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ESSENTIAL DIMENSION LOWERING MAPPINGS HAVING DENSE DEFICIENCY SET

BY

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ABSTRACT. Two classes of surjective maps $f: S^m \to S^n$ that are one-to-one over the image of a dense set are constructed. We show that for $m, n \geq 3$ there is a monotone surjection $f: S^m \to S^n$ that is one-to-one over the image of a dense set; and for $3 \leq n \leq m \leq 2n-3$, each element of $\pi_m(S^n)$ can be represented as a monotone surjection $f: S^m \to S^n$ that is one-to-one over the image of a dense set.

1. Introduction. The present paper should be considered as a continuation of the study of surjective maps between spheres that are "one-to-one over the image of a dense set". (A surjection $f: X \to Y$ is *one-to-one over the image of a dense set* if there exists a dense set $D \subseteq X$ such that for each $y \in f(D)$, $\# f^{-1}(y) = 1$; $\# =$ cardinality.)

First inconceivable examples of such maps were constructed by J. J. Walsh [**Wa 5**]. Specifically, for any pair $n \geq 3$, $d \geq 2$ of integers, Walsh has built a monotone, surjective map $f: S^n \to S^n$ of degree d that is one-to-one over the image of a dense set.

More recently, in [**Be-Wa**], it has been established that for any $m, n \geq 2$ there is a surjection $f: S^m \to S^n$ that is one-to-one over the image of a dense set. By construction this map is not monotone and factors through a 1-dimensional compactum, and hence it is null-homotopic; even more, it has no stable values. (A point $y \in Y$ is a *stable value* of a map $f: X \to Y$ between metric spaces if there exists an open cover $\mathcal{U}$ of Y so that for every $\mathcal{U}$-approximation f' to f, y is in the image of f'.)

In this paper we show:

(a) For $m, n \geq 3$ there is a *monotone* surjection $f: S^m \to S^n$ that is one-to-one over the image of a dense set.

(b) For $3 \leq n \leq m \leq 2n-3$ each element of $\pi_m(S^n)$ can be represented as a *monotone* surjection $f: S^m \to S^n$ that is one-to-one over the image of a dense set. In particular, if $3 \leq n \leq m \leq 2n-3$ and $\pi_m(S^n) \neq 0$ (e.g., $\pi_{n+1}(S^n) = Z_2$), there is a monotone, *essential* map $f: S^m \to S^n$ that is one-to-one over the image of a dense set (and hence, all values of f are stable).

The techniques used in the paper stem from D. Wilson [**Wi 1, Wi 2**] and J. J. Walsh [**Wa 1–Wa 5**]. Mappings are constructed by making use of "defining sequences". Although the necessary definitions are given, and in that respect the

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paper is self-contained, familiarity with [**Wa 5**] is desirable. We follow the notation developed in that paper.

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2. Preliminaries. For any family P of subsets of a set X, and for any $A \subseteq X$, we define

$$\operatorname{St}(A, P) = \bigcup \{p \in P \colon p \cap A \neq \varnothing\}.$$

Following [**Wa 5**], by a (*stratified*) *partition* on a closed PL n-manifold N we mean a collection $P = \{p_1, \ldots, p_k\}$ of closed subsets of N that cover N with the following properties.

(P1) Each $p \in P$ is a PL n-submanifold (with boundary) of N.

(P2) If $p_{i(1)}, \ldots, p_{i(t)}$ are mutually distinct elements of P, then $p_{i(1)} \cap \cdots \cap p_{i(t)}$ is either empty or an $(n - t + 1)$-dimensional PL submanifold of the boundary of $p_{i(1)} \cap \cdots \cap p_{i(t-1)}$.

Observe that $p_{i(1)} \cap \cdots \cap p_{i(t)} \neq \varnothing$ has empty boundary if and only if $p \cap p_{i(1)} \cap \cdots \cap p_{i(t)} = \varnothing$ for all $p \in P - \{p_{i(1)}, \ldots, p_{i(t)}\}$.

If L is any triangulation of N, by J_i^N denote the standard handlebody decomposition of N associated with the ith barycentric subdivision $\beta^i L$ of L:

$$J_i^N = \{\operatorname{St}(v, \beta^{i+1} L) \colon v \text{ is a vertex of } \beta^i L\}.$$

It is easy to see that J_i^N satisfies (P1) and (P2). For $i \geq 1$ and $j = \operatorname{St}(v, \beta^{i+1} L) \in J_i^N$, define the *index* of j, $\operatorname{Ind}(j)$, to be equal to k if v is the barycenter of a k-simplex in $\beta^i L$.

Let M^m, N^n be PL manifolds, and P, Q partitions on M, N respectively. We say that a function $T \colon P \to Q$ is *admissible*, provided:

(A1) T is a bijection;

(A2) for all $p_{i(1)}, \ldots, p_{i(t)} \in P$,

$$p_{i(1)} \cap \cdots \cap p_{i(t)} \neq \varnothing \Rightarrow T(p_{i(1)}) \cap \cdots \cap T(p_{i(t)}) \neq \varnothing;$$

(A3) for all $p, p' \in P$,

$$p \cap p' \neq \varnothing \Rightarrow T(p) \cap T(p') \neq \varnothing\,.$$

Let L be any triangulation of N. If $T \colon P \to J$ is a triple satisfying (A2) and (A3), where $J = J_i^N$ is the handlebody decomposition of N associated with $\beta^i L$, by an *induced map* we mean any map $h \colon M \to N$ with $h(p) \subseteq T(p)$ for all $p \in P$. We can define h by the "backward induction" on t, requiring that $h(p_{i(1)} \cap \cdots \cap p_{i(t)}) \subseteq T(p_{i(1)}) \cap \cdots \cap T(p_{i(t)})$. Since each nonempty intersection of elements of J is an absolute retract, the inductive step "goes through". The same fact establishes that any two induced maps are homotopic (see [**Wa 5**]), which enables us to talk about *the* induced map.

A sequence of triples $\{T_i \colon P_i \to J_i\}_{i=0}^{\infty}$ is a *defining sequence* provided, for all $i \geq 0$:

(DS1) $J_i = J_i^N$ is the handlebody decomposition of N associated with $\beta^i L$;

(DS2) P_i is a partition on M;

(DS3) T_i is an admissible function;

(DS4) for all $p \in P_i$, $p' \in P_{i+1}$,

$$p \cap p' \neq \emptyset \Leftrightarrow \operatorname{Int}(p \cap p') \neq \emptyset \Leftrightarrow T_i(p) \cap T_{i+1}(p') \neq \emptyset .$$

The reader should find establishing the following result a useful exercise.

