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SURGERY IN DIMENSION FOUR AND NONCOMPACT 5-MANIFOLDS

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ABSTRACT. This paper describes a precise relationship between the problems of completing surgery in dimension four and finding boundaries for noncompact 5-manifolds.

Two outstanding problems of low-dimensional topology are completing 4-dimensional surgery and finding boundaries for noncompact 5-dimensional manifolds. Although no complete theory is known for either problem, some interesting results have been found. (See **[1, 4, 5, 10]** and Addendum.) In this paper we describe a complete relationship between these two problems. Consequently, knowledge of 4-dimensional surgery yields information about noncompact 5-manifolds and conversely.

All manifolds and maps in this paper are assumed to be smooth. Each manifold is provided with a basepoint and, when appropriate, basepaths will be chosen to given 2-spheres without explicit mention.

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1. Statement of results. Let (f, b): $(M, \partial M) \to (Y, X)$ be a degree-one normal map in the sense of Wall **[11]**, where M is a compact connected 4-manifold and (Y, X) is a Poincaré pair (not necessarily simple). If $X \neq \varnothing$, then we require that $f|\partial M$: $\partial M \to X$ induces an isomorphism of homology groups with $\Lambda = \mathbf{Z}[\pi_1(Y)]$-coefficients. Also, we assume that f is 2-connected, $K_2(M)$ is a free Λ-module, and the surgery obstruction $\sigma(f, b) \in L_4^h(\pi, \omega)$ is zero.

If $H \subset K_2(M)$ is a subkernel, we will say that H is *representable* in M if we can find disjoint framed embedded 2-spheres $S_1, \ldots, S_r \subset \operatorname{int} M$ such that the inclusion $M - \bigcup_i S_i \to M$ induces an isomorphism of fundamental groups, and the classes of $S_1, \ldots, S_r$ in $H_2(M; \Lambda) \cong \pi_2(M)$ form a basis over Λ for H. The subkernel H is *stably representable* if, for some integer $k \geqslant 0$, $H \oplus \langle e_1, \ldots, e_k\rangle$ is representable in $M \# k(S^2 \times S^2)$. Here e_i denotes the class of $S^2 \times 1 \subset i$th-summand $S^2 \times S^2$, and $\langle e_1, \ldots, e_k\rangle$ is the submodule generated by $e_1, \ldots, e_k$. We regard $H \oplus \langle e_1, \ldots, e_k\rangle$ as

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a subkernel after performing surgery on (f, b) to "kill" k null-homotopic circles in int M.

Now suppose that H_1 and H_2 are two subkernels of $K_2(M)$. Then any Λ-module isomorphism $H_1 \to H_2$ extends to an automorphism α of $K_2(M)$ [**11**, Corollary 5.3.1]. In fact, if we identify $K_2(M)$ with the standard kernel, then α represents a well-defined element $\bar{\alpha}$ of $L_5^h(\pi, \omega)$. Assume that H_1 is representable. We can perform (relative) surgery on (f, b) and obtain (f', b') where $f'\colon P \to Y$ is a homotopy equivalence. In the next section, we will prove the following

PROPOSITION 1. *The subkernel H_2 is stably representable iff there exists a normal cobordism (F, B)* rel ∂ *from (f', b') to (f'', b''), where f'' is a homotopy equivalence and the surgery obstruction $\sigma(F, B)$ is precisely $\bar{\alpha}$.*

COROLLARY 2. *If $\bar{\alpha} = 0$, then H_2 is stably representable.*

REMARK. If we replace $L_n^h(\pi, \omega)$ everywhere by $L_n^s(\pi, \omega)$, then after making the obvious necessary modifications the corresponding conclusion of Proposition 1 can be obtained from the results of [**1**, Appendix].

