TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 287, Number 2, February 1985

PRO-LIE GROUPS

BY
R. W. BAGLEY, T. S. WU AND J. S. YANG

ABSTRACT. A topological group G is pro-Lie if G has small compact normal subgroups K such that G/K is a Lie group. A locally compact group G is an *L-group* if, for every neighborhood U of the identity and compact set C, there is a neighborhood V of the identity such that $gHg^{-1} \cap C \subset U$ for every $g \in G$ and every subgroup $H \subset V$. We obtain characterizations of pro-Lie groups and make several applications. For example, every compactly generated L-group is pro-Lie and a compactly generated group which can be embedded (by a continuous isomorphism) in a pro-Lie group is pro-Lie. We obtain related results for factor groups, nilpotent groups, maximal compact normal subgroups, and generalize a theorem of Hofmann, Liukkonen, and Mislove [**4**].

Introduction. We divide the paper into two sections. In §1 we consider some rather general properties of locally compact groups and establish their relationships to pro-Lie groups. In §2 we obtain some related results and consider pro-Lie groups in more special settings. We obtain several characterizations of pro-Lie groups, the last one in terms of the bounded part and periodic part of the group. In this paper we are concerned only with those groups which are locally compact.

The following background is taken directly from the referee's report.

The application of Lie group theory is the basis of most of the finer results on the structure theory of locally compact groups.

Historically, the first half of this century saw topological group theory concentrate on the interaction of Lie group theory and the general theory of compact and locally compact groups. The guiding principle was one of Hilbert's problems formulated at the turn of the century (the famed number FIVE). Even before this program was completed, Iwasawa published a key paper in 1949 on the structure theory of locally compact groups which are projective limits of Lie groups, i.e. those groups which now are called—somewhat unfortunately from the view point of good English—pro-Lie groups. The vitality of Iwasawa's ideas continues to influence writers in the area up to these days, as is, e.g., exemplified by this paper. The fine-structure theory of locally compact groups flourished through the sixties. While this work did not show the raw power of penetration of the work of Gleason, Montgomery, and Yamabe, which in the early fifties led to the solution of Hilbert's Fifth Problem, it contributed much to the insight into the structure of more special classes of locally compact groups through the contributions of Grosser, Hofmann, Moskowitz, Mostert and their followers and students. In the meantime one observes a steady, although somewhat slower, flow of results on locally compact groups, and their structure theory springs from the work of researchers in the field, while more

Received by the editors November 23, 1983 and, in revised form, April 9, 1984.
1980 *Mathematics Subject Classification*. Primary 22D05.

vigorous endeavors go into the representation theory of locally compact groups and, notably, Lie groups.

Structural results on locally compact groups are the deeper and more successful, the closer the groups under consideration are to Lie groups. Thus, pro-Lie groups and their characterization must necessarily play a central role. This leads to the emergence of various classes of locally compact groups becoming significant through their relationship to pro-Lie groups. The research presented in this paper is no exception in this regard. It is perhaps most significantly different from the traditional line insofar as it emphasizes a wider class of topological groups than those which are locally compact projective limits of Lie groups, namely, groups whose Lie group quotients separate the points. This condition is strictly weaker than being pro-Lie—even within the class of locally compact groups!—and it is strictly stronger than having enough representations into Lie groups to separate the points. Among several other results, this paper contributes to the relationship between these so-called residual Lie groups and pro-Lie groups.

1. Some characterizations of pro-Lie groups.

DEFINITION 1.1. A locally compact topological group G is an *M-group* if and only if for every neighborhood U of the identity there is a neighborhood W of the identity such that $gHg^{-1} \subset U$ for every $g \in G$ and every subgroup $H \subset W$. A locally compact topological group G is an *L-group* if and only if, for every neighborhood U of the identity and every compact set C, there is a neighborhood W of the identity such that $gHg^{-1} \subset U \cup (G - C)$, equivalently $gHg^{-1} \cap C \subset U$, for every $g \in G$ and subgroup $H \subset W$.

The following characterizations of M-groups and L-groups are proved easily.

