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THE MACKEY TOPOLOGY AND COMPLEMENTED SUBSPACES OF LORENTZ SEQUENCE SPACES $d(w,p)$ FOR $0<p<1$

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ABSTRACT. In this paper we continue the study of Lorentz sequence spaces $d(w,p)$, $0 < p < 1$, initiated by N. Popa [**8**]. First we show that the Mackey completion of $d(w,p)$ is equal to $d(v,1)$ for some sequence v. Next, we prove that if $d(w,p) \not\subset l_1$, then it contains a complemented subspace isomorphic to l_p. Finally we show that if $\lim n^{-1}(\sum_{i=1}^{n} w_i)^{1/p} = \infty$, then every complemented subspace of $d(w,p)$ with symmetric bases is isomorphic to $d(w,p)$.

I. Introduction. A *p-norm*, $0 < p \leq 1$, on a vector space X is a map $x \mapsto \|x\|$ such that:

1. $\|x\| > 0$ if $x \neq 0$.
2. $\|tx\| = |t|\,\|x\|$ for all $x \in X$ and all scalars t.
3. $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$.

Let $B = \{x \in X \colon \|x\| \leq 1\}$; then the family $\{rB\}_{r>0}$ is a base of neighbourhoods of zero for a Hausdorff locally bounded vector topology on X (see [**9**]). If X is complete, we say that X is a *p-Banach space.*

The *Mackey topology* μ of a locally bounded space X with separating dual is the strongest locally convex topology on X which is weaker than the original one (see [**10**]). It is easy to see that this normable topology is generated by neighbourhoods $\{r\,\overline{\operatorname{conv}}\,B\}_{r>0}$. The Minkowski functional of the set $\overline{\operatorname{conv}}\,B$ is called the *Mackey norm* on X. The completion of the space (X,μ) is called the *Mackey completion* of X and denoted by $\hat{X}$. The extension of the Mackey norm to $\hat{X}$ is denoted by $\|\cdot\|\hat{}$.

For every subset E of ω (= the space of all scalar sequences) we denote

$$E^{+} = \{x = (x_i) \in E \colon x_i \geq 0 \text{ for } i = 1, 2, \ldots\}$$

and

$$E^{++} = \{x \in E^{+} : x \text{ is nonincreasing}\}.$$

Let $0 < p < \infty$ and let $w = (w_i) \in l_\infty^{++} \backslash l_1$. For $x = (x_i) \in \omega$ we define

$$\|x\|_{w,p} = \sup_{\pi} \left(\sum_{i=1}^{\infty} |x_{\pi(i)}|^p w_i \right)^{1/p},$$

where π ranges over all permutations of the positive integers. The space $d(w,p) = \{x \in \omega \colon \|x\|_{w,p} < \infty\}$ equipped with the locally bounded vector topology induced by $\|\cdot\|_{w,p}$ is called the *Lorentz sequence space.*

It is well known that $d(w,p)$ is a p-Banach space for $0 < p < 1$ and a Banach space for $p \geq 1$. Moreover, the sequence of unit vectors (e_i) is a symmetric basis of

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$d(w,p)$. From the assumption $w \in l_\infty^{++}\backslash l_1$ follows that $d(w,p) \subset c_0$. Therefore for every $x = (x_i) \in d(w,p)$ there exists a nonincreasing rearrangement $x^* = (x_i^*)$ of x (i.e. a nonincreasing sequence obtained from $(|x_i|)$ by a suitable permutation of the integers) and $\|x\|_{w,p} = (\sum_{i=1}^\infty x_i^{*p} w_i)^{1/p}$.

Observe that $d(w,p) \approx l_p$ if and only if $w \notin c_0$ (cf. [**6**, p. 176]).

The first topic of the present paper is the Mackey topology of $d(w,p)$, $0 < p < 1$. Using a representation of the dual of $d(w,p)$, N. Popa [**8**] proved that the Mackey completion of $d(w,p)$ ($p = 1/k$, $k \in \mathbf{N}$, and w satisfies some additional conditions) is isomorphic to $d(v,1)$ for a suitable sequence v. In §3 we show that the above theorem holds for any Lorentz sequence space $d(w,p)$, $0 < p < 1$. Our result is obtained without determining any dual space.

The last part of our paper is devoted to the study of complemented subspaces of $d(w,p)$, $0 < p < 1$.

