

AN APPLICATION OF FLOWS TO TIME SHIFT AND TIME REVERSAL IN STOCHASTIC PROCESSES¹

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ABSTRACT. A simple proposition (Theorem 1) on flows allows the investigation of random time shift and time reversal in Markov processes without assuming any regularity of paths. Theorem 5 is a generalization of Nagasawa's time reversal theorem and Theorem 4 generalizes a recent result of Gettoor and Glover.

1. Flows.

1.1. We consider a flow in a measurable space (Ω, \mathcal{F}) that is a family of transformations θ_t , $t \in \mathbf{R}$, such that $\theta_s \theta_t = \theta_{s+t}$ for all $s, t \in \mathbf{R}$ and $\{(\omega, \omega): \theta_t \omega \in A\} \in \mathcal{B}_{\mathbf{R}} \times \mathcal{B}$ for all $A \in \mathcal{F}$ ($\mathcal{B}_{\mathbf{R}}$ is the Borel σ -algebra in \mathbf{R}).

We put $(\theta_t Y)(\omega) = Y(\theta_t \omega)$, for every function Y and

$$(P\theta_t)(A) = P(\theta_t^{-1}A), \quad A \in \mathcal{F},$$

$$\bar{P} = \int_{\mathbf{R}} P\theta_t dt,$$

for every measure P . We denote by \mathcal{A} the collection of all sets $A \in \mathcal{F}$ such that $\theta_t^{-1}A = A$ for all $t \in \mathbf{R}$. A function Y is measurable with respect to the σ -algebra \mathcal{A} if and only if it is \mathcal{F} -measurable and invariant under all transformations θ_t . Obviously $\bar{P}(A) = 0$ or $+\infty$ for all $A \in \mathcal{A}$. Nevertheless we prove

THEOREM 1. *If \bar{P} is σ -finite (over \mathcal{F}), it determines uniquely the values $P(A)$ for all $A \in \mathcal{A}$.*

1.2. As a tool we use stationary times. A *stationary time* τ is a measurable mapping from Ω to the extended real line $[-\infty, +\infty]$ such that $\theta_t \tau = \tau - t$ for all $t \in \mathbf{R}$.

Suppose that \mathcal{F} coincides with its completion with respect to the class of all probability measures. Then the first hitting time of a set $A \in \mathcal{F}$,

$$\tau_A^+ = \inf\{t: \theta_t \omega \in A\},$$

and the last exit time from A ,

$$\tau_A^- = \sup\{t: \theta_t \omega \in A\}$$

are stationary times² (the measurability follows from [5, Chapter 3, §1]).

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² We put $\inf \phi = +\infty$, $\sup \phi = -\infty$.

In general, every stationary time τ is the first hitting time of $\{\tau < 0\}$ and it is the last exit time from $\{\tau > 0\}$.

We denote the indicator function for the set $\{\tau \in \mathbf{R}\}$ by κ_τ . Let ρ be a positive Borel function on \mathbf{R} such that $\int \rho(t) dt = 1$. We put $\rho(-\infty) = \rho(+\infty) = 0$. We note that $\theta_t \rho(\tau) = \rho(\tau - t)$. Hence

$$\kappa_\tau = \int \rho(\tau - t) dt = \int \theta_t \rho(\tau) dt$$

and, for every \mathcal{A} -measurable Y ,

$$(1.1) \quad P\kappa_\tau Y = PY \int \theta_t \rho(\tau) dt = \int P\theta_t(Y\rho(\tau)) = \bar{P}\rho(\tau)Y.$$

1.3. Suppose that there exists a strictly positive function F such that

$$(1.2) \quad F_s = \int_{-\infty}^s \theta_t F dt \quad \text{is } P\text{-integrable for some } s \in \mathbf{R}.$$

Put

$$\tau_a = \inf\{s: F_s \geq a\}, \quad a > 0.$$

Note that $P\{\tau_a = -\infty\} = 0$ and $\{\tau_a = +\infty\} = \{F_\infty \leq a\}$. Hence $\kappa_{\tau_a} = 1_{F_\infty > a} \uparrow 1$ P -a.s. as $a \downarrow 0$. By (1.1),

$$(1.3) \quad PY = \lim_{a \downarrow 0} \bar{P}Y\rho(\tau_a).$$

The restriction of measure P to \mathcal{A} can be recovered from \bar{P} by formula (1.3). An analogous formula can be written if

$$(1.4) \quad F^s = \int_s^{+\infty} \theta_t F dt \quad \text{is } P\text{-integrable for some } s \in \mathbf{R}.$$

Both conditions (1.2) and (1.4) are satisfied if \bar{P} is σ -finite.