REMARK. A locally compact topological group G is an M-group if, for each pair of nets $\{x_\alpha\}$ and $\{g_\alpha\}$ in G, the net $\{g_\alpha x_\alpha g_\alpha^{-1}\}$ converges to the identity whenever $\{X_\alpha\}$ converges to the identity, where X_α is the group generated by x_α. If, for each pair $\{x_\alpha\}$, $\{g_\alpha\}$, the net $\{g_\alpha x_\alpha g_\alpha^{-1}\}$ either converges to the identity or fails to converge whenever $\{X_\alpha\}$ converges to the identity, then G is an L-group.

Topological groups which are SIN-groups, that is, have small neighborhoods of the identity, invariant under all the inner automorphisms, have been studied a great deal (see, for example, [**3**]). Since Lie groups in general are not SIN-groups, it is natural to consider a more general class than the class of SIN-groups. We demonstrate that the class of M-groups is a natural one in this respect.

In [**2**, Theorem 3], we showed that a compactly generated N-group is SIN. An N-group is a group G such that, for each neighborhood U of the identity and compact set C, there is a neighborhood V such that $gVg^{-1} \subset U \cup (G - C)$ for every $g \in G$. Essentially the same proof in [**2**] referred to above establishes the following proposition. *By "compactly generated" (CG) we mean generated by a compact neighborhood of the identity.* In particular, compactly generated groups are locally compact.

PROPOSITION 1.1. *If G is a compactly generated L-group, then G is an M-group.*

DEFINITION 1.2. A topological group G is *pro-Lie* if G has small compact normal subgroups K such that G/K is a Lie group.

For easy reference we include the following diagram, indicating relationships between the classes of groups under consideration. The numbers in brackets indicate references of papers in which the corresponding implications are established.

$$
\begin{tikzcd}[column sep=huge, row sep=large]
N \arrow[r, Rightarrow] \arrow[d, Rightarrow, shift left, "/" marking, "{[2]}"] & L \arrow[d, Rightarrow, shift left, "/" marking, "{[1]}"] & \text{locally compact residual Lie} \arrow[l, Rightarrow, "\text{Theorem 1.3}"'] \arrow[d, Rightarrow, shift left, "/" marking, "{[1]}"] \\
\mathrm{SIN} \arrow[r, Rightarrow] \arrow[u, Rightarrow, shift left] & M \arrow[r, Leftrightarrow, "\text{Theorem 1.2}"] \arrow[u, Rightarrow, shift left] & \text{pro-Lie} \arrow[u, Rightarrow, shift left] \\
\mathrm{CGN} \arrow[r, Rightarrow] \arrow[u, Rightarrow, "{[2]}"'] & \mathrm{CGL} \arrow[u, Rightarrow, "\text{Proposition 1.1}"'] & \text{CG residual Lie} \arrow[l, Rightarrow] \arrow[u, Rightarrow, "\text{Theorem 1.4}"']
\end{tikzcd}
$$

The following characterization of pro-Lie groups is the main theorem of this section. It generalizes part of Theorem 2.11 [**3**].

THEOREM 1.2. *A locally compact group is pro-Lie if and only if it is an M-group.*

PROOF. The "only if" part follows from the fact that a Lie group has no small subgroups.

To prove that a locally compact M-group is pro-Lie, we let G be locally compact and G_1 an open subgroup of G which is pro-Lie [**6**, p. 174]. Thus, there is a neighborhood base $\{V_\alpha\}$ at the identity of G and compact subgroups $\{K_\alpha\}$ such that $K_\alpha \subset V_\alpha$, K_α is normal in G_1, and G_1/K_α is a Lie group. Let V be an arbitrary neighborhood of the identity of G. There is a compact neighborhood U of the identity and a compact subgroup H contained in V such that every subgroup contained in U is contained in H [**6**, p. 172]. If G is an M-group, then $gK_{\alpha_0}g^{-1} \subset U$ for every $g \in G$ and some α_0. Otherwise there exist $\{k_\alpha\}$ and $\{g_\alpha\}$ such that $k_\alpha \in K_\alpha$ and $g_\alpha k_\alpha g_\alpha^{-1} \notin U$. This is a contradiction since $\{K_\alpha\}$ converges to the identity. We let $K = K_{\alpha_0}$, satisfying the above condition. It follows that $gKg^{-1} \subset H$ for every $g \in G$ and, hence, $\bigcup\{gKg^{-1} \colon g \in G\}$ generates a compact subgroup $N \subset V$ which is normal in G. Since $G_1/N = G_1/NK$ and G_1/K is a Lie group, G_1/N is a Lie group. Therefore G/N is a Lie group and G is pro-Lie as desired.