It is well known that every Lorentz sequence space $d(w,p)$, $p \geq 1$, has complemented subspace isomorphic to l_p (see [**6**, Proposition 4.e.3]). N. Popa [**8**] showed that unlike the case $p \geq 1$ there are spaces $d(w,p)$, $0 < p < 1$, which contain no complemented subspaces isomorphic to l_p and conjectured that it is true for each $d(w,p)$, $0 < p < 1$. In §4 we prove that if $\inf_n n^{-1}(\sum_{i=1}^n w_i)^{1/p} = 0$ (i.e. $d(w,p) \not\subset l_1$, see Proposition 1), then $d(w,p)$ has complemented subspace isomorphic to l_p. Moreover, if $\lim_{n\to\infty} n^{-1}(\sum_{i=1}^n w_i)^{1/p} = \infty$, then every complemented subspace of $d(w,p)$ with symmetric basis is isomorphic to $d(w,p)$.

Throughout the paper we denote by $B_{w,p}$ the closed unit ball in $d(w,p)$, $\mathbf{R}^n = \operatorname{span}\{e_i\}_{i=1}^n$, $B_{w,p}^n = B_{w,p} \cap \mathbf{R}^n$, $n = 1,2,\dots$. In addition we denote $S_n(x) = x_1 + \cdots + x_n$, $n = 1,2,\dots$, $S_0(x) = 0$ for any sequence $x = (x_i) \in \omega$.

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II. Technical results. In this section we assume that $0 < p \leq 1$, $w = (w_i) \in l_\infty^{++}\backslash l_1$, $\sigma_k = S_k(w)^{1/p}$, $f_k = \sigma_k^{-1}\sum_{i=1}^k e_i$ for $k = 1,2,\dots$, and $f_0 = 0$.

LEMMA 1. *Let $\|\cdot\|_n$ be the norm on $\mathbf{R}^n$ defined by*

$$\|x\|_n = \sum_{i=1}^n |x_i|(\sigma_i - \sigma_{i-1}) \quad \textit{for } x = (x_i) \in \mathbf{R}^n,$$

and let

$$B^n = \{x = (x_i) \in \mathbf{R}^n \colon \|x\|_n \leq 1\}, \qquad n \in \mathbf{N}.$$

Then:

(a) $(B^n)^{++} = \operatorname{conv}\{f_k \colon k = 0,1,\dots,n\}$.

(b) $(B_{w,p}^n)^{++} \subset (B^n)^{++}$.

(c) *Let $0 < p < 1$ and $x = (x_i) \in (\mathbf{R}^n)^{++}$. Then $\|x\|_n = \|x\|_{w,p} = 1$ if and only if $x = f_k$ for some $k = 1,2,\dots,n$.*

(d) *If $p = 1$, then $(B_{w,p}^n)^{++} = (B^n)^{++}$.*

PROOF. (a) Every point $x \in \mathbf{R}^n$ may be written in the form

$$x = \sum_{i=1}^{n-1} \sigma_i(x_i - x_{i+1})f_i + \sigma_n x_n f_n.$$

In addition, for each $x \in (\mathbf{R}^n)^+$,

$$\|x\|_n = \sum_{i=1}^{n-1} \sigma_i(x_i - x_{i+1}) + \sigma_n x_n.$$

Therefore every $x \in (\mathbf{R}^n)^{++}$ with $\|x\|_n = 1$ is a convex combination of the vector $f_1, \dots, f_n$. It implies $(B^n)^{++} \subset \operatorname{conv}\{f_0, f_1, \dots, f_n\}$.

We observe that $\|f_k\|_{w,p} = \|f_k\|_n = 1$ for $k = 1, 2, \dots, n$. Both the sets $(\mathbf{R}^n)^{++}$ and B^n are convex, so

$$\operatorname{conv}\{f_0, \dots, f_k\} \subset B^n \cap (\mathbf{R}^n)^{++} = (B^n)^{++}.$$

(b) By the proof of (a) and by concavity of the function $x \mapsto \|x\|_{w,p}^p$ on $(\mathbf{R}^n)^{++}$,

$$\|x\|_{w,p}^p \geq \sum_{i=1}^{n-1} \sigma_i(x_i - x_{i+1})\|f_i\|_{w,p}^p + \sigma_n x_n \|f_n\|_{w,p}^p = \|x\|_n$$

$$\text{for } x \in (\mathbf{R}^n)^{++}, \ \|x\|_n = 1.$$

Thus (b) follows from homogeneity of the functionals $\|\cdot\|_n$ and $\|\cdot\|_{w,p}$.

(c) Since the function $x \mapsto \|x\|_{w,p}^p$ is strictly concave on $(\mathbf{R}^n)^{++}\setminus\{0\}$, $0 < p < 1$, the assertion (c) is clear.

(d) It is enough to observe that $\|x\|_{w,1} = \|x\|_n$ for every $x \in (\mathbf{R}^n)^{++}$.

