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A WEIGHTED INEQUALITY FOR THE MAXIMAL BOCHNER-RIESZ OPERATOR ON $\mathbf{R}^2$

BY

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ABSTRACT. For $f \in \mathscr{S}(\mathbf{R}^2)$, let $(T_R^\alpha f)\hat{}(\xi) = (1 - |\xi|^2 R^2)_+^\alpha \hat{f}(\xi)$. It is a well-known theorem of Carleson and Sjölin that T_1^α defines a bounded operator on L^4 if $\alpha > 0$. In this paper we obtain an explicit weighted inequality of the form

$$\int \sup_{0<R<\infty} |T_R^\alpha f(x)|^2 w(x)\,dx \leqslant \int |f|^2 P_\alpha w(x)\,dx,$$

with P_α bounded on L^2 if $\alpha > 0$. This strengthens the above theorem of Carleson and Sjölin. The method gives information on the maximal operator associated to general suitably smooth radial Fourier multipliers of $\mathbf{R}^2$.

1. Introduction. For $f \in \mathscr{S}(\mathbf{R}^2)$ and $\alpha > 0$, let

$$(T_R^\alpha f)\hat{}(\xi) = \left(1 - |\xi|^2 R^2\right)_+^\alpha \hat{f}(\xi),$$

where $R \in \mathbf{R}^+$ and $\hat{}$ denotes the Fourier transform. Let

$$T_*^\alpha f(x) = \sup_{0<R<\infty} |T_R^\alpha f(x)|.$$

In [1] it was shown that $\|T_*^\alpha f\|_p \leqslant C_{p,\alpha}\|f\|_p$ for all $\alpha > 0$ provided that $2 \leqslant p \leqslant 4$ (a result which extended the theorem of Carleson and Sjölin [3]). In this note we establish a weighted inequality which provides a stronger form of the above result on the maximal Bochner-Riesz operator.

THEOREM 1. *Let $2 \leqslant q < \infty$ and $\alpha > 0$. Then there exists an operator $P = P_{q,\alpha}$ which is bounded on L^q such that*

$$\int |T_*^\alpha f(x)|^2 w(x)\,dx \leqslant \int |f(x)|^2 Pw(x)\,dx$$

whenever w is a nonnegative function in L^q.

Several comments are in order.

1. It suffices to prove the theorem in the case $q = 2$. For if $P_{2,\alpha}$ fulfills the requirements of the theorem in the case $q = 2$, $P_{q,\alpha}$, defined by $P_{q,\alpha}w = [P_{2,\alpha}(w^{q/2})]^{2/q}$, does so for L^q. That $P_{q,\alpha}$ is bounded on L^q is clear, and we may interpolate with change of measure between the estimates

$$\int |T_*^\alpha f|^2 w^j \leqslant \int |f|^2 (P_{2,\alpha}w)^j, \qquad j = 0, 1,$$

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to obtain the inequality of Theorem 1. (This observation was pointed out to the author by Professor J. Garnett.)

2. The proof of Theorem 1 is constructive—that is, an explicit formula for P_w in terms of w is given. Previously, Rubio de Francia [7] had been able to show that for each nonnegative w in L^2, there exists a nonnegative w' in L^2 with $\|w'\|_2 \leqslant C_\alpha \|w\|_2$ such that

$$\sup_{0<R<\infty} \int |T_R^\alpha f(x)|^2 w(x)\, dx \leqslant \int |f(x)|^2 w'(x)\, dx.$$

However, the proof gave no information on how to construct w' from w. Since an earlier version of this paper was prepared, it has come to the author's notice that Córdoba [5] has constructed from each nonnegative w in L^2 a nonnegative w' in L^2, with $\|w'\|_2 \leqslant C_\alpha \|w\|_2$, such that

$$\int |T_1^\alpha f(x)|^2 w(x)\, dx \leqslant \int |f(x)|^2 w'(x)\, dx.$$

The definition of w' given by Córdoba was not invariant under dilations, so the full strength of Rubio's inequality was not obtained.

