the following procedure. First we form, for $\varepsilon$ in $\mathbf{R}^+$, $K_\varepsilon f$:

$$K_\varepsilon f(x) = \int_{\mathbf{R}^n \setminus B_\varepsilon} k(x, y) f(x - y)\, dy$$

where $B_\varepsilon$ is the ball of radius $\varepsilon$. Then we try to let $\varepsilon$ tend to 0. There are two natural questions: when does $K_\varepsilon f$ converge in norm and when does it converge pointwise, as $\varepsilon$ tends to 0? The pointwise convergence will, of course, be controlled by the maximal operator $K^{\#}f = \sup_{\varepsilon \in \mathbf{R}^+} |K_\varepsilon f|$.

THEOREM 4.1. *Suppose that $k$ is a singular integral kernel, and that for some $q$ greater than $2(n - 1)/n$,*

$$\left(\int_S dy' |k(x, y')|^q\right)^{1/q} \leqslant B, \qquad x \in \mathbf{R}^n.$$

*Then $\|K^{\#}f\|_2 \leqslant C_q \|f\|_2 f \in L^2(\mathbf{R}^n)$, and $K_\varepsilon f$ converges pointwise and in $L^2(\mathbf{R}^n)$ to an $L^2(\mathbf{R}^n)$-function denoted $Kf$.*

PROOF. The arguments presented by Aguilera and Harboure (op. cit., pp. 567 and 570) show that it suffices to prove *a priori* estimates for $K^{\#}f$. We may also suppose that $q \leqslant 2$.

By using spherical harmonics, we may write

$$k(x, y) = \sum_{m \in \mathbf{N}^*} \sum_{j \in D_m} b_{j,m}(x) k_{j,m}(y)$$

where

$$b_{j,m}(x) = \omega^{-1} \int_S dy'\, k(x, y') \overline{Y}_{j,m}(y')$$

and

$$k_{j,m}(y) = |y|^{-n} Y_{j,m}(y').$$

Since $q > 2(n - 1)/n$, we may choose $\beta$ in $\mathbf{R}$ such that

$$-1/2 < \beta < (1/q - 1/2)(1 - n).$$

From Lemma 3.3, with $r = 2$, and the hypothesis,

$$\left[\sum_{m \in \mathbf{N}^*} m^{2\beta} \sum_{j \in D_m} |b_{j,m}(x)|^2\right]^{1/2} \leqslant C_{q,\beta} B.$$

Now

$$
\begin{aligned}
|K_\varepsilon f(x)| &= \left| \sum_{m\in\mathbf{N}^*} \sum_{j\in D_m} b_{j,m(x)} \int_{\mathbf{R}^n\setminus B_\varepsilon} dy\, k_{j,m}(y) f(x-y) \right| \\
&\leqslant \left[ \sum_{m\in\mathbf{N}^*} m^{2\beta} \sum_{j\in D_m} \left| b_{j,m}(x) \right|^2 \right]^{1/2} \\
&\quad \cdot \left[ \sum_{m\in\mathbf{N}^*} m^{-2\beta} \sup_{\varepsilon\in\mathbf{R}^+} \sum_{j\in D_m} \left| \int_{\mathbf{R}^n\setminus B_\varepsilon} dy\, k_{j,m}(y) f(x-y) \right|^2 \right]^{1/2} \\
&\leqslant C_{q,\beta} B \left[ \sum_{m\in\mathbf{N}^*} m^{-2\beta} \sup_{\varepsilon\in\mathbf{R}^+} \sum_{j\in D_m} \left| \int_{\mathbf{R}^n\setminus B_\varepsilon} dy\, k_{j,m}(y) f(x-y) \right|^2 \right]^{1/2}.
\end{aligned}
$$

Since the right-hand side is now independent of $\varepsilon$, it also majorises $K^\# f$. The proof will be finished by showing that

$$
\left\| \sup_{\varepsilon\in\mathbf{R}^+} \left( \sum_{j\in D_m} \left| \int_{\mathbf{R}^n\setminus B_\varepsilon} dy\, k_{j,m}(y) f(\cdot - y) \right|^2 \right)^{1/2} \right\|_2 \leqslant C \log(2+m) m^{-1} \|f\|_2, \tag{3}
$$

for then

$$
\begin{aligned}
&\left\| \left[ \sum_{m\in\mathbf{N}^*} m^{-2\beta} \sup_{\varepsilon\in\mathbf{R}^+} \sum_{j\in D_m} \left| \int_{\mathbf{R}^n\setminus B_\varepsilon} dy\, k_{j,m}(y) f(\cdot - y) \right|^2 \right]^{1/2} \right\|_2 \\
&= \left[ \sum_{m\in\mathbf{N}^*} m^{-2\beta} \left\| \sup_{\varepsilon\in\mathbf{R}^+} \left( \sum_{j\in D_m} \left| \int_{\mathbf{R}^n\setminus B_\varepsilon} dy\, k_{j,m}(y) f(\cdot - y) \right|^2 \right)^{1/2} \right\|_2^2 \right]^{1/2} \\
&\leqslant \left[ \sum_{m\in\mathbf{N}^*} m^{-2\beta} C^2 \log^2(2+m) m^{-2} \|f\|_2^2 \right]^{1/2} \leqslant C \|f\|_2,
\end{aligned}
$$

as required.