COROLLARY 1. *If $y = (y_i) \in (\mathbf{R}^n)^+$ and $S_k(y) \leq \sigma_k$ for $k = 1, \dots, n$, then*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p w_i\right)^{1/p} \quad \textit{for every } x = (x_i) \in (\mathbf{R}^n)^{++}.$$

PROOF. Corollary 1 follows immediately from Lemma 1(b).

COROLLARY 2. *For every $x = (x_i) \in (\mathbf{R}^n)^{++}$,*

$$\left(\sum_{i=1}^{n-1} x_i^p w_i\right)^{1/p} + (\sigma_n - \sigma_{n-1})x_n \leq \left(\sum_{i=1}^{n} x_i^p w_i\right)^{1/p}.$$

PROOF. It suffices to apply Corollary 1 with $\tilde{w}_1 = S_{n-1}(w)$, $\tilde{w}_2 = w_n$, $\tilde{x}_1 = (\sum_{i=1}^{n-1} x_i^p w_i)^{1/p}\sigma_{n-1}$, $\tilde{x}_2 = x_n$, $y_1 = \sigma_{n-1}$, and $y_2 = \sigma_n - \sigma_{n-1}$.

PROPOSITION 1. *Let $0 < p \leq 1$, $w = (w_i)$ and $v = (v_i)$ belong to $l_\infty^{++}\setminus l_1$. Then*

$$d(w,p) \subset d(v,1) \quad \textit{if and only if} \ \inf_n \frac{S_n^{1/p}(w)}{S_n(v)} > 0.$$

In particular

$$d(w,p) \subset l_1 \quad \textit{if and only if} \ \inf_n n^{-1} S_n^{1/p}(w) > 0.$$

PROOF. If $d(w,p) \subset d(v,1)$, then, by the closed graph theorem, the inclusion map is continuous. Moreover, $\|f_n\|_{w,p} = 1$ for $n = 1, 2, \dots$. Thus

$$\sup_n \|f_n\|_{v,1} = \sup_n \frac{S_n(v)}{S_n^{1/p}(w)} < +\infty.$$

If $\inf_n S_n^{1/p}(w)/S_n(v) > 0$, then, by Corollary 1, $d(w,p) \subset d(v,1)$.

LEMMA 2. *Let* $\inf_n \sigma_n/n = 0$. *Then there exist an increasing sequence of integers* (n_k) *and a sequence of positive numbers* $q = (q_n) \in c_0$ *such that:*

(a) $S_n(q) \le \sigma_n$ *for* $n = 1, 2, \ldots$.

(b) $S_{n_k}(q) = \sigma_{n_k}$ *for* $k = 1, 2, \ldots$.

(c) *The sequence* $(S_n(q)/n)$ *is nonincreasing.*

PROOF. We define (n_k) by induction taking $n_1 = 1$ and

$$n_{k+1} = \inf\left\{n > n_k : \frac{\sigma_n}{n} < \frac{\sigma_{n_k}}{n_k}\right\}, \qquad k = 1, 2, \ldots.$$

Put $Q_n = n\sigma_{n_k}/n_k$ for $n_k \le n < n_{k+1}$, $k = 1, 2, \ldots$, $q_n = Q_n - Q_{n-1}$ for $n = 1, 2, \ldots$, and $Q_0 = 0$. The assertions (a), (b) and (c) follow immediately from the construction.

LEMMA 3. *If* $q = (q_n) \in c_0^+$ *and* $(S_n(q)/n) \in \omega^{++}$, *then* $S_n(q) \le S_n(q^*) \le 2S_n(q)$ *for* $n = 1, 2, \ldots$.

PROOF. Evidently $S_n(q) \le S_n(q^*)$. We define

$$A = \{i \in \{1, \ldots, n\} : q_i^* = q_j \text{ for some } j > n\}.$$

Since the sequence $(S_n(q)/n)$ is nonincreasing, $q_{n+1} \le S_n(q)/n$ for $n = 1, 2, \ldots$. Thus, if $i \in A$ and $q_i^* = q_j$ for $j > n$, so $q_i^* = q_j \le S_{j-1}(q)/(j-1) \le S_n(q)/n$. Therefore

$$S_n(q^*) = \sum_{i \in A} q_i^* + \sum_{\substack{i \le n \\ i \notin A}} q_i \le |A| \frac{S_n(q)}{n} + S_n(q) \le 2S_n(q).$$

LEMMA 4. *Let* $\lim_{n\to\infty} \sigma_n/n = +\infty$ *and let* $x_m = (x_{mi})$ *be a normalized sequence in* $d(w, p)$. *Then* $\lim_{m\to\infty} \|x_m\|_{c_0} = 0$ *implies* $\lim_{m\to\infty} \|x_m\|_{l_1} = 0$.