3. It would be of some interest to establish a version of Theorem 1 in which the operator P did not depend upon q. In particular, the conjecture (see [**8**, p. 7]) that P may be realised as (essentially) $\sum_{k=0}^{\infty} 2^{-k\alpha} M_2^{k/2}$ remains unsolved. Here, as subsequently, M_N denotes the maximal function corresponding to averages over rectangles in N uniformly distributed directions in $\mathbf{R}^2$. The relevant fact about M_N here is that $\|M_N f\|_2 \leqslant C(\log 3N)^\beta \|f\|_2$, where C and β are absolute constants. See Córdoba [**4**]. On the other hand, it can be shown that the operator $P_{2,\alpha}$ of Theorem 1 satisfies $\|P_{2,\alpha} w\|_q \leqslant C_{q,\alpha} \|w\|_q$ when $2 \leqslant q \leqslant 4$. See the remark at the end of §2.

Theorem 1 follows from Theorem 2 below. For $\alpha > \frac{1}{2}$ let $\hat{\mathcal{O}}^\alpha(\xi) = (1 - |\xi|)_+^{\alpha-1}|\xi|$. Let

$$\mathcal{G}^\alpha(f)^2(x) = \int_0^\infty |\mathcal{O}_t^\alpha * f(x)|^2\, dt/t,$$

where if ψ is a function defined on $\mathbf{R}^2$, $\psi_t(x) = t^{-2}\psi(x/t)$.

THEOREM 2. *If $\alpha > \frac{1}{2}$ then there exists an operator P bounded on L^2 such that*

$$\int \mathcal{G}^\alpha(f)^2(x) w(x)\, dx \leqslant C_\alpha \int |f(x)|^2 Pw(x)\, dx$$

whenever w is a nonnegative function in L^2.

In fact, Theorem 2 leads to a generalisation of Theorem 1. For $\alpha > \frac{1}{2}$ let $\mathcal{L}_\alpha^2$ denote the usual space of Bessel potentials on $\mathbf{R}$ (see, for example, [**9**, Chapter 5]) transformed under the change of variables $s \mapsto \exp s$. Thus, if m is defined on $(0, \infty)$, $\|m\|_{\mathcal{L}_\alpha^2} = m\|(\log(\cdot))\|_{L_\alpha^2}$. If we let

$$(S_R^m f)\hat{\ }(\xi) = m(|\xi| R)\hat{f}(\xi) \quad \text{and} \quad S_*^m f(x) = \sup_{0<R<\infty} |S_R^m f(x)|,$$

we have the following inequality [**2**], valid when $\alpha > \frac{1}{2}$:

$$S_*^m f(x) \leqslant C_\alpha \|m\|_{\mathcal{L}_\alpha^2} \mathcal{G}^\alpha(f)(x).$$

Consequently, we obtain Theorem 3.

THEOREM 3. *Let $\alpha > \frac{1}{2}$. Then there exists an operator P bounded on L^2 such that if m is a radial function on $\mathbf{R}^2$ whose restriction to $(0, \infty)$ lies in $\mathscr{L}^2_\alpha$, then*

$$\int |S_*^m f(x)|^2 w(x)\,dx \leqslant C_\alpha \|m\|_{\mathscr{L}^2_\alpha} \int |f(x)|^2 P w(x)\,dx$$

whenever w is a nonnegative function in L^2.

Theorem 1 may be deduced from Theorem 3 as follows. Let

$$\left(1-|\xi|^2\right)_+^\lambda = \phi_0(|\xi|) + \left[\left(1-|\xi|^2\right)_+^\lambda - \phi_0(|\xi|)\right],$$

where ϕ_0 is a C^∞ function of compact support in $[-\frac{3}{4}, \frac{3}{4}]$, agreeing with $(1-r^2)_+^\lambda$ on $[-\frac{1}{2}, \frac{1}{2}]$. Then the maximal operator corresponding to ϕ_0 is dominated by the Hardy-Littlewood maximal function, while $(1-r^2)_+^\lambda - \phi_0(r)$ belongs to $\mathscr{L}^2_\alpha$ if $\alpha < \lambda + \frac{1}{2}$.

Theorem 2 follows from the following lemma, proved in §2 below.

