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THE DETERMINATION OF THE LIE ALGEBRA ASSOCIATED TO THE LOWER CENTRAL SERIES OF A GROUP

BY

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This paper is dedicated to Professor Wilhelm Magnus

ABSTRACT. In this paper we determine the Lie algebra associated to the lower central series of a finitely presented group in the case where the defining relators satisfy certain independence conditions. Other central series, such as the lower p-central series, are treated as well.

1. Statement of results. The main purpose of this paper is to determine the Lie algebra associated to the lower central series of a group, thus extending the results of [**6**, **7**] to groups defined by more than one relator. The methods apply to other central series such as the lower p-central series.

The lower central series of a group G is the sequence of subgroups G_n $(n \geqslant 1)$ defined inductively by

$$G_1 = G, \qquad G_{n+1} = [G, G_{n+1}],$$

where $[G, G_{n+1}]$ denotes the subgroup of G generated by the commutators $[x, y] = x^{-1}y^{-1}xy$ with $x \in G$, $y \in G_n$. The associated graded abelian group $\mathrm{gr}(G) = \oplus_{n \geqslant 1}\mathrm{gr}_n(G)$, where $\mathrm{gr}_n(G) = G_n/G_{n+1}$, has the structure of a graded Lie algebra over the ring $\mathbf{Z}$ of integers, the bracket operation in $\mathrm{gr}(G)$ being induced by the commutator operation in G (cf. [**2**, **9**, **11**, **12**]). The construction of the Lie algebra $\mathrm{gr}(G)$ uses only the fact that (G_n) is a sequence of subgroups of G with the following properties:

(i) $G_1 = G$,
(ii) $G_{n+1} \subseteq G_n$,
(iii) $[G_n, G_k] \subseteq G_{n+k}$.

Such a family of subgroups of G is called a *(central) filtration* of G.

Let F be the free group on the N letters $x_1, \ldots, x_N$, and let (F_n) be the lower central series of F. If ξ_i is the image of x_i in $\mathrm{gr}_1(F) = F/[F, F]$, then the Lie algebra

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$L = \operatorname{gr}(F)$ associated to (F_n) is a free Lie algebra with basis $\xi_1,\ldots,\xi_N$ (cf. loc. cit.). If $x \in F$, $x \neq 1$, there is a largest integer $n = \omega(x) \geqslant 1$ such that $x \in F_n$. This integer is called the *weight* of x (with respect to (F_n)); the image of x in $\operatorname{gr}_n(F)$ is called the initial form of x (with respect to (F_n)). (If $x = 1$, its initial form is defined to be the zero element of L.)

Let $r_1,\ldots,r_t \in F$ and let $\rho_1,\ldots,\rho_t$ be their initial forms with respect to the lower central series of F. Let $\mathfrak{r} = (\rho_1,\ldots,\rho_t)$ be the ideal of L generated by $\rho_1,\ldots,\rho_t$ and let U be the enveloping algebra of $L/\mathfrak{r}$. Then $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is a U-module via the adjoint representation. Let $\mathfrak{g}$ be the Lie algebra associated to the lower central series of $G = F/R$, where $R = (r_1,\ldots,r_t)$ is the normal subgroup of F generated by the elements $r_1,\ldots,r_t$. In general, $\mathfrak{g} \neq L/\mathfrak{r}$ unless the relators $r_1,\ldots,r_t$ satisfy certain independence conditions (cf. [**7**, **8**] for the case $t = 1$).