COROLLARY 1. *A compactly generated L-group is pro-Lie. A locally compact* SIN*-group is pro-Lie* (*see Theorem* 2.11 *of* [**3**]).

COROLLARY 2. *If G is a locally compact L-group and has a compact normal subgroup K such that G/K is a Lie group, then G is pro-Lie.*

PROOF. It is easy to see that G is an M-group. Thus the corollary follows immediately from the theorem.

DEFINITION 1.3. A locally compact group G is a *residual Lie group* if there is a collection of closed normal subgroups $\{H_\alpha\}$ such that $\bigcap H_\alpha = e$ and G/H_α is a Lie group for each α.

Obviously every pro-Lie group is a residual Lie group, but simple examples show that the converse does not hold. We see by Theorem 1.4 below that the converse does hold for compactly generated groups. This is an extension of the Main Approximation Theorem [**6**, p. 175].

THEOREM 1.3. *If G is a topological group which has a collection of closed normal subgroups $\{H_\gamma\}$ such that $\bigcap H_\gamma = e$ and G/H_γ is an M-group for each γ, then G is an L-group.*

PROOF. If G is not an L-group, there is a net $\{X_\alpha\}$ of subgroups of G which converges to the identity of G and $\{g_\alpha x_\alpha g_\alpha^{-1}\}$ converges to $x \neq e$ for some net $\{g_\alpha\}$, where $x_\alpha \in X_\alpha$. We choose a closed normal subgroup H such that $x \notin H$ and G/H is an M-group. Now $g_\alpha x_\alpha g_\alpha^{-1} H$ converges to H; so $xH = H$, a contradiction, which implies that G is an L-group.

COROLLARY. *If G is a residual Lie group, then G is an L-group.*

THEOREM 1.4. *If G is a compactly generated residual Lie group, then G is pro-Lie.*

PROOF. The theorem follows from Theorem 1.3, Proposition 1.1 and Theorem 1.2.

With Corollary 1 of Theorem 1.2 we established the fact that compactly generated L-groups are pro-Lie. Another result of this sort is Proposition 1.5 below which characterizes pro-Lie groups among L-groups. We also noticed that residual Lie groups are L-groups.

PROPOSITION 1.5. *A topological group G is pro-Lie if and only if G is an L-group and has a compact neighborhood U of the identity such that, for some neighborhood V of the identity, $gHg^{-1} \subset U$ for every $g \in G$ and subgroup $H \subset V$.*

PROOF. Let W be any neighborhood of the identity. If G is an L-group, there is a neighborhood W_1 of the identity such that $W_1 \subset V$ and $gHg^{-1} \cap U \subset W$ for every $g \in G$ and subgroup $H \subset W_1$, where U and V are the neighborhoods referred to in the proposition. By hypothesis $gHg^{-1} \subset U$. Thus $gHg^{-1} \subset W$ for $g \in G$. This proves that G is an M-group, and it follows that G is pro-Lie.

For the converse let G be pro-Lie and let U be a compact neighborhood of the identity. There is a compact normal subgroup $K \subset U$ such that G/K is a Lie group. Thus there is a neighborhood V of the identity such that every subgroup H contained in V is also contained in K. Thus $gHg^{-1} \subset K \subset U$ for every $g \in G$.