Now given any degree-one normal map (f, b) as above with $\sigma(f, b) = 0$ in $L_4^h(\pi, \omega)$, we construct a noncompact 5-manifold W with a single end ε, tame with vanishing collar obstruction. (See [**8**] for terminology.) As an application of the above proposition we prove

THEOREM 3. *If ε has a collar in W, then (f, b) has a solution; i.e., (f, b) is normally cobordant* rel ∂ *to (f', b'), where f' is a homotopy equivalence.*

Conversely, if (f, b) has a solution (f', b'), then an element $\bar{\alpha}$ in $L_5^h(\pi, \omega)$ is defined. The end ε has a collar in W iff there exists a normal cobordism (F, B) rel ∂ *from (f', b') to (f'', b'') such that f'' is a homotopy equivalence and the surgery obstruction $\sigma(F, B)$ is precisely $\bar{\alpha}$.*

COROLLARY 4. *If $L_5^h(\pi, \omega) = 0$ (e.g., $\pi_1(Y) = 1$), then (f, b) has a solution iff ε has a collar in W.*

If f is already a homotopy equivalence, then we can show

COROLLARY 5. *Let β be any element of $L_5^h(\pi, \omega)$. Then there exists a noncompact 5-manifold W as above such that the end ε has a collar in W iff there exists a normal cobordism (F, B)* rel ∂ *from (f, b) to (f', b') such that f' is a homotopy equivalence and $\sigma(F, B) = \beta$.*

Finally, let W be any noncompact 5-manifold such that ∂W (possibly empty) is diffeomorphic to the interior of a compact manifold. Suppose that ε is a tame end of W with vanishing collar obstruction. We produce a degree-one normal map (f, b) as above such that the conclusion of Theorem 3 is valid.

2. Proof of Proposition 1. Let H_1, H_2 be subkernels of $K_2(M)$ and suppose that H_1 is representable in M. Let α be any automorphism of $K_2(M)$ that restricts to an isomorphism from H_1 to H_2.

LEMMA 6. *If α is an element of $RU_r(\Lambda)$ and r is odd, then H_2 is representable in M. (See* [**11**, *Chapter* 6] *for the definition of $RU_r(\Lambda)$.)*

PROOF. Since H_1 is representable in M, there exist disjoint framed embedded 2-spheres $S_1, \ldots, S_r \subset \operatorname{int} M$, as in §1, whose classes $a_1, \ldots, a_r$ in $H_2(M; \Lambda)$ form a basis for H_1 over Λ. Extend $a_1, \ldots, a_r$ to a basis $a_1, \ldots, a_r, a_1', \ldots, a_r'$ for $K_2(M)$ such that $\lambda(a_i, a_j) = \lambda(a_i', a_j') = 0$, $\lambda(a_i, a_j') = \delta_{ij}$ and $\mu(a_i) = \mu(a_i') = 0$, $1 \leqslant i, j \leqslant r$. Using [**2**, Lemma 2.7 and identities at the bottom of p. 406], we can express α with respect to the basis

$$\alpha = \begin{pmatrix} R & 0 \\ R & (R^*)^{-1} \end{pmatrix} \cdot \alpha_1 \cdots \alpha_l,$$

where R is contained in $\mathrm{GL}(r, \Lambda)$ and each α_i is of the form

$$\text{(i)} \quad \begin{pmatrix} I & P \\ 0 & I \end{pmatrix} \quad \text{or} \quad \text{(ii)} \quad \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Here $P = Q - Q^*$ for some matrix Q.

If $l = 0$, then $a_1, \ldots, a_r$ generate H_2 and we are done. Assume that $l > 0$. If a_l is of type (i), then replace $a_1', \ldots, a_r'$ by $\alpha_l(a_1'), \ldots, \alpha_l(a_r')$, respectively. This has the effect of reducing l by one. If α_l is of type (ii), then by [7] we can represent the classes $a_1', \ldots, a_r'$ by framed immersed 2-spheres $S_1', \ldots, S_r' \subset \operatorname{int} M$ such that $S_i \cap S_j' = \varnothing$ if $i \neq j$ and $S_i \cap S_i'$ consists of a single point of transverse intersection, $1 \leqslant i, j \leqslant r$. By [**7**, Proposition 2.2], $S_1', \ldots, S_r'$ are regularly homotopic to disjoint embedded 2-spheres. Moreover, the regular homotopy can be chosen so that it does not introduce new intersection points with $S_1, \ldots, S_r$. Consequently, S_1, $S_1', \ldots, S_r$, S_r' form r disjoint wedges of 2-spheres. Interchange S_i and S_i' (a_i and a_i'). Again l is reduced by one. Inductively we can assume that $l = 0$. □