PROPOSITION 2.1 (SEE [**Wa 5**]). *Let* $\{T_i\colon P_i \to J_i\}_{i=0}^{\infty}$ *be a defining sequence.*

(i) *Setting* $h^{-1}(y) = \bigcap_{i=0}^{\infty} \operatorname{St}(p_i, P_i)$ *for any choice of* $p_i \in P_i$ *with* $T_i(p_i) \ni y$ *defines a surjective map* $h\colon M \to N$. *Moreover,* $\operatorname{Int}\operatorname{St}(p_i, P_i) \supseteq \operatorname{St}(p_{i+1}, P_{i+1})$ $(i = 0, 1, 2, \ldots)$.

(ii) *If* $h_i\colon M \to N$ *is the map induced by* $T_i\colon P_i \to J_i$, *then* $h = \lim_{i\to\infty} h_i$ *and* $h_0 \simeq h_1 \simeq h_2 \simeq \cdots \simeq h$.

(iii) *If each* $p \in P_i$, $i \geq 0$, *is connected, then* h *is a monotone map.*

(iv) *If for each* $i \geq 1$ *and each* $j \in J_i$ *with* $\operatorname{Ind}(j) = n$ *there exists a PL m-cell* $B \subseteq M$ *with*

$$\operatorname{St}(T_i^{-1}(j), P_{i+1}) \subseteq \operatorname{Int} B \subseteq \operatorname{St}(T_i^{-1}(j), P_i),$$

then the points $y \in N$ *for which* $h^{-1}(y) \subseteq M$ *is a cellular set form a dense subset of* N.

To construct interesting maps between manifolds using 2.1, we have to produce defining sequences. The major step consists of generating a triple $T_{i+1}\colon P_{i+1} \to J_{i+1}$ from a triple $T_i\colon P_i \to J_i$ previously constructed. To make the notation easier, the triple $T_i\colon P_i \to J_i$ will be denoted by $T\colon P \to J$, and the triple $T_{i+1}\colon P_{i+1} \to J_{i+1}$ by $\tilde{T}\colon \tilde{P} \to \tilde{J}$. Coherently, we will rename the subdivision $\beta^i L$ and again call it L. Hence

$$J = \{\operatorname{St}(v, \beta L)\colon v \text{ is a vertex of } L\},$$
$$\tilde{J} = \{\operatorname{St}(v, \beta^2 L)\colon v \text{ is a vertex of } \beta L\}.$$

The construction of $\tilde{P}$ is in two stages. We define an intermediate triple $\hat{T}\colon \hat{P} \to \tilde{J}$. *Warning.* $\hat{P}$ will be a partition of M, and $\hat{T}$ will satisfy (A2) and (A3), but not necessarily (A1).

The elements of $\hat{P}$ will be indexed by the set S of all collections $\{p_{i(1)}, \ldots, p_{i(t)}\} \subseteq P$ that have nonempty intersections. (These intersections, in Walsh's terminology, are called the *strata* of P.)

The collection $\hat{P} = \{p_s,\ s \in S\}$ will satisfy the following properties.

(H1) $\hat{P}$ is a partition on M.

(H2) $p_{s(1)}, \ldots, p_{s(t)} \in \hat{P}$ have nonempty intersection if and only if $\{s(1), \ldots, s(t)\} \subseteq S$ is well-ordered with respect to inclusion.

(H3) For any $p \in P$ and $p_s \in \hat{P}$,

$$p \cap p_s \neq \emptyset \Leftrightarrow \operatorname{Int}(p \cap p_s) \neq \emptyset \Leftrightarrow p \in s.$$

(H4) For any $p \in P$, p and $p_{\{p\}}$ are homeomorphic.

Still following [**Wa 5**], we construct the elements $p_s \in \hat{P}$ as follows (see Figure 1). Let K be a triangulation of M so that each stratum $\bigcap s$, $s \in S$, is a full subcomplex of K. Define the *core* of $s \in S$ by

$$c(s) = \bigcup \left\{\tau\colon \tau \text{ is a simplex of } \beta K \text{ contained in } \bigcap s - \partial\Big(\bigcap s\Big)\right\}.$$

Finally, set

$$p_s = \bigcup \{\operatorname{St}(v, \beta^2 K)\colon v \in c(s) \text{ is a vertex of } \beta K\}.$$

Observe that, by choosing a sufficiently small triangulation K:

(H5) Given neighborhoods $U(p)$ of $p \in P$, we can arrange that $\mathrm{St}(p, \hat{P}) \subseteq U(p)$ for all $p \in P$.

We can also define the function $\hat{T}\colon \hat{P} \to \tilde{J}$ by $\hat{T}(p_s) = \mathrm{St}(v, \beta^2 L) \in \tilde{J}$, where v is determined as follows. If $s = \{p_{i(1)}, \dots, p_{i(t)}\}$ and $T(p_{i(r)}) = \mathrm{St}(v_r, \beta L)$, then v is the barycenter of the simplex whose vertices are $v_1, \dots, v_t$. Property (H2) implies that $\hat{T}$ satisfies (A2) and (A3). It is evident that $\hat{T}$ is a one-to-one function (but not necessarily a surjection).

We will "repair" the triple $\hat{T}\colon \hat{P} \to \tilde{J}$ to get the triple $\tilde{T}\colon \tilde{P} \to \tilde{J}$, but the reparation will depend on the desired properties of the function $h\colon M \to N$ determine by the defining sequence. The "reparation process", as well as the construction of the triple $T_0\colon P_0 \to J_0$, is explained in detail in forthcoming sections.

REMARK. If the partiion P is the standard handlebody decomposition corresponding to a triangulation K of the manifold M, then the partition $\hat{P}$ constructed above is (up to an ambient isotopy) the standard handlebody decomposition of M corresponding to the barycentric subdivision K' of K. This fact will be implicitly used in the sequel.

3. Essential maps. The purpose of this section is to establish the following

PROPOSITION 3.1. *Let $f\colon S^m \to S^n$ be any map, and let $3 \le n \le m \le 2n-3$. Then there exists a surjective monotone map $h\colon S^m \to S^n$ homotopic to f such that the set $\{y \in S^n\colon h^{-1}(y)$ is cellular in $S^m\}$ is dense in S^n.*

A routine consequence of 3.1 is the result announced in the Introduction.