In [**3**] the class [IN] of groups was defined as those which have a compact invariant neighborhood of the identity. A topological group of this class is not necessarily SIN nor residual Lie as is shown by the example obtained as the semidirect product of the group $K = \{0,1\}^Z$, with the group Z acting by shifts. We note that K is a compact neighborhood of the identity which is invariant under all the inner automorphisms of the group. This group is not an L-group. As a matter of fact we have the following corollary of Proposition 1.5.

COROLLARY. *If G is an L-group in the class* [IN], *then G is pro-Lie.*

Clearly, if G is a locally compact SIN-group, then G is an L-group and in the class [IN]. Thus, by the corollary, every localy compact SIN-group is pro-Lie (see Theorem 2.11 in [**3**]). However, an MAP-group is not necessarily pro-Lie [**2**, Example 2].

It is easy to see that a locally compact group which is embeddable in a pro-Lie group is itself a residual Lie group. By "embeddable" we mean there is a continuous isomorphism into.

In [**1**] an example of a locally compact residual Lie group which is not embeddable in a pro-Lie group is given. Thus, in general, "pro-Lie" and "residual Lie" are quite distinct. However, Theorem 1.4 established the equivalence of "pro-Lie" and "residual-Lie" for compactly generated groups. We have the following corollary of that result.

PROPOSITION 1.6. *If G is a compactly generated group which is embeddable in a pro-Lie group, then G is pro-Lie.*

PROOF. Since G is embeddable in a pro-Lie group and G is locally compact, G is a residual Lie group. It follows from Theorem 1.4 that G is pro-Lie.

As mentioned above, Lie groups in general are not SIN, but SIN groups are pro-Lie. We see by the next proposition that, for totally disconnected locally compact groups, the classes of pro-Lie groups and SIN-groups coincide.

PROPOSITION 1.7. *A locally compact totally disconnected group is pro-Lie if and only if it is a* SIN*-group.*

PROOF. The "if" part has already been established. For the converse, we assume that G is pro-Lie and let U be a neighborhood of the identity. There is a neighborhood W of the identity such that $gHg^{-1} \subset U$ for every $g \in G$ and every subgroup $H \subset W$. Since G is totally disconnected there is a compact open subgroup K of G contained in W [**6**, p. 54]. Thus $\bigcup gKg^{-1}$ is an invariant neighborhood of the identity which is contained in U.

COROLLARY 1. *If G is a totally disconnected, compactly generated L-group, then G is* SIN.

COROLLARY 2. *If G is a locally compact pro-Lie group, then there is an open normal subgroup G_1 such that G_1/G_0 is compact.*

PROOF. Since G is pro-Lie, G/G_0 is pro-Lie. It follows from Proposition 1.7 that G/G_0 is SIN and therefore has an open compact normal subgroup. The inverse image of this subgroup under the canonical map of G to G/G_0 is the desired group G_1.

We have a partial converse to Corollary 2 in the following.

PROPOSITION 1.8. *If G is a locally compact L-group* (*or residual Lie group*) *and has an open normal subgroup G_1 such that G_1/G_0 is compact, then G is pro-Lie.*

PROOF. There is a maximal compact normal subgroup K of G_1 and G_1/K is a Lie group [**2**]. The proof is completed by application of Corollary 2, Theorem 1.2.

For locally compact topological groups G the properties, pro-Lie, SIN, and Lie are preserved under the canonical maps into factor groups G/H where H is a closed normal subgroup of G. This is not the case with N-groups [**2**], and as far as we know may not be the case with residual Lie groups or L-groups. However, we have the following result for L-groups, and later we prove a similar result for residual Lie groups.