1.4. Let τ be a stationary time. For every function $Z(\omega)$ we denote by $\theta_\tau Z$ the function $Y(\omega)$ defined by the formula

$$Y(\omega) = \begin{cases} Z(\theta_{\tau(\omega)}\omega) & \text{if } \tau(\omega) \in \mathbf{R}, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that Y is invariant and therefore formulas (1.1) and (1.3) are applicable to Y . We conclude

If P_1 and P_2 are two measures on (Ω, \mathcal{F}) and if $\bar{P}_1 = \bar{P}_2$ is a σ -finite measure, then $P_1\theta_\tau = P_2\theta_\tau$ for all stationary times τ .

1.5. The definition of the measure \bar{P} makes sense if an arbitrary locally compact group G acts on (Ω, \mathcal{F}) (the Lebesgue measure on \mathbf{R} should be replaced by a left- or right-invariant measure on G). Formula (1.1) and its proof remain valid if τ is a G -valued function defined on an invariant set Ω_τ such that $\theta_g \tau(\omega) = \tau(\omega)g^{-1}$ for all $g \in G$. (We put $\kappa_\tau = 1_{\Omega_\tau}$ and we set $\rho(\tau) = 0$ if $\tau(\omega)$ is not defined.)

However the construction in subsection 1.3 is not applicable if $G \neq \mathbf{R}$. It remains an open problem for which groups G Theorem 1 is true.

2. Random time shift in Markov processes.

2.1. Let $p(s, x; t, B)$ be a Markov transition function in (E, \mathcal{E}) . We put

$$\mu T_t^s(B) = \int \mu(dx) p(s, x; t, B); \quad T_t^s f(x) = \int p(s, x; t, dy) f(y).$$

Suppose that a σ -finite measure ν_t is given for every $t \in \mathbf{R}$. We say that $\nu = \{\nu_t\}$ is an *entrance rule* if, for all $B \in \mathcal{E}$,

$$\nu_s T_t^s(B) \leq \nu_t(B) \quad \text{and} \quad \nu_s T_t^s(B) \uparrow \nu_t(B) \quad \text{as } s \uparrow t.$$

ν is an *entrance rule at time r* if $\nu_t = 0$ for $t \leq r$ and $\nu_s T_t^s = \nu_t$ for $r < s < t$.³ In an analogous way, we define an *entrance rule at time $-\infty$* . Every entrance rule has a representation

$$(2.1) \quad \nu = \int_{[-\infty, +\infty]} \nu^r \sigma(dr)$$

where ν^r is an entrance rule at time r . We note that $\sigma\{-\infty\} = 0$ if and only if $\nu_s T_t^s \downarrow 0$ as $t \uparrow \infty$ for some s (see [1, Formula (5.5)]).

A family $h = \{h^t, t \in \mathbf{R}\}$ of positive \mathcal{E} -measurable functions is called an *exit rule* if

$$T_t^s h^t \leq h^s \quad \text{and} \quad T_t^s h^t \uparrow h^s \quad \text{as } t \downarrow s.$$

All concepts related to entrance rules have natural analogs for exit rules.

The following result is due to Kuznetsov [4]. Suppose that (E, \mathcal{E}) is a standard Borel space and let an entrance rule μ and an exit rule h satisfy the condition

$$(2.2) \quad \mu_t(h^t = +\infty) = 0 \quad \text{for all } t.$$

Then there exists a stochastic process (X_t, P_μ^h) on a random time interval (α, β) such that, for every $t \in \mathbf{R}$,

$$(2.3) \quad P_\mu^h(\alpha < t, X_t \in dy, t < \beta) = \mu_t(dy) h^t(y),$$

and for all $t_1 < \dots < t_n \in \mathbf{R}$,

$$(2.4) \quad \begin{aligned} P_\mu^h(\alpha < t_1, X_{t_1} \in dy_1, \dots, X_{t_n} \in dy_n, t_n < \beta) \\ = \mu_{t_1}(dy_1) p(t_1, y_1; t_2, dy_2) \cdots p(t_{n-1}, y_{n-1}; t_n, dy_n) h^{t_n}(y_n). \end{aligned}$$

The measure P_μ^h is σ -finite.