THEOREM 3.2. *For any map $f\colon S^m \to S^n$, $3 \le n \le m \le 2n-3$, there exists a surjective monotone map $g\colon S^m \to S^n$ homotopic to f that is one-to-one over the image of a dense set.*

PROOF. Let $h\colon S^m \to S^n$ be a map whose existence is promised by 3.1. We "carefully shrink countably many cellular fibers of h" in order to obtain the sought-after map $g\colon S^m \to S^n$. The shrinking process can be described as follows.

Let $U_1, U_2, \dots$ be a countable basis of open sets of S^m. Choose a fiber F of h with $F \cap U_1 \neq \emptyset$, and pick a cellular fiber C of h in a "small" *connected* neighborhood V of F (here we use the fact that h is a monotone map). Let $\lambda\colon S^m \to S^m$ be a surjection whose only nondegenerate point-preimage is C. We can arrange that $\lambda(C) \in U_1$ and $\lambda =$ identity off of V. Then $g_1 = h\lambda^{-1}$ is a monotone surjection "close" to h, and one of the fibers of g_1 is a point in U_1. In a similar fashion we produce monotone maps $g_2, g_3, \dots$ such that g_{i+1} is "close" to g_i, it agrees with g_i off of a "small" neighborhood of a fiber of g_i, and g_{i+1} has degenerate point-preimages in each of the sets $U_1, \dots, U_{i+1}$.

Exercising sufficient control on all choices made, and carefully interpreting the quoted words in the preceding paragraph, we can arrange that the sequence g_1, $g_2, \dots$ converges to a monotone map $g\colon S^m \to S^n$ homotopic to h that is one-to-one over the image of a dense set.

Before giving a proof of Proposition 3.1, we state and prove an interesting corollary of Theorem 3.2.

In what follows, E^r denotes Euclidean r-dimensional space. Observe that the homogeneity properties of E^r establish that the set of all stable values of a surjection $f\colon X \to E^r$ is open in E^r.

COROLLARY 3.3. *Let $m, n \geq 2$ be integers.*

(i) *If $\pi_{m-1}(S^{n-1}) = 0$, then any surjection $f\colon E^m \to E^n$ that is one-to-one over the image of a dense set has no stable values.*

(ii) *If $3 \leq n \leq m \leq 2n-3$ and $\pi_{m-1}(S^{n-1}) \neq 0$, then there exists a proper monotone surjection $f\colon E^m \to E^n$ that is one-to-one over the image of a dense set and has all values stable.*

PROOF. (i) In view of the observation made before the statement of Corollary 3.3, it suffices to prove that if $\# f^{-1}(y) = 1$, then y is not a stable value of f. Let B be a "small" ball around $f^{-1}(y)$. The assumption about the homotopy group reveals that $f|\partial B$ is a null-homotopic map in a "small" deleted neighborhood of y. Redefine$f$ in $\operatorname{Int} B$, using the homotopy, to get an approximation f' to f whose image misses y.

(ii) By Freudenthal's Suspension Theorem (see [**Sp**, p. 458]), $\pi_{m-1}(S^{n-1}) \cong \pi_m(S^n)$. Application of 3.2 gives an *essential* monotone surjection $f\colon S^m \to S^n$ that is one-to-one over the image of a dense set. Pick $y \in S^n$ such that $\# f^{-1}(y) = 1$. Then $f/\colon S^m - f^{-1}(y) \to S^n - y$ is a proper monotone surjection that is one-to-one over the image of a dense set. All values of $f/$ are stable since the opposite would violate the fact that f is essential.

PROOF OF 3.1. We construct a defining sequence $\{T_i\colon P_i \to J_i\}_{i=0}^{\infty}$ with the following additional properties.

(E1) For mutually distinct elements $p_{i(1)}, \ldots, p_{i(t)} \in P_i$,

$$p_{i(1)} \cap \cdots \cap p_{i(t)} \neq \varnothing \Leftrightarrow t \leq n+1 \text{ and } T(p_{i(1)} \cap \cdots \cap T(p_{i(t)})) \neq \varnothing$$

$(i = 0, 1, \ldots)$.

(E2) Each element $p \in P_i$ is $(m-n)$-connected $(i = 0, 1, \ldots)$.

(E3) If $j \in J_i$, $\operatorname{Ind}(j) = n$, then $T_i^{-1}(j)$ is a PL m-ball $(i = 1, 2, \ldots)$.

As announced in §2, we construct the defining sequence by induction. Suppressing indices, we start with an admissible function $T\colon P \to J$ satisfying (E1) and (E2) (produced following the inductive analysis). Let K be a triangulation of S^m such that all strata of P are full subcomplexes of K. Let $\hat{T}\colon \hat{P} \to \tilde{J}$ be the triple constructed in §2, $\hat{P} = \{p_s,\ s \in S\}$. Observe that (E1) implies that $\hat{T}$ satisfies (A1)–(A3). Also, (H2) implies that $\hat{T}$ satisfies (E1).

We now "repair" the triple $\hat{T}\colon \hat{P} \to \tilde{J}$ to get another triple $\tilde{T}\colon \tilde{P} \to \tilde{J}$ which satisfies (E2) and (E3). We want to maintain all properties that $\hat{T}\colon \hat{P} \to \tilde{J}$ already satisfies. For all $s \in S$ choose $p(s) \in s$; if possible, choose $p(s)$ so that $\operatorname{Ind} T(p(s)) = n$. A quick remark: each $s \in S$ contains at most one p with $\operatorname{Ind} T(p) = n$. We interrupt the proof to introduce some notation.

For a compactum X, denote by $C(X) = X \times [0,1]/(x_1, 1) \sim (x_2, 1)$ the *cone* over X. We identify $X = X \times \{0\} \subseteq C(X)$. Name $\frac{1}{2}C(X) = X \times [0, \frac{1}{2}] \subset C(X)$ the *bottom half of the cone* over X. If A is a subcomplex of K, by $A^{(r)}$ we denote the r-skeleton of A with respect to K. Finally, "$\approx$" means "PL homeomorphic".

For all $s \in S$ with $\# s > 1$ choose a polyhedron $X_s \subseteq S^m$ containing $c(s)$ with the following properties.