PROPOSITION 1.9. *If G is a locally compact L-group and K is a compact normal subgroup, then G/K is an L-group.*

PROOF. We let G_1 be the group generated by VK where V is a compact neighborhod of the identity. Then G_1 is an open compactly generated L-group. Thus G_1 is pro-Lie by Proposition 1.1 and Theorem 1.2. Let U be a neighborhood of the identity of G and C a compact subset of G. We prove there is a neighborhood V of the identity such that $gHg^{-1} \cap CK \subset UK$ for every $g \in G$ and subgroup $H \subset VK$. This will prove that G/K is an L-group. Since G_1 is pro-Lie and G is an L-group, there is a neighborhood W of the identity and a compact subgroup $N \subset W$ such that N is normal in G_1, G_1/N is a Lie group, and $gMg^{-1} \cap CK \subset U$ for every $g \in G$ and subgroup $M \subset W$. Since G_1/NK is a Lie group there is a neighborhood V of the identity such that $V \subset W$ and every subgroup contained in VNK is contained in NK. Let $H \subset VK$. Then $gNg^{-1} \cap CK \subset U$ and $gHg^{-1} \cap CK \subset gNKg^{-1} \cap CK \subset UK$. This completes the proof.

PROPOSITION 1.10. *If G is a locally compact, σ-compact residual Lie group and K is a compact normal subgroup, then G/K is residual Lie.*

PROOF. Let $g \in G$ such that $g \notin K$. Choose a neighborhood V of the identity of G such that $g \notin VK$. Let $\{H_\alpha\}$ be a collection of closed normal subgroups of G such that $\bigcap H_\alpha = e$ and G/H_α is a Lie group for each α. Since $G - V$ is σ-compact, there is a countable collection $\{H_i\}$ from $\{H_\alpha\}$ such that $\bigcap H_i \subset V$. There is no loss of generality in assuming that the sequence $\{H_i\}$ is monotone decreasing by inclusion. Thus $\bigcap H_i K \subset VK$ and, for some H in $\{H_i\}$, we have $g \notin HK$ and G/HK is a Lie group. This completes the proof.

2. Properties of locally compact groups related to pro-Lie groups. In this section we consider pro-Lie groups in more specific settings than those of §1. We show that a locally compact, σ-compact group G is pro-Lie if G/K is pro-Lie, where K is a compact normal subgroup on which the inner automorphisms of G restricted to K are equicontinuous. We prove that a compactly generated nilpotent group has a maximal compact normal subgroup. If a locally compact L-group has a maximal compact normal subgroup, then it is pro-Lie. Our Theorem 2.6 is a generalization of Theorem 4 of [**4**].

As opposed to the situation for Lie groups, a topological group G may fail to be pro-Lie even though G/K is pro-Lie for a closed normal pro-Lie subgroup K. As a matter of fact the semidirect product G, referred to after the proof of Proposition 1.5, has the compact normal subgroup K and G/K is discrete. Yet G is not residual Lie. However, there are pertinent results; for example, Theorem 2.2 below.

DEFINITION 2.1. A closed normal subgroup K of a topological group G is *equinormal* if the group of inner automorphisms of G restricted to K is equicontinuous.

Let $Z(K, G)$ denote the centralizer of K in G.

LEMMA 2.1. *If G is a locally compact, σ-compact group and K is a compact equinormal Lie subgroup of G, then $Z(K,G)K$ has finite index in G.*

PROOF. Let $\eta\colon G \to A(K)$, the automorphism group of K, where $\eta(g) = i_g$, restricted to K, and i_g is the inner automorphism of G determined by g. Since

$A(K)/I(K)$ is discrete when $A(K)$ has the usual topology, which makes it a topological group [**5**, Lemma 1.1], and $\eta(G)$ is equicontinuous on K, $\eta^{-1}(I(K)) = G_1$ has finite index in G. Thus $Z(K,G)K$ has finite index in G since $G_1 \subset Z(K,G)K$.

THEOREM 2.2. *If G is a locally compact, σ-compact group, and K is a compact equinormal subgroup such that G/K is pro-Lie, then G is pro-Lie.*

PROOF. Since K is compact and equinormal, there is an arbitrarily small compact subgroup N of K which is normal in G and K/N is a Lie group. Let $G_1 = G/N$ and $K_1 = K/N$. Now $Z(K_1,G_1)K_1/K_1$ is pro-Lie since it is a closed subgroup of $G_1/K_1 \cong G/K$. It follows that $Z(K_1,G_1)/Z(K_1,G_1) \cap K_1$ is pro-Lie since it is σ-compact and has as a continuous isomorphic image $Z(K_1,G_1)K_1/K_1$. Let $Z = Z(K_1,G_1) \cap K_1$. Using the proof of Theorem 7 of [**4**] on the quotient exact sequence

$$(1) \to Z \to Z(K_1,G_1) \to Z(K_1,G_1)/Z \to (1),$$

we see that $Z(K_1,G_1)$ is pro-Lie. It follows that $Z(K_1,G_1)K_1$ is pro-Lie, and by Lemma 2.1, G_1 is pro-Lie. The conclusion follows since N was taken to be arbitrarily small.