THEOREM 1. *If* (i) *$L/\mathfrak{r}$ is a free $\mathbf{Z}$-module and* (ii) *$\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is a free U-module on the images of $\rho_1,\ldots,\rho_t$, then $\mathfrak{g} = L/\mathfrak{r}$.*

For any prime p and any abelian group M we let $M(p) = M/pM$. If, in addition, N is a subgroup of M, we let $N_M(p)$ be the image of $N(p)$ in $M(p)$. Conditions (i) and (ii) in Theorem 1 are equivalent to the following condition:

(iii) *For any prime p, $\mathfrak{r}_L(p)/[\mathfrak{r}_L(p),\mathfrak{r}_L(p)]$ is a free $U(p)$-module on the images of $\rho_1,\ldots,\rho_t$.*

In fact, condition (iii) implies that the rank of the nth homogeneous component of $\mathfrak{g}(p) = \mathfrak{g} \otimes \mathbf{Z}/p\mathbf{Z}$ is independent of p and hence that $\mathfrak{g}$ is a free $\mathbf{Z}$-module (cf. [**6**, **7**]). A formula for the rank of $\mathfrak{g}_n$ can also be given (cf. loc. cit.).

EXAMPLE 1. $t = N - 1$, $\rho_1 = [\xi_1,\xi_2]$, $\rho_2 = [\xi_2,\xi_3],\ldots,\rho_{N-1} = [\xi_{N-1},\xi_N]$.

EXAMPLE 2. $t = N - 1$, $\rho_1 = [\xi_1,\xi_2]$, $\rho_2 = [\xi_1,\xi_3],\ldots,\rho_{N-1} = [\xi_1,\xi_N]$.

EXAMPLE 3. $N = 3$, $t = 2$, $\rho_1 = [\xi_3,[\xi_1,\xi_2]]$, $\rho_2 = [\xi_2,[\xi_1,\xi_3]]$.

EXAMPLE 4. $N = 3$, $t = 2$, $\rho_1 = [\xi_1,\xi_2]$, $\rho_2 = [[\xi_1,\xi_3],\xi_2]$.

As a by-product of the proof, we obtain the following result:

THEOREM 2. *Let Γ be the integral group ring of G, let I be the augmentation ideal of Γ, and let $\operatorname{gr}(\Gamma) = \oplus_{n\geqslant 0} I^n/I^{n+1}$ be the graded algebra associated to the I-adic filtration of Γ. Then, under the conditions* (i) *and* (ii), *we have* $\operatorname{gr}(\Gamma) = U$.

Theorem 1 can be used to determine the structure of the lower central series quotients of certain link groups (cf. [**3**, **4**, **5**]). The proof of Theorem 1 requires the introduction of more general filtrations (see §2), and is proved in this more general context (see §3). The examples are treated in §4. In this section we also give an example to show that the theorem is not true under the hypotheses suggested in [**12**]. In §5 we obtain analogous results for the lower p-central series. The above results are also true for pro-p-groups with virtually the same proofs; one only has to replace $\mathbf{Z}$ by $\mathbf{Z}_p$ (the ring of p-adic integers), subgroups by closed subgroups, and the group ring by the completed group algebra over $\mathbf{Z}_p$. For example, if $G = F/R$ satisfies the conditions of Theorem 1, and G is the pro-p-completion of G, then $\operatorname{gr}(G) = \operatorname{gr}(G) \otimes \mathbf{Z}_p$. The techniques used in the proofs are contained in [**6** and **7**]; that they yield the above results does not seem to have been noticed.

2. The (x, τ)-filtration of the free group F. Let A be the Magnus algebra of formal power series in the noncommutative indeterminates $X_1, \dots, X_N$ with coefficients in $\mathbf{Z}$. The homomorphism of F into the group of units of A defined by $x_i \mapsto 1 + X_i$ is injective and can be extended to an injective homomorphism of the group ring $\Lambda = \mathbf{Z}[F]$ into A (see [**2**]). We identify Λ (and hence F) with its image in A. If $\tau_1, \dots, \tau_N$ are integers $\geqslant 1$, we define a valuation w of A by setting

$$w\Big(\sum a_{i_1,\dots,i_k} X_{i_1} \cdots X_{i_k}\Big) = \operatorname{Inf}\big\{\tau_{i_1} + \cdots + \tau_{i_k} : a_{i_1,\dots,i_k} \neq 0\big\}.$$