(a) $X_s \subseteq p(s) \cap (p_s \cup \operatorname{Int} p_{\{p(s)\}})$;
(b) $X_s \cap \partial p(s) = c(s)$;
(c) $(X_s, X_s \cap p_s, c(s)) \approx ((c(s)) \cup C((c(s))^{(m-n)}), c(s) \cup \frac{1}{2}C((c(s))^{(m-n)}), c(s))$, and
(d) if $s_1 \neq s_2$, then $X_{s_1} \cap X_{s_2} = \emptyset$.

Sets X_s exist, since $\dim C(c(s)^{(m-n)}) \leq m - n + 1$, and $2(m - n + 1) < m$. The same inequality, coupled with (c) and the fact that each p_s with $\#s = 1$ is $(m-n)$-connected (see (H4)), testifies that $p_{\{p(s)\}} \setminus \bigcup\{X_{s'} \colon s' \in S,\ \#s' > 1\}$ is still $(m-n)$-connected for all $s \in S$.

Let K' be a subdivision of K such that all mentioned subsets of S^m are full subcomplexes with respect to K'. Let N_s be the second derived neighborhood of X_s in $p(s)$ with respect to K'. Define

$$\tilde{p}_s = \begin{cases} p_s \cup N_s & \text{if } \#s > 1, \\ p_s \setminus \bigcup\{\operatorname{Int} N_{s'},\ s' \in S,\ \#s' > 1\} & \text{if } \#s = 1. \end{cases}$$

The reader can easily verify that $\tilde{P} = \{\tilde{p}_s,\ s \in S\}$ is a partition on S^m, that $\tilde{T} \colon \tilde{P} \to \tilde{J}$ defined by $\tilde{T}(\tilde{p}_s) = \hat{T}(p_s)$ is an admissible function, and that the triple $\tilde{T} \colon \tilde{P} \to \tilde{J}$ satisfies (E1) and (E2). (Note that p_s collapses to $c(s)$; by adding the cone over $c(s)^{(m-n)}$, we "killed" first $(m-n)$ homotopy groups).

If $\#s = n + 1$, then $\tilde{p}_s$ is a regular neighborhood of $X_s \approx C(c(s))$ (since $\dim(c(s)) = m - n$), and hence $\tilde{p}_s$ is an m-ball. This establishes (E3). From (a) and (H3), it easily follows that T and $\tilde{T}$ satisfy (DS4).

Observe that, by our choice of $p(s)$, $s \in S$, we have $\operatorname{St}(p, \tilde{P}) = \operatorname{St}(p, \hat{P})$, for all $p \in P$ with $\operatorname{Ind} T(p) = n$. Thus, taking into account (H5), we get:

(E4) Given neighborhoods $U(p)$ of $p \in P$ with $\operatorname{Ind} T(p) = n$, we can arrange that $\operatorname{St}(p, \tilde{P}) \subseteq U(p)$ for all such p.

If $h \colon S^m \to S^n$ is the map associated with a defining sequence $\{T_i \colon P_i \to J_i\}_{i=0}^{\infty}$ satisfying (E1)–(E3), then, by 2.1(iii), h is a monotone surjection, and using 2.1(iv), together with (E4), we see that we can arrange that the set $\{y \in S^n \colon h^{-1}(y)$ is cellular$\}$ is dense in S^n. Indeed, let B_j be a regular neighborhood of $T_i^{-1}(j)$ contained in $\operatorname{St}(T_i^{-1}(j), P_i)$. By (E3), B_j is an m-ball. By (E4) we can arrange that $\operatorname{St}(T_i^{-1}(j), P_{i+1}) \subseteq \operatorname{Int} B_j$.

To finish the proof of 3.1, in view of 2.1(ii), we need to construct a triple $T_0 \colon P_0 \to J_0$ satisfying (E1) and (E2) such that the induced map $h_0 \colon S^m \to S^n$ is homotopic to f. By Freudenthal's Suspension Theorem [**Sp**, p. 458], there is a map $f' \colon S^{m-1} \to S^{n-1}$ whose suspension $\Sigma f' \colon S^m \to S^n$ is homotopic to f. Without loss of generality, we may assume that f' is a surjective simplicial map, with respect to some triangulations K_0, L_0 of S^{m-1}, S^{n-1} respectively. Then the map $\Sigma f' \colon \Sigma K_0 \to \Sigma L_0$ is simplicial. To suppress unnecessary symbols, rename it as $f \colon K \to L$. Let

$$p_v = \bigcup\{\operatorname{St}(w, \beta K) \colon f(w) = v,\ w \text{ is a vertex of } K\}$$

and set $P = \{p_v \colon v \text{ is a vertex of } L\}$. Then P is a partition on S^m, and the function $T \colon P \to J_0$ given by $T(p_v) = \operatorname{St}(v, \beta L)$ satisfies (A1)–(A3) and (E1).

We now "repair" the triple $T \colon P \to J_0$ to get a new triple $T_0 \colon P_0 \to J_0$ satisfying, in addition, (E2). The strategy is the same as for obtaining $\tilde{T}$ from $\hat{T}$. Observe that if σ is a suspension vertex of L, then p_σ is an m-ball. Moreover, p_σ intersects all elements of P except for $p_\tau \in P$, where τ is the other suspension vertex. If $p \in P - \{p_\sigma, p_\tau\}$, then p collapses to $p \cap p_\sigma$. For each $p \in P - \{p_\sigma, p_\tau\}$ choose a polyhedron $A_p \subseteq p \cap p_\sigma$ such that $\dim A_p \leq m - n$ and the pair $(p \cap p_\sigma, A_p)$ is $(m-n)$-connected. We can arrange that $A_{p(1)} \cap A_{p(2)} = \emptyset$ if $p(1) \neq p(2)$. (We can take A_p to be the $(m-n)$-skeleton of a shrunk copy of $p \cap p_\sigma$.) Next, embed the cones $C(A_p)$ into p_σ to obtain polyhedra $Y(p)$, $p \in P - \{p_\sigma, p_\tau\}$. We can arrange that $Y(p) \cap \partial p_\sigma = A_p$ for all $p \in P - \{p_\sigma, p_\tau\}$ and, by general positioning, that $Y(p) \cap Y(p') = \emptyset$ for $p \neq p'$ (we are in the range of dimensions where $2(m-n+1) < m$). Let K' be a subdivision of K such that all mentioned subsets of S^m are full subcomplexes of K'. If N_v is the second derived neighborhood of $Y(P_v)$ in p_σ, set

$$\tilde{p}_v = \begin{cases} p_v \cup N_v, & v \text{ is a vertex of } K_0, \\ p_\sigma - \bigcup\{\operatorname{Int} p_{v'},\ v' \text{ is a vertex of } K_0\}, & v = \sigma, \\ p_\tau, & v = \tau. \end{cases}$$

Then $P_0 = \{\tilde{p}_v : v \text{ is a vertex of } K\}$ is a partition on S^m, and $T_0 : P_0 \to J_0$ defined by $T_0(\tilde{p}_v) = T(p_v)$ satisfies (A1)–(A3), (E1) and (E2). If $h_0 : S^m \to S^n$ is a map induced by $T_0 : P_0 \to J_0$, then h_0 and f are $\mathcal{U}$-close, where $\mathcal{U} = \{\operatorname{St}(j, J_0),\ j \in J_0\}$ is a closed cover of S^n such that each nonempty intersection of elements of $\mathcal{U}$ is an absolute retract (in fact, it is a PL ball). Hence (see [**Wa 5**]) h_0 and f are homotopic maps.

