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ANALYTIC OPERATOR ALGEBRAS (FACTORIZATION AND AN EXPECTATION)

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ABSTRACT. Let M be a σ-finite von Neumann algebra and $\{\alpha_t\}_{t\in\mathbf{T}}$ a periodic flow on M. The algebra of analytic operators in M is $\{a \in M\colon \mathrm{sp}_\alpha(a) \subseteq \mathbf{Z}_+\}$ and is denoted $H^\infty(\alpha)$. We prove that every invertible operator $a \in H^\infty(\alpha)$ can be written as $a = ub$, where u is unitary in M and $b \in H^\infty(\alpha) \cap H^\infty(\alpha)^{-1}$. We also prove inner-outer factorization results for $a \in H^\infty(\alpha)$.

Another result represents $H^\infty(\alpha)$ as the image of a certain nest subalgebra (of a von Neumann algebra that contains M) via a conditional expectation. As corollaries we prove a distance formula and an interpolation result for the case where M is an injective von Neumann algebra.

1. Introduction. In this paper we intend to study some aspects of analyticity in operator algebras. In [**1**] W. Arveson presented the theory of subdiagonal algebras as a noncommutative analogoue of the theory of weak* Dirichlet algebras.

In [**6**] R. Loebl and P. Muhly have shown that some subdiagonal algebras arise as algebras of analytic operators with respect to a flow on a von Neumann algebras. Similar results were obtained independently by S. Kawamura and J. Tomiyama in [**5**]. Let M be a von Neumann algebra and $\{\alpha_t\}_{t\in\mathbf{R}}$ a σ-weakly continuous representation of $\mathbf{R}$ as *-automorphisms of M (i.e. α is a flow on M). Let $H^\infty(\alpha)$ be the set $\{a \in M\colon \mathrm{sp}_\alpha(a) \subseteq [0,\infty)\}$, where $\mathrm{sp}_\alpha(a)$ is Arveson's spectrum of a with respect to α.

It was shown in [**6**] that $H^\infty(\alpha)$ is an algebra and if M is $\mathbf{R}$-finite, then $H^\infty(\alpha)$ is a maximal subdiagonal algebra. This class of algebras, obtained in this manner from flows on von Neumann algebras, contains the analytic crossed products (also called nonsefladjoint crossed products) and the nest subalgebras (see [**6**] for details). The structure of these algebras $H^\infty(\alpha)$ was further studied by several authors (see [**7–15**]).

In this paper we continue the study of the algebras $H^\infty(\alpha)$ with the assumption that the flow α is periodic and M is a σ-finite von Neumann algebra.

The main result of §3 is Theorem 3.10 which shows that every invertible operator $a \in M$ can be written as $a = ub$, where u is unitary in M and $B \in H^\infty(\alpha) \cap H^\infty(\alpha)^{-1}$. This result was proved in [**1**] for finite maximal subdiagonal algebras. We, however, do not assume that M is finite. The result is also related to a factorization result of an operator with respect to a nest algebra (see [**3**, Theorem 3.3]) and we will comment on this later (see the discussion following Proposition 3.7).

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On the way to the main result we prove some factorization results for vectors in H (where M is represented on a Hilbert space H with a separating and cyclic vector ξ such that $a \to \langle a\xi, \xi\rangle$ is an α-invariant state on M) that extend the classical inner-outer factorization (when $M = L^\infty(\mathbf{T})$ and α acts by translation, $H^\infty(\alpha)$ is the classical Hardy space $H^\infty(\mathbf{T})$). This is also used to obtain results on subspaces of H that are invariant under $H^\infty(\alpha)$ and are generated by a single vector in H.

In §4 we introduce an expectation from a certain von Neumann algebra R_0' onto M such that it maps a certain nest subalgebra of R_0' onto $H^\infty(\alpha)$. If $H = L^2(\mathbf{T})$, $M = L^\infty(\mathbf{T})$ and $H^\infty(\alpha) = H^\infty(\mathbf{T})$, then $R_0' = B(H)$ (the algebra of all bounded operators on H) and this expectation is the one used by Arveson in [**3**, §5]. We believe that this expectation might help to "carry" results that can be proved for nest algebras, or nest subalgebras (of von Neumann algebras) to the algebra $H^\infty(\alpha)$. As examples we prove a distance formula (Corollary 4.7) and an interpolation result (Corollary 4.8).

2. Preliminaries. Let M be a σ-finite von Neumann algebra acting on a Hilbert space H and let $\{\alpha_t\}_{t\in\mathbf{R}}$ be a periodic flow (i.e. a periodic σ-weakly continuous representation of $\mathbf{R}$ as *-automorphisms of M). We assume that the period is 2π and write $\mathbf{T}$ for the interval $[0, 2\pi]$ identified with the unit circle. Since $\mathbf{T}$ is compact, the map ε_0, defined by $\varepsilon_0(x) = \int_0^{2\pi} \alpha_t(x)\, dt$, $x \in M$, is a well-defined, linear, σ-weakly continuous map from M onto the fixed point algebra, M_0, of M with respect to $\{\alpha_t\}$ (dt denotes the normalized Lebesgue measure on $\mathbf{T}$). In fact, ε_0 is a faithful normal expectation from M onto M_0 such that $\varepsilon_0 \circ \alpha_t = \varepsilon_0$, $t \in \mathbf{T}$. By choosing a faithful normal state ϕ_0 of M_0 and letting ϕ be $\phi_0 \circ \varepsilon_0$ we get a faithful normal state ϕ on M that is $\{\alpha_t\}$-invariant (i.e. $\phi \circ \alpha_t = \phi$, $t \in \mathbf{T}$). Considering the Gelfand-Naimark-Segal construction of ϕ, we may suppose that M has a separating and cyclic vector $\xi \in H$ such that $\phi(x) = \langle x\xi, \xi\rangle$ is an $\{\alpha_t\}$-invariant state on M. The algebra M, the flow α, the space H and the vector ξ, as above, will be fixed throughout the paper.

We now define a *canonical pair* to be a pair $\{B, \eta\}$ with the following properties:

(1) B is a σ-finite von Neumann algebra acting on H.

(2) ξ is a separating and cyclic vector for B.

(3) $\{\eta_t\}_{t\in\mathbf{T}}$ is a periodic flow on B.

(4) $\langle a\xi, \xi\rangle = \langle \eta_t(a)\xi, \xi\rangle$ for $t \in \mathbf{T}$, $a \in B$.

The discussion above shows that $\{M, \alpha\}$ is a canonical pair. Property (4) (applied to $\{M, \alpha\}$) shows that we can define unitary operators $\{W_t\colon t \in \mathbf{T}\}$ such that $W_t a\xi = \alpha_t(a)\xi$, $t \in \mathbf{T}$, $a \in M$. Let us now write R for the commutant of M and let γ be the flow on R defined by $\gamma_t(a) = W_t a W_t^*$, $t \in \mathbf{T}$, $a \in R$. Then $\langle \gamma_t(a)\xi, \xi\rangle = \langle W_t a W_t^* \xi, \xi\rangle = \langle a\xi, \xi\rangle$, $a \in R$. Hence $\{R, \gamma\}$ is a canonical pair.

The two canonical pairs $\{M, \alpha\}$ and $\{R, \gamma\}$ will be fixed throughout the paper. In the remainder of this section we will describe some of the structure of a canonical pair and set up notation. Since we will be mostly interested in the pairs $\{M, \alpha\}$ and $\{R, \gamma\}$, we will introduce the notation with respect to $\{M, \alpha\}$ and in parentheses the analogous notation for $\{R, \gamma\}$. Of course, for all the results that will be introduced or defined for $\{M, \alpha\}$, there are analogous results for $\{R, \gamma\}$.

For each $n \in \mathbf{Z}$ we define a σ-weakly continuous linear map ε_n on M (and analogously λ_n for R) by

$$\varepsilon_n(x) = \int_0^{2\pi} e^{-itn}\alpha_t(x)\,dt, \qquad x \in M.$$

Let M_n be $\varepsilon_n(M)$ (and $R_n = \lambda_n(R)$). Then it is easy to check that $M_n = \{x \in M\colon \alpha_t(x) = e^{int}x,\ t \in \mathbf{T}\}$.

For each $n \in \mathbf{Z}$ define a projection f_n by

$$f_n = \sup\{uu^*\colon u \text{ is a partial isometry in } M_n\}$$

$$(g_n = \sup\{uu^*\colon u \text{ is a partial isometry in } R_n\}).$$

Then, by [**12**, Lemma 2.2] the f_n lie in $Z(M_0)$ (the centre of M_0). The following lemma appears in [**11**].

LEMMA 2.1. (1) *For every $n, m \in \mathbf{Z}$, $M_nM_m \subseteq M_{n+m}$, $M_n^* = M_{-n}$.*

(2) *Let $x \in M_n$ and let $x = v|x|$ be the polar decomposition of x.*
Then $v \in M_n$ and $|x| \in M_0$.

The following result can be found in [**14**, Proposition 2.3 and Theorem 2.4]. Although it was assumed there that the algebra M is finite, this assumption was not used for the proof of this result.

PROPOSITION 2.2. *Fix $n \in \mathbf{Z}$. Then there is a sequence $\{v_{n,m}\}_{m=1}^{\infty}$ of partial isometries in M_n with the following properties:*

(1) *$v_{n,m}^*v_{n,j} = 0$ if $m \neq j$,*

(2) *$\sum_{m=1}^{\infty} v_{n,m}v_{n,m}^* = f_n$,*

(3) *$M_n = \sum_{m=1}^{\infty} v_{n,m}M_0$; i.e. every $x \in M_n$ can be written as $\sum_{m=1}^{\infty} v_{n,m}x_m$ for some $x_m \in M_0$, where the sum converges to the σ-weak operator topology.* (*For $\{R, \gamma\}$ we write $\{u_{n,m}\}_{m=1}^{\infty} \subseteq R_n$.*)

For a flow η on a von Neumann algebra B, let $B^{\eta}(S)$ denote the spectral subspace associated with S; i.e. $B^{\eta}(S) = \{x \in B\colon \mathrm{sp}_{\eta}(x) \subseteq S\}$.

We write $H^{\infty}(\alpha)$ for $M^{\alpha}(\{n \in \mathbf{Z}\colon n \geq 0\})$ and $H_0^{\infty}(\alpha)$ for $M^{\alpha}(\{n \in \mathbf{Z}\colon n > 0\})$. (Similarly $H^{\infty}(\gamma)$ and $H_0^{\infty}(\gamma)$ are defined.)

We have, for $n \in \mathbf{Z}$, $M^{\alpha}(\{n\}) = M_n$ and, for $x \in M$, $\mathrm{sp}_{\alpha}(x) = \{n \in \mathbf{Z}\colon \varepsilon_n(x) \neq 0\}$. When it causes no confusion we write H^{∞} in place of $H^{\infty}(\alpha)$.

The following result appears in [**12**, Theorem 2.4].

PROPOSITION 2.3. (1) $H^{\infty}(\alpha) = \{x \in M\colon \varepsilon_n(x) = 0$ *for* $n < 0\}$.

(2) *$H^{\infty}(\alpha)$ is the σ-weakly closed subalgebra of M generated by M_0 and all partial isometries in $\bigcup\{M_n\colon n > 0\}$.*

In fact, by [**9**, Theorem 1] M is linearly spanned by $\bigcup\{M_n\colon n \in \mathbf{Z}\}$ in the σ-weak operator topology.