PROPOSITION 2.3. *If G is a compactly generated nilpotent group, then G has a maximal compact normal subgroup K and G/K is a Lie group.*

PROOF. It follows from [**4**, Theorem 9] that G is pro-Lie. Thus, the proof can be reduced to the case that G is a Lie group. In this case G/G_0 is discrete. Thus, if G/G_0 has a maximal finite subgroup, then G has a maximal compact group, hence a maximal compact normal subgroup. The proof can be completed by showing that a finitely generated nilpotent group has a maximal finite subgroup. This follows from the discrete version of Proposition 8 of [**4**], using induction on the nilpotency class of G.

We now prove a theorem which is a generalization of Theorem 4 of [**4**]. We first prove two lemmas.

LEMMA 2.4. *Let G be a closed normal pro-Lie subgroup of a locally compact group F. Let G_1 be any normal subgroup of G such that G_1/G_0 is compact and let Q be the maximal compact normal subgroup of G_1. Then $\overline{F_0G}$ is contained in ZG, where Z is the centralizer of Q relative to $\overline{F_0G}$ and $\overline{F_0G}$ is contained in the normalizer in F of every compact normal subgroup K of G.*

PROOF. We first show that G_1 is normal in $\overline{F_0G}$. If $g \in G_1$, then $\{fgf^{-1} : f \in F_0\}$ is connected and intersects gG_0, a component of G. Thus $fgf^{-1} \in gG_0 \subset G_1$ for each $f \in F_0$. It follows that G_1 is normal in $\overline{F_0G}$ since G_1 is normal in G. Since Q is maximal compact normal in G_1, Q is normal in $\overline{F_0G}$. Let $\{\varphi_f : f \in F_0\}$ be the automorphisms of Q which are the restrictions of inner automorphisms determined by the elements of F_0. By [**5**, Theorem 1] each φ_f is an inner automorphism of Q. Thus, for each $f \in F_0$, there is $q \in Q$ such that $\varphi_f(x) = qxq^{-1}$ for all $x \in Q$, i.e., $fxf^{-1} = qxq^{-1}$ for all $x \in Q$. It follows that $q^{-1}f \in Z$. Thus $F_0 \subset ZQ$ and $\overline{F_0G} \subset ZG$, as desired. If K is compact normal in G, choose G_1 so that $G_0K \subset G_1$; then $K \subset Q$ and the last assertion follows.

LEMMA 2.5. *If G is a compactly generated pro-Lie closed normal subgroup of a locally compact group F, then $\overline{F_0 G}$ is pro-Lie.*

PROOF. There is no loss of generality in taking $F = \overline{F_0 G}$. Let $G \subset K$, normal in F, be such that K/G is maximal compact normal in F/G, a connected locally compact group. Further, F/K is a connected Lie group without nontrivial compact normal subgroups. By 2.4 every compact normal subgroup of K is normal in F. It follows that F is pro-Lie since K is.

THEOREM 2.6. *If F is a locally compact group, G is a compactly generated closed pro-Lie normal subgroup, and F_1 is any open subgroup of F containing F_0 and G such that $F_1/\overline{F_0 G}$ is compact, then F_1 is pro-Lie.*

PROOF. By Lemma 2.5, $\overline{F_0 G}$ is pro-Lie. Since $F_1/\overline{F_0 G}$ is compact, it follows from the theorem below that $F_1$is pro-Lie.