This finishes the proof of 3.1.

4. Monotone maps. In §3 we have shown that, in certain range of dimensions, there exist essential, monotone maps $f : S^m \to S^n$ that are one-to-one over the image of a dense set. In this section we show how to construct monotone (inessential) surjections $f : S^m \to S^n$ that are one-to-one over the image of a dense set for any $m, n \geq 3$. If $m > n \geq 4$, the existence of such maps follows from 3.2. Indeed, let $f_i : S^i \to S^{i-1}$ be a monotone surjection that is one-to-one over the image of a dense set ($i = n+1,\ n+2, \dots, m$). Let $g_i : S^i \to S^i$ be a homeomorphism intermingling the two pertinent (countable) dense subsets of S^i ($i = n+1,\ n+2, \dots, m-1$). Then the composition $f_{n+1} g_{n+1} \cdots f_{m-1} g_{m-1} f_m : S^m \to S^n$ is a map with the desired properties. However, we want to present an independent proof that also works for $3 \leq m \leq n$ or $n = 3$.

THEOREM 4.1. *For any $m, n \geq 3$ there exists a monotone surjection $h : S^m \to S^n$ that is one-to-one over the image of a dense set.*

In the proof we need

LEMMA 4.2. *Let $h : X \to Y$ be a surjective map between compact metric spaces. If each nonempty open set in X contains a fiber of h, then h is one-to-one over the image of a dense set.*

PROOF. Suppose not. Let $F_\varepsilon = \bigcup\{h^{-1}(y) : y \in Y,\ \operatorname{diam} h^{-1}(y) \geq \varepsilon\}$. Then F_ε is a closed set for any $\varepsilon > 0$, and $\bigcup\{F_\varepsilon, \varepsilon > 0\}$ has nonempty interior. By Baire's Category Theorem [**Du**, p. 250], there exists $\varepsilon > 0$ such that F_ε has nonempty interior. Let $U \subseteq F_\varepsilon$ be a nonempty open set with $\operatorname{diam} U < \varepsilon$. Then U does not contain any fibers of h, contrary to the hypothesis.

PROOF OF 4.1. We construct a defining sequence $\{T_i\colon P_i \to J_i\}_{i=0}^{\infty}$ with the following properties.

(M1) Each $p \in P_i$ is connected, $i = 0, 1, 2, \ldots$.

(M2) If $p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)} \in P_i$ are mutually distinct elements, then $p_{i(1)} \cap p_{i(2)} \cap p_{i(3)} \cap p_{i(4)} = \varnothing$.

As in §3, we show first how to construct the triple $T_{i+1}\colon P_{i+1} \to J_{i+1}$ from the triple $T_i\colon P_i \to J_i$ already constructed. As before, we suppress indices, starting with an admissible function $T\colon P \to J$ satisfying (M1) and (M2). Let $\hat{T}\colon \hat{P} \to \tilde{J}$ be the triple constructed in §2. Observe that, although it is one-to-one, T will *never* be onto (this is the whole point of the construction).

We have to create new elements of P that will correspond to elements of $\tilde{J} - \operatorname{Im}\hat{T}$, as well as make all elements of $\hat{P}$ connected. Observe that if $j \in \tilde{J}$ with $\operatorname{Ind}(j) \leq 1$, then (by (A3)) $j \in \operatorname{Im}\hat{T}$; and if $\operatorname{Ind}(j) \geq 3$, then (by (M2)) $j \notin \operatorname{Im}\hat{T}$.

First "connect up" all components of elements of $\hat{P}$. The only important property of $\hat{P}$ we use here is that for each disconnected $p_s \in \hat{P}$ there is a *connected* element $\hat{p} \in \hat{P}$ such that each component of p_s intersects $\hat{p}$. (If $p \in s$, $\hat{p} = p_{\{p\}}$ would do; see (H4).) Fixing a triangulation K' of S^m such that all pertinent subsets of S^m are (full) subcomplexes, for each disconnected $p \in \hat{P}$ choose a PL arc α_p lying in a connected element $c(p) \in \hat{P}$ such that $\alpha_p \cap \partial c(p)$ is a finite set intersecting each component of p. We can also arrange that different arcs are disjoint, and lie in the complement of the $(m-2)$-skeleton of K' (here we use $m \geq 3$). Choose a subdivision K'' of K' such that α_p's are subcomplexes with respect to K'', and let N_p be the second derived neighborhood of α_p in $c(p)$. Finally, set

$$p^0 = \begin{cases} p \cup N_p & \text{if } p \in \hat{P} \text{ is disconnected,} \\ p - \bigcup\{N_{p'}\colon p' \in \hat{P} \text{ is disconnected}\} & \text{if } p \in \hat{P} \text{ is connected.} \end{cases}$$

Set $\hat{P}^0 = \{p^0,\ p \in \hat{P}\}$. Then $\hat{T}^0\colon \hat{P}^0 \to \tilde{J}$ defined by $\hat{T}^0(p_s^0) = \hat{T}(p_s)$ is a triple satisfying (A2), (A3), (M1) and (M2).

Now, we create new elements so that $\hat{T}^0$ can be extended to a bijection.