For a subset $S \subseteq H$ let $[S]$ denote the closed linear subspace spanned by S. For $m \neq n$, $a \in M_n$, $b \in M_m$, we have $\langle a\xi, b\xi\rangle = \langle \alpha_t(b^*a)\xi, \xi\rangle$ for each $t \in \mathbf{T}$ and therefore $\langle a\xi, b\xi\rangle = \langle \varepsilon_0(b^*a)\xi, \xi\rangle = 0$. Hence $\{[M_n\xi]\}_{n\in\mathbf{Z}}$ is an orthogonal family of subspaces. Let E_n be the projection onto $[M_n\xi]$. Then $E_n \in M_0'$ and $\sum E_n = I$ (as ξ is a cyclic vector for M and $[M\xi] = \sum_n \bigoplus [M_n\xi]$). Note also that $W_t = \sum_{n=-\infty}^{\infty} e^{int}E_n$ is the spectral decomposition of $\{W_t\}_{t\in\mathbf{T}}$ (hence we also have that E_n is the projection onto $[R_n\xi]$) and $E_n a\xi = \varepsilon_n(a)\xi$ for $n \in \mathbf{Z}$, $a \in M$. We write H^2 for $\sum_{n=0}^{\infty} \bigoplus [M_n\xi]$ and H_0^2 for $\sum_{n=1}^{\infty} \bigoplus [M_n\xi]$. For $n \in \mathbf{Z}$ we let P_n be the orthogonal projection onto $\sum_{m=n}^{\infty} \bigoplus [M_n\xi]$; i.e. $P_n = \sum_{m=n}^{\infty} E_m$ (and P_n is the projection onto $[M^\alpha[n,\infty)\xi] = [R^\gamma[n,\infty)\xi]$). Then $P_{n+1} \le P_n$ for $n \in \mathbf{Z}$, $\bigvee_{n=-\infty}^{\infty} P_n = I$, and $\bigwedge_{n=-\infty}^{\infty} P_n = 0$.

With the partial isometries $\{v_{n,m} \colon n \in \mathbf{Z},\ m \ge 1\}$ defined as in Proposition 2.2, we can define maps $\{\beta_n\}_{n\in\mathbf{Z}}$ on M_0' by the formula

$$\beta_n(T) = \sum_{m=1}^{\infty} v_{n,m} T v_{n,m}^*$$

(and on $R_0', \eta_n(T) = \sum_{m=1}^{\infty} u_{n,m} T u_{n,m}^*$).

We summarize some of the properties of $\{\beta_n\}_{n\in\mathbf{Z}}$ that will be used later in the following (for its proof see [**15**, Lemma 2.4]).

LEMMA 2.4. *Fix $n, m \in \mathbf{Z}$.*

(1) *β_n is a well-defined *-homomorphism from M_0' onto $f_n M_0'$ whose restriction to $f_{-n}M_0'$ is a *-isomorphism onto $f_n M_0'$.*

(2) *Suppose $Q \in M_0'$ is a projection. Then*

$$\beta_n(Q) = \bigvee \{uQu^* \colon u \in M_n \text{ is a partial isometry}\}.$$

(3) *For $T \in M_0'$, $\beta_n\beta_m(T) = f_n\beta_{n+m}(T)$.*

(4) *For $T \in M_0'$, T lies in M' $(= R)$ if and only if $\beta_j(T) = f_jT$ for each $j \in \mathbf{Z}$.*

(5) *$\beta_n(E_0) = E_n$ (hence $\beta_m(E_n) = \beta_m\beta_n(E_0) = f_nE_{n+m}$ and, consequently, $\beta_m(P_n) = f_nP_{n+m}$).*

The following notation and definitions will be used later:

1. A projection $Q \in M_0'$ is said to be an *M-wandering projection* if, for each $n \in \mathbf{Z}$, $Q\beta_n(Q) = 0$ (note that this implies that $\beta_n(Q)\beta_m(Q) = 0$ for $n \ne m$). The set of all the M-wandering projections in M_0' will be denoted $\mathcal{P}_1$.
2. For $Q \in \mathcal{P}_1$ we let $\sigma(Q)$ be $\sum_{n=0}^{\infty} \beta_n(Q)$ and we write $\mathcal{P}_3$ for $\sigma(\mathcal{P}_1)$.
3. A closed subspace $\mathcal{M}$ of H is called invariant if $ax \in \mathcal{M}$ for each $a \in H^\infty(\alpha)$ and $x \in \mathcal{M}$. Let us denote by $\mathcal{P}_2$ the set of all orthogonal projections whose range is an invariant subspace (as $M_0 \subseteq H^\infty(\alpha)$, $\mathcal{P}_2 \subseteq M_0'$).
4. For $P \in \mathcal{P}_2$ we let $\delta(P)$ be $P - \vee\{\beta_n(P) \colon n > 0\}$.

The following lemma was proved in [**14**].

LEMMA 2.5. *If $P \in \mathcal{P}_2$, then $\delta(P) \in \mathcal{P}_1$ and*

$$P = \sigma(\delta(P)) + \bigwedge_{n>0} \bigvee_{m\ge n} \beta_m(P).$$

The projection $\bigwedge_{n>0}\bigvee_{m\ge n}\beta_m(P)$ lies in M' and $P \in \mathcal{P}_3$ iff $\bigwedge_{n>0}\bigvee_{m\ge n}\beta_m(P) =$ 0.

Note that if $Q, P \in \mathcal{P}_2$ and $Q \le P \in \mathcal{P}_3$ $(= \sigma(\mathcal{P}_1))$ then $Q \in \mathcal{P}_3$ since $\bigwedge_{n>0}\bigvee_{m\ge n}\beta_n(Q) \le \bigwedge_{n>0}\bigvee_{m\ge n}\beta_n(P) = 0$. In particular any $Q \in \mathcal{P}_2$ that satisfies $Q \le P_0$ (i.e. $Q(H)$ is an invariant subspace, contained in H^2) is in $\mathcal{P}_3$.

LEMMA 2.6. *$H^\infty(\alpha) = M \cap \mathrm{Alg}\{P_n \colon n \in \mathbf{Z}\}$; i.e. $a \in H^\infty(\alpha)$ if and only if $a \in M$ and $P_naP_n = aP_n$ for each $n \in \mathbf{Z}$.*

PROOF. Immediate from [**6**, Corollary 2.14]. □

LEMMA 2.7. *Let $c(E_n)$ be the central support of E_n (as a projection in M_0'). Then $f_n = c(E_n)$.*

PROOF. Since $f_n \in Z(M_0)$ and $f_n E_n = E_n$ we have $f_n \geq c(E_n)$. Suppose z is a nonzero projection in $Z(M_0)$ such that $zE_n = 0$ and $z \leq f_n$. Then there is a nonzero partial isometry $v \in M_n$ such that $vv^* \leq z$. Since $zE_n = 0$ and $v\xi \in E_n(H)$, $zv\xi = 0$ and $vv^*v\xi = 0$. Hence $v\xi = 0$. Since ξ is a separating vector and $v \neq 0$ we get a contradiction. Therefore $f_n = c(E_n)$. □

3. Factorization and invariant subspaces. For $g \in H$ let $\mathcal{M}(g)$ be $[H^\infty g]$ and $P(g)$ be the projection onto $\mathcal{M}(g)$. Then $P(g) \in \mathcal{P}_2$.

PROPOSITION 3.1. *For $g \in H$ the following two conditions are equivalent:*

(1) *$P(g) \in \mathcal{P}_3$.*

(2) *There is a partial isometry $U \in R$ such that $U^*g \in H^2$ and $UU^*g = g$.*

Moreover,

(3) *If (2) is satisfied, then we can choose U in such a way that $U^*U \in R_0$ and $\mathcal{M}(U^*g) = \mathcal{M}(E_0U^*g)$.*

(4) *If (1) is satisfied and $P(g) = \sigma(Q)$, $Q \in \mathcal{P}_1$, then $[M_0Qg] = Q(H)$ and $\beta_n(Q)(H) = [M_nQg]$ for $n \in \mathbf{Z}$.*

PROOF. (2) *implies* (1). Suppose that $f = U^*g \in H^2$ and $Uf = g$ for some partial isometry $U \in R$. Then $P(g) = UP(f)U^*$. But $P(g) \leq P_0$, hence $P(f) \in \mathcal{P}_3$. Now if $P(f) = \sigma(Q)$, $Q \in P_1$, then $P(g) = \sigma(UQU^*)$ and $UQU^* \in P_1$.

(1) *implies* (2). Suppose now that $P(g) \in \mathcal{P}_3$, hence $P(g) = \sigma(\delta(P(g)))$, and let Q be $\delta(P(g))$. Let a be in M_n for some $n > 0$. Then $a = \sum v_{n,m}x_m$ for some x_m in M_0 and $aQg = \sum v_{n,m}Qx_mg \in \beta_n(Q)(H)$. Since Q is M-wandering, $h = Qg$ is a wandering vector in the sense that $\langle ah, h\rangle = 0$ for $a \in M$ with $\varepsilon_0(a) = 0$. Hence $\langle ah, h\rangle = \langle \varepsilon_0(a)h, h\rangle$ for all $a \in M$. The map $a \to \langle ah, h\rangle$ is a normal positive linear functional on M_0. Consider M_0 as acting on $E_0(H) = [M_0\xi]$; then ξ is a cyclic and separating vector for M_0 and thus (by [**4**, Theorem 4, Part III, Chapter 1]) there is some $h_0 \in [M_0\xi]$ such that $\langle ah_0, h_0\rangle = \langle ah, h\rangle$ for every $a \in M_0$. For $a \in M_n$, $n \neq 0$, $ah_0 \in E_n(H)$ and $\langle ah_0, h_0\rangle = 0$ $(= \langle ah, h\rangle)$. Hence $\langle ah_0, h_0\rangle = \langle ah, h\rangle$ for each $a \in M$.

Now define a partial isometry U on H by extending the isometric map $ah_0 \to ah$, $a \in M$. Then the initial subspace of U is $[Mh_0]$ and the final subspace is $[Mh]$.

Since, for $b \in M$, $Ubah_0 = bah = bUah_0$, $U \in M' = R$.

Since $W_tah_0 = \alpha_t(a)W_th_0 = \alpha_t(a)h_0$ for $a \in M$, $t \in \mathbf{T}$, $W_t[Mh_0] = [\alpha_t(M)h_0] = [Mh_0]$ for each $t \in \mathbf{T}$. Hence $W_tU^*UW_t = U^*U$. Thus $U^*U \in R_0$. Let f be U^*g.

It is left to show that, with this choice of U, we get $\mathcal{M}(f) = \mathcal{M}(E_0f)$. For this, note that we have:

(a)

$$[M_0(1-Q)g] = (1-Q)[M_0g] \subseteq \sum_{n=1}^{\infty} \beta_n(Q)(H)$$

(as $[H^\infty g] = \sum_{n=0}^{\infty} \beta_n(Q)(H)$).

(b) $[M_0Qg] \subseteq Q(H)$.

(c) $[H_0g] \subseteq \sum_{n=1}^{\infty} \beta_n(Q)(H)$ (as $a\beta_n(Q)g \in \beta_{n+k}(Q)(H)$ for $a \in M_k$).