LEMMA. *Let Φ be a smooth real-valued bump function supported in $[-1, 1]$. Let $\phi(\xi) = \Phi((|\xi| - 1)/\delta)$ for small $\delta > 0$. Let $\hat{\psi} = \phi$. Then there exists an operator $Q = Q_\delta$ such that*

(i)

$$\int_0^\infty \int |\psi_t * f(x)|^2 w(x)\,dx\,\frac{dt}{t} \leqslant \int |f(x)|^2 Q_\delta w(x)\,dx$$

whenever w is a nonnegative test function, and

(ii)

$$\|Q_\delta w\|_2 \leqslant C\delta(\log(3/\delta))^\beta \|w\|_2.$$

(Here and subsequently, β will denote a positive absolute constant, and C will be a positive constant depending only possibly on $\max_{0 \leqslant j \leqslant 3} \|\Phi^{(j)}\|_\infty$; C and β may not be the same at each occurrence.)

To obtain Theorem 2 from this lemma, we merely write

$$\hat{\mathcal{O}}^\alpha(\xi) = (1-|\xi|)_+^{\alpha-1}|\xi| = \sum_{k=0}^\infty 2^{-k(\alpha-1)}\phi_k(|\xi|),$$

with ϕ_k smooth, supported in $[1-2^{-k}, 1-2^{-k-2}]$ and satisfying $\|\phi_k^{(j)}\|_\infty \leqslant C2^{kj}$, and letting $\hat{\psi}^k(\xi) = \phi_k(|\xi|)$, we observe that if $\alpha > 1/2$,

$$\begin{aligned}
\int \mathscr{G}^\alpha(f)^2 w &= \int \int_0^\infty |\mathcal{O}_t^\alpha * f(x)|^2 w(x)\,\frac{dt}{t}\,dx \\
&= \int \int_0^\infty \left| \left(\sum_{k=0}^\infty 2^{-k(\alpha-1)} \psi_t^k \right) * f(x) \right|^2 w(x)\,\frac{dt}{t}\,dx \\
&\leqslant \left\{ \sum_{k=0}^\infty 2^{-k(\alpha-1)} \left(\int \int_0^\infty |\psi_t^k * f(x)|^2 w(x)\,\frac{dt}{t}\,dx \right)^{1/2} \right\}^2 \\
&\leqslant \left\{ \sum_{k=0}^\infty 2^{-k(\alpha-1)} \left(\int |f(x)^2 Q_{2^{-k}} w(x)\,dx \right)^{1/2} \right\}^2 \\
&\leqslant \frac{C}{(\alpha-1/2)} \int |f(x)|^2 \sum_{k=0}^\infty 2^{-k(\alpha-3/2)} Q_{2^{-k}} w(x)\,dx.
\end{aligned}$$

Thus P is realised as $\sum_{k=0}^{\infty} 2^{-k(\alpha-3/2)} Q_{2^{-k}}$, and

$$\|Pw\|_2 \leqslant C \sum_{k=0}^{\infty} 2^{-k(\alpha-3/2)} 2^{-k} k^{\beta} \|w\|_2 \leqslant C\|w\|_2.$$

2. Proof of the lemma. From now on we consider Φ and δ, hence ϕ, to be fixed. We assume $\delta = 1/N^2$, where N is a power of 2.

We need to recall that in **[1]** we constructed a covering of $\mathbf{R}^2 - \{0\}$ by rectangles $\{S_k^j\}$ such that $\sum \chi_{S_k^j} \leqslant 25$, together with a partition of unity $\{\beta_k^j\}$ subordinate to that covering. If the distance between the centre of S_k^j and the origin is d, then S_k^j has sidelengths comparable to δd and $\delta^{1/2} d$, and is oriented so that the direction of the longer side is approximately perpendicular to the line joining the origin to the centre of S_k^j. Thus S_k^j lies in an annulus A_k centred at the origin of width approximately $\delta 2^{k\delta}$ and subtends an angle of approximately $\delta^{1/2}$ at the origin. In each large annulus $\{2^l \leqslant |\xi| \leqslant 2^{l+1}\}$, $l \in \mathbf{Z}$, there are $1/\delta$ smaller annuli A_k. Let γ_k^j be a smooth multiplier supported in $2S_k^j$ (the rectangle with the same centre and orientation as S_k^j but twice the sidelengths) and satisfying the same estimates as β_k^j, and let $(B_k^j f)\hat{}(\xi) = \gamma_k^j(\xi)\hat{f}(\xi)$. Let $\hat{\rho}(\xi_1, \xi_2)$ be a smooth bump function of ξ_1, supported in $[\frac{1}{2}, 4] \cup [-4, -\frac{1}{2}]$ and identically one on $[1,2] \cup [-2,-1]$, and let $g(f)(x) = (\sum_{k\in\mathbf{Z}} |\rho_{2^k} * f(x)|^2)^{1/2}$. The proof of Proposition 4 of **[1]** shows us that