For any integer $n \geqslant 0$ let $A_n = \{u \in A : w(u) \geqslant n\}$. Then $A_0 = A$, $A_{n+1} \subseteq A_n$ and $A_m \cdot A_n \subseteq A_{m+n}$ which implies that A_n is an ideal of A. Hence $\operatorname{gr}(A) = \bigoplus_{n \geqslant 0} \operatorname{gr}_n(A)$, where $\operatorname{gr}_n(A) = A_n/A_{n+1}$, has a natural structure of a graded ring. If ξ_i is the image of X_i in $\operatorname{gr}_n(A)$, where $n = \tau_i$, then $\operatorname{gr}(A)$ is the ring of noncommutative polynomials in the ξ_i with coefficients in $\mathbf{Z}$. If L is the Lie subring of $\operatorname{gr}(A)$ generated by the ξ_i, then L is the free Lie algebra over $\mathbf{Z}$ with basis $\xi_1, \dots, \xi_N$.

If for $n > 0$ we set $F_n = (1 + A_n) \cap F$, we obtain a filtration (F_n) of F (cf. [**2**]). We call this filtration the (x, τ)-filtration of F. If $\operatorname{gr}(F)$ is the Lie algebra associated to this filtration, the mapping $F \to A$ defined by $x \mapsto x - 1$, induces an injective Lie algebra homomorphism of $\operatorname{gr}(F)$ into $\operatorname{gr}(A)$, where the bracket operation in $\operatorname{gr}(A)$ is defined by $[u, v] = uv - vu$. We use this isomorphism to identify $\operatorname{gr}(F)$ with its image in $\operatorname{gr}(A)$. If $\operatorname{gr}(\Lambda)$ is the graded ring associated to the filtration (Λ_n) of Λ, where $\Lambda_n = A_n \cap \Lambda$, then

$$L \subseteq \operatorname{gr}(F) \subseteq \operatorname{gr}(\Lambda) \subseteq \operatorname{gr}(A).$$

It follows that $\operatorname{gr}(\Lambda) = \operatorname{gr}(A)$. Let T_n $(n \geqslant 1)$ be the set of elements of the form x_i^e with $e = \pm 1$, $\tau_i = 1$, and define subsets S_n of F inductively as follows: $S_1 = T_1$, and for $n \geqslant 1$, $S_n = T_n \cup T_n'$, where T_n' is the set of elements of the form $[x, y]^e$ with $e = \pm 1$, $x \in S_p$, $y \in S_q$, $p + q = n$. Let $\tilde{F}_n$ be the subgroup of F generated by the S_k with $k \geqslant n$. Then $\tilde{F}_1 = \tilde{F}$, $\tilde{F}_{n+1} \subseteq \tilde{F}_n$, and an easy calculation using the formulae

$$(1) \qquad [x, yz] = [x, z][x, y][[x, y], z],$$

$$(2) \qquad [xy, z] = [x, z][[x, z], y][x, y]$$

shows that $[\tilde{F}_n, \tilde{F}_k] \subseteq \tilde{F}_{n+k}$. If $\tau_i = 1$ for all i, then $(\tilde{F}_n)$ is the lower central series of F. If $\tilde{L}$ is the Lie algebra associated to $(\tilde{F}_n)$, then the inclusions $\tilde{F}_n \subseteq F_n$ induce a canonical homomorphism of $\tilde{L}$ into $\operatorname{gr}(F)$, which must necessarily be injective by the Poincaré-Birkhoff-Witt theorem since $\tilde{L}$ is generated by $\tilde{\xi}_1, \dots, \tilde{\xi}_N$, where $\tilde{\xi}_i$ is the image of x_i in $\operatorname{gr}_n(F)$ with $n = \tau_i$. It follows that $\tilde{F}_n = F_n$ for all $n \geqslant 1$, and hence that $L = \tilde{L} = \operatorname{gr}(F)$.