This implies that $Q(H) = Q[H^\infty g] = Q[M_0 g] = [M_0 Qg] = [M_0 h]$. Also, for $n \in \mathbf{Z}$, $[M_n h] = V\{[vM_0 h]\colon v \in M_n \text{ is a partial isometry}\} = \beta_n(Q)(H)$. Hence $[H^\infty g] = [H^\infty h]$. But

$$\mathcal{M}(f) = [H^\infty f] = [H^\infty U^* g] = U^*[H^\infty g] = U^*[H^\infty h] = [H^\infty h_0],$$

and

$$\begin{aligned} f - h_0 = U^*(g - Qg) = U^*(1-Q)g &\in U^* \sum_{n=1}^{\infty} \beta_n(Q)(H) \\ &= U^* \sum_{n=1}^{\infty} \bigoplus [M_n h] = \sum_{n=1}^{\infty} \bigoplus [M_h h_0] \subseteq \sum_{n=1}^{\infty} E_n(H). \end{aligned}$$

Hence $h_0 = E_0 f$ and $[H^\infty f] = [H^\infty h_0] = [H^\infty E_0 f]$. □

The following proposition shows the uniqueness of the above representation.

PROPOSITION 3.2. *Suppose $g \in H$ and $g = U_1 f^1 = U_2 f^2$ such that, for $i = 1, 2$,*

(1) *U_i is a partial isometry in R with initial subspace $[Mf^i]$,*

(2) *$[H^\infty f^i] = [H^\infty E_0 f^i] \subseteq H^2$.*

Then $U_2^* U_1$ and $U_1^* U_2$ lie in R_0.

PROOF. Since g has such a representation, $P(g) = \sum_{n=0}^{\infty} \beta_n(Q)$ for some $Q \in P_1$. We first claim that $U_i^* Qg \in E_0(H)$, $i = 1, 2$. Fix $i = 1$ or 2 and let h_0 be $U_i^* Qg$. Then $h_0 = U_i^* Qg \in U_i^*[H^\infty g] = [H^\infty f^i] \subseteq H^2$. Using Proposition 3.1(4) we see that $Q(H) = [M_0 Qg]$, $\beta_n(Q)(H) = [M_n Qg]$ for $n \in \mathbf{Z}$, and $[H^\infty g] = \sum_{n=0}^{\infty} \beta_n(Q)(H) = [H^\infty Qg]$. Hence $[H^\infty E_0 f^i] = [H^\infty f^i] = U_i^*[H^\infty g] = U_i^*[H^\infty Qg] = [H^\infty h_0]$ and consequently

$$[H_0^\infty E_0 f^i] = [H_0^\infty [H^\infty E_0 f^i]] = [H_0^\infty [H^\infty h_0]] = [H_0^\infty h_0].$$

We then have $[M_0 h_0] = [M_0 E_0 f^i]$ since

$$[H^\infty E_0 f^i] = [M_0 E_0 f^i] \oplus [H_0^\infty E_0 f^i]$$

and

$$\begin{aligned} [H^\infty h_0] &= U_i^*[H^\infty Qg] = U_i^*([H_0^\infty Qg] \oplus [M_0 Qg]) \\ &= [H_0^\infty h_0] \oplus [M_0 h_0] = [H_0^\infty E_0 f^i] \oplus [M_0 h_0]. \end{aligned}$$

In particular $h_0 \in E_0(H)$. Now,

$$\begin{aligned} f^i - h_0 = U_i^*(1-Q)g &\in U_i^* \sum_{n=1}^{\infty} \beta_n(Q)(H) = U_i^* \sum_{n=1}^{\infty} \bigoplus [M_n Qg] \\ &= \sum_{n=1}^{\infty} \bigoplus [M_n h_0] \subseteq \sum_{n=1}^{\infty} E_n(H). \end{aligned}$$

Hence $h_0 = E_0 f^i$.

Let F_n^i be the projection onto $[M_n h_0]$. Then $U_i[M_n h_0] = [M_n Qg] = \beta_n(Q)(H)$. Hence $U_i F_n^i U_i^* = \beta_n(Q)$. As the initial subspace of U_i is $[Mf^i] = [M[H^\infty f^i]] = [M[H^\infty h_0]] = [Mh_0] = \sum_{n=-\infty}^{\infty} F_n^i(H)$, $U_i E_n U_i^* = U_i F_n^i U_i^* = \beta_n(Q)$. In particular $U_1 E_n U_1^* = U_2 E_n U_2^*$ for each $n \in \mathbf{Z}$. Consequently $U_1 W_t U_1^* = U_2 W_t U_2^*$ for each $t \in \mathbf{T}$ and $U_2^* U_1 W_t U_1^* U_1 = U_2^* U_2 W_t U_2^* U_1$. But $U_1^* U_1, U_2^* U_2 \in R_0$, hence $U_2^* U_1 W_t = W_t U_2^* U_1$, $t \in \mathbf{T}$. Thus $U_2^* U_1 \in R_0$. □

LEMMA 3.3. *Suppose g lies in H^2. Then there is a partial isometry $U \in H^\infty(\gamma)$ such that $U^*U \in R_0$, $U^*g \in H^2$ and $\mathcal{M}(U^*g) = \mathcal{M}(E_0U^*g)$.*

PROOF. Since $g \in H^2$, $P(g) \in \mathcal{P}_3$ and we can construct a partial isometry $U \in R$ as in Proposition 3.1. It is left to show that U, as constructed there, lies in $H^\infty(\gamma)$. Let $P(g)$ be $\sum_{n=0}^{\infty} \beta_n(Q)$, for some $Q \in \mathcal{P}_1$, then, from the way U was constructed, $UE_0f = Qg$ where $f = U^*g$. Therefore $U[M_nE_0f] = [M_nQg] \subseteq [M_nH^2] \subseteq P_n(H)$. Since the initial subspace of U is $\sum_{n=-\infty}^{\infty} \bigoplus [M_nE_0f]$, $UE_nU^* \leq P_n$ and, consequently, $UP_nU^* \leq P_n$. Hence $U \in R \cap \mathrm{Alg}\{P_n : n \in \mathbf{Z}\} = H^\infty(\gamma)$ (by Lemma 2.6). $\square$

COROLLARY 3.4. *Let g be in H^2. Then there are partial isometries $U \in H^\infty(\gamma)$, $V \in H^\infty(\alpha)$, such that:*

(1) *U^*g and V^*g lie in H^2.*

(2) *$UU^*g = VV^*g = g$.*

(3) *$U^*U \in R_0$, $V^*V \in M_0$.*

(4) *$[H^\infty(\alpha)U^*g] = [H^\infty(\alpha)E_0U^*g]$ and $[H^\infty(\gamma)V^*g] = [H^\infty(\gamma)E_0V^*g]$.*

PROOF. The assertions for U follow from Lemma 3.3 and the assertions for V follow from Lemma 3.3 applied to $\{R,\gamma\}$ in place of $\{M,\alpha\}$. $\square$

Let M be the algebra $L^\infty(\mathbf{T})$ acting on $L^2(\mathbf{T})$ and let $\{\alpha_t\}_{t\in\mathbf{T}}$ be defined by translations. Then $H^\infty(\alpha)$ is the classical algebra H^∞. In this case $M = R$ and $\alpha = \gamma$. An element $f \in H^2$ is called outer, in this case, if $[H^\infty f] = H^2$. But this is equivalent to $[H^\infty f] = [H^\infty E_0 f] \neq 0$ (note that in this case, E_0f is just the zeroth Fourier coefficient of f). Hence Lemma 3.3 is the inner-outer factorization in the classical case. It is also known, in this case, that $f \in H^2$ is outer if and only if $\exp \int \log|f|\,dt = |E_0f| \neq 0$. Since, by Szëgo's theorem, $\exp\int \log|f|\,dt = \inf\{\|f-g\| : g \in [H_0^\infty f]\}$, $f \in H^2$ is outer if and only if $|E_0f| = \inf\{\|f-g\| : g \in [H_0^\infty f]\} \neq 0$.

In our more general setting we say that $f \in H^2$ is *R-outer* if $[H^\infty(\alpha)f] = [H^\infty(\alpha)E_0f] \neq \{0\}$ and *M-outer* if $[H^\infty(\gamma)f] = [H^\infty(\gamma)E_0f] \neq \{0\}$. Also we say that $U \in R$ is *R-inner* if U is a partial isometry in $H^\infty(\gamma)$ with $U^*U \in R_0$ and $V \in M$ is *M-inner* if V is a partial isometry in $H^\infty(\alpha)$ with $V^*V \in M_0$. Then Corollary 3.4 presents the M-inner-outer and the R-inner-outer factorizations (which are the classical inner-outer factorization if $M = L^\infty(\mathbf{T})$ and α acts by translations). The next proposition is analogous to the characterization of an outer function, mentioned above, in the classical case.

PROPOSITION 3.5. *For $f \in H^2$ the following conditions are equivalent:*

(1) *f is R-outer (resp. f is M-outer).*

(2) *$[H^\infty(\alpha)f] = N_0H^2$, where N_0 is a nonzero projection in R_0 (resp. $[H^\infty(\gamma)f] = N_0H^2$, $0 \neq N_0 \in M_0$).*

(3) *$\|E_0f\| = \inf\{\|(1-a)f\| : a \in H_0^\infty(\alpha)\} \neq 0$ (resp. $\|E_0f\| = \inf\{\|(1-a)f\| : a \in H_0^\infty(\gamma)\} \neq 0$).*

PROOF. It will suffice to prove the R-version.

(1) *implies* (2). Suppose $f \in H^2$ is R-outer. Then $[H^\infty(\alpha)f] = [H^\infty(\alpha)E_0f] = \sum_{n=0}^{\infty} \bigoplus [M_nE_0f]$. Let F_0 be the projection onto $[M_0E_0f]$. Then $\beta_n(F_0)(H) =$

$[M_n E_0 f]$ for all $n \in \mathbf{Z}$. Let N_0 be $\sum_{n=-\infty}^{\infty} \beta_n(F_0)$. Then

$$\beta_m(N_0) = f_m \sum_{n=-\infty}^{\infty} \beta_{n+m}(F_0) = f_m N_0 \quad \text{for each } m \in \mathbf{Z}.$$

Hence $N_0 \in R$. Also, as $\beta_n(F_0) \leq E_n$ for each $n \in \mathbf{Z}$, $N_0 \in \{E_n \colon n < \mathbf{Z}\}'$. Thus $N_0 \in R_0$. Now $N_0 H^2 = \sum_{n=0}^{\infty} \beta_n(F_0)(H) = [H^\infty(\alpha) E_0 f] = [H^\infty(\alpha) f]$.