Now assume that $H_2 \oplus \langle e_1, \ldots, e_k \rangle$ is representable in $N = M \# k(S^2 \times S^2)$. By hypothesis H_1 is representable in M. Choose disjoint framed embedded 2-spheres $S_1, \ldots, S_{r+k} \subset \operatorname{int} N$, as in §1, such that classes of $S_1, \ldots, S_r$ form a basis for H_1 while $S_i = S^2 \times 1 \subset i$th-summand $S^2 \times S^2$ for $i > r$. Construct a cobordism V_1 by attaching 3-handles to $N \times I$ along these 2-spheres in $N \times 1$. Notice that V_1 is a relative cobordism between N and P. Similarly, form a cobordism V_2 by attaching 3-handles to $N \times I$ along 2-spheres in $N \times I$ whose classes form a basis for $H_2 \oplus \langle e_1, \ldots, e_k \rangle$. Identify V_1 and V_2 along their common boundary part N in order to obtain V. It is possible to extend (f', b') over V to obtain a normal cobordism (F, B) rel ∂. One verifies as in [**11**, p. 66] that $\sigma(F, B) = \bar{\alpha}$.

Conversely, assume that the cobordism (F, B) exists where $F\colon V \to Y \times I$. After doing surgery on circles inside int V, we may assume that the inclusions $P = \partial_- V \to V$, $\partial_+ V \to V$ induce isomorphisms of fundamental groups. Consequently, by the proof of [**6**, Theorem 8.1] we can find a handlebody-decomposition $V = P \times I \cup h_1^2 \cup \cdots \cup h_m^2 \cup h_1^3 \cup \cdots \cup h_m^3$ containing handles of index 2 and 3 only. The number of 2-handles is equal to the number of 3-handles since P and $\partial_+ V$ are homotopy equivalent. After adding cancelling pairs of 2- and 3-handles, if necessary, we can assume that $m = r + k$, $k \geqslant 0$, and that m is odd. Then $N \approx P \# m(S^2 \times S^2)$

and the classes of the belt 2-spheres $S_1, \ldots, S_m$ of $h_1^2, \ldots, h_m^2$, respectively, generate a subkernel of $K_2(N)$. Let $H_2' \subset K_2(N)$ be the subkernel generated by

$$\partial\left(\text{core 3-disk of } h_i^3\right), \qquad 1 \leqslant i \leqslant m.$$

Recall that P was first obtained from M by performing surgery on f: $(M, \partial M) \to (Y, X)$. We can write $M \approx P \# r(S^2 \times S^2)$ where now $S_1, \ldots, S_r$ generate the subkernel H_1. Identify N with $M \# k(S^2 \times S^2)$. Then any isomorphism $H_2' \to H_2 \oplus \langle e_1, \ldots, e_k \rangle$ extends to an automorphism of $K_2(N)$ which, for sufficiently large m, must be an element of $RU_m(\Lambda)$. Since m is odd and H_2' is representable (using the 2-spheres $\partial(\text{core } h_i^3)$) the subkernel $H_2 \oplus \langle e_1, \ldots, e_k \rangle$ is representable in $M \# k(S^2 \times S^2)$ by Lemma 6. □

3. Applications to noncompact 5-manifolds. Let (f, b): $(M, \partial M) \to (Y, X)$ be a degree-one normal map such that f is 2-connected, $K_2(M)$ is a free Λ-module and $\sigma(f, b) = 0$ in $L_4^h(\pi, \omega)$. We construct a noncompact 5-manifold W as follows: First choose any subkernel $H \subset K_2(M)$ with basis $a_1, \ldots, a_r$ such that $\lambda(a_i, a_j) = 0$ and $\mu(a_i) = 0$, $1 \leqslant i, j \leqslant r$. Let $U_1 = M \times I \cup h_1^2 \cup \cdots \cup h_r^2$, where each 2-handle h_i^2 is attached along with a null-homotopic circle in int $M \times 1$ so that $\partial_- U_1 = M$ while $\partial_+ U_1 = N$, where $N = M \# r(S^2 \times S^2)$. Using [7] we can find disjoint framed embedded 2-spheres $S_1, \ldots, S_r \subset \text{int } N$ representing $a_1, \ldots, a_r$, respectively, such that the inclusion $N - \bigcup_i S_i \to N$ induces an isomorphism of fundamental groups. Let $V_1 = N \times I \cup h_1^3 \cup \cdots \cup h_r^3$, where h_i^3 is a 3-handle attached to $N \times 1$ along S_i with its framing. Identify U_1 and V_1 along their common boundary part N in order to obtain W_1. Then W_1 is a relative cobordism from M to a 4-manifold $\partial_+ W_1$ which we will denote by M_1. It is possible to extend (f, b) to a normal cobordism