THEOREM 4 [**4**]. *If F is a locally compact group and G is a compactly generated pro-Lie closed normal subgroup such that F/G is compact, then F is pro-Lie.*

Our final theorem characterizes pro-Lie groups in terms of the bounded and periodic parts of the group. We let $B(G)$, the bounded part of G, be the set of all elements whose conjugacy classes are relatively compact, and $P(G)$, the periodic part of G, be the set of all elements of G contained in compact subgroups of G. By [**7**, Theorem 4], $B(G) \cap P(G)$ is a normal subgroup of G.

Before proving the theorem we prove three lemmas. For the lemmas as well as the theorem we let L be the closure of the subgroup generated by the union of the collection of all compact normal subgroups of G.

LEMMA 2.7. *If L, $B(G)$, and $P(G)$ are as defined above and G is pro-Lie, then:*

(1) *L has a compact subgroup which is open in L and $L = \bigcup\{K \triangleleft G\colon K \text{compact}\}$.*
(2) *Every compact subset of L is contained in a compact subgroup of L.*
(3) *$L_0 \subset B(G) \cap P(G)$.*

PROOF. Let $\{K_\alpha\}$ be the collection of all compact normal subgroups of G. Let G_1 be an open normal subgroup of G such that G_1/G_0 is compact, and let K be the maximal compact normal subgroup of G_1. It follows that $\bigcup K_\alpha \cap G_1 \subset K$. Since G_1 is open, $\overline{\langle \bigcup K_\alpha \rangle} \cap G_1 = \overline{\langle \bigcup K_\alpha \rangle \cap G_1} \subset K$, where $\langle \bigcup K_\alpha \rangle$ is the group generated by $\bigcup K_\alpha$. Thus $L \cap K = L \cap G_1$ and (1) is proved. Clearly $L = \bigcup K_\alpha$. Now suppose X is a compact subset of L. Since $L \cap K$ is open in L, we have $\langle \bigcup K_\alpha \rangle L \cap K = L$. Thus $X \subset F(L \cap K)$, where $F \subset \langle \bigcup K_\alpha \rangle$ and F is finite. It follows that $X \subset K_0(L \cap K)$ for some compact subgroup K_0 of $\langle \bigcup K_\alpha \rangle$. This proves (2); (3) follows from the fact that L_0 is compact and normal in G. The compactness of L_0 follows because L contains a compact open subgroup.

LEMMA 2.8. *If G is pro-Lie, then G/L is a Lie group, every compact subset of L generates a compact normal subgroup of G, and $L = B(G) \cap P(G)$.*

PROOF. If H is any compact normal subgroup of G such that G/H is a Lie group, then $H \subset L$ and G/L is a Lie group. For the remaining part of the lemma we can assume that G is a Lie group. By 2.7(1), $L \subset B(G) \cap P(G)$ and $L = B(G) \cap P(G)$

by Corollary 5.6 of [**7**]. By 2.7(1) a compact subset of L is contained in a compact normal subgroup. This completes the proof.

If B is a subgroup of G, we let $N_G(B)$ be the normalizer of B, that is $N_G(B) = \{g \in G\colon gBg^{-1} \subset B\}$.

LEMMA 2.9. *If B is a compact normal subgroup of $B(G) \cap P(G)$, then $G_0 \subset N_G(B)$.*

PROOF. Let $g \in G_0$ and $x \in B \subset B(G) \cap P(G)$. There is a compact subgroup H containing x which is normal in G [**7**, Corollary 5.6]. By Theorem 1 of [**5**] there is an element $y \in H$ such that $gyx = xgy$, since $g \in G_0$ induces an automorphism on H which is an inner automorphism of H. It follows that $g \in N_G(B)H$. The conclusion of the lemma follows since $H \subset B(G) \cap P(G)$ and B is normal in $B(G) \cap P(G)$.