(2) *implies* (3). Suppose $[H^\infty(\alpha) f] = N_0 H^2$. Then

$$[H_0^\infty(\alpha) f] = [H_0^\infty(\alpha)[H^\infty(\alpha) f]] = [H_0^\infty(\alpha) N_0 H^2] = N_0 H^2.$$

Hence

$$\inf\{\|(1-a)f\| \colon a \in H_0^\infty(\alpha)\} = \inf\{\|f - g\| \colon g \in N_0 H^2\}.$$

Since $f = N_0 f$,

$$\inf\{\|(1-a)f\| \colon a \in H_0^\infty(\alpha)\} = \inf\{\|f - g\| \colon g \in H_0^2\} = \|E_0 f\|.$$

If $\|E_0 f\| = 0$ then $f \in H_0^2$ and $[H^\infty(\alpha) f] \subseteq H_0^2$. Hence $N_0 H^2 \subseteq H_0^2$. But $N_0 E_0 \subseteq E_0$ (as $N_0 \in R_0$). Therefore

$$N_0 = \sum_{n=-\infty}^{\infty} N_0 E_n = \sum_{n=-\infty}^{\infty} N_0 \beta_n(E_0) = \sum_{n=-\infty}^{\infty} \beta_n(N_0 E_0) = 0.$$

(3) *implies* (1). Suppose $\|E_0 f\| = \inf\{\|(1-a)f\| \colon a \in H_0^\infty(\alpha)\}$. Let P be the projection onto $[H_0^\infty(\alpha) f]$. Then $P \leq P_1$ and $(1-P)f = E_0 f + (P_1 - P)f$. Hence

$$(\inf\{\|(1-a)f\| \colon a \in H_0^\infty(\alpha)\})^2 = \|(1-P)f\|^2 = \|E_0 f\|^2 + \|(P_1 - P)f\|^2.$$

Thus $P_1 f = Pf$ and $E_0 f = (1-P)f \in [H^\infty(\alpha) f]$. It is left to show that $f \in [H^\infty(\alpha) E_0 f]$. Let F_m be the projection onto $[M_m E_0 f]$. Then $F_m \leq E_m$ and $\sum_{m=0}^{\infty} F_m(H) = [H^\infty(\alpha) E_0 f]$. Suppose $E_m f \in [H^\infty(\alpha) E_0 f]$ for all $0 \leq m < j$ (note that $E_0 f \in [H^\infty(\alpha) E_0 f]$). Then for each $a \in H_0^\infty(\alpha)$, $E_j a f = \sum_{m=-\infty}^{\infty} E_j a E_m f = \sum_{m=0}^{j-1} E_j a E_m f$. But $a E_m f \in [H_0^\infty(\alpha)[H^\infty(\alpha) E_0 f]] \subseteq [H^\infty(\alpha) E_0 f]$. Hence

$$E_j a E_m f \in E_j [H^\infty(\alpha) E_0 f] = E_j \sum_{m=0}^{\infty} F_m(H) = F_j(H) \subseteq [H^\infty(\alpha) E_0 f].$$

Therefore $E_j a f \in [H^\infty(\alpha) E_0 f]$. This induction argument shows that $E_j a f \in [H^\infty(\alpha) E_0 f]$ for all $a \in H_0^\infty(\alpha)$, $j \geq 0$. Hence $[H_0^\infty(\alpha) f] \subseteq [H^\infty(\alpha) E_0 f]$. Since $f = E_0 f + P_1 f = E_0 f + Pf$ and $Pf \in [H_0^\infty(\alpha) f]$, we are done. □

COROLLARY 3.6. *Suppose $g \in H$. Then the invariant subspace $[H^\infty(\alpha) g]$ can be written as the orthogonal sum of a reducing subspace $\mathcal{M}_0$ (i.e. $a\mathcal{M}_0 \subseteq \mathcal{M}_0$ for all $a \in M$) and a space of the form UH^2, where U is a partial isometry in R with $U^*U \in R_0$.*

PROOF. Let $P(g)$ be the projection onto $[H^\infty(\alpha) g]$. Then

$$P(g) = \sum_{n=0}^{\infty} \beta_n(\delta(P(g))) + F_0,$$

where F_0 is a projection in R, orthogonal to $\sum_{n=-\infty}^{\infty} \beta_n(\delta(P(g)))$. Hence

$$[H^\infty(\alpha)(1-F_0)g] = (1-F_0)[H^\infty g] = \sum_{n=0}^{\infty} \beta_n(Q)(H)$$

(for $Q = \delta(P(g)) \in \mathcal{P}_1$). Hence $(1-F_0)g$ has a representation $(1-F_0)g = Uf$ as described in Proposition 3.1. Thus

$$[H^\infty(\alpha)(1-F_0)g] = U[H^\infty(\alpha)f] = U[H^\infty(\alpha)E_0 f] = UH^2$$

and

$$[H^\infty(\alpha)g] = [H^\infty(\alpha)F_0 g] \oplus UH^2 = F_0(H) \oplus UH^2. \quad \square$$

We now define $a \in M$ (resp. $a \in R$) to be *M-outer* (resp. *R-outer*) if $a\xi$ is M-outer (resp. R-outer).

PROPOSITION 3.7. *For $a \in M$ the following conditions are equivalent.*

(1) *There is a partial isometry $U \in M$ such that $U^*a \in H^\infty(\alpha)$ and $UU^*a = a$.*

(2) *Let P be the projection onto $[aH^2]$. Then $P = \sum_{n=0}^{\infty} \eta_n(Q)$ for some R-wandering projection $Q \in R_0'$.*

(3) $\bigcap_n [aP_n(H)] = \{0\}$.

(4) *There is a partial isometry $U \in M$ such that $U^*a \in H^\infty(\alpha)$ is M-outer and $UU^*a = a$.*

Moreover, if (4) is satisfied and U_1, U_2 are two such partial isometries such that the initial subspace of U_i is $[U_i^*aH]$, then $U_1^*U_2$ and $U_2^*U_1$ lie in M_0.

PROOF. Note that $[aH^2] = [aH^\infty(\gamma)\xi] = [H^\infty(\gamma)a\xi]$ and $[U_i^*aH] = [U_i^*aR\xi] = [RU_i^*a\xi]$. Hence the equivalence of (1), (2) and (4) and the uniqueness statement follow from Propositions 3.1 and 3.2 (with $\{M,\alpha\}$ replaced by $\{R,\gamma\}$). We now show that (2) is equivalent to (3). For this note that $P = \sum_{n=0}^{\infty} \eta_n(Q)$ for some R-wandering projection $Q \in R_0'$ if and only if $\bigwedge_{m=0} \bigvee_{n\geq m} \eta_n(P) = 0$ (Lemma 2.5). But

$$\bigvee_{n\geq m} \eta_n(P) = \bigvee_{n\geq m} \left(\bigvee\{uPu^*\colon u \in R_n \text{ is a partial isometry}\}\right)$$

$$= \bigvee \left\{ uPu^*\colon u \text{ is a partial isometry in } \bigcup_{n\geq m} M_n \right\}.$$

Since $R^\gamma[m,\infty)$ is generated by the partial isometries in $\bigcup_{n\geq m} M_n$ (as a σ-weakly closed linear space), $\bigvee_{n\geq m} \eta_n(P)$ is the projection onto

$$[R^\gamma[m,\infty)P(H)] = [R^\gamma[m,\infty)[H^\infty(\gamma)a\xi]] = [R^\gamma[m,\infty)a\xi] = [aP_m(H)]$$

for $m \geq 0$. Hence $\bigwedge_{m=0} \bigvee_{n\geq m} \eta_n(P) = 0$ if and only if $\bigcap_{m>n} [aP_m(H)] = \{0\}$. Since $\bigcap_{m\in\mathbf{Z}} [aP_m(H)] = \{0\}$ if and only if $\bigcap_{m>0} [aP_m(H)] = \{0\}$, we are done. $\square$

The previous proposition presents the factorization for operators $a \in M$ and its uniqueness. Consider now the nest algebra $\mathcal{A} = \mathrm{Alg}\{P_n\colon n \in \mathbf{Z}\}$ in $B(H)$. In [**3**, Theorem 3.3] Arveson presented a factorization for operators A in $B(H)$ (that satisfy a certain condition) as $A = UB$, where U is a partial isometry in $B(H)$ with $U^*U \in \mathcal{A} \cap \mathcal{A}^*$ and B is outer (in the sense that $[AP_n(H)] = [AH] \cap P_n(H)$ and the projection onto $[AH]$ lies in $\mathcal{A} \cap \mathcal{A}^*$). The condition for $A \in B(H)$ to have such a factorization is $\bigcap_n [AP_n(H)] = \{0\}$. Hence Proposition 3.7 shows that $a \in M$

satisfies our condition for factorization if and only if it satisfies Arveson's condition for factorization (with respect to $\mathcal{A} = \mathrm{Alg}\{P_n : n \in \mathbf{Z}\}$). The next result shows that $b \in H^\infty(\alpha)$ is M-outer if and only if it is outer in Arveson's terminology.

PROPOSITION 3.8. *For $b \in H^\infty(\alpha)$ the following statements are equivalent:*
(1) *b is M-outer.*
(2) $[bH^2] = [\varepsilon_0(b)H^2] \neq \{0\}$.
(3) $[bP_n(H)] = [\varepsilon_0(b)P_n(H)]$ *for each $n \in \mathbf{Z}$ and* $[\varepsilon_0(b)H^2] \neq \{0\}$.
(4) $[bH^2] = QH^2$ *for some nonzero projection $Q \in M_0$.*
(5) $0 \neq \|\varepsilon_0(b)\xi\| = \inf\{\|b\xi - g\| : g \in [bH_0^2]\}$.
(6) *The projection onto $[bH]$ lies in M_0 and, for each $n \in \mathbf{Z}$, $[bP_n(H)] = [bH] \cap P_n(H)$.*

PROOF. The equivalence of (1), (2), (4) and (5) follows from Proposition 3.5 with the observations that

$$[bH^2] = [bH^\infty(\gamma)\xi] = [H^\infty(\gamma)b\xi],$$

$$[\varepsilon_0(b)H^2] = [\varepsilon_0(b)H^\infty(\gamma)\xi] = [H^\infty(\gamma)\varepsilon_0(b)\xi] = [H^\infty(\gamma)E_0 b\xi],$$

and

$$[bH_0^2] = [bH_0^\infty(\gamma)\xi] = [H_0^\infty(\gamma)b\xi].$$

(2) *implies* (3).

$$\begin{aligned}[pP_n(H)] &= [bR^\gamma[n,\infty)\xi] = [R^\gamma[n,\infty)b\xi] = [R^\gamma[n,\infty)[H^\infty(\gamma)b\xi]] \\ &= [R^\gamma[n,\infty)[H^\infty(\gamma)\varepsilon_0(b)\xi]] = [R^\gamma[n,\infty)\varepsilon_0(b)\xi] = [\varepsilon_0(b)P_n(H)].\end{aligned}$$

(3) implies (2) is trivial.

(3) *implies* (6). $[bH] = \bigvee\{[bP_n(H)] : n \in \mathbf{Z}\} = \bigvee\{[\varepsilon_0(b)P_n(H)] : n \in \mathbf{Z}\} = [\varepsilon_0(b)H]$. Hence the projection onto $[bH]$ lies in M_0. Also $[bH] \cap P_n(H) = P_n[bH] = P_n[\varepsilon_0(b)H] = [\varepsilon_0(b)P_n(H)] = [bP_n(H)]$.

(6) *implies* (4). Let $Q \in M_0$ be the projection onto $[bH]$. Then $[bH^2] = [bH] \cap P_0(H) = Q(H) \cap P_0(H) = QP_0(H) = QH^2$. □

LEMMA 3.9. *Suppose $a \in H^\infty(\alpha)$ is invertible; then a is M-outer if and only if $a^{-1} \in H^\infty(\alpha)$.*

PROOF. Suppose $a^{-1} \in H^\infty(\alpha)$; then $[aH^2] = H^2$ and, hence, a is outer by Proposition 3.8(4). Suppose now that a is invertible and M-outer. Then $H = [aH] = [\varepsilon_0(a)H]$. Hence $\xi \in [\varepsilon_0(a)H]$ and, as $P_0\xi = \xi$, $\xi \in P_0[\varepsilon_0(a)H] = [\varepsilon_0(a)H^2] = [aH^2]$. Hence there are $\{x_n\} \subseteq H^\infty(\alpha)$ such that $ax_n\xi \to \xi$ and, by applying a^{-1}, $x_n\xi \to a^{-1}\xi$. But $x_n\xi \in H^2$, hence $a^{-1}\xi \in H^2$. For $n < 0$, $\varepsilon_n(a^{-1})\xi = E_n a^{-1}\xi = 0$. Thus $a^{-1} \in H^\infty(\alpha)$. □

The following result was proved in [**3**] for nest algebras and in [**1**] for maximal finite subdiagonal algebras.