PROOF. We can assume that $x_m = x_m^*$ for every $m \in \mathbf{N}$. Fix $\varepsilon > 0$. There is $n_0 \in \mathbf{N}$ such that $2n/\varepsilon \le \sigma_n$ for every $n \ge n_0$. Let

$$y_i = \begin{cases} 0 & \text{if } i < n_0, \\ 2/\varepsilon & \text{if } i \ge n_0. \end{cases}$$

Then $S_k(y) \le \sigma_k$ for every $k \in \mathbf{N}$. From Corollary 1 follows

$$\sum_{i=1}^n x_{mi} y_i \le \left(\sum_{i=1}^n x_{mi}^p w_i\right)^{1/p} \le \|x_m\|_{w,p} = 1, \qquad n, m = 1, 2, \ldots.$$

Thus

$$\frac{2}{\varepsilon} \sum_{i=n_0}^{\infty} x_{mi} \le 1 \quad \text{for } m = 1, 2, \ldots.$$

Finally

$$\sum_{i=n_0}^{\infty} x_{mi} \le \frac{\varepsilon}{2} \quad \text{for } m = 1, 2, \ldots.$$

LEMMA 5. *Let $0 < p < 1$ and $x = (x_i) \in d(w,p)^{++}$. If $\|x\|_{w,p} = \widehat{\|x\|}_{w,p} = 1$, then $x = f_k$ for some $k = 1, 2, \ldots$.*

PROOF. Let $x^{(n)} = \sum_{i=1}^n x_i e_i$ and let $\|\cdot\|_n$ be as in Lemma 1. Every point f_k is of the form $f_k = (\alpha, \alpha, \ldots, \alpha, 0, \ldots)$ for some $\alpha > 0$. Suppose that $x \neq f_k$ for $k = 1, 2, \ldots$. Then there is $l \in \mathbf{N}$ such that $x_{l-1} > x_l > 0$. Therefore by Lemma 1(b) $\|x^{(l)}\|_l \leq \|x^{(l)}\|_{w,p}$ and by Lemma 1(c) we see that the equality cannot hold. Thus for some $\varepsilon > 0$ we have

$$\|x^{(l)}\|_l \leq \|x^{(l)}\|_{w,p} - \varepsilon.$$

From this, using Corollary 2, we get by induction

$$\widehat{\|x^{(n)}\|}_{w,p} \leq \|x^{(n)}\|_n \leq \|x^{(n)}\|_{w,p} - \varepsilon \quad \text{for } n \geq 1.$$

Thus $\widehat{\|x\|}_{w,p} \leq \|x\|_{w,p} - \varepsilon$.

III. The Mackey topology of $d(w,p)$, $0 < p < 1$.

THEOREM 1. *Let $0 < p < 1$ and $w = (w_i) \in l_\infty^{++} \backslash l_1$. Then there exists a sequence $v = (v_i) \in l_\infty^{++} \backslash l_1$ such that $d(w,p) \subset d(v,1)$ and the Mackey topology of $d(w,p)$ is induced from $d(v,1)$.*

The sequence $v \in c_0$ if and only if $\inf_n n^{-1} S_n^{1/p}(w) = 0$.

PROOF. If $\inf_n n^{-1} S_n^{1/p}(w) > 0$, then by Proposition 1 $d(w,p) \subset l_1 = d(v,1)$ for $v = (1, 1, \ldots)$. By [**8**, Proposition 3.4], the Mackey topology of $d(w,p)$ is induced from l_1.

Let $\inf_n n^{-1} S_n^{1/p}(w) = 0$. We choose sequences $(n_k) \subset \mathbf{N}$ and (q_n) according to Lemma 2. Put $v_n = q_n^*$, $n = 1, 2, \ldots$.

We will show that

$$(*) \qquad B_{v,1}^n \subset \operatorname{conv} B_{w,p}^n \subset 2B_{v,1}^n \quad \text{for every } n \in \mathbf{N}.$$

Indeed, by Lemma 3, $S_k(v) = S_k(q^*) \leq 2S_k(q) \leq 2S_k^{1/p}(w)$, for $k = 1, 2, \ldots$. Thus, using Corollary 1 with $y_k = \frac{1}{2} v_k$, we obtain $(B_{w,p}^n)^{++} \subset 2(B_{v,1}^n)^{++}$. Hence the right inclusion follows from the convexity of $B_{v,1}$.

It is obvious that if $(B_{v,1}^n)^{++} \subset \operatorname{conv} B_{w,p}^n$, then the left inclusion holds. Since $(B_{v,1}^n)^{++} = \operatorname{conv}\{g_j : j = 0, 1, \ldots, n\}$, where $g_j = S_j^{-1}(v) \sum_{i=1}^j e_i$, $g_0 = 0$ (see Lemma 1(a) and (b)), it suffices to prove that $g_j \in \operatorname{conv} B_{w,p}^n$ for $j = 1, \ldots, n$.