Let $\tilde{J} - \operatorname{Im}\hat{T}^0 = \{j_1, j_2, \ldots, j_r\}$. We can order this set so that $k \leq l$ implies $\operatorname{Ind}(j_k) \leq \operatorname{Ind}(j_l)$. We define a sequence $\{\hat{T}^k\colon \hat{P}^k \to \tilde{J}\}_{k=1}^{r}$ of triples satisfying (A2), (A3), (M1), (M2) and

(a_k) $\tilde{J} - \operatorname{Im}\hat{T}^k = \{j_{k+1}, j_{k+2}, \ldots, j_r\}$;

(b_k) for any $p \in P$, $p' \in \hat{P}^k$,

$$p \cap p' \neq \varnothing \Leftrightarrow \operatorname{Int}(p \cap p') \neq \varnothing \Leftrightarrow T(p) \cap \hat{T}^k(p') \neq \varnothing\,.$$

Clearly, $\hat{T}^0\colon \hat{P}^0 \to \tilde{J}$ satisfies (a_0) and (b_0). Finally, we set $\{\tilde{T}\colon \tilde{P} \to \tilde{J}\} = \{\hat{T}^r\colon \hat{P}^r \to \tilde{J}\}$.

An argument for the inductive step is as follows. Assume the triple $\hat{T}^{k-1}\colon \hat{P}^{k-1} \to \tilde{J}$ satisfies (A2), (A3), (M1), (M2), (a_{k-1}) and (b_{k-1}). Define

$$B = \bigcup\{p \in \hat{P}^{k-1}\colon \hat{T}^{k-1}(p) \cap j_k \neq \varnothing\}.$$

Claim 1. B is connected.

Indeed, let $p_1, p_2 \in \hat{P}^{k-1}$ with $\hat{T}^{k-1}(p_i) \cap j_k \neq \varnothing$, $i = 1, 2$. Let v_1, v_2 be the two vertices of βL such that $\hat{T}^{k-1}(p_i) = \operatorname{St}(v_i, \beta^2 L)$, $i = 1, 2$. Similarly, let v be the vertex of βL, such that $j_k = \mathrm{St}(v, \beta^2 L)$. Let $\sigma, \sigma_1, \sigma_2$ be the simplexes of L whose barycenters are v, v_1, v_2 respectively. Since $\mathrm{St}(v, \beta^2 L) \cap \mathrm{St}(v_i, \beta^2 L) \neq \emptyset$, σ and σ_i are comparable simplexes, $i = 1, 2$ (i.e. one is a face of the other). Choose a sequence $w_1, w_2, \ldots, w_l$ of vertices of σ, such that w_1 is a vertex of σ_1, and w_l is a vertex of σ_2. Say, $\sigma_1 = \langle w_1, a_1, \ldots, a_q \rangle$, $\sigma_2 = \langle w_l, b_1, \ldots, b_u \rangle$. In the sequence $\langle w_1, a_1, \ldots, a_{q-1}, a_q \rangle, \langle w_1, a_1, \ldots, a_{q-1} \rangle, \ldots, \langle w_1, a_1 \rangle, \langle w_1 \rangle, \langle w_1, w_2 \rangle, \langle w_2 \rangle, \langle w_2, w_3 \rangle, \langle w_3 \rangle, \ldots, \langle w_{l-1}, w_l \rangle, \langle w_l \rangle, \langle w_l, b_1 \rangle, \ldots, \langle w_l, b_1, \ldots, b_u \rangle$ any two consecutive simplexes of L are comparable. Let $j^1, j^2, \ldots, j^x$ be the corresponding sequence of elements of J (determined by the barycenters of the simplexes in the sequence). Note that (1) any two consecutive elements in this sequence intersect, (2) we can arrange the vertices of σ_1 and σ_2 so that each element in the sequence intersects j_k, and (3) by our choice of indexing elements of $J - \mathrm{Im}\,\hat{T}^0$ and the fact that $\mathrm{Ind}(j) \leq 1$ implies $j \in \mathrm{Im}\,T^0$, each element in the sequence is in $\mathrm{Im}\,\hat{T}^{k-1}$. Now by (A3) and (M1), $(\hat{T}^{k-1})^{-1}(j^1), (\hat{T}^{k-1})^{-1}(j^2), \ldots, (\hat{T}^{k-1})^{-1}(j^x)$ is a sequence of connected sets whose union contains p_1, p_2 and itself is contained in B such that any two consecutive elements intersect. Hence, B is connected.

Claim 2. If $p \in P$ and if $T(p) \cap j_k \neq \emptyset$, then $\mathrm{Int}(B \cap p) \neq \emptyset$.

Indeed, if $j_k = \mathrm{St}(v, \beta^2 L)$, and if v is a barycenter of a simplex $\sigma = \langle a_1, \ldots, a_l \rangle$ of L, then $T(p) = \mathrm{St}(a_i, \beta L)$ for some i, $1 \leq i \leq l$. But then

$$p' = (\hat{T}^{k-1})^{-1}(\mathrm{St}(a_i, \beta^2 L)) \subseteq B$$

and $\mathrm{Int}(p' \cap p) \neq \emptyset$ (by (b_{k-1})).

Following the well-established pattern, once again choose a triangulation K_k of S^m such that all relevant subsets of S^m are (full) subcomplexes. Let α be a PL arc in $\mathrm{Int}\,B$ that intersects all sets of the form p or $\mathrm{Int}(B \cap p')$ for some $p \in \hat{P}^{k-1}$ with $\hat{T}^{k-1}(p) \cap j_k \neq \emptyset$ or some $p' \in P$ with $T(p') \cap j_k \neq \emptyset$. By Claims 1 and 2 such an arc exists. We can also arrange that it misses the $(m-2)$-skeleton of K_k. Let K'_k be a subdivision of K_k such that α is a subcomplex, and let N be the second derived neighborhood of α (in B). Define $A(p) = p - \mathrm{Int}\,N$ for all $p \in \hat{P}^{k-1}$. Setting $\hat{P}^k = \{A(p) \colon p \in \hat{P}^{k-1}\} \cup \{N\}$ and $\hat{T}^k(A(p)) = \hat{T}^{k-1}(p)$, $T^k(N) = j_k$ defines a triple $\hat{T}^k \colon \hat{P}^k \to \tilde{J}$. The reader should observe that this triple satisfies (A2), (A3), (M1), (M2), (a_k) and (b_k).

We now proceed with the description of a number of improvements on the construction of a triple $\tilde{T} \colon \tilde{P} \to \tilde{J}$. For convenience, we use the following notation. If $p_{i(1)}, \ldots, p_{i(t)} \in P$ with $\bigcap_{r=1}^{t} T(p_{i(r)}) \neq \emptyset$, then by $A(p_{i(1)}, \ldots, p_{i(t)})$ we denote the element of $\tilde{P}$ such that

$$T(A(p_{i(1)}, \ldots, p_{i(t)})) = \mathrm{St}(v, \beta^2 L),$$

where v is the barycenter of the simplex of L whose vertices are determined by "centers" of $T(p_{i(1)}, \ldots, T(p_{i(t)}))$.