THEOREM 3.10. *Suppose $a \in M$ is invertible. Then we can write $a = ub$, where u is unitary in M and $b \in H^\infty(\alpha) \cap H^\infty(\alpha)^{-1}$.*

PROOF. First note that if $x \in \bigcap_n [aP_n(H)] = \bigcap_n \{ay : y \in P_n(H)\}$, then $a^{-1}x \in \bigcap_n P_n(H) = \{0\}$. Hence $\bigcap_n [aP_n(H)] = \{0\}$ and we can apply Proposition 3.7 to write $a = wc$, where w is a partial isometry in M, $w^*wc = c$ and $c \in H^\infty(\alpha)$ is M-outer. Now let $\varepsilon_0(c) = v|\varepsilon_0(c)|$ be the polar decomposition of $\varepsilon_0(c)$, let $b \in H^\infty(\alpha)$

be v^*c, and let u be wv. The final projection of v is the projection onto $[\varepsilon_0(c)H]$. The initial subspace of w is $[cH]$ and, since c is M-outer, $[cH] = [\varepsilon_0(c)H]$. Therefore $u = wv$ is a partial isometry in M with initial subspace $[|\varepsilon_0(c)|H]$ and final subspace $[aH]$. We have $a = wc = bu$ and $[bH^2] = v^*[cH^2] = v^*[\varepsilon_0(c)H^2] = [\varepsilon_0(b)H^2]$. Hence b is outer and has the further propety that $\varepsilon_0(b) \geq 0$.

Now note that, since $a = ub$ is invertible, b is bounded below. Hence $\operatorname{Ker} b$ $(= \{x \in H\colon bx = 0\}) = \{0\}$ and, for each closed linear subspace $K \subseteq H$, $\{bx\colon x \in K\}$ is closed. If $\varepsilon_0(b)x = 0$ for some $x \in E_0(H)$, then $0 \neq bx \in H_0^2 \cap [bH^2] = [bH_0^2]$ (as b is M-outer) and, since $[bH_0^2] = \{bx\colon x \in H_0^2\}$, there is some $y \in H_0^2$ such that $bx = by$. Since $\operatorname{Ker} b = \{0\}$, $x = y = 0$. Hence $\operatorname{Ker} \varepsilon_0(b) = \{0\}$.

Let y be in $[\varepsilon_0(b)E_0(H)] \subseteq [\varepsilon_0(b)H^2] = [bH^2] = \{bz\colon z \in H^2\}$. Then $y = bz$ for some $z \in H^2$. But $y \in E_0(H)$, hence $y = \varepsilon_0(b)E_0 z \in \{\varepsilon_0(b)z\colon z \in E_0(H)\}$. Consequently the range of $\varepsilon_0(b)$, as an operator on $E_0(H)$, is closed. We know also that $\operatorname{Ker} \varepsilon_0(b) = \{0\}$ and that $\varepsilon_0(b) \geq 0$. Therefore $\varepsilon_0(b)$ is invertible. Thus $H = [\varepsilon_0(b)H] = [bH] = \{bx\colon x \in H\}$. Hence b is invertible. By the previous lemma $b \in H^\infty(\alpha) \cap H^\infty(\alpha)^{-1}$. □

COROLLARY 3.11. *Every invertible positive operator $a \in M$ can be factored in the form $a = b^*b$, where $b \in H^\infty(\alpha) \cap H^\infty(\alpha)^{-1}$.*

PROOF. Let $a^{1/2}$ be the positive square root of a. Then $a^{1/2}$ is invertible and by the last theorem we can write $a^{1/2} = ub$ for some partial isometry $u \in M$ such that $u^*ub = b$ and $b \in H^\infty(\alpha) \cap H^\infty(\alpha)^{-1}$. Hence $a = a^{1/2}a^{1/2} = b^*u^*ub = b^*b$. □

Note that, although most of our analysis depends on the special representation of M that was chosen (with a cyclic and separating vector ξ and such that $\{M, \alpha\}$ is a canonical pair), Theorem 3.10 and Corollary 3.11 do not depend on the representation and hold in general.

4. An expectation and its application. In this section we construct an expectation from the von Neumann algebra R_0' (that contains M) onto M that maps the nest subalgebra $R_0' \cap \operatorname{Alg}\{P_n\colon n \in \mathbf{Z}\}$ onto $H^\infty(\alpha)$. We will apply it to prove two results (a distance estimate and an interpolation result).

Before constructing the expectation we want to replace α by another flow, $\tilde{\alpha}$, with a special property. For this we will need the following discussion which will be summarized in Lemma 4.5.

For $n \in \mathbf{Z}$ let $c_n \in Z(M_0)$ be defined as follows

$$c_n = \begin{cases} f_n \sum_{m=0}^{n-1} f_m, & n > 0, \\ 0, & n = 0, \\ -f_n \sum_{m=n}^{-1} f_m, & n < 0. \end{cases}$$

Let $\tilde{W}_t$, $t \in T$, be the unitary operator $\sum_{n=-\infty}^{\infty} \exp(itc_n)E_n$. Then, it was shown in [**15**, Lemma 3.8] that $\{\tilde{W}_t\}_{t \in T}$ implements a flow $\tilde{\alpha}$ on M with $\tilde{\alpha}_t(a) = \exp(itc_n)a$ for $a \in M_n$.

Let $c_n = \sum_{m=-\infty}^{\infty} m h_{m,n}$ be the spectral decomposition of c_n, $n \in \mathbf{Z}$. Then $h_{m,n} \neq 0$ only if $0 \leq m \leq n$. Also note that $h_{0,n} = 1 - f_n$. For $a \in M_n$, $a = f_n a$ and one easily gets from this that $\operatorname{sp}_{\tilde{\alpha}}(a) \subseteq \{1, 2, \ldots, n\}$ (if $n > 0$), $\operatorname{sp}_{\tilde{\alpha}}(a) = 0$ (if $n = 0$) and $\operatorname{sp}_{\tilde{\alpha}}(a) \subseteq \{n, n+1, \ldots, -1\}$ (if $n < 0$). The following lemma follows immediately.

LEMMA 4.1. *$H^\infty(\alpha)=H^\infty(\tilde\alpha), H_0^\infty(\alpha)=H_0^\infty(\tilde\alpha)$ and $M^{\tilde\alpha}(\{0\})=M_0$.* $\square$

We have

$$\tilde W_t=\sum_n \exp(itc_n)E_n=\sum_n\sum_m \exp(itm)h_{m,n}E_n=\sum_m e^{itm}\sum_n h_{m,n}E_n.$$

Hence, if we write $\tilde E_n$ for the projection onto $M^{\tilde\alpha}(\{n\})$ (to be denoted by $\tilde M_n$), then $\{\tilde E_n\}$ are the spectral projections of $\tilde W_t$ and we have

$$\tilde E_m=\sum_n h_{m,n}E_n.$$

(In fact $\tilde E_m=\sum_{n=m}^\infty h_{m,n}E_n$ if $m>0$, $\tilde E_m=\sum_{n=-\infty}^\infty h_{m,n}E_n$ if $m<0$ and $\tilde E_0=E_0$.)

LEMMA 4.2. *For $j,m,n\in\mathbf Z$ with $m\neq n$, $j\neq 0$, we have $h_{j,m}h_{jn}=0$.*

PROOF. We will assume that $j>0$. The proof for $j<0$ is similar. As $j>0$ we can suppose that $m>n>0$. Let q be $h_{j,m}h_{j,n}$. Then $c_mq=c_nq=jq$ and $q\le f_nf_m$. But then

$$\begin{aligned} jq&=\left(\sum_{i=0}^{m-1}f_i\right)q=\left(\sum_{i=0}^{n-1}f_i\right)q+\left(\sum_{i=n}^{m-1}f_i\right)q\\ &\ge\left(\sum_{i=0}^{n-1}f_i\right)q+f_nq=\left(\sum_{i=0}^{n-1}f_i\right)q+q\end{aligned}$$

and $(\sum_{i=0}^{n-1}f_i)q=jq$. This leads to a contradiction if $q\neq 0$. $\square$

LEMMA 4.3. (1) *$\sum_{j=1}^n h_{m,j}\ge\sum_{j=1}^n h_{m+1,j}$ for $n>0$ and $m>0$.*

(2) *$\sum_{j=n}^{-1}h_{m-1,j}\le\sum_{j=n}^{-1}h_{m,j}$ for $n<0$ and $m<0$.*

PROOF. We will prove assertion (1). The proof of (2) is similar and will be omitted. Fix $n>0$, $m>0$ and $n\ge j\ge 1$. Then $c_jh_{m+1,j}=(m+1)h_{m+1,j}$. As $c_j=f_j\sum_{k=0}^{j-1}f_k$, we can write $h_{m+1,j}$ as a sum of orthogonal projections $\{p_s\}_{s=2}^j$ in $Z(M_0)$ such that $f_j(\sum_{k=0}^{s-1}f_k)p_s=(m+1)p_s$ and $f_j(\sum_{k=0}^{s-2}f_k)p_s=mp_s$. Hence $f_jf_{s-1}p_s=p_s$ and $c_{s-1}p_s=f_{s-1}(\sum_{k=0}^{s-2}f_k)p_s=f_j(\sum_{k=0}^{s-2}f_k)p_s=mp_2$. Thus $p_s\le h_{m,s-1}\le\sum_{i=1}^n h_{m,i}$. As $h_{m+1,j}=\sum_{s=2}^j p_s$, $h_{m+1,j}\le\sum_{i=1}^n h_{m,i}$. Since this holds for each $1\le j\le n$ we are done. $\square$

Now recall that $\tilde E_m=\sum_n h_{m,n}E_n$ and let $\tilde f_n$ be the projections (in $Z(\tilde M_0)=Z(M_0)$) defined by $\tilde f_n=\sup\{uu^*\colon u$ is a partial isometry in $\tilde M_n\}$, $n\in\mathbf Z$. By Lemma 2.7, $\tilde f_n=c(\tilde E_n)$ and, thus, $\tilde f_n=\sum_j h_{n,j}f_j$.