Up to this point, our proof follows those of Calderón and Zygmund [3] and of Aguilera and Harboure [2]. Our approach now diverges from theirs, in that we have the machinery of the previous sections at our disposal.

By a theorem of Bochner, presented by Stein [15, p. 72], if

$$
h_j(x) = Y_{j,m}(x')|x|^{-n} 1_{\mathbf{R}^n\setminus B_1}(x), \qquad x \in \mathbf{R}^n,
$$

then $\hat{h}_j(\xi) = g_m(\xi) Y_{j,m}(\xi')$, $\xi \in \mathbf{R}^n$, for some radial function $g_m$ in $L^2(\mathbf{R}^n)$. On the other hand,

$$
\hat{k}_{j,m}(\xi) = \gamma_{m,0} Y_{j,m}(\xi'), \qquad \xi \in \mathbf{R}^n,
$$

($\gamma_{m,0}$ is defined in Lemma 1.1), and so $h_j = \phi_m * k_{j,m}$ where $\hat{\phi}_m = \gamma_{m,0}^{-1} g_m$. Clearly $\phi_m \in L^2(\mathbf{R}^n)$. Observe that

$$
\sum_{j\in D_m} h_j * \bar{k}_{j,m} = \sum_{j\in D_m} \phi_m * k_{j,m} * \bar{k}_{j,m}
$$

and

$$\sum_{j\in D_m}\left(k_{j,m}*\bar{k}_{j,m}\right)^{\wedge}\xi)=(-1)^m\gamma_{m,0}^2\sum_{j\in D_m}Y_{j,m}(\xi)\overline{Y}_{j,m}(\xi)$$
$$=(-1)^m\gamma_{m,0}^2 d_m,$$

and so

$$\phi_m=(-1)^m\gamma_{m,0}^{-2}d_m^{-1}\sum_{j\in D_m}h_j*\bar{k}_{j,m}\tag{4}$$

and

$$\hat{\phi}_m(\xi)=\gamma_{m,0}^{-1}d_m^{-1}\sum_{j\in D_m}\hat{h}_j(\xi)\overline{Y}_{j,m}(\xi').\tag{5}$$

As a consequence of (4), we see that $\phi_m$ is $C^\infty$ off $S^{n-1}$. More importantly, by putting $\tilde{h}_j=k_{j,m}1_B$, we see that

$$\phi_m=(-1)^m\gamma_{m,0}^{-2}d_m^{-1}\sum_{j\in D_m}\left(k_{j,m}*\bar{k}_{j,m}-\tilde{h}_j*\bar{k}_{j,m}\right)$$
$$=\delta_0-(-1)^m\gamma_{m,0}^{-2}d_m^{-1}\sum_{j\in D_m}\tilde{h}_j*\bar{k}_{j,m},$$

$\delta_0$ being the Dirac measure at 0, from which it follows that

$$\phi_m(x)=O\left(|x|^{-n-1}\right)\quad\text{if } |x|>1,$$

and in particular $\phi_m\in L^1(\mathbf{R}^n)$.

Normalized dilation is such that $(\phi_m)_\varepsilon*k_{j,m}=k_{j,m}1_{\mathbf{R}^n\setminus B_\varepsilon}$, and so

$$\sup_{\varepsilon\in\mathbf{R}^+}\left[\sum_{j\in D_m}\left|\int_{\mathbf{R}^n\setminus B_\varepsilon}dy\,k_{j,m}(y)f(\cdot-y)\right|^2\right]^{1/2}$$
$$=\sup_{\varepsilon\in\mathbf{R}^+}\left[\sum_{j\in D_m}\left|(\phi_m)_\varepsilon*k_{j,m}*f(\cdot)\right|^2\right]^{1/2}.$$

This maximal function may be treated using Theorem 2.1. It is necessary to calculate $\int_{\mathbf{R}^n}dy\,\phi_m(y)|y|^{iu}$, which requires a little work.