$$\int \sum_{j,k} |B_k^j f|^2 w \leqslant \frac{C}{(s-1)^{\beta}} \int g(f)^2 \left(M_1^2 M_N w^s\right)^{1/s} \tag{1}$$

whenever $s > 1$. (In that proof M_N appeared raised to the third power, but since we are using *smooth* cutoff functions γ_k^j, we can dispose of two of these powers.) The operator g, being a vector-valued singular integral, satisfies the inequality

$$\int g(f)^2 u \leqslant \frac{C}{(s-1)^{\beta}} \int |f|^2 (Mu^s)^{1/s} \tag{2}$$

for all $s > 1$, since $(Mu^s)^{1/s}$, $s > 1$, is an A_1 weight with constant not exceeding $C/(s-1)^{\beta}$. (Here M is the Hardy-Littlewood maximal function which is dominated by the strong maximal function M_1.) Combining (1) and (2) yields

$$\int \sum_{j,k} |B_k^j f|^2 w < \frac{C}{(s-1)^{\beta}} \int |f|^2 \left(M_1^3 M_N w^s\right)^{1/s} \tag{3}$$

for $s > 1$.

Construction of Q. For f and w in the Schwartz class, say, and w nonnegative, we see that

$$\begin{aligned}
&\int_0^{\infty} \int |\psi_t * f(x)|^2 w(x)\, dx\, \frac{dt}{t} \\
&\quad = \int_0^{\infty} \int \psi_t * f(x)\, \overline{\psi_t * f(x)}\, w(x)\, dx\, \frac{dt}{t} \\
&\quad = \int_0^{\infty} \int \sum_{\substack{j,j'\\k,k'}} T_{kt}^j f(x)\, \overline{T_{k't}^{j'} f(x)}\, w(x)\, dx\, \frac{dt}{t} \\
&\quad = \int_0^{\infty} \int \sum_{\substack{j,j'\\k,k'}} T_{kt}^j f(x)\, \overline{T_{k't}^{j'} f(x)}\; \overline{R_{kk'}^{jj'} w(x)}\, dx\, \frac{dt}{t},
\end{aligned}$$

where $(T_{kt}^j f)\hat{}(\xi) = \phi(t\xi)\beta_k^j(\xi)\hat{f}(\xi)$, and $(R_{kk'}^{jj'}w)\hat{}(\xi) = \hat{w}(\xi)$ if $\xi \in S_k^j - S_{k'}^{j'}$. (We have left a certain latitude in the definition of $R_{kk'}^{jj'}$ which we exploit later.) Observe now that, for a given t, there are at most three consecutive values of k for which T_{kt}^j is not the zero operator for all j. Applying Parseval's relation once more, and the Cauchy-Schwarz inequality in j, j', k, k', we have

$$\begin{aligned}
&\int_0^\infty \int |\psi_t * f(x)|^2 w(x)\, dx \frac{dt}{t} \\
&\quad = \int_0^\infty \int \sum_{k,k'} \sum_{|j-j'|\geqslant 2} T_{kt}^j \overline{T_{k't}^{j'} f}\, \overline{R_{kk'}^{jj'} w}\, dt \frac{dt}{t} \\
&\qquad + \int_0^\infty \int \sum_{k,k'} \sum_{|j-j'|\leqslant 1} T_{kt}^j f \overline{T_{k't}^{j'} f}\, w\, dx \frac{dt}{t} \\
&\quad \leqslant 3\int_0^\infty \int \Bigg(\sum_{\substack{j,j' \\ k,k'}} \left|T_{kt}^j f T_{k't}^{j'} f\right|^2 \Bigg)^{1/2} \Bigg(\sup_{|k-k'|\leqslant 2} \sum_{|j-j'|\geqslant 2} \left|R_{kk'}^{jj'} w\right|^2 \Bigg)^{1/2} dx \frac{dt}{t} \\
&\qquad + 9\int_0^\infty \int \sum_{j,k} \left|T_{kt}^j f\right|^2 w\, dx \frac{dt}{t} \\
&\quad = 3\int_0^\infty \int \sum_{j,k} \left|T_{kt}^j f(x)\right|^2 Aw(x)\, dx \frac{dt}{t},
\end{aligned}$$

where

$$Aw(x) = \Bigg(\sup_{|k-k'|\leqslant 2} \sum_{|j-j'|\geqslant 2} \left|R_{kk'}^{jj'} w(x)\right|^2 \Bigg)^{1/2} + 3w(x).$$