For $n>0$, $\tilde f_n=\sum_{j=1}^\infty h_{n,j}$ (as $h_{n,j}=h_{n,j}f_j$ for $n\neq 0$) and for $n<0$, $\tilde f_n=\sum_{j=-\infty}^{-1}h_{n,j}$. Also $\tilde f_0=I$. Using Lemma 4.3 we immediately have

LEMMA 4.4. *For $n\ge 0$, $\tilde f_{n+1}\le\tilde f_n$ and for $n<0$, $\tilde f_n\le\tilde f_{n+1}$.*

When a canonical pair $\{B,\eta\}$ satisfies the property of Lemma 4.4 (for the corresponding projections) we say that it satisfies the *roof condition.*

Now let $\tilde{P}_n$ be $\sum_{j=n}^{\infty} \tilde{E}_j$ for $n \in \mathbf{Z}$ (or, equivalently, $\tilde{P}_n$ is the orthogonal projection onto $[M^{\tilde{\alpha}}[n,\infty)\xi]$). Then, for $n \in \mathbf{Z}$,

$$\tilde{P}_n = \sum_{j=n}^{\infty} \tilde{E}_j = \sum_{j=n}^{\infty} \sum_i h_{j,i} E_i = \sum_i \left(\sum_{j=n}^{\infty} h_{j,k} \right) E_i.$$

Fix $n > 0$ and let $r_i = \sum_{j=n}^{\infty} h_{j,i}$ and $t_i = r_i + \bigvee_{j=n}^{i-1} r_j(1-f_i)$ for $i \geq n$ (where $t_n = r_n$). Then, as $E_i \leq f_i$ for each $i \in \mathbf{Z}$,

$$\tilde{P}_n = \sum t_i E_i.$$

From the definition of $\{c_m\}$, we have $r_i \leq f_i$ and $f_i f_{i+k} r_i \leq r_{i+k}$, for $i > 0$ and $k > 0$. Hence, for $i > n$ and $k > 0$, $r_i \leq r_{i+k} + (1 - f_{i+k}) r_i$. It follows that $t_i \leq t_{i+1}$ for $i \geq n$. Hence for $n > 0$,

$$(*) \qquad \tilde{P}_n = t_n P_n + (t_{n+1} - t_n) P_{n+1} + \cdots.$$

For $n < 0$,

$$\tilde{P}_n = \sum_i \left(\sum_{j=n}^{\infty} h_{j,i} \right) E_i = \sum_{i=-\infty}^{n-1} \left(\sum_{j=n}^{\infty} h_{j,i} \right) E_i + P_n$$
$$= \sum_{i=-\infty}^{n-1} \left(\sum_{j=n}^{-1} h_{j,i} \right) E_i + P_n.$$

Let $r'_i = \sum_{j=n}^{-1} h_{j,i}$. Then we have $r'_{i-k} \leq f_{i-k}$ and $f_i f_{i-k} r'_{i-k} \leq r'_i$ for $i < 0$, $k > 0$. Hence $f_i r'_{i-k} \leq r'_i$ and $r'_{i-k} \leq r'_i + (1 - f_i) r'_{i-k}$. Let t'_i be $r'_i + (1 - f_i)(\bigvee_{j=-\infty}^{i-1} r'_j)$ and, then, $t'_i \leq t'_{i+1}$ for $i \leq -2$ because $r'_i \leq r'_{i+1} + (1 - f_{i+1}) r'_i \leq t'_{i+1}$ and, for $j < i$,

$$(1 - f_i) r'_j \leq (1 - f_i) r'_{i+1} + (1 - f_i)(1 - f_{i+1}) r'_j \leq t'_{i+1}.$$

As $r'_i E_i = t'_i E_i$ we have,

$$(**) \quad \tilde{P}_n = \sum_{i=-\infty}^{n-1} t'_i E_i + P_n = (1 - t'_{-1}) P_n + (t'_{-1} - t'_{-2}) P_{n-1} + (t'_{-2} - t'_{-3}) P_{n-2} + \cdots.$$

We wish now to get expressions similar to (*) and (**) with the roles of $\{P_n\}$ and $\{\tilde{P}_n\}$ interchanged. Since $\tilde{E}_m = \sum_m h_{m,n} E_n$ and $h_{m,j} h_{m,k} = 0$ for $j \neq k$, we have $\tilde{E}_m E_n = h_{m,n} E_n = h_{m,n} \tilde{E}_m$. Hence $E_n = \sum_m h_{m,n} \tilde{E}_m$ and $P_n = \sum_{j=n}^{\infty} (\sum_n h_{m,j} \tilde{E}_n) = \sum_m (\sum_{j=n}^{\infty} h_{m,j}) \tilde{E}_m$. Now fix $n > 0$. Then

$$P_n = \sum_{m=1}^{\infty} \left(\sum_{j=n}^{\infty} h_{m,j} \right) \tilde{E}_m.$$

Let q_m, for $m > 0$, be $1 - \sum_{j=1}^{\infty} h_{m,j}$. Then $q_m \tilde{E}_m = 0$ because $\tilde{E}_m = \sum_{n=1}^{\infty} h_{m,n} E_n$ and

$$q_m + \sum_{j=n}^{\infty} h_{m,j} = 1 - \sum_{i=1}^{n-1} h_{m,j} \leq 1 - \sum_{j=1}^{n-1} h_{m+1,j} = q_{m+1} + \sum_{j=n}^{\infty} h_{m,j},$$

where the inequality follows from Lemma 4.3. Let s_m be $q_m + \sum_{j=n}^{\infty} h_{m,j}$, $m > 0$. Then

$$P_n = \sum_{m=1}^{\infty} s_m \tilde{E}_m = s_1 \tilde{P}_1 + (s_2 - s_1)\tilde{P}_2 + (s_3 - s_2)\tilde{P}_3 + \cdots$$

for $n > 0$.

Now fix $n < 0$. We have

$$P_n = \tilde{P}_0 + \sum_{m=n}^{-1} \left(\sum_{j=n}^{\infty} h_{m,j} \right) \tilde{E}_m = \tilde{P}_0 + \sum_{m=n}^{-1} \left(\sum_{j=n}^{-1} h_{m,j} \right) \tilde{E}_m.$$

Let s'_m be $\sum_{j=n}^{-1} h_{m,j}$ for $m < 0$. Then we have, by Lemma 4.3, that $s'_{m-1} \leq s'_m$. Hence

$$P_n = (1 - s'_{-1})\tilde{P}_0 + (s'_{-1} - s'_{-2})\tilde{P}_{-1} + (s'_{-2} - s'_{-3})\tilde{P}_{-2} + \cdots + s'_n \tilde{P}_n$$

for $n < 0$.

We summarize all this in the following

LEMMA 4.5. *There is a canonical pair $\{M, \tilde{\alpha}\}$ with the following properties:*

(1) $H^\infty(\alpha) = H^\infty(\tilde{\alpha})$ *and* $M^{\tilde{\alpha}}(\{0\}) = M_0$ $(= M^\alpha(\{0\}))$.

(2) $\tilde{\alpha}$ *satisfies the roof condition* (*see Lemma* 4.4).

(3) *For* $n \in \mathbf{Z}$, $\tilde{P}_n = \sum_m t_{m,n} P_n$, *where* $\{t_{m,n}\}_{m=-\infty}^{\infty}$ *is an orthogonal family of projections in* $Z(M_0)$. *For* $n \geq 0$, $\tilde{P}_n \leq P_n$.

(4) *For* $n \in \mathbf{Z}$, $P_n = \sum s_{m,n} \tilde{P}_n$, *where* $\{s_{m,n}\}_{m=-\infty}^{\infty}$ *is an orthogonal family of projections in* $Z(M_0)$. *For* $n \leq 0$, $P_n \leq \tilde{P}_n$.

Consequently,

(5) $R \cap \mathrm{Alg}\{P_n \colon n \in \mathbf{Z}\} = R \cap \mathrm{Alg}\{\tilde{P}_n \colon n \in \mathbf{Z}\}$.

(6) $M'_0 \cap \mathrm{Alg}\{P_n \colon n \in \mathbf{Z}\} = M'_0 \cap \mathrm{Alg}\{\tilde{P}_n \colon n \in \mathbf{Z}\}$.

(*Here* $\tilde{P}_n$ *is the projection onto* $[M^{\tilde{\alpha}}[n, \infty)\xi]$).

Let B be a von Neumann algebra and let C be a von Neumann subalgebra of B. An *expectation* of B onto C is a positible linear map ψ from B onto C such that $\psi(I) = I$ and $\psi(bc) = \psi(b)c$ for every $b \in B$, $c \in C$.

If $\psi \colon B \to C$ is an expectation onto C, then ψ is bounded (in fact $\|\psi\| = 1$), $\psi \circ \psi = \psi$, C is the set of fixed points of ψ and $\psi(b)^*\psi(b) \leq \psi(b^*b)$ for $b \in B$ (see [**1**, Appendix] for details).

THEOREM 4.6. *There exists an expectation* $\psi \colon M'_0 \to R$ (*resp.* $\psi' \colon R'_0 \to M$) *such that*

$$\psi(M'_0 \cap \mathrm{Alg}\{P'_n \colon n \in \mathbf{Z}\}) = R \cap \mathrm{Alg}\{P_n \colon n \in \mathbf{Z}\} \quad (= H^\infty(\gamma))$$

(*resp.* $\psi'(R'_0 \cap \mathrm{Alg}\{P_n \colon n \in \mathbf{Z}\}) = M \cap \mathrm{Alg}\{P_n \colon n \in \mathbf{Z}\} = H^\infty(\alpha)$).

PROOF. It will suffice to prove the existence of ψ. Using Lemma 4.5 (and its notations) we have $M^{\tilde{\alpha}}(\{0\}) = M_0$, $M'_0 \cap \mathrm{Alg}\{P_n \colon n \in \mathbf{Z}\} = M'_0 \cap \mathrm{Alg}\{\tilde{P}_n \colon n \in \mathbf{Z}\}$ and $R \cap \mathrm{Alg}\{P_n \colon n \in \mathbf{Z}\} = R \cap \mathrm{Alg}\{\tilde{P}_n \colon n \in \mathbf{Z}\}$. Hence we can prove the theorem for $\tilde{\alpha}$ in place of α or, equivalently, assume that α satisfies the roof condition (i.e., $f_{n+1} \leq f_n$ for $n \geq 0$ and $f_{n-1} \leq f_n$ for $n < 0$). In the rest of the proof we will assume that α satisfies the roof condition and we write f_+ for $\bigwedge\{f_n \colon n > 0\}$, f_- for $\bigwedge\{f_n \colon n < 0\}$ and f_∞ for $f_+ \vee f_-$.

Since $\mathbf{Z}$ is an amenable group we can assign to each $g \in l^\infty(\mathbf{Z})$ a number $m(g) \in \mathbf{C}$ such that $g \to m(g)$ is a linear functional, $\inf\{g(n): n \in \mathbf{Z}\} \leq m(g) \leq \operatorname{Sup}\{g(n): n \in \mathbf{Z}\}$ for each real valued $g \in l^\infty(\mathbf{Z})$ and $m(g_k) = m(g)$, where $g_k(s) = g(k+s)$, $k, s \in \mathbf{Z}$. In fact, we can choose a state ρ on the algebra $l^\infty(\mathbf{Z}_+)/c_0$ and let $m(g)$ be the value of ρ on the coset, in $l^\infty(\mathbf{Z}_+)/c_0$, of $g' \in l^\infty(\mathbf{Z}_+)$, where $g'(n) = (1/(2n+1))\sum_{i=-n}^{n} g(i)$, $n \geq 0$. If for some real valued $g \in l^\infty(\mathbf{Z})$, $g'(n) \geq a(n)$ for $n \geq 0$ and $a(n) \to a_0 \in \mathbf{R}$ as $n \to \infty$, then $m(g) = \rho(g' + c_0) \geq \rho(a + c_0) = \rho(a_0 + c_0) = a_0$.