THEOREM 2.10. *A locally compact group G is pro-Lie if and only if there is a collection $\{B_\alpha\}$ of compact normal subgroups of $B(G) \cap P(G)$ such that $\bigcap B_\alpha = e$, $B(G) \cap P(G)/B_\alpha$ is a Lie group, and $G/N_G(B_\alpha)$ is compact. Under these circumstances the following hold:*

(1) *$B(G) \cap P(G)$ is closed,*

(2) *$G/(B(G) \cap P(G))$ is a Lie group.*

PROOF. For the "if" part assume B is a compact normal subgroup of $B(G) \cap P(G)$ such that $B(G) \cap P(G)/B$ is a Lie group and $G/N_G(B)$ is compact. Then $G/N_G(B)$ is totally disconnected since $G_0 \subset N_G(B)$ by 2.9. Thus $G/N_G(B)$ is finite since it is a Lie group. Let $G = FN_G(B)$ where F is finite. Then $B_1 = \bigcap\{gBg^{-1}\colon g \in F\}$ is a compact normal subgroup such that G/B_1 is a Lie group. It follows that G is pro-Lie.

To complete the proof we note that (1) and (2) follow from Lemma 2.8, and if G is pro-Lie the subgroups B_α can be chosen as normal subgroups of G.

As mentioned earlier, an example of a residual Lie group which is not embeddable in a pro-Lie group is given in [**1**]. The referee suggests the following general construction, which yields examples of the same sort.

Let $\{G_j\colon j \in J\}$ be a family of discrete groups, each of which is a semidirect product of an infinite normal subgroup N_j and a finite subgroup F_j in such a fashion that N_j is contained in the group $[N_j, F_j]$ generated by all commutators $[n, f] = nfn^{-1}f^{-1}$, with $n \in N_j$, $f \in F_j$. Then any normal subgroup of G_j containing F_j must contain $[N_j, F_j]$, hence N_j and also F_j, hence all of G_j.

Now let G be the subgroup of $\prod_{j\in J} G_j$ of all $(g_j)_{j\in J}$, almost all of whose components g_j are in F_j, and declare the subgroup $\prod_{j\in J} F_j$ in its Tychonoff topology to be open in G. The projections of the product induce on G quotient maps onto the discrete groups G_j; hence G is residually discrete (hence residually Lie). If $f\colon G \to L$ is a continuous injective morphism into a pro-Lie group then we find in L arbitrarily small open identity neighborhoods V containing a unique maximal subgroup N which is compact and normal in L. Then $U = f^{-1}(V)$ is an identity neighborhood of G with a unique maximal subgrojup $M = f^{-1}(N)$ which is MAP and normal in G. If J is infinite, then every identity neighborhood of G contains at least one F_j (identified with its canonical image in G). It follows that M contains such an F_j and then, since M is normal, also all of G_j by the choice of G_j, N_j and F_j. The point of the construction is then to choose G_j, N_j, and F_j so that, in addition, F_j is not MAP, in order that G will not allow a continuous injection into a pro-Lie group.

ACKNOWLEDGEMENT. The authors would like to thank the referee and Professor K. H. Hofmann for their help, which resulted in significant improvements.

REFERENCES

1. R. W. Bagley and T. S. Wu, *Maximal compact normal subgroups and pro-Lie groups,* Proc. Amer. Math. Soc. (to appear).
2. R. W. Bagley, T. S. Wu and J. S. Yang, *On a class of topological groups more general than SIN groups*, Pacific J. Math. (to appear).
3. S. Grosser and M. Moskowitz, *Compactness conditions in topological groups*, J. Reine Angew. Math. **246** (1971), 1-40.
4. K. H. Hofmann, J. R. Liukkonen and M. W. Mislove, *Compact extensions of compactly generated nilpotent groups are pro-Lie*, Proc. Amer. Math. Soc. **84** (1982), 443–448.
5. K. Iwasawa, *On some types of topological groups*, Ann. of Math. (2) **50** (1949), 507–558.
6. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York, 1965.
7. S. P. Wang, *Compactness properties of topological groups*, Trans. Amer. Math. Soc. **154** (1971), 301–314.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MIAMI, CORAL GABLES, FLORIDA 33124

DEPARTMENT OF MATHEMATICS, CASE WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO 44106

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SOUTH CAROLINA 29208