3. Proof of Theorem 1. We shall prove Theorem 1 in the more general context of the (x, τ)-filtration. Therefore, let (F_n) be the (x, τ)-filtration of F. Let $R_n = R \cap F_n$, and let $\operatorname{gr}(R)$ be the Lie algebra associated to the filtration (R_n) of R. Identifying $\operatorname{gr}(R)$ with its image in $\operatorname{gr}(F)$, the ideal $\mathfrak{r}$ is contained in $\operatorname{gr}(R)$. An easy inductive argument shows that $\mathfrak{r} = \operatorname{gr}(R)$ if and only if the induced homomorphism

$$\theta : \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}] \to \operatorname{gr}(R)/[\operatorname{gr}(R), \operatorname{gr}(R)]$$

is surjective (and hence bijective). Let U and U' be respectively the enveloping algebras of $\mathfrak{g} = \operatorname{gr}(F)/\mathfrak{r}$ and $\operatorname{gr}(G) = \operatorname{gr}(F)/\operatorname{gr}(R)$. The canonical homomorphism $\psi\colon U \to U'$ is surjective and compatible with θ; i.e., for all $x \in \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$, $u \in U$,

$$\theta(u \cdot x) = \psi(u) \cdot \theta(x).$$

Let $M = R/[R, R]$, and let M_n be the image of R_n in M. Then (M_n) is a filtration of M and we have $\operatorname{gr}(M) = \operatorname{gr}(R)/\operatorname{gr}([R, R])$, where $\operatorname{gr}([R, R])$ is the Lie algebra associated to the filtration $([R, R]_n)$, where $[R, R]_n = [R, R] \cap F_n$. Since $\operatorname{gr}(M)$ is an abelian Lie algebra, we have a canonical surjection

$$\theta'\colon \operatorname{gr}(R)/[\operatorname{gr}(R), \operatorname{gr}(R)] \to \operatorname{gr}(M).$$

Let Γ be the group ring of G over $\mathbf{Z}$, and let Γ_n be the image of Λ_n under the canonical homomorphism of Λ onto Γ. Let $\operatorname{gr}(\Gamma)$ be the graded ring associated to the filtration (Γ_n) of Γ. If $\mathscr{R}$ is the ideal of $\operatorname{gr}(\Lambda)$ generated by $\operatorname{gr}(R)$, then U' is canonically isomorphic to $\operatorname{gr}(\Lambda)/\mathscr{R}$, and the kernel of the canonical homomorphism of $\operatorname{gr}(\Lambda)$ onto $\operatorname{gr}(\Gamma)$ contains $\mathscr{R}$. Hence we obtain a surjective homomorphism ψ': $U' \to \operatorname{gr}(\Gamma)$. In addition, $\operatorname{gr}(M)$ is a $\operatorname{gr}(\Gamma)$-module since $\Gamma_n \cdot M_k \subseteq M_{n+k}$, and θ' is compatible with ψ'.

We now show that θ and θ' are bijective. The proof is by induction on the degrees. Suppose then that θ and θ' are bijective in degrees $n < k$. Since $\mathfrak{r}_n = \operatorname{gr}_n(R)$ for $n < e = \min\{\omega(r_1), \dots, \omega(r_t)\}$, we may assume that $k \geqslant e$.

That θ' is bijective in degree k follows exactly as in [7] since, by assumption, $\operatorname{gr}(F)/\mathfrak{r}$ is a free $\mathbf{Z}$-module. The homomorphism θ is injective in degree k since the bijectivity of θ for $n < k$ implies that $\mathfrak{r}_n = \operatorname{gr}_n(R)$ for $n < k$; hence $[\mathfrak{r}, \mathfrak{r}]_k = [\operatorname{gr}(R), \operatorname{gr}(R)]_k$, since both sides are known once we know $\mathfrak{r}_n$ and $\operatorname{gr}_n(R)$ for $n < k$.