Now the dt/t measure of $\{t | T_{kt}^j \neq 0\}$ is dominated by δ for each fixed (k, j), and integrating by parts as in **[1]** shows that $|T_{kt}^j f(x)| \leqslant CL_k^j * |f|(x)$, with C independent of t and depending only on $\max_{0\leqslant j\leqslant 3}\|\Phi^{(j)}\|_\infty$. Here, $\int L_k^j \leqslant 1$ and $L_k^j * |f|(x) \leqslant CM_N f(x)$. Let γ_k^j be a smooth bump function supported in $2S_k^j$ and identically one on S_k^j, so that if $(B_k^j f)\hat{}(\xi) = \gamma_k^j(\xi)\hat{f}(\xi)$ then $T_{kt}^j f(x) = T_{kt}^j B_k^j f(x)$. Thus,

$$\begin{aligned}
&\int_0^\infty \int |\psi_t * f(x)|^2 w(x)\, dx \frac{dt}{t} \\
&\qquad \leqslant C\delta \int \sum_{j,k} \left(L_k^j * |B_k^j f|(x) \right)^2 Aw(x)\, dx \\
&\qquad \leqslant C\delta \int \sum_{j,k} \left(L_k^j * |B_k^j f|^2 \right)(x) Aw(x)\, dx \\
&\qquad = C\delta \int \sum_{j,k} |B_k^j f|^2(x) L_k^j * Aw(x)\, dx \\
&\qquad \leqslant C\delta \int \sum_{j,k} |B_k^j f|^2 M_N Aw \\
&\qquad \leqslant \frac{C\delta}{(s-1)^\beta} \int |f|^2 \left\{ M_1^3 M_N (M_N Aw)^s \right\}^{1/s}
\end{aligned}$$

for each $s > 1$, by (3). We now choose $s = 1 + 1/\log(3/\delta)$, so that

$$\int_0^\infty \int |\psi_t * f(x)|^2 w(x)\, dx \frac{dt}{t}$$
$$\leqslant \int |f|^2 \left[C\delta \left(\log \frac{3}{\delta}\right)^\beta \left\{ M_1^3 M_N (M_N A w)^{1+(\log 3/\delta)^{-1}} \right\}^{1/(1+\log 3/\delta)^{-1}} \right]$$

This completes our construction of Q satisfying condition (i) of the lemma.

Boundedness of Q. A very simple interpolation argument shows that since M_N satisfies

$$\|M_N f\|_2 \leqslant C(\log 3N)^\beta \|f\|_2 \quad \text{and} \quad \|M_N f\|_{3/2} \leqslant CN\|f\|_{3/2},$$

then M_N is bounded on L^p with $p = 2/(1 + (\log(3/\delta))^{-1})$ with norm still no larger than $O((\log 3N)^\beta)$. To complete the proof of the lemma, then, it remains to show that $\|Aw\|_2 \leqslant C(\log 3/\delta)^\beta \|w\|_2$, to which task we now turn ourselves; without loss of generality it suffices to obtain a similar estimate for

$$A'w(x) = \left(\sup_k \sum_{|j-j'| \geqslant 2} \left| R_k^{jj'} w(x) \right|^2 \right)^{1/2},$$

where, for notational simplicity, we have written $R_{kk}^{jj'}$ as $R_k^{jj'}$.