$$(F_1, B_1) \text{ rel } \partial, \quad \text{where } F_1\colon (W_1; M, N, M_1) \to \left(Y \times I; Y \times 0, Y \times \tfrac{1}{2}, Y \times 1\right).$$

Moreover $F_1 | M_1$ is 2-connected and $K_2(M_1)$ has a subkernel with basis $b_1, \ldots, b_r$ corresponding to the belt 2-spheres of $h_1^2, \ldots, h_r^2$. Let H_1 be a dual subkernel; i.e., a subkernel of $K_2(M_1)$ with basis $b_1', \ldots, b_r'$ such that $\lambda(b_j', b_j') = 0$, $\mu(b_i') = 0$ and $\lambda(b_i, b_j') = \delta_{ij}$, $1 \leqslant i, j \leqslant r$. Using the subkernel H_1, construct a cobordism $W_2 = U_2 \cup V_2$ as before. Inductively construct W_i and define $W = \bigcup_i W_i$, where W_i and

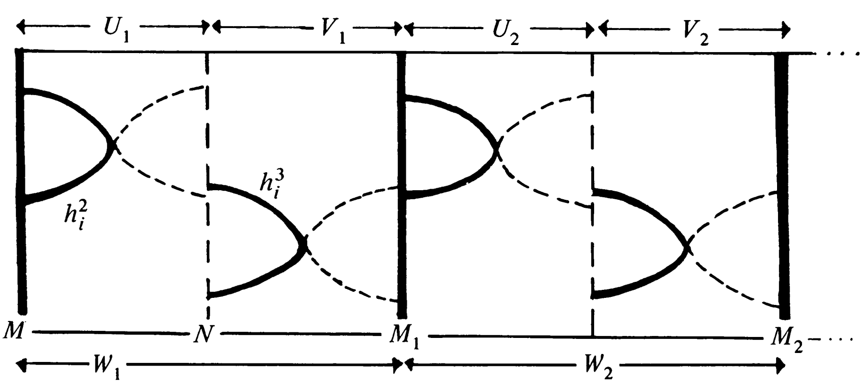

FIGURE 1

W_{i+1} are identified along their common boundary part $\partial_+ W_i \approx \partial_- W_{i+1}$. (See Figure 1.) If $\partial M \neq \varnothing$, then round the resulting corners. Notice that if $\partial M \neq \varnothing$, then $\partial W \approx M \cup \{\text{open collar of } \partial M\}$.

The subsets $\bigcup_{i \geqslant k} W_i$, $k \geqslant 1$, determine a unique end ε of W. Clearly π_1 is stable at ε and the natural maps $\pi_1(\varepsilon) \to \pi_1(W) \cong \pi_1(M)$ are isomorphisms. A straightforward calculation shows that $H_q(W, M; \Lambda) = 0$ if $q \neq 3$, while $H_3(W, M; \Lambda)$ is a free Λ-module of rank r. In fact, the image of the boundary homomorphism ∂: $H_3(W, M; \Lambda) \to H_2(M; \Lambda)$ is precisely the subkernel $H \subset K_2(M)$. It follows by **[8**, Lemma 6.2] that ε is a tame end with vanishing collar obstruction. (This obstruction is the equivalence class of the Λ-module $H_3(W, M; \Lambda)$ in $\tilde{K}_0(\pi_1(\varepsilon))$, the projective class group of $\pi_1(\varepsilon)$.)

REMARK. There are other ways to construct a noncompact 5-manifold which satisfies the conclusion of Theorem 3. For example, we can cross (f, b) with S^1, complete surgery and take the appropriate infinite cyclic cover. Then let W be a closed neighborhood of one of the two ends.