Fix $j \in \{1, \ldots, n\}$. We find n_k such that $n_k \leq j < n_{k+1}$. Let $\mathcal{C}$ be the family of all subsets of cardinality n_k in the set $\{1, \ldots, j\}$. We define

$$x_C = S_{n_k}^{-1/p}(w) \sum_{i \in C} e_i \quad \text{for some } C \in \mathcal{C}.$$

We have $\|x_C\|_{w,p} = 1$ and

$$\begin{aligned}\frac{1}{|\mathcal{C}|}\sum_{C\in\mathcal{C}} x_C &= \binom{j}{n_k}^{-1} S_{n_k}^{-1}(w)\sum_{C\in\mathcal{C}}\sum_{i\in C} e_i \\ &= \binom{j}{n_k}^{-1}\binom{j-1}{n_k-1} S_{n_k}^{-1/p}(w)\sum_{i=1}^{j} e_i \\ &= \frac{n_k}{j} S_{n_k}^{-1/p}(w)\sum_{i=1}^{j} e_i = S_j^{-1}(q)\sum_{i=1}^{j} e_i \\ &= \frac{S_j(q^*)}{S_j(q)} g_j.\end{aligned}$$

Thus $(S_j(q^*)/S_j(q))g_j \in \operatorname{conv} B_{w,p}^n$. Since $S_j(q) \leq S_j(q^*)$ and the set $\operatorname{conv} B_{w,p}^n$ is balanced, $g_j \in \operatorname{conv} B_{w,p}^n$. Therefore the assertion $(*)$ holds. Thus the Mackey topology of $d(w,p)$ and the $d(v,1)$-topology coincide on the subspace of all finitely supported sequences. Since this subspace is dense in $d(w,p)$, these two topologies coincide on $d(w,p)$.

If $\inf_n n^{-1}S_n^{1/p}(w) = 0$, then $v \in c_0$ by Lemma 2.

As a simple application of Theorem 1 we obtain the representation of the dual $d(w,p)'$ of $d(w,p)$, $0 < p < 1$.

COROLLARY 3. *Let $0 < p < 1$, $w = (w_i) \in l_\infty^{++}\backslash l_1$. Then*

(a) $$d(w,p)' = l_\infty \quad \textit{if } \inf_n \frac{S_n^{1/p} w}{n} > 0;$$

(b) $$d(w,p)' = \left\{ y \in c_0\colon \sup_n \frac{S_n(y^*)}{S_n^{1/p}(w)} < +\infty \right\} =: E(w,p) \quad \textit{if } \inf_n \frac{S_n^{1/p}(w)}{n} = 0.$$

PROOF. If $\inf_n S_n^{1/p}(w)/n > 0$, then by Theorem 1 $d(\widehat{w,p}) = l_1$, so $d(w,p)' = l_\infty$. Let $\inf_n S_n^{1/p}(w)/n = 0$. Then by Theorem 1 there exists $v = (v_i) \in c_0^{++}\backslash l_1$ such that $d(\widehat{w,p}) = d(v,1)$. Therefore by Proposition 1 $\sup_n S_n(v)/S_n^{1/p}(w) < +\infty$. By [**4**, Theorem 11], $d(v,1) = \{y \in c_0\colon \sup_n S_n(y^*)/S_n(v) < +\infty\}$. Hence $d(w,p)' = d(v,1)' \subset E(w,p)$.

The inclusion $E(w,p) \subset d(w,p)'$ follows directly from Corollary 1.

REMARK 1. Theorem 1 and Corollary 3 are respectively extensions of Theorem 6.3 and Proposition 6.1 in [**8**].

IV. Complemented subspaces of $d(w,p)$, $0 < p < 1$.

THEOREM 2. *Let $0 < p < 1$ and let $w = (w_i) \in c_0^{++}\backslash l_1$. If $\inf_n S_n^{1/p}(w)/n = 0$, then there is a positive continuous projection from $d(w,p)$ onto a sublattice order isomorphic to l_p.*

PROOF. First we construct by induction an increasing sequence of integers $\{n_k\}_{k=0}^\infty$ and a sequence $q = (q_i) \in \omega^+$ such that the following conditions are satisfied for all $k \geq 0$:

$$\left(\sum_{i=n_k+1}^{j} w_i\right)^{1/p} \geq \sum_{i=n_k+1}^{j} q_i \quad \text{for } n_k < j \leq n_{k+1}; \tag{1}$$

$$k \leq \left(\sum_{i=n_k+1}^{n_{k+1}} w_i\right)^{1/p} = \sum_{i=n_k+1}^{n_{k+1}} q_i; \tag{2}$$

$$\text{the sequence } \left(\sum_{i=n_k+1}^{j} \frac{q_i}{j-n_k}\right)_{j=n_k+1}^{n_{k+1}} \text{ is nonincreasing;} \tag{3}$$

$$\left(\sum_{i=1}^{n_{k+1}-n_k} w_i\right)^{1/p} \leq 2\left(\sum_{i=n_k+1}^{n_{k+1}} w_i\right)^{1/p}. \tag{4}$$