First, by the properties of $\phi_m$, Plancherel's theorem, and the formula (5), we have that

$$\int_{\mathbf{R}^n}dy\,\phi_m(y)|y|^{iu}=\lim_{\delta\to0+}\int_{\mathbf{R}^n}dy\,\phi_m(y)|y|^{iu-\delta}$$
$$=\lim_{\delta\to0+}\int_{\mathbf{R}^n}d\xi\,\hat{\phi}_m(\xi)\gamma_{0,n+iu-\delta}|\xi|^{\delta-iu-n}$$
$$=\lim_{\delta\to0+}d_m^{-1}\gamma_{m,0}^{-1}\gamma_{0,n+iu-\delta}\sum_{j\in D_m}\int_{\mathbf{R}^n}d\xi\hat{h}_j(\xi)\overline{Y}_{j,m}(\xi')|\xi|^{\delta-iu-n}.$$

Continuing, by using Plancherel's theorem again, we find that

$$\begin{aligned}\sum_{j\in D_m} &\int_{\mathbf{R}^n} d\xi\, \hat{h}_j(\xi)\overline{Y}_{j,m}(\xi')|\xi|^{\delta-iu-n}\\ &= \sum_{j\in D_m}\int_{\mathbf{R}^n} dy\, h_j(y)(-1)^m\gamma_{m,n+iu-\delta}^{-1}\overline{Y}_{j,m}(y')|y|^{iu-\delta}\\ &= (-1)^m\gamma_{m,n+iu-\delta}^{-1}\sum_{y\in D_m}\int_S dy'\int_1^\infty dt\, t^{iu-\delta-1}|Y_{j,m}(y')|^2\\ &= (-1)^m\gamma_{m,n+iu-\delta}^{-1}\omega d_m(\delta-iu)^{-1}.\end{aligned}$$

and consequently we may conclude that

$$\begin{aligned}(2\pi\omega)^{-1}\int_{\mathbf{R}^n} dy\,\phi_m(y)|y|^{iu} &= (2\pi)^{-1}(-1)^m\gamma_{m,0}^{-1}\gamma_{0,n+iu}\gamma_{m,n+iu}^{-1}(-iu)^{-1}\\ &= a_u^m,\quad \text{say}.\end{aligned}$$

In order to apply Theorem 2.1, we must study $a_u^m$. First, we consider its behaviour for small $u$ ($u\in J$). Later we shall estimate $|a_u^m|$ for large $u$ ($u\in\mathbf{R}\setminus J$).

Let $M$ be the integral part of $(m-1)/2$, and let $\delta$ be $m-2M$. Then, by the recurrence formula for the $\Gamma$-function,

$$\begin{aligned}a_u^m &= (4\pi)^{-1}\pi^{-n/2}\frac{\Gamma(n/2+m/2)}{\Gamma(m/2)}\cdot\frac{\Gamma(n/2+iu/2)}{\Gamma(1-iu/2)}\cdot\frac{\Gamma(m/2-iu/2)}{\Gamma(n/2+m/2+iu/2)}\\ &= (4\pi)^{-1}\pi^{-n/2}\prod_{j=0}^{M-1}\frac{(n/2+\delta/2+j)(\delta/2+j-iu/2)}{(\delta/2+j)(n/2+\delta/2+j+iu/2)}\\ &\quad\cdot\frac{\Gamma(n/2+\delta/2)\Gamma(n/2+iu/2)\Gamma(\delta/2-iu/2)}{\Gamma(\delta/2)\Gamma(1-iu/2)\Gamma(n/2+\delta/2+iu/2)}\\ &= \prod_{j=0}^{M-1}\frac{(n+\delta+2j)(\delta+2j-iu)}{(\delta+2j)(n+\delta+2j+iu)}\cdot a_n^\delta.\end{aligned}$$

Since $\delta$ is either 1 or 2, $a_u^\delta$ is analytic in $J$. Evidently, if $|u|\leqslant\pi$ and $j\in\mathbf{N}$,

$$\begin{aligned}\left|\frac{(n+\delta+2j)(\delta+2j-iu)}{(\delta+2j)(n+\delta+2j+iu)}\right| &= \left|\left(1-\frac{iu}{\delta+2j}\right)\left(1+\frac{iu}{n+\delta+2j}\right)^{-1}\right|\\ &\leqslant\left|1-\frac{i\pi}{\delta+2j}\right|\leqslant 1+5(\delta+2j)^{-2}.\end{aligned}$$

Similarly, for the same range of $u$ and $j$, for the derivative ($w.r.t.u$), we have

$$\left\|\left[\frac{(n+\delta+2j)(\delta+2j-iu)}{(\delta+2j)(n+\delta+2j+iu)}\right]'\right\|\leqslant 22(\delta+2j)^{-1}.$$

Consequently,

$$\sup_{|u|\leqslant\pi}\left|\prod_{j=1}^{M}\frac{(n+\delta+2j)(\delta+2j-iu)}{(\delta+2j)(n+\delta+2j+iu)}\right|\leqslant C$$

and

$$\sup_{|u|\leqslant\pi}\left|\left[\prod_{j=1}^{M}\frac{(n+\delta+2j)(\delta+2j-iu)}{(\delta+2j)(n+\delta+2j+iu)}\right]'\right|\leqslant C\log(2+m),$$

for some absolute constant $C$. Elementary Fourier analysis now shows that it is possible to write

$$a_u^m=\int_{\mathbf{R}}d\mu(v)e^{-2\pi iuv},\qquad u\in J,$$

where $\int_{\mathbf{R}}|d\mu(v)|\leqslant C\,log(2+m)$.