2.2. Suppose that a transition function p is stationary. Excessive measures and functions can be characterized as entrance and exit rules independent of t . Let μ be an entrance rule. If the measure

$$\bar{\mu} = \int_{\mathbf{R}} \mu_t dt$$

is σ -finite, it is excessive. To every exit rule h there corresponds an excessive function

$$\bar{h} = \int_{\mathbf{R}} h^t dt.$$

³ In the literature, entrance rules at time 0 are called entrance laws.

We assume that the process $X_t(\omega)$ can be chosen in such a way that there exists a flow θ_t , $t \in \mathbf{R}$, with the properties: $\theta_t \alpha = \alpha - t$, $\theta_t \beta = \beta - t$ and $\theta_t X_u = X_{u+t}$.⁴

A random time shift of the process X_t is defined on the space $\Omega_\tau = \{\omega: -\infty < \tau < +\infty\}$ by the formula

$$(2.5) \quad \tilde{X}_t(\omega) = X_{\tau(\omega)+t}(\omega).$$

Suppose that an entrance rule μ and an exit rule h satisfy the conditions

$$(2.6) \quad \bar{\mu} \text{ is } \sigma\text{-finite and } \bar{h} < \infty \quad \bar{\mu}\text{-a.e.}$$

Then the conditions of Kuznetsov's theorem are satisfied for three pairs $(\bar{\mu}, \bar{h})$, $(\bar{\mu}, h)$ and (μ, \bar{h}) and therefore, the measures $P_{\bar{\mu}}^{\bar{h}}$, $P_{\bar{\mu}}^h$ and $P_{\mu}^{\bar{h}}$ are defined and σ -finite. We note that

$$\overline{P_{\bar{\mu}}^h} = \overline{P_{\bar{\mu}}^{\bar{h}}} = P_{\bar{\mu}}^{\bar{h}}.$$

By applying Theorem 1 and subsection 1.4, we get the following result:

THEOREM 2. *Measures $P_{\bar{\mu}}^h$ and $P_{\mu}^{\bar{h}}$ coincide on all invariant sets. If τ is a stationary time, then*

$$(2.7) \quad P_{\bar{\mu}}^h \theta_\tau = P_{\mu}^{\bar{h}} \theta_\tau.$$

The shift \tilde{X}_t of X_t defined by formula (2.5) has identical laws under $P_{\bar{\mu}}^h$ and $P_{\mu}^{\bar{h}}$.

3. Backward transition functions and duality. Theorem of Nagasawa and theorem of Gettoor and Glover.

3.1. Along with forward transition functions $p(s, x; t, dy)$ we consider *backward transition functions* $q(s, dx; t, y)$. We say that a forward transition function p and a backward transition function q are *m-related* if

$$(3.1) \quad m_s(dx) p(s, x; t, dy) = q(s, dx; t, y) m_t(dy) \quad \text{for all } s < t.$$

It follows from this relation that m is an entrance rule for both p and q .

If $g = \{g_t\}$ is an exit rule for q , then $(g \circ m)_t(dx) = g_t(x) m_t(dx)$ is an entrance rule for p , and all entrance rules for p which are absolutely continuous with respect to m can be represented in this form.

To every statement on forward transition functions there corresponds a statement on backward transition functions. The "backward" version of Kuznetsov's theorem is as follows. Let $\nu = \{\nu'\}$ be an entrance rule and $g = \{g_t\}$ be an exit rule for a backward transition function q . If

$$(3.2) \quad \nu'(g_t = +\infty) = 0 \quad \text{for all } t,$$

then there exists a stochastic process (X_t, Q_g^ν) on a random time interval (α, β) such that, for every $t \in \mathbf{R}$,

$$(3.3) \quad Q_g^\nu(\alpha < t, X_t \in dy, t < \beta) = g_t(y) \nu'(dy)$$

⁴ (The σ -algebra \mathcal{F} in Ω is generated by the sets $\{X_t \in B\}$, $t \in \mathbf{R}$, $B \in \mathcal{E}$.) Counterexamples show that Theorems 2 through 5 are not true without this assumption.

and, for all $t_1 < \dots < t_n \in \mathbf{R}$,

$$(3.4) \quad \begin{aligned} Q_g^\nu(\alpha < t_1, X_{t_1} \in dy_1, \dots, X_{t_n} \in dy_n, t_n < \beta) \\ = g_{t_1}(y_1) q(t_1, dy_1; t_2, y_2) \cdots q(t_{n-1}, dy_{n-1}; t_n, y_n) \nu^{t_n}(dy_n). \end{aligned}$$

The measure Q_g^ν is σ -finite.