We start with $n_0 = 0$, $q_0 = 0$. Suppose that n_k has been already defined for some $k \geq 0$. Since $w \notin l_1$, there is $r \in \mathbf{N}$, $r \geq n_k$ such that for every $n > r$

$$\left(\sum_{i=1}^{n-n_k} w_i\right)^{1/p} \leq 2\left(\sum_{i=n_k+1}^{n} w_i\right)^{1/p}.$$

Applying Lemma 2 to the sequence $(w_i)_{i=n_k+1}^{\infty}$ we can find $n_{k+1} > r$ and $(q_i)_{i=n_k+1}^{n_{k+1}}$ such that (1), (2) and (3) hold. As $n_{k+1} > r$, the same is true of (4).

Let

$$f_k = \left(\sum_{i=n_k+1}^{n_{k+1}} w_i\right)^{-1/p} \sum_{i=n_k+1}^{n_{k+1}} e_i, \qquad k = 0,1,2,\ldots.$$

It follows from (4) that $\|f_k\|_{w,p} \leq 2$.

Now we define the projection P: $d(w,p) \to \overline{\text{span}}\{f_k\}_{k=0}^{\infty}$ by

$$P(x) = \sum_{k=0}^{\infty}\left(\sum_{n_k+1}^{n_{k+1}} x_i q_i\right) f_k, \quad \text{where } x = (x_i) \in d(w,p).$$

Let $x = (x_i) \in d(w,p)$ and let $(\hat{x}_i)_{i=n_k+1}^{n_{k+1}}$ and $(\hat{q}_i)_{i=n_k+1}^{n_{k+1}}$, $k = 0,1,\ldots$, be respectively nonincreasing rearrangements of the sequences $(|x_i|)_{i=n_k+1}^{n_{k+1}}$ and $(q_i)_{i=n_k+1}^{n_{k+1}}$. Using (3) and Lemma 3 we have

$$\sum_{i=n_k+1}^{1} \hat{q}_i \leq 2 \sum_{i=n_k+1}^{1} q_i, \qquad l = n_k+1,\ldots,n_{k+1}.$$

Thus by (1) and Corollary 1 we get

$$\begin{aligned}\|Px\|_{w,p}^p &\leq 2^p \sum_{k=0}^{\infty}\left|\sum_{i=n_k+1}^{n_{k+1}} x_i q_i\right|^p \leq 2^p \sum_{k=0}^{\infty}\left|\sum_{i=n_k+1}^{n_{k+1}} \hat{x}_i \hat{q}_i\right|^p \\ &\leq 2^{p+1} \sum_{k=0}^{\infty}\left(\sum_{i=n_k+1}^{n_{k+1}} \hat{x}_i^p w_i\right) \leq 2^{p+1} \sum_{i=1}^{\infty} x_i^{*p} w_i = 2^{p+1}\|x\|_{w,p}^p.\end{aligned}$$

Thus P is continuous. By (2) and [**8**, Lemma 3.1] there is a strictly increasing sequence (j_k) such that (f_{j_k}) is equivalent to the canonical basis of l_p. Therefore the desired result follows from unconditionality of the basic sequence (f_k).

REMARK 2. Theorem 2 solves Problems 3 and 3a in [**8**].

COROLLARY 4. *If* $\inf_n n^{-1} S_n^{1/p}(w) = 0$, *then* $d(w,p) \oplus l_p$ *is isomorphic to* $d(w,p)$, $0 < p < 1$.

PROOF. By Theorem 2, $d(w,p) = X \oplus l_p$ for some F-space X. Therefore

$$d(w,p) = X \oplus l_p = X \oplus l_p \oplus l_p = d(w,p) \oplus l_p.$$

COROLLARY 5. *Let* $0 < p < 1$, *and* $\inf_n n^{-1} S_n^{1/p}(w) = 0$. *Then* $d(w,p)$ *has uncountably many mutually nonequivalent unconditional bases.*

PROOF. It is enough to know that $d(w,p)$ has at least two mutually nonequivalent bases (cf. [**6**, p. 118]). Thus our result follows from Corollary 4.

In the proof of the next theorem we use the same ideas as in [**7**, Theorem 2.3].