Since $|g'(n)| \leq \operatorname{Sup}_i |g(i)|$, $n \geq 0$, we see that

$$|m(g)| = |\rho(g' + c_0)| \leq \|g' + c_0\| \leq \|g\|_\infty.$$

Now let T be an operator in M_0' and x, y vectors in H. Then $g_{x,y}(n) = \langle \beta_n(T)x, y\rangle$ defines a function $g_{x,y} \in l^\infty(\mathbf{Z})$ (as $\|\beta_n(T)\| \leq \|T\|$). For each $x, y \in H$ let $[x, y]$ be $m(g_{x,y})$. Then

$$|[x, y]| = |m(g_{x,y})| \leq \|g_{x,y}\|_\infty \leq \|T\|\,\|x\|\,\|y\|.$$

Hence there is an operator $\psi_0(T)$ on H such that $\langle \psi_0(T)x, y\rangle = m(g_{x,y})$ for $x, y \in H$, and $\|\psi_0(T)\| \leq \|T\|$.

For a unitary operator $u \in M_0$, $\langle u^*\psi_0(T)ux, y\rangle = m(g_{ux,uy}) = m(g_{x,y}) = \langle \psi_0(T)x, y\rangle$ as $g_{ux,uy} = \langle \beta_n(T)ux, uy\rangle = \langle u\beta_n(T)x, uy\rangle = g_{x,y}$, for all $x, y \in H$. Hence $\psi_0(T) \in M_0'$.

To find $\beta_m(\psi_0(T))$ for $m \in \mathbf{Z}$, consider the operator $v_{m,k}v_{m,k}^*\beta_m(\psi_0(T)) = v_{m,k}\psi_0(T)v_{m,k}^*$ for $k \geq 1$. Then

$$\langle v_{m,k}\psi_0(T)v_{m,k}^*x, y\rangle = \langle \psi_0(T)v_{m,k}^*x, v_{m,k}^*y\rangle = m(g),$$

where

$$\begin{aligned} g(n) &= \langle \beta_n(T)v_{m,k}^*x, v_{m,k}^*y\rangle = \langle v_{m,k}\beta_n(T)v_{m,k}^*x, y\rangle = \langle v_{m,k}v_{m,k}^*\beta_m(\beta_n(T))x, y\rangle \\ &= \langle f_m\beta_{m+n}(T)x, v_{m,k}v_{m,k}^*y\rangle = \langle \beta_{m+n}(T)x, v_{m,k}v_{m,k}^*y\rangle. \end{aligned}$$

Hence, as $m(g)$ is translation-invariant,

$$\langle v_{m,k}\psi_0(T)v_{m,k}^*x, y\rangle = \langle \psi_0(T)x, v_{m,k}v_{m,k}^*y\rangle = \langle v_{m,k}v_{m,k}^*\psi_0(T)x, y\rangle.$$

Since this holds for all $x, y \in H$, $v_{m,k}v_{m,k}^*\beta_m(\psi_0(T)) = v_{m,k}v_{m,k}^*\psi_0(T)$ for each $k \geq 1$. Hence $\beta_m(\psi_0(T)) = f_m\beta_m(\psi_0(T)) = f_m\psi_0(T)$. By Lemma 2.4(4), $\psi_0(T) \in R$.

Since each β_n is positive and linear and so is m, ψ_0 is a positive linear map from M_0' into R.

Suppose $T \in M_0' \cap \operatorname{Alg}\{P_n: n \in \mathbf{Z}\}$. Then, for $n, m \in \mathbf{Z}$,

$$\begin{aligned} \beta_n(T)P_m &= \beta_n(T)f_nP_m = \beta_n(T)\beta_n(P_{m-n}) = \beta_n(TP_{m-n}) = \beta_n(P_{m-n}TP_{m-n}) \\ &= \beta_n(P_{m-n})\beta_n(T)\beta_n(P_{m-n}) = P_m\beta_n(T)P_m. \end{aligned}$$

Consequently, for $x, y \in H$ and $m \in \mathbf{Z}$, $\langle \psi_0(T)P_mx, y\rangle = m(g)$, where $g(n) = \langle \beta_n(T)P_mx, y\rangle = \langle \beta_n(T)P_mX, P_my\rangle$. Hence $\langle \psi_0(T)P_mx, y\rangle = \langle \psi_0(T)P_mx, P_my\rangle$. It follows that

$$\psi_0(M_0' \cap \operatorname{Alg}\{P_n: n \in \mathbf{Z}\}) \subseteq R \cap \operatorname{Alg}\{P_n: n \in \mathbf{Z}\}.$$

Suppose now that $T \in R$ and $S \in M_0'$. Then, for $n \in \mathbf{Z}$, $\beta_n(TS) = T\beta_n(S)$ and, therefore $\langle \psi_0(TS)x, y\rangle = m(g)$, where $g(n) = \langle T\beta_n(S)x, y\rangle = \langle \beta_n(S)x, T^*y\rangle$. Thus $\psi_0(TS) = T\psi_0(S)$.

Note, however, that ψ_0 need not be an expectation since we might have $\psi_0(I) \neq I$ (one can easily find examples where $\mathrm{sp}(\alpha)$ is finite and then $\psi_0(I) = 0$ since $\beta_n(I) = 0$ for all but finitely many n's). As $I \in \mathrm{Alg}\{P_n\} \cap (\mathrm{Alg}\{P_n\})^*$, $\psi_0(I) \in \mathrm{Alg}\{P_n\} \cap (\mathrm{Alg}\{P_n\})^* = \{P_n : n \in \mathbf{Z}\}'$. Hence $\psi_0(I) \in R_0$. Also $\psi_0(I)T = \psi_0(T) = T\psi_0(I)$ for $T \in R$; hence $\psi_0(I) \in M \cap R_0 = Z(M_0) \cap R_0$.

We now let q_i be $f_i - f_{i+1}$ if $i \geq 0$ and $f_{i+1} - f_i$ if $i < 0$. Then $I = \sum_{i=0}^{\infty} q_i + f_+ = \sum_{i=-\infty}^{-1} q_i + f_-$. For $i \geq 0$, $j < 0$ let q_{ij} be $q_i q_j$. Then

$$\sum_{i \geq 0, j<0} q_{ij} = \sum_{i \geq 0} q_i \left(\sum_{j<0} q_j\right) = \sum_{i\geq 0} q_i(I - f_-) = (1 - f_+)(1 - f_-) = 1 - f_\infty.$$

Also note that, for $i \geq 0$, $j < 0$ and $n \in \mathbf{Z}$,

$$(*) \qquad q_{ij} f_n = (f_i - f_{i+1})(f_{i+1} - f_j) f_n = \begin{cases} 0 & \text{if } n > i \text{ or } n \leq j, \\ q_{ij} & \text{if } j < n \leq i. \end{cases}$$

Now, for fixed $i \geq 0$, $j < 0$ and $x, y \in H$, $\langle q_{ij}\psi_0(I)x, y\rangle = m(g)$, where $g(n) = \langle q_{ij} f_n x, y\rangle$. But

$$\frac{1}{2N+1} \sum_{k=-N}^{N} \langle q_{ij} f_k x, y\rangle = \frac{i-j}{2N+1} \langle q_{ij}x, y\rangle \to 0 \quad \text{as } N \to \infty.$$

Hence $\langle q_{ij}\psi_0(I)x, y\rangle = m(g) = 0$. This holds for all x, y and, therefore, $q_{ij}\psi_0(I) = 0$ for each $i \geq 0$, $j < 0$. Summing over i, j we find that $(1 - f_\infty)\psi_0(I) = 0$. Hence $\psi_0(I) = f_\infty \psi_0(I)$.

For $x \in H$ we have $\langle \psi_0(I) f_+ x, x\rangle = m(g)$, where $g(n) = \langle f_n f_+ x, x\rangle$. For $n \geq 0$, $g(n) = \langle f_+ x, x\rangle$ and, therefore

$$\begin{aligned} g'(n) &= \frac{1}{2N+1} \sum_{n=-N}^{N} \langle f_n f_+ x, x\rangle \\ &\geq \frac{1}{2N+1}(N+1)\langle f_+ x, x\rangle \to \frac{1}{2}\langle f_+ x, x\rangle \quad \text{as } N \to \infty. \end{aligned}$$

Thus $\langle \psi_0(I) f_+ x, x\rangle \geq \frac{1}{2}\langle f_+ x, x\rangle$. Hence $\psi_0(I) f_+ \geq \frac{1}{2} f_+$. Similarly $\psi_0(I) f_- \geq \frac{1}{2} f_-$ and consequently $f_\infty \psi_0(I) \geq \frac{1}{2} f_\infty$. In particular f_∞ is the range projection of $\psi_0(I)$ (hence lies in R_0). Therefore there is some $h \in Z(M_0) \cap R_0$ such that $h\psi_0(I) = f_\infty$ (and $h \geq 0$).

Let $\psi_1(T)$ be $h\psi_0(T)$ for each $T \in M_0'$. Then ψ_1 has the following properties (which follow from the properties of ψ_0 and h):

(1) ψ_1 is a linear, positive map from M_0' into R (note: $\psi_1(T) = h^{1/2}\psi_0(T)h^{1/2}$).

(2) $\psi_1(ST) = \psi_1(S)T$ for $T \in R$, $S \in M_0'$.

(3) ψ_1 maps $M_0' \cap \mathrm{Alg}\{P_n : n \in \mathbf{Z}\}$ into $R \cap \mathrm{Alg}\{P_n : n \in \mathbf{Z}\}$.

(4) $\psi_1(T) = f_\infty T$ for $T \in R$.

Next we construct another map, ψ_2, from M_0' into R with properties (1)–(3) as above (with ψ_1 replaced by ψ_2) and

(4′) $\psi_2(T) = (1 - f_\infty)T$ for $T \in R$.

The sum $\psi_1 + \psi_2$ is the required expectation from M_0' into R.

Now fix $T \in M_0'$, $i \geq 0$ and $j < 0$. Let T_{ij} be $(1/(i-j))\sum_{n=-\infty}^{\infty}\beta_n(T)q_{ij}$. By $(*)$ the sum has at most $i-j$ nonzero terms and it defines an operator $T_{ij} \in q_{ij}M_0'q_{ij}$ (note that $\beta_n(T)q_{ij} = q_{ij}\beta_n(T)$). Define $\psi_2(T)$ to be $\psi_2(T) = \sum_{i\geq 0, j<0} T_{ij}$. As $\{q_{ij}\colon i \geq 0,\ j < 0\}$ is an orthogonal family of projections, and $T_{ij} \in q_{ij}M_0'q_{ij}$, $\psi_2(T)$ is a well-defined operator in M_0' and $\psi_2(T) = (1-f_\infty)\psi_2(T)(1-f_\infty)$ (since $1 - f_\infty = \sum q_{ij}$).

For $m \in \mathbf{Z}$, $T \in M_0'$, $i \geq 0$ and $j < 0$ we have

$$\beta_m(T_{ij}) = \frac{1}{i-j}\sum_{n=-\infty}^{\infty}\beta_m\beta_n(T)\beta_m(q_{ij}) = f_m\frac{1}{i-j}\sum_n \beta_{n+m}(T)\beta_m(q_{ij}).$$

But

$$\begin{aligned}\beta_m(q_{ij}) &= \beta_m(q_iq_j) = \beta_m((f_i - f_{i+1})(f_{j+1} - f_j)) \\ &= f_m(f_{i+m} - f_{i+m+1})(f_{j+m+1} - f_{j+m}) \\ &= \begin{cases} f_m q_{i+m,j+m} & \text{if } i+m \geq 0,\ j+m < 0, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Hence, for $-i \leq m < -j$,

$$\beta_m(T_{ij}) = f_m\frac{1}{i-j}\sum_n \beta_{m+n}(T)q_{i+m,j+m} = f_mT_{i+m,j+m}$$

and for $m \geq -j$ or $m < -i$, $\beta_m(T_{ij}) = 0$.