(i) Given $p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)} \in P$ with $T(p_{i(1)}) \cap T(p_{i(2)}) \cap T(p_{i(3)}) \cap T(p_{i(4)}) \neq \emptyset$, we can arrange that there exists a PL arc $\alpha \subset \mathrm{Int}\,p_{i(1)}$ such that α intersects the interior of each of the following elements of $\tilde{P}$, and no other elements of $\tilde{P}$: $A(p_{i(1)}), A(p_{i(2)}, p_{i(2)}), A(p_{i(1)}, p_{i(2)}, p_{i(3)}), A(p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)})$.

The trick is first to specify an arc $\alpha \subset \mathrm{Int}\,p_{i(1)}$ that meets "right" elements of $\hat{P}$. In the process of "connecting up", we can choose arcs to miss α, and (choosing a small triangulation of S^m) we can arrange that the "connecting tubes" miss α. Consequently, α hits the "right" elements of $\hat{P}^0$. In the inductive process of constructing partitions $\hat{P}^1, \ldots, \hat{P}^r = \tilde{P}$, we can choose the relevant arcs either to hit or to miss α (according to the nature of the partition element that is about to be constructed).

(ii) Along with the hypotheses as in (i), assume that U is an open set in S^m and $U \cap p_{i(1)} \neq \emptyset$. Then we can arrange that α (which satisfies the conclusion of (i)) is contained in U.

Indeed, if α is any arc as in (i), we can find a PL homeomorphism $\psi\colon S^m \to S^m$ such that $\psi =$ identity off of $\operatorname{Int} p_{i(1)}$ and $\psi(\alpha) \subset U$. Then $\tilde{P}' = \{\psi(\tilde{p})\colon \tilde{p} \in \tilde{P}\}$ and $\alpha' = \psi(\alpha)$ satisfy all conclusions of (ii).

(iii) Given $p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)} \in P$ as in (i) and an arc $\alpha \subset \bigcup_{t=1}^{4} \operatorname{Int} p_{i(t)}$ intersecting each $\operatorname{Int} p_{i(t)}$, $t = 1, 2, 3, 4$, we can arrange that $A(p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)})$ is contained in a prechosen neighborhood U of α.

Using an argument of the same type as in (i), we can arrange that α has all properties as an arc serving as a guide for constructing $A(p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)})$. Then it remains to choose a small triangulation of S^m to get the required containment (in the inductive process, already "born" elements cannot "grow").

The improvements (ii) and (iii), applied to

$$A(p_{i(1)}), A(p_{i(1)}, p_{i(2)}), A(p_{i(1)}, p_{i(2)}, p_{i(3)}), A(p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)}),$$

coupled together yield the following (here we use $n \geq 3$).

(iv) For any nonempty open set $U \subseteq S^m$, in the construction of the defining sequence $\{T_i\colon P_i \to J_i\}_{i=0}^{\infty}$, whenever P_i is given, we can arrange (by carefully choosing P_{i+1} and P_{i+2}) that P_{i+2} contains an element contained in U.

(v) Given $p \in P$ and an open set $U \subseteq S^m$ with $p \subseteq U$, we can arrange that $\operatorname{St}(A(p), \tilde{P}) \subseteq U$.

Indeed, using (H5), we can arrange that $\operatorname{St}(p, \hat{P}) \subseteq U$. In the "connecting up" process, we choose $c(p')$ to be $p_{\{p''\}}$ for some $p'' \in P$. In this way we get $\operatorname{St}(\hat{p}, \hat{P}^0) \subseteq U$ where $\hat{p} = p_{\{p\}}$. Since $\hat{p}$ is connected, we have $\hat{p}^0 \subseteq \hat{p}$ and hence $\operatorname{St}(\hat{p}^0, \hat{P}^0) \subseteq U$. Inductively assume that $\operatorname{St}(\hat{p}^{k-1}, \hat{P}^{k-1}) \subseteq U$, where $\hat{p} \in \hat{P}^{k-1}$ "comes" from $\hat{p} \in \hat{P}$. If $j_k \cap \hat{T}^{k-1}(\hat{p}^{k-1}) = \emptyset$, we have $\operatorname{St}(\hat{p}^k, \hat{P}^k) \subseteq \operatorname{St}(\hat{p}^{k-1}, \hat{P}^{k-1}) \subseteq U$. So assume that $j_k \cap \hat{T}^{k-1}(\hat{p}^{k-1}) \neq \emptyset$. The corresponding set B defined along the inductive argument can be written as $B = B_1 \cup B_2$, where $B_1 = \bigcup\{p' \in \hat{P}^{k-1}\colon p' \subseteq B,\ p' \cap \hat{p}^{k-1} \neq \emptyset\}$, and $B_2 = \bigcup\{p' \in \hat{P}^{k-1}\colon p' \cap \hat{p}^{k-1} = \emptyset\}$. Then $B_1 \subseteq U$ and each $p' \subseteq B_2$ hits B_1. Consequently, if we replace the set B in the inductive argument by the set $B' = B_1 \cup \{$the collection of all components of $B_2 \cap U$ that hit $B_1\}$, the constructed element N will be contained in U, and hence $\operatorname{St}(\hat{p}^k, \hat{P}^k) \subseteq U$.

Improvements (iv) and (v) give the following:

Let $\{U_1, U_2, U_3, \ldots\}$ be a countable basis of open sets for the topology on S^m. Then we can arrange that, for each i, there exists $p(i) \in P_{3i}$ with $\operatorname{St}(p(i), P_{3i}) \subseteq U_i$.

Since by 2.1(i) each of the sets of the form $\operatorname{St}(p, P)$, $p \in P$, contains a fiber of the map $h\colon S^m \to S^n$ determined by such a defining sequence, we conclude that h satisfies the hypotheses of 4.2, and hence it is one-to-one over the image of a dense set. (M1) implies that h is monotone.