Suppose that p and q are m -related and that h is an exit rule for p and g is an exit rule for q . It follows from (3.3) and (3.4) that

$$(3.5) \quad P_{g \circ m}^h = Q_g^{h \circ m}.$$

3.2. If a backward transition function q is stationary, then we write

$$q_t(dx, y) = q(t, dx; 0, y), \quad t < 0.$$

If p and q are both stationary and if $m_t = m$ for all t , then the condition (3.1) can be rewritten as follows

$$(3.6) \quad m(dx) p_t(x, dy) = q_{-t}(dx, y) m(dy).$$

Suppose that μ is an entrance rule for p , ν is an entrance rule for q and let $\bar{\mu}$ and $\bar{\nu}$ be σ -finite and absolutely continuous with respect to m . The densities $u = d\bar{\mu}/dm$ and $v = d\bar{\nu}/dm$ can be chosen to be q -excessive and p -excessive respectively. By (3.5),

$$\overline{P_\mu^\nu} = P_\mu^\nu = P_{u \circ m}^\nu = Q_u^{\nu \circ m} = Q_u^{\bar{\nu}} = \overline{Q_u^\nu},$$

and Theorem 1 and subsection 1.4 imply

THEOREM 3. *Measures P_μ^ν and Q_u^ν coincide on all invariant sets. For every stationary time τ , $P_\mu^\nu \theta_\tau = Q_u^\nu \theta_\tau$. The shifted process $\tilde{X}_t = X_{\tau+t}$ has identical laws under P_μ^ν and Q_u^ν .*

We write $(X_t, P) \cong (X'_t, P')$ if stochastic processes (X_t, P) and (X'_t, P') have identical laws (i.e., if they have the same finite-dimensional distributions).

3.3. Formula

$$(3.7) \quad \hat{p}_t(x, dy) = q_{-t}(dy, x)$$

determines a stationary forward transition function which is in weak duality with $p_t(x, dy)$ in the sense of Gettoor and Sharpe [3]. We note that $(\tilde{X}_{-t}, Q_u^{\nu*}) \cong (X_t, \hat{P}_\nu^u)$ where an entrance rule ν for \hat{p} and an entrance rule ν^* for q are related by the formula $(\nu^*)'_t = \nu_{-t}$. Analogously, $(u^*)_t = u^{-t}$. We can assume without any substantial loss of generality that there exists a measurable transformation r of the space (Ω, \mathcal{F}) such that $X_t(r\omega) = X_{-t}(\omega)$. If τ is a stationary time, then so is $\tau^*(\omega) = -\tau(\omega)$. Theorem 3 implies

THEOREM 4. *Let stationary transition functions p and \hat{p} be in weak duality relative to m . Let μ be an entrance rule for p and ν be an entrance rule for \hat{p} . Suppose that $\bar{\mu}$ and $\bar{\nu}$ are σ -finite and absolutely continuous with respect to m . Let u and v be excessive functions (for p and \hat{p} respectively) such that $\bar{\mu} = u \circ m$, $\bar{\nu} = v \circ m$. Then, for every stationary time τ , $(X_{\tau-t}, P_\mu^\nu) \cong (X_{\tau^*+t}, \hat{P}_\nu^u)$.*

This is a generalization of a theorem of Gettoor and Glover (see [2, Theorem (6.5)]) who have considered the situation when $\tau = \beta$,

$$\begin{aligned}\mu_t &= \mu_0 P_t & \text{for } t > 0, \\ \nu_t &= \nu_0 \hat{P}_t & \text{for } t > 0, \\ \mu_t &= \nu_t = 0 & \text{for } t \leq 0.\end{aligned}$$

In this situation $\beta^* = \alpha = 0$ \hat{P}_ν^u -a.s. and $(X_{\beta-t}, P_\mu^\nu) \cong (X_t, \hat{P}_\nu^u)$. Moreover

$$u = d(\mu_0 G)/dm, \quad v = d(\nu_0 \hat{G})/dm$$

where

$$G = \int T_t dt, \quad \hat{G} = \int \hat{T}_t dt.$$

(Actually, in [2] the process $X_{(\beta-t)^+}$ rather than $X_{\beta-t}$ is considered and therefore certain regularity conditions for paths are required.)