Suppose that $m > 0$. Then (by $(*)$) $f_mq_{i+m,j+m} = 0$ if $-m \leq i < 0$ and, therefore,

$$\begin{aligned}\beta_m(\psi_2(T)) &= \sum_{i\geq 0, j<0} f_mT_{i+m,j+m} \\ &= \sum_{i+m\geq 0, j+m<0} f_mT_{i+m,j+m} = f_m\psi_2(T).\end{aligned}$$

Similarly $\beta_m(\psi_2(T)) = f_m\psi_2(T)$ for $m < 0$. Hence $\psi_2(T) \in R$ for each $T \in M_0'$.

Suppose now that $T \in M_0' \cap \mathrm{Alg}\{P_n\colon n \in \mathbf{Z}\}$. Then, for $i \geq 0$, $j < 0$, $T_{ij} \in \mathrm{Alg}\{P_n\colon n \in \mathbf{Z}\}$ (since, as was shown before, $\beta_m(T) \in \mathrm{Alg}\{P_n\colon n \in \mathbf{Z}\}$ for each $m \in \mathbf{Z}$ and $q_{ij} \in M_0$). Hence $\psi_2(M_0' \cap \mathrm{Alg}\{P_n\colon n \in \mathbf{Z}\}) \subseteq R \cap \mathrm{Alg}\{P_n\colon n \in \mathbf{Z}\}$.

For $T \in R$, $\beta_N(T) = f_nT$ for each $n \in \mathbf{Z}$. Hence

$$T_{ij} = \frac{1}{i-j}\sum_n f_nTq_{ij} = \frac{1}{i-j}\sum_n f_nq_{ij}T$$

for each $i \geq 0$, $j < 0$. Using $(*)$,

$$T_{ij} = \frac{1}{i-j}\sum_{n=j+1}^{i} q_{ij}T = q_{ij}T.$$

Hence $\psi_2(T) = (1 - f_\infty)T$.

Therefore ψ_2 satisfies the following properties:

(1′) ψ_2 is a positive linear map from M_0' into R.

(2′) $\psi_2(ST) = \psi_3(S)T$ for $S \in M_0'$, $T \in R$ as

$$\psi_2(ST)q_{ij} = (1-j)^{-1}\sum_n \beta_n(ST)q_{ij} = (1-j)^{-1}\sum \beta_n(S)Tq_{ij} = \psi_2(S)Tq_{ij}.$$

(3′) $\psi_2(M_0' \cap \mathrm{Alg}(P_n\colon n \in \mathbf{Z}\}) \subseteq R \cap \mathrm{Alg}\{P_n\colon n \in \mathbf{Z}\}$.

(4′) $\psi_2(T) = (1 - f_\infty)T$ for $T \in R$.

Then $\psi = \psi_1 + \psi_2$ is the required expectation. □

Suppose R_0' is an injective von Neumann algebra. Then there is an expectation ϕ from $B(H)$ onto R_0'. Since $\{P_n\colon n \in \mathbf{Z}\} \subseteq R_0'$, ϕ maps $\mathrm{Alg}\{P_n\colon n \in \mathbf{Z}\}$ onto $R_0' \cap \mathrm{Alg}\{P_n\colon n \in \mathbf{Z}\}$. Let ϕ_0 be the map $\psi' \circ \phi$ (where ψ' is the expectation from R_0' onto M whose existence was proved in Theorem 4.6). Then ϕ_0 is an expectation onto M and $\phi_0(\mathrm{Alg}\{P_n\colon n \in \mathbf{Z}\}) = H^\infty(\alpha)$.

COROLLARY 4.7. *Suppose R_0' is injective. Then, for $a \in M$,* $\mathrm{dist}(a, H^\infty(\alpha)) = \|(1-P_0)aP_0\|$. (*Here* $\mathrm{dist}(a, H^\infty(\alpha))$ *is* $\inf\{\|a-b\|\colon b \in H^\infty(\alpha)\}$.)

PROOF. As Arveson shows in [**3**], for any algebra $\mathcal{A}$ ($\subseteq B(H)$) and operator $a \in B(H)$, $\mathrm{dist}(a, \mathcal{A}) \geq \mathrm{Sup}\{\|(1-P)aP\|\colon P \in \mathrm{Lat}\,\mathcal{A}\}$ (where $\mathrm{Lat} = \{P\colon P$ is a projection such that $PbP = bP$ for all $b \in \mathcal{A}\}$). Hence

$$\begin{aligned}\|(1-P_0)aP_0\| &\leq \mathrm{Sup}\{\|(1-P_n)aP_n\|\colon n \in \mathbf{Z}\}\\ &\leq \mathrm{Sup}\{\|(1-P)aP\|\colon P \in \mathrm{Lat}\,H^\infty(\alpha)\} \leq \mathrm{dist}(a, H^\infty(\alpha)) \quad \text{for } a \in M.\end{aligned}$$

It will suffice, therefore, to show

$$\|(1-P_0)aP_0\| \geq \mathrm{Sup}\{\|(1-P_n)aP_n\|\colon n \in \mathbf{Z}\} \geq \mathrm{dist}(a, H^\infty(\alpha)).$$

Let ϕ_0 be the expectation from $B(H)$ onto M as in the discussion preceding this corollary. By Arveson's distance formula [**3**],

$$\mathrm{dist}(a, \mathrm{Alg}\{P_n\colon n \in \mathbf{Z}\}) = \mathrm{Sup}\{\|(1-P_n)aP_n\|\colon n \in \mathbf{Z}\}.$$

Let us write t for the right-hand side of this equation and, for $\varepsilon > 0$, we can find $a_\varepsilon \in \mathrm{Alg}\{P_n\colon n \in \mathbf{Z}\}$ such that $\|a - a_\varepsilon\| \leq t + \varepsilon$. Applying ϕ_0 we get $\|a - \phi_0(a_\varepsilon)\| = \|\phi_0(a - a_\varepsilon)\| \leq \|a - a_\varepsilon\| \leq t + \varepsilon$. As $\phi_0(a_\varepsilon) \in H^\infty(\alpha)$ and, the choice of $\varepsilon > 0$ is arbitrary, we have $\mathrm{dist}(a, H^\infty(\alpha)) \leq t$.

Now fix $\varepsilon > 0$. Then there are $n \in \mathbf{Z}$ and $x \in P_n(H)$ such that $\|(1-P_n)ax\| > t - \varepsilon$ and $\|x\| \leq 1$. Let F be the orthogonal projection onto $[H^\infty(\alpha)x]$. Then $F \leq P_n$ and $t - \varepsilon < \|(1-P_n)ax\| \leq \|(1-F)aFx\|$.

By Corollary 3.6 there is a partial isometry $U \in R$ such that $U^*U \in R_0$ and $F = UP_0U^*$ (since $F \leq P_n$, $F \in \mathcal{P}_3$). Hence

$$\begin{aligned}t - \varepsilon &< \|(1-F)aFx\| = \|(1-UP_0U^*)aUP_0U^*x\| = \|U(1-P_0)U^*aUP_0U^*x\|\\ &= \|U(1-P_0)aP_0U^*x\| \leq \|(1-P_0)aP_0\|.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $t \leq \|(1-P_0)aP_0\|$. □

COROLLARY 4.8. *Suppose R_0' is injective. Let $\{a_i\colon 1 \leq i \leq N\}$ in $H^\infty(\alpha)$ satisfy, for some $\varepsilon > 0$,*

$$\sum_{i=1}^N \|(1-P_n)a_ix\|^2 \geq \varepsilon^2\|(1-P_n)x\|^2$$

for all $n \in \mathbf{Z}$, $x \in H$. *Then there are* $\{b_i: 1 \le i \le N\} \subseteq H^\infty(\alpha)$ (*that can be chosen so that* $\|b_i\| \le 4N\varepsilon^{-3}$ *if* $\|a_i\| \le 1$ *for each* $1 \le i \le N$) *such that* $\sum_{i=1}^N b_i a_i = I$.

PROOF. Consider $\{a_i\}_{i=1}^N$ as elements of $\mathrm{Alg}\{P_n: n \in \mathbf{Z}\}$. Then by [**3**, Theorem 4.3] there are $\{b_i'\}_{i=1}^N$ (with $\|b_i'\| \le 4N\varepsilon^{-3}$ if $\|a_i\| \le 1$ for each $1 \le i \le N$) such that $\sum b_i' a_i = I$ and $\{b_i'\}_{i=1}^N \subseteq \mathrm{Alg}\{P_n: n \in \mathbf{Z}\}$. Applying the expectation ϕ_0 (from $B(H)$ onto M, as in the discussion preceding Corollary 4.7) we get $\sum \phi_0(b_i')a_i = I$ and $\|\phi_0(b_i')\| \le \|b_i'\|$. Let b_i be $\phi_0(b_i')$ and we are done. □

REMARK. Note that R_0' is injective if and only if M is. Indeed, if M is injective then so is R. But $R_0 = \varepsilon_0(R)$ and consequently R_0 and R_0' are injective von Neumann algebras. On the other hand if R_0' is injective, then it follows from Theorem 4.6 that M is injective too.

REFERENCES

1. W. B. Arveson, *Analyticity in operator algebras*, Amer. J. Math. **89** (1967), 578–642.
2. ——, *On groups of automorphisms of operator algebras*, J. Funct. Anal. **15** (1974), 217–243.
3. ——, *Interpolation problems in nest algebras*, J. Funct. Anal. **20** (1975), 208–233.
4. J. Dixmier, *Von Neumann algebras*, North-Holland, Amsterdam, 1981.
5. S. Kawamura and J. Tomiyama, *On subdiagonal algebras associated with flows in operator algebras*, J. Math. Soc. Japan **29** (1977), 73–90.
6. R. I. Loebl and P. S. Muhly, *Analyticity and flows in von Neumann algebras*, J. Funct. Anal. **29** (1978), 214–252.
7. M. McAsey, P. S. Muhly and K.-S. Saito, *Nonselfadjoint crossed products (invariant subspaces and maximality)*, Trans. Amer. Math. Soc. **248** (1979), 381–409.
8. ——, *Nonselfadjoint crossed products*. II, J. Math. Soc. Japan **33** (1981), 485–495.
9. K.-S. Saito, *The Hardy spaces associated with a periodic flow on a von Neumann algebra*, Tôhoku Math. J. **29** (1977), 69–75.
10. ——, *Invariant subspaces for finite maximal subdiagonal algebras*, Pacific J. Math. **93** (1981), 431–434.
11. ——, *Nonselfadjoint subalgebras by compact abelian actions on finite von Neumann algebras*, Tôhoku Math. J. **34** (1982), 485–494.
12. ——, *Spectral resolutions of invariant subspaces by compact abelian group actions on von Neumann algebras*, preprint.
13. B. Solel, *The invariant subspace structure of nonselfadjoint crossed products*, Trans. Amer. Math. Soc. **279** (1983), 825–840.
14. ——, *Invariant subspaces for algebras of analytic operators associated with a periodic flow on a finite von Neumann algebra*, preprint.
15. ——, *Algebras of analytic operators associated with a periodic flow on a von Neumann algebra*, preprint.

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