For $|j - j'| \geqslant 2$, let

$$r_k(j, j') = \sup\{ |\xi - \xi'| : \xi, \xi' \in S_k^j - S_k^{j'}, \text{with } \xi \text{ and } \xi' \text{ lying on the same line segment passing through the origin} \},$$

and let

$$t_k(j, j') = \sup\{ |\xi - \xi'| : \xi, \xi' \in S_k^j - S_k^{j'}, \text{with } \xi \text{ and } \xi' \text{ on the same line segment which is tangent to a circle centred at the origin} \}.$$

(These are the 'radial' and 'tangential' lengths of $S_k^j - S_k^{j'}$.) Let the eccentricity $e_k(j, j')$ of $S_k^j - S_k^{j'}$ be $t_k(j, j')/r_k(j, j')$. For each $k \in \mathbf{Z}$ and each $m \in \mathbf{Z}$ let $\mathscr{B}^m = \{(j, j') | 2^m < e_k(j, j') \leqslant 2^{m+1}\}$. Notice that $\mathscr{B}^m$ is independent of k. Observe that the eccentricities range between $1/N$ and N; thus $\mathscr{B}^m$ is nonempty only when $|m| \leqslant \log N$. Now

$$\int \left\{ \sup_{(j,j')} \left| R_k^{jj'} w \right|^2 \right\} dx = \int \sup_k \left\{ \sum_m \sum_{(j,j') \in \mathscr{B}^m} \left| R_k^{jj'} w \right|^2 \right\} dx$$
$$\leqslant C(\log N)^\beta \max_{|m| \leqslant \log N} \int \sup_k \left\{ \sum_{(j,j') \in \mathscr{B}^m} \left| R_k^{jj'} w \right|^2 \right\} dx.$$

Thus it is sufficient to obtain a bound for

$$\int \sup_k \left\{ \sum_{(j,j') \in \mathscr{B}^m} \left| R_k^{jj'} w \right|^2 \right\} dx$$

which is independent of m.

For each m, then, let $\mathscr{G}_\lambda^m = \{A_\lambda | \lambda \in \Lambda_m\}$ be the collection of 'rectangles' formed by the concentric circles $\{|\xi| = 2^{p/2^m N}\}_{p \in \mathbf{Z}}$ and the uniformly distributed rays $\{\arg \xi = 2\pi p/N\}, p = 0, 1, \ldots, N-1$. Let $(A_\lambda g)\hat{}(\xi) = \chi_{A_\lambda}(\xi)\hat{g}(\xi)$.

We now make use of our freedom in defining $R_k^{jj'}w$. If $(j, j') \in \mathscr{B}^m$, let $(R_k^{jj'}w)\hat{}(\xi) = \Sigma\chi_{A_\lambda}(\xi)\hat{w}(\xi)$, the sum being taken over all $\lambda \in \Lambda_m$ such that $A_\lambda \cap (S_k^j - S_k^{j'}) \neq \varnothing$. By the construction of $\{A_\lambda\}$, there will be at most four nonzero terms in this sum, and C. Fefferman's observation [6] that for a given k, the sets $\{S_k^j - S_k^{j'}\}_{j,j'}$ are "almost disjoint" in the sense that $\Sigma_{j,j'}\chi_{S_k^j - S_k^{j'}} \leqslant C$ yields

$$\sum_{(j,j') \in \mathscr{B}^m} \left|R_k^{jj'}w\right|^2 \leqslant C \sum_{\lambda \in \Lambda_m} \left|A_\lambda w\right|^2$$

for all k. If we now take the supremum over $k \in \mathbf{Z}$, we obtain the desired result since

$$\left\| \left(\sum_{\lambda \in \Lambda_m} |A_\lambda w|^2 \right)^{1/2} \right\|_2 = \|w\|_2,$$

$\Sigma\chi_{A_\lambda}$ being identically equal to one almost everywhere on $\mathbf{R}^2$.

REMARK. The above computations concerning the boundedness of Q show that

$$\|Q_\delta w\|_q \leqslant C\delta(\log(3/\delta))^\beta \|w\|_q$$

will hold provided that $q \geqslant 2$ and

$$\left\| \left(\sum |A_\lambda w|^2 \right)^{1/2} \right\|_q \leqslant C(\log N)^\beta \|w\|_q$$

with A_λ as above and $|m| \leqslant \log N$. In the case $q = 2$ this is trivial; however, it remains true when $2 \leqslant q \leqslant 4$. The endpoint case $m = \log N$ is in [1] (in fact inequality (3) above is the required estimate) and the case $m = -\log N$ is implicit in the article of Córdoba in the proceedings of the conference in harmonic analysis in honour of Antoni Zygmund held at Chicago (W. Beckner et al., eds., Wadsworth, 1982). The other cases are handled similarly; the middle case $m = 0$ is substantially simpler.

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