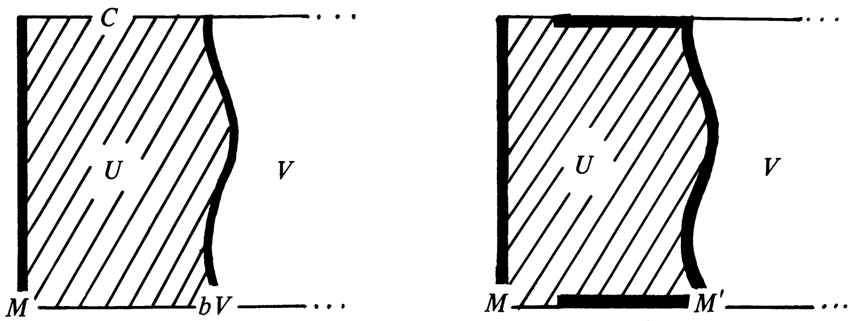

FIGURE 2

PROOF OF THEOREM 3. Assume that ε has a collar V in W. Then V is a connected neighborhood of ε that is a closed submanifold of W. The frontier bV of V is a compact submanifold (possibly with boundary) and $V \approx bV \times [0, \infty)$. Let U denote the closure in W of $W - V$. If $\partial M = \varnothing$, then U is a (relative) cobordism between M and M', where $M' = bV$. If $\partial M \neq \varnothing$, then $C = \partial U - \{\operatorname{int} M \cup \operatorname{int} bV\}$ is an h-cobordism between ∂M and $\partial(bV)$. In this case, let $M' = bV \cup \{C\text{-open collar of } \partial M\}$. Then U is again a relative cobordism between M and M'. (See Figure 2.)

It is easy to verify that the inclusions $M \to U$, $M' \to U$ induce isomorphisms of fundamental groups. Consequently, we can find a handlebody decomposition

$$U = M \times I \cup k_1^2 \cup \cdots \cup k_s^2 \cup k_1^3 \cup \cdots \cup k_t^3$$

containing handles of index 2 and 3 only. Let N denote the middle-level 4-manifold. Extend (f, b) to a normal cobordism

$$(F, B) \operatorname{rel} \partial, \quad \text{where } F\colon (U; M, N, M') \to \left(Y \times I; Y \times 0, Y \times \tfrac{1}{2}, Y \times 1\right).$$

We claim that (F, B) is a normal cobordism from (f, b) to (f', b') where f': $M' \to Y$ is a homotopy equivalence. To see this, let

$$S_i = \partial\left(\text{core 3-disk of } k_i^3\right), \qquad i \leqslant i \leqslant t,$$

and let c_i be the class of S_i in $H_2(N;\ \Lambda)$. The claim will follow from the proof of [**11**, Lemma 5.7] together with the following observation:

LEMMA 7. *The classes $c_1, \ldots, c_t$ generate a subkernel of $K_2(N)$.*

PROOF. Since $S_1, \ldots, S_t$ are disjointly embedded, we have $\lambda(c_i, c_j) = 0$ and $\mu(c_i) = 0$, $1 \leqslant i, j \leqslant t$. Also since the inclusion $M' \to U$ induces an isomorphism of fundamental groups, it follows that the inclusion $N - \bigcup_i S_i \to N$ also induces an isomorphism of fundamental groups. Consequently, we can find 2-spheres $S_1', \ldots, S_t' \subset \operatorname{int} N$ such that $S_i \cap S_j' = \varnothing$ if $i \neq j$ and consists of a single point of transverse intersection if $i = j$. Let c_i' be the class of S_i' in $H_2(N;\ \Lambda)$. It may happen that c_i' is not contained in $K_2(N)$. If this is the case, alter S_i' as follows: Since f_*': $H_2(M';\ \Lambda) \to H_2(Y;\ \Lambda)$ must be surjective, there exists a class y_i in $H_2(M';\ \Lambda)$ such that $f_*'(y_i) = -(F|N)_*(c_i')$. Represent y_i by an immersed 2-sphere $T_i \subset \operatorname{int} M'$. In fact, we can choose $T_i \subset \operatorname{int} M_0'$, where $M_0' = M' - \{$attaching circles of dual 2-handles of $k_1^3, \ldots, k_t^3\}$. Regard N as obtained from M_0' by attaching copies of $D^2 \times S^2$. Then $T_i \subset N - \bigcup_i S_i$ and we can assume that T_i misses $S_1, \ldots, S_t$. Add T_i to S_i' via connected sum taken along a suitable arc in N so that $(F|N)_*(c_i') = 0$; i.e., c_i' is contained in $K_2(N)$. Without loss of generality we can assume that the normal bundle of S_i' in N is trivial. Complete the argument following [**11**, Chapter 5]. Choose μ_i in $\mu(c_i')$ and make the change of basis