THEOREM 3. *Let* $0 < p < 1$, $w = (w_i) \in l_\infty^{++} \backslash l_1$. *If* $\lim_{n\to\infty} S_n^{1/p}(w)/n = \infty$, *then each infinite-dimensional complemented subspace of* $d(w,p)$ *contains a subspace* Y *which is isomorphic to* $d(w,p)$ *and complemented in* $d(w,p)$.

PROOF. Let P be a continuous projection from $d(w,p)$ onto an infinite-dimensional subspace X of $d(w,p)$. Since $\lim_{n\to\infty} S_n^{1/p}(w)/n = \infty$, by Theorem 1 $d(\widehat{w,p}) = l_1$. Because X is complemented in $d(w,p)$, so its Mackey topology is also induced from l_1. Since the l_1-closure of $\operatorname{conv}\{P(e_i) : i \in \mathbf{N}\}$ is a neighbourhood of zero in $\hat{X}$, the set $\{P(e_i) : i \in \mathbf{N}\}$ is not precompact in l_1. Therefore, using the standard gliding hump method, we can construct a strictly increasing sequence of the integers (n_k) and sequences of vectors (y_k) and (z_k) such that:

(1) $y_k = P(e_{n_{2k+1}} - e_{n_{2k}})$;

(2) $z_k = \sum_{i\in A_k} t_i e_i$ is a block basic sequence;

(3) $\sum_{k=1}^{\infty} \|y_k - z_k\|_{w,p}^p < 1$;

(4) $0 < C_1 \le \|z_k\|_{l_1} \le \|z_k\|_{w,p} \le C_2$ for $k \in \mathbf{N}$, where C_1, C_2 are some constants.

By Lemma 4 we have $\inf_k \max_{i\in A_k} |t_i| > 0$. Since (e_k) is symmetric and P is continuous, the sequence (z_k) is equivalent to (e_k). Thus, as in [**3**], we may define a continuous projection Q by

$$Q(x) = \sum_{n=1}^{\infty} \frac{x_{i_n}}{t_{i_n}} z_n \quad \text{if } x = (x_i) \in d(w,p),$$

where $i_n \in A_n$ and $|t_{i_n}| = \max\{|t_i| : i \in A_n\}$, $n = 1, 2, \ldots$. Using a stability theorem (cf. [**6**, Proposition 1.a.9] and [**7**, Proposition 1.2]) we conclude that $\overline{\operatorname{span}}\{P(e_{n_{2k+1}}) - P(e_{n_{2k}})\}_{k\ge k_0}$ is isomorphic to $d(w,p)$ and complemented in $d(w,p)$.

Our next result is an easy consequence of Theorem 3 and Pełczyński's decomposition method.

COROLLARY 6. *Let* $0 < p < 1$ *and* $w = (w_i) \in l_\infty^{++} \backslash l_1$. *If* $\lim_{n\to\infty} S_n^{1/p}(w)/n = \infty$, *then every infinite-dimensional complemented subspace of* $d(w,p)$ *with symmetric basis is isomorphic to* $d(w,p)$.

COROLLARY 7. *Let* $0 < p < 1$, $w = (w_i) \in c_0^{++}\backslash l_1$ *and* $\lim_{n\to\infty} S_n^{1/p}(w)/n = \infty$. *Then* $d(w,p)$ *contains a closed subspace* X *nonisomorphic to* l_p *and* $d(w,p)$ *such that* $\hat{X} \approx l_1$.

PROOF. It follows from Corollary 6 that $d(w,p) \oplus l_p \not\approx d(w,p)$. Moreover $d(w,p) \oplus l_p$ is isomorphic to some subspace Z of $d(w,p) \oplus d(w,p) \approx d(w,p)$. Since $l_p \oplus d(w,p) = l_1 \oplus l_1 \approx l_1$ we get $\hat{Z} \approx l_1$.

REMARK 3. Corollary 7 solves partially Problem 2 in [**8**].

PROPOSITION 2. *Let* $0 < p < 1$, $w = (w_i) \in l_\infty^{++}\backslash l_1$ *and* $w_1 < S_n^{1/p}(w)/n$ *for* $n > 1$. *If* $P\colon d(w,p) \mapsto Y \subset d(w,p)$ *is a constructive projection, then* $Y = \overline{\operatorname{span}}\{e_i\colon i \in A\}$ *for some set* $A \subset N$.

PROOF. We can assume that $w_1 = 1$. Since $1 < n^{-1}S_n^{1/p}(w)$, by Theorem 1 and Corollary 1, we have $d(\widehat{w,p}) = l_1$ and $(B_{w,p}^n)^{++} \subset B_{l_1}$, $n = 1, 2, \ldots$. Thus $B_{w,p} \subset B_{l_1}$ and

$$\hat{B} = \overline{\operatorname{conv}}^{l_1} B_{w,p} \subset B_{l_1} = \overline{\operatorname{conv}}^{l_1}\{c_i\colon i = 1, 2, \ldots\} \subset \overline{\operatorname{conv}}^{l_1} B_{w,p} = \hat{B},$$

where $\hat{B} = \{x \in l_1\colon \|x\|\widehat{_{w,p}} \le 1\}$.