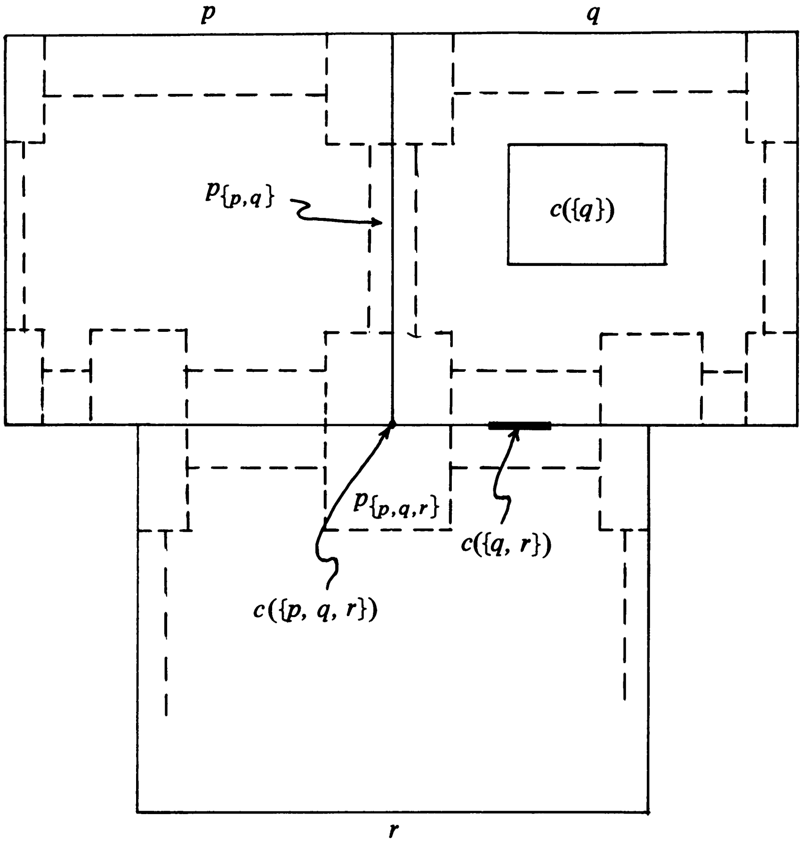

FIGURE 1

To finish the proof of 4.1, we need to construct a triple $T_0: P_0 \to J_0$ satisfying (A1)–(A3) and (M1), (M2). As in §3, we take advantage of the fact that spheres are suspensions.

Let L' be any triangulation of S^{n-1}. Then there exists a partition P of S^{m-1} and an admissible map $T: P \to J$, $J = \{\mathrm{St}(v, \beta L'): v$ is a vertex of $L'\}$ such that if $p_{i(1)}, p_{i(2)}, p_{i(3)} \in P$ are distinct, then $p_{i(1)} \cap p_{i(2)} \cap p_{i(3)} = \varnothing$. We now "suspend" the triple $T: P \to J$. Let J_0 be the standard handlebody decomposition of S^n corresponding to the triangulation $L = \Sigma L'$. P is a partition of $S^{m-1} \subset S^m$. Let P_0' be the partition of S^m consisting of slightly "thickened" copies of $p \in P$ together with two m-balls corresponding to the suspension points. Defining $T_0': P_0' \to J_0$ in the obvious way, the reader should realize that this triple satisfies (A1)–(A3) and (M2). It remains to "connect up" elements of P_0'. Observing that both m-balls in P_0', corresponding to two suspension points, intersect all components of all disconnected elements of P_0', the author leaves this as an exercise.

This completes the proof of 4.1.

REMARK 4.3. In [**Be-Wa**] it is shown that a map $f: S^m \to S^2$ that is one-to-one over the image of a dense set is far from being monotone. Using the technique of this section, one can construct a map $f: S^m \to S^2$ that is one-to-one over the image of a dense set, thus giving an alternative proof of the result in [**Be-Wa**]. One finds a defining sequence $\{T_i\colon P_i \to J_i\}_{i=0}^{\infty}$ with the additional property:

(U) If $p_{i(1)}, p_{i(2)}, p_{i(3)} \in P$ are mutually distinct, then $p_{i(1)} \cap p_{i(2)} \cap p_{i(3)} = \varnothing$.

The argument is slightly easier than the one given in the case of monotone maps, since one does not have to worry about "connecting up" various components.

REMARK 4.4. Maps $h\colon S^m \to S^n$ constructed in 4.1 and 4.3, are totally unstable (i.e. they have no stable values). In fact $h\colon S^m \to S^n$ can be approximated by a map $h_i\colon S^m \to S^n$ induced by the admissible function $T_i\colon P_i \to J_i$, which, in turn, can be approximated by a map $h_i'\colon S^m \to S^n$ induced by the triple $\hat{T}_i\colon \hat{P}_i \to \tilde{J}_i$. If $T_i\colon P_i \to J_i$ satisfies (M2), then $\operatorname{Im} h_i'$ is contained in the union of all elements j of J_i with $\operatorname{Ind}(j) \leq 2$, and hence it is contained in a regular neighborhood of the 2-skeleton of $\beta^i L$. Consequently, $h\colon S^m \to S^n$ constructed as in 4.1 can be approximated by maps that factor through 2-dimensional polyhedra.

REFERENCES

[**Be-Wa**] M. Bestvina and J. J. Walsh, *Mappings between Euclidean spaces that are one to one over the image of a dense subset*, Proc. Amer. Math. Soc. **91** (1984), 449–455.

[**Du**] J. Dugundji, *Topology*, Allyn & Bacon, Boston, Mass., 1966.

[**Sp**] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[**Wa 1**] J. J. Walsh, *Monotone and open mappings on manifolds*. I, Trans. Amer. Math. Soc. **209** (1975), 419–432.

[**Wa 2**] ——, *Light open and open mappings on manifolds*. II, Trans. Amer. Math. Soc. **217** (1976), 271–284.

[**Wa 3**] ——, *Isotoping mappings to open mappings*, Trans. Amer. Math. Soc. **250** (1979), 121–145.

[**Wa 4**] ——, *Extending mappings to monotone mappings*, Houston J. Math. **3** (1977), 579–592.

[**Wa 5**] ——, *Mappings with dense deficiency set*, Michigan Math. J. **28** (1981), 117–127.

[**Wi 1**] D. C. Wilson, *Open mappings on manifolds and a counterexample to the Whyburn conjecture*, Duke Math. J. **40** (1973), 705–716.

[**Wi 2**] ——, *Monotone mappings of manifolds onto cells*, Proc. First Conf. Monotone and Open Mappings (SUNY, Binghamton, 1970), SUNY, Binghamton, New York, 1971, pp. 37–54.

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