$$c_j'' = c_j' - \left(c_j \mu_j + \sum_{i<j} c_i \lambda\left(c_i, c_j'\right)\right).$$

We see that $\lambda(c_i'', c_j'') = 0$, $\mu(c_i'') = 0$ and $\lambda(c_i, c_j'') = \delta_{ij}$, $1 \leqslant i, j \leqslant t$. Thus $c_1, \ldots, c_t$ generate a subkernel of $K_2(N)$. □

Conversely, suppose that (f, b) has a solution. Then there exists a normal cobordism $(F, B) \operatorname{rel} \partial$ from (f, b) to (f', b'), where f': $P \to Y$ is a homotopy equivalence. Assume that F: $(V;\ M, P) \to (Y \times I;\ Y \times 0, Y \times 1)$. After doing surgery on circles inside $\operatorname{int} V$ we may assume without loss of generality that the inclusions $M \to V$, $P \to V$ induce isomorphisms of fundamental groups. Consequently, we can find a handlebody decomposition $V = M \times I \cup k_1^2 \cup \cdots \cup k_s^2 \cup k_1^3 \cup \cdots \cup k_t^3$ containing handles of index 2 and 3 only. Let N denote the middle-level 4-manifold. Then $N \approx M \# s(S^2 \times S^2)$. Let

$$S_i = \partial\left(\text{core 3-disk of } k_i^3\right), \qquad 1 \leqslant i \leqslant t.$$

Then the classes of $S_1, \ldots, S_t$ in $H_2(N;\ \Lambda)$ generate a subkernel $H_1 \subset K_2(N)$.

Now attach s 2-handles to W along null-homotopic circles in $\operatorname{int} M$. Call the resulting 5-manifold W'. Notice that $\partial W'$ can be identified with $N \cup \{$open collar of $\partial N\}$. Moreover, the image of $H_3(W', \partial W';\ \Lambda) \cong H_3(W, \partial W;\ \Lambda) \oplus \Lambda^s$ under the boundary homomorphism ∂: $H_3(W', \partial W';\ \Lambda) \to H_2(\partial W';\ \Lambda) \cong H_2(N;\ \Lambda)$ is the subkernel $H \oplus \langle e_1, \ldots, e_s\rangle \subset K_2(N)$. Let H_2 be a subkernel of $K_2(N)$ which is dual with respect to intersection pairing.

If it is the case that H_2 is representable in N, then form W'' by attaching 3-handles to W' along framed 2-spheres in $\operatorname{int} N$ (as in §1) whose classes form a basis for H_2. Then $\partial W'' = N' \cup \{\text{open collar of } \partial N\}$, where N' is obtained from N by surgery. A direct computation reveals that the inclusion $\partial W'' \to W''$ is a homotopy equivalence. By [9, Theorem 1.6] $W'' \approx N' \times [0, \infty)$. It follows immediately that the end of W has a collar.

If H_2 is stably representable, then after adding sufficiently many cancelling pairs of 2- and 3-handles to the cobordism V, we can assume that H_2 is in fact representable. In the general case, we do not know whether or not H_2 is representable. Proposition 1 completes the argument. □

In order to prove Corollary 5, represent β by an element B in $\mathrm{SU}_r(\Lambda)$ for some r. Perform surgery on the identity map of M to "kill" r null-homotopic circles. We obtain a 2-connected degree-one normal map

$$(f, b), \quad \text{where } f\colon \big(M \# r(S^2 \times S^2), \partial M\big) \to (M, \partial M).$$

The kernel K_2 is a sum of r standard planes. Regard B as an automorphism of K_2. If a_i denotes the class of $S^2 \times 1 \subset i$th-summand $S^2 \times S^2$, then the classes $B(a_1), \ldots, B(a_r)$ generate a subkernel H. Construct the manifold W as above using (f, b) and H. □

Finally, we consider an arbitrary noncompact 5-manifold W. For simplicity, we will assume that ∂W is compact (possibly empty). When ∂W is noncompact but diffeomorphic to the interior of a compact manifold, the following arguments can still be used. (See [8].)