It is easiest to study $a_u^m$ for $u$ large by using the asynptotic formula for the $\Gamma$-function (see, for instance, E. T. Whittaker and G. N. Watson [**20**, 12.33]). The details of the estimation are rather tedious but straightforward, so we shall omit them. The fruits of our labours are the following inequalities:

$$|a_u^m|\leqslant C|u|^{n/2-1}\qquad\text{if }|u|\leqslant m,$$

$$|a_u^m|\leqslant Cm^{n/2}|u|^{-1}\qquad\text{if }|u|\geqslant m.$$

We may apply Theorem 2.1, taking $p$ to be 2 and $b(u)$ to be $a_u^m$. It follows that

$$\left\|\sup_{\varepsilon\in\mathbf{R}^+}\left[\sum_{j\in D_m}|(\phi_m)*f|^2\right]^{1/2}\right\|_2\leqslant C\log(2+m)\left\|\left[\sum_{j\in D_m}|k_{j,m}*f|^2\right]^{1/2}\right\|_2$$

$$\leqslant C\log(2+m)m^{-1}\|f\|_2,$$

by Lemma 1.1. This is the estimate (3) needed to finish the proof. □

## References


1. N. E. Aguilera, *Maximal operators associated to radial functions in* $L^2(\mathbf{R}^2)$, Proc. Amer. Math. Soc. **80** (1980), 283–286.

2. N. E. Aguilera and E. O. Harboure, *Some inequalities for maximal operators*, Indiana Univ. Math. J. **29** (1980), 559–576.

3. A. P. Calderón and A. Zygmund, *On a problem of Mihlin*, Trans. Amer. Math. Soc. **78** (1955), 209–224. See also Addenda to the paper "*On a problem of Mihlin*", Trans. Amer. Math. Soc. **84** (1957), 559–560.

4. ______, *On singular integrals with variable kernels*, Applicable Anal. **7** (1978), 221–238.

5. R. R. Coifman and G. Weiss, *Extension of Hardy spaces and their uses in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.

6. A. Córdoba, *The Kakeya maximal function and the spherical summation multipliers*, Amer. J. Math. **99** (1977), 1–22.

7. M. Cowling, *On Littlewood-Paley-Stein theory*, Rend. Circ. Mat. Palermo Suppl. **1** (1981), 1–20.

8. ______, *Harmonic analysis on semigroups*, Ann. of Math. (2) **117** (1983), 267–283.

9. N. Dunford and J. T. Schwartz, *Linear operators*. Part I: *General theory*, Interscience, New York and London, 1958.

10. C. Fefferman and E. M. Stein, *$H^p$ spaces of several variables*, Acta Math. **129** (1972), 137–193.

11. D. Goldberg, *A local Hardy space*, Duke Math. J. **46** (1979).

12. M. de Guzmán, *Differentiation of integrals in* $\mathbf{R}^n$, Lecture Notes in Math., Vol. 481, Springer-Verlag, Berlin and New York, 1975.

13. J. Marcinkiewicz and A. Zygmund, *Quelques inégalités pour les opérations linéaires*, Fund. Math. **32** (1939), 115–121.

14. ______, *Note on the class* $L\log L$, Studia Math. **31** (1969), 305–310.

15. ______, *Singular integrals and differentiability properties of functions*, Princeton Math. Series, no. 30, Princeton Univ. Press, Princeton, N.J., 1970.

16. ______, *Maximal functions: spherical means*, Proc. Nat. Acad. Sci. U.S.A. **73** (1976), 2174–2175.

17. E. M. Stein, *Some problems in harmonic analysis*, Proc. Sympos. Pure Math., vol. 35, Part 1, Amer. Math. Soc., Providence, R.I., 1979, pp. 3–20.

18. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Math. Series, no. 32, Princeton Univ. Press, Princeton, N.J., 1971.

19. J.-O. Strömberg, *Maximal functions associated to rectangles with uniformly distributed directions*, Ann. of Math. (2) **107** (1978), 399–402.

20. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th ed., reprinted, Cambridge Univ. Press, Cambridge, 1973.

SCHOOL OF MATHEMATICS, UNIVERSITY OF NEW SOUTH WALES, KENSINGTON, N.S.W. 2033, AUSTRALIA

ISTITUTO DI MATEMATICA, VIA L. B. ALBERTI 4, 16132 GENOVA, ITALY