Therefore $\|\cdot\|\widehat{_{w,p}} = \|\cdot\|_{l_1}$.

Hence a continuous extension $\hat{P}$ of P is a contractive projection in $l_1 = d(\widehat{w,p})$. By [**5**, Chapter 6, §17, Theorem 3] (see also [**6**, Theorem 2.a.4]),

$$\hat{P}(x) = \sum_{j=1}^{m} h_j(x) u_j,$$

where $\{u_j\}_{j=1}^m$ are vectors of norm 1 in l_1 ($m = \dim Y$ is either an integer or ∞), $u_j = \sum_{i\in A_j} t_i e_i$, with $A_j \cap A_k = \varnothing$ for $j \neq k$ and $\{h_j\}_{j=1}^m \subset l_1'$ satisfy $\|h_j\|_\infty = h_j(u_j) = 1$, $j = 1, 2, \ldots$.

Since for every $x \in d(w,p)$ and $j = 1, 2, \ldots$,

$$\|x\|_{w,p} \ge \|Px\|_{w,p} = \|\hat{P}x\|_{w,p} \ge \|h_j(x)u_j\|_{w,p},$$

so $u_j \in d(w,p)$ and $Q_j(x) := h_j(x)u_j$ is a contractive projection from $d(w,p)$ onto a one-dimensional subspace $\operatorname{span}\{u_j\}$.

Therefore $\|u_j\|_{w,p} = \|u_j\|\widehat{_{w,p}} = 1$. By Lemma 5, $u_j^* = f_k$ for some $k = 1, 2, \ldots$. Since $1 < S_n^{1/p}(w)/n$ for $n > 1$, $\|f_k\|\widehat{_{w,p}} < \|f_k\|_{w,p}$ if $k > 1$. Thus $u_j^* = e_1$, $j = 1, 2, \ldots$.

COROLLARY 8. *Let* $0 < p < 1$, $w = (w_i) \in c_0^{++}\backslash l_1$ *and* $w_1 < S_n^{1/p}(w)/n$ *for* $n > 1$. *Then* l_p *is not isomorphic to the range of a contractive projection in* $d(w,p)$.

REMARK 4. Corollary 8 is an extension of Theorem 5.5 in [**8**].

V. Open problems and remarks. If $\underline{\lim}_{n\to\infty} S_n^{1/p}(w)/n = 0$, then by Theorem 2 there exists a continuous projection P from $d(w,p)$ onto a subspace isomorphic to l_p. Moreover, if $\lim_{n\to\infty} S_n^{1/p}(w)/n = \infty$, then by Theorem 3 no subspace isomorphic to l_p is complemented in $d(w,p)$.

PROBLEM 1. Let $0 < p < 1$ and $0 < \underline{\lim}_{n\to\infty} S_n^{1/p}(w)/n < \infty$. Is there a continuous projection from $d(w,p)$ onto a subspace isomorphic to l_p?

PROBLEM 2. Let $0 < p < 1$ and $\underline{\lim}_{n\to\infty} S_n^{1/p}(w)/n = 0$. Is there a contractive projection from $d(w,p)$ onto a subspace isomorphic to l_p?

The next result is an extension of Theorem 3.8 in [**8**].

PROPOSITION 3. *Each symmetric basis* (y_k) *of* $d(w,p)$ $(0 < p < 1)$ *is equivalent to the canonical basis* (e_k) *of* $d(w,p)$.

PROOF. Using the standard gliding hump method we can find a strictly increasing sequence of natural numbers (n_k) such that the sequence $x_k = y_{n_{2k}} - y_{n_{2k+1}}$ is equivalent to a block basic sequence $z_k = \sum_{i\in A_k} b_i e_i$. Since x_k is symmetric and equivalent to (y_k), by [**8**, Lemma 3.1] $\inf_k \max_{i\in A_k} |b_i| > 0$. Hence (y_k) dominates (e_k). If we interchange the roles of (e_k) and (y_k) we deduce the equivalence of these bases.

If $\underline{\lim}_{n\to\infty} S_n^{1/p}(w)/n = 0$, then $d(w,p)$ has uncountable many mutually nonequivalent unconditional bases. However the above proposition and Corollary 6 suggest the following

PROBLEM 3. Let $0 < p < 1$ and $\lim_{n\to\infty} S_n^{1/p}(w)/n = \infty$. Are every two unconditional bases in $d(w,p)$ equivalent?

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