Assume that ε is a tame end of W with vanishing collar obstruction. By [8], we can find neighborhoods V of ε (in the complement of any prescribed compact set) which satisfy the following conditions:

(i) The neighborhood V is a connected manifold having compact connected boundary and just one end.

(ii) The natural maps $\pi_1(\varepsilon) \to \pi_1(V)$ are isomorphisms.

(iii) The inclusion $\partial V \to V$ induces an isomorphism $\pi_1(\partial V) \to \pi_1(V)$.

(iv) The Λ-module $H_q(V, \partial V; \Lambda) = 0$ and $H_3(V, \partial V; \Lambda)$ is a finitely generated free Λ-module.

In fact, we can choose neighborhoods V_1, V_2 as above with $V_2 \subset \operatorname{int} V_1$ such that the cobordism

$$U = \operatorname{closure}(V_1 - \operatorname{int} V_2)$$

has a handlebody-decomposition

$$U = \partial V_1 \times I \cup h_1^2 \cup \cdots \cup h_r^2 \cup h_1^3 \cup \cdots \cup h_s^3,$$

where the classes of the cores of some of the 3-handles, say $h_1^3, \ldots, h_r^3$ generate $H_3(V_1, \partial V_1; \Lambda)$. The classes of the remaining cores are mapped isomorphically by

$$\partial\colon H_3(V_1, \partial V_1; \Lambda) \to H_2(\partial V_1; \Lambda)$$

onto image ∂. Let $V = V_2 \cup \{\text{dual 2-handles of } h_{r+1}^3, \ldots, h_s^2\}$. Denote ∂V by M. A straightforward calculation shows that V satisfies conditions (i)–(iv) above and $H_3(V, M; \Lambda)$ is a free Λ-module of rank r with generators in 1-1 correspondence with $h_1^2, \ldots, h_r^2$. Let $a_1, \ldots, a_r$ be the image of these generators under $\partial\colon H_3(V, M; \Lambda) \to H_2(M; \Lambda)$. Then $\lambda(a_i, a_j) = 0$ and $\mu(a_i) = 0$, $1 \leqslant i, j \leqslant r$. It is not difficult to check that the middle-level 4-manifold of U is diffeomorphic to $\partial V_1 \# r(S^2 \times S^2)$. (This is also a consequence of [**10**, Lemma 3.2]). Thus we can find classes $a_1', \ldots, a_r'$ in $H_2(M; \Lambda)$ such that $\lambda(a_i', a_j') = 0$, $\lambda(a_i, a_j') = \delta_{ij}$, $\mu(a_i') = 0$ for all i, j and the classes $a_1, \ldots, a_r, a_1', \ldots, a_r'$ generate a free submodule K of $H_2(M; \Lambda)$. In fact, K is a direct summand of $H_2(M; \Lambda)$ which is orthogonal with respect to intersection pairing. By elementary techniques we can construct a Poincaré pair (Y, X) and degree-one normal map (f, b), $f\colon (M, \partial M) \to (Y, X)$ such that f is 2-connected and $K_2(M)$ is precisely K. It is easy to check that all arguments in the proof of Theorem 3 apply to (f, b) and the noncompact 5-manifold V. $\square$

Addendum. Shortly after I completed this paper, Simon Donaldson [**3**] proved that there does not exist any smooth closed 1-connected spin 4-manifold with nontrivial definite intersection pairing. This immediately implies the existence of unobstructed surgery problems in dimension four with no solutions. (Such an example can be obtained by attempting to surger the three hyperbolic pairs of the Kummer surface.) Consequently we can now give a handlebody construction, as in §3 above, of a smooth noncompact 5-manifold W with a single end ε, tame with vanishing collar obstruction, such that ε has no collar neighborhood in W. In fact, ε is 1-connected at infinity. Michael Freedman [**4**], in his recent work on Casson handles, has shown that such an end actually has a topological (i.e., nonsmooth) collar neighborhood.

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