3.4. We say that a random time τ is a *renewal time* for a Markov process (X_t, P) if (\tilde{X}_t, P) where \tilde{X}_t is defined by formula (2.5) has the same transition function as (X_t, P) . If (X_t, P) is a strong Markov process with a stationary transition function, then every optional time is a renewal time.

Markov process (X_t, P_ν^u) corresponding by Kuznetsov's theorem to a stationary transition function p , an entrance rule ν and an excessive function u has the stationary transition function

$$\begin{aligned}p_t^u(x, dy) &= u(x)^{-1} p_t(x, dy) u(y) \quad \text{for } 0 < u(x) < \infty, \\ p_t^u(x, B) &= 1_B(x) \quad \text{for } u(x) = 0 \text{ or } \infty.\end{aligned}$$

We call p^u the u -transform of p .

The following result is an immediate implication of Theorem 4:

THEOREM 5. *Let $p, \hat{p}, \mu, \nu, u, v$ have the same meaning as in Theorem 4. If a stationary time τ is a renewal time for (X_t, \hat{P}_ν^u) then $(X_{\tau-t}, P_\mu^\nu)$ is a Markov process with a stationary transition function \hat{p}^ν (the ν -transform of \hat{p}).*

Theorem 5 is a generalization of Nagasawa's theorem [6].

4. A property of invariant measures.

4.1. Let P be an invariant measure for a flow θ_t . For every positive measurable Y we put

$$\bar{Y} = \int_{\mathbf{R}} \theta_t Y dt.$$

We note that if τ is a stationary time, then

$$\begin{aligned}(4.1) \quad P\kappa_\tau Y &= PY \int \theta_t \rho(\tau) dt = \int P\theta_t(\theta_{-\tau} Y \rho(\tau)) dt \\ &= P \int \theta_{-\tau} Y \rho(\tau) dt = P\bar{Y} \rho(\tau)\end{aligned}$$

(cf. (1.1)).

4.2. An invariant measure P is called *dissipative* if $\bar{Y} < \infty$ P -a.e. for every positive P -integrable Y , and it is called *conservative* if $\bar{Y} = 0$ or ∞ P -a.e. for every positive measurable Y . It is well known (see, e.g., [7, §V.5]) that every σ -finite invariant measure P can be represented as the sum of a dissipative measure P_d and a conservative measure P_c which are mutually singular.

THEOREM 6. *Let P be a dissipative invariant measure. If $\bar{Y}_1 = \bar{Y}_2$ P -a.e., then $PY_1 = PY_2$.*

Indeed, let F be a strictly positive P -integrable function. Since $\bar{F} < \infty$ P -a.e., the condition (1.2) is satisfied and there exist stationary times τ_a such that $\kappa_{\tau_a} \uparrow 1$ as $a \downarrow 0$. By (4.1),

$$PY_i = \lim_{a \downarrow 0} P\bar{Y}_i \rho(\tau_a), \quad i = 1, 2.$$

4.3. Let P be an arbitrary σ -finite invariant measure. There exists a partition of Ω into disjoint sets Ω_c and Ω_d such that $P_c(\Omega_d) = P_d(\Omega_c) = 0$. Let F be a P -integrable function which is strictly positive on Ω_c and equal to 0 on Ω_d . We have $\bar{F} = 0$ or ∞ P_c -a.e., $\bar{F} = 0$ P_d -a.e. and therefore $\bar{F} = 2\bar{F}$ P -a.e. On the other hand, $P(F) \neq P(2F)$ if $P_c \neq 0$. Therefore the statement of Theorem 6 is true *only* if P is dissipative.

REMARK. It follows from subsections 4.2 and 4.3 that a σ -finite invariant measure P is dissipative if and only if it satisfies condition (1.2) (or (1.4)).

4.4. A measure P_m^1 corresponding, by Kuznetsov's theorem, to a stationary transition function p , an excessive measure m and the exit rule $h = 1$ is invariant with respect to the flow θ . It is dissipative if and only if $Gf < \infty$, m -a.e. where G is defined in subsection 3.3.

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