To show that θ is surjective in degree k it suffices to show that $\theta'' = \theta' \circ \theta$ is surjective in degree k. If $e_i = \omega(r_i)$, we may assume that $e_i \leqslant e_j$ for $i \leqslant j$ and that $e_i > k$ for $i > s$. Let β be a nonzero element of $\operatorname{gr}_k(M)$, and let $b \in M_k$ be an element whose image in $\operatorname{gr}_k(M)$ is β. If $\bar{r}_i$ is the image of r_i in M, we have

$$b = v_1 \cdot \bar{r}_1 + v_2 \cdot \bar{r}_2 + \cdots + v_s \cdot \bar{r}_s,$$

where $v_i \in \Gamma$, the group ring of G. Since $\Gamma_i \cdot M_j \subseteq M_{i+j}$, we can choose b so that the above expression for b involves only those terms $v_i \cdot \bar{r}_i$ with $\omega_\Gamma(v_i) + e_i \leqslant k$ ($\omega_\Gamma(v) = \operatorname{Sup}\{n\colon v \in \Gamma_n\}$). Since b does not belong to M_{k+1}, this expression is not empty. Let f be the smallest integer of the form $\omega_\Gamma(v_i) + e_i$, and let S be the set of integers i with $\omega_\Gamma(v_i) + e_i = f$. Let u_i be a homogeneous element of U with $\psi''(u_i) = \bar{v}_i$, where $\psi'' = \psi' \circ \psi$ and $\bar{v}_i$ is the image of v_i in $\operatorname{gr}_n(\Gamma)$ with $n = f - e_i$. Let $\bar{\rho}_i$ be the image of ρ_i in $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$, and let $\xi = \sum u_i \cdot \bar{\rho}_i$ ($i \in S$). If $f < k$, we have $\theta(\xi) = 0$; hence $\xi = 0$, since $\deg(\xi) = f$ and θ is injective in degree f. But this contradicts the fact that $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a free U-module. Hence $f = k$ and $\beta = \theta''(\xi)$ which implies that surjectivity of θ'' in degree k.

From the proof it follows that the homomorphism ψ': $\operatorname{gr}(\Lambda)/\mathscr{R} \to \operatorname{gr}(\Gamma)$ is bijective. Since $\mathfrak{r} = \operatorname{gr}(R)$, we have $U = \operatorname{gr}(\Lambda)/\mathscr{R}$ which yields Theorem 2.

4. The examples. We shall need the following result:

LEMMA. *Let k be a commutative ring, and let $L(X)$ be the free Lie algebra over k on the set X. Let S be a subset of X, and let $\mathfrak{a}$ be the ideal of $L(X)$ generated by $X - S$.*

Then $\mathfrak{a}$ is a free Lie algebra over k with basis consisting of the elements $\mathrm{ad}(s_1)\mathrm{ad}(s_2) \cdots \mathrm{ad}(s_n)(x)$ *with $n \geqslant 0$, $s_i \in S$, and $x \in X - S$.*

COROLLARY. *If W is the enveloping algebra of $L(X)/\mathfrak{a}$, then $\mathfrak{a}/[\mathfrak{a}, \mathfrak{a}]$ is a free W-module with basis the images of the elements of $X - S$.*

The proof of the lemma can be found in [**2**, §2, Proposition 10]. The corollary follows since $L/\mathfrak{a} = L(S)$.

We now show that the given systems of defining relators satisfy conditions (i) and (ii) of the theorem. Let $\mathfrak{s}$ be the ideal of L generated by ξ_1. Then, by the lemma, $\mathfrak{s}$ is a free Lie algebra over **Z** with basis consisting of the elements

$$\mathrm{ad}(\xi_{i_1})\mathrm{ad}(\xi_{i_2}) \cdots \mathrm{ad}(\xi_{i_n})(\xi_1),$$

with $n \geqslant 0$, $2 \leqslant i_1, i_2, \ldots, i_n \leqslant N$. In Examples 2, 3 and 4, the relators $\rho_1, \ldots, \rho_t$ lie in $\mathfrak{s}$ and the elements

$$\mathrm{ad}(\xi_{i_1})\mathrm{ad}(\xi_{i_2}) \cdots \mathrm{ad}(\xi_{i_n})(\rho_i)$$

with $n \geqslant 0$, $2 \leqslant i_1, i_2, \ldots, i_n \leqslant N$, $1 \leqslant i \leqslant t$, are part of a basis for $\mathfrak{s}$. Since these elements generate $\mathfrak{r}$ as an ideal of $\mathfrak{s}$, it follows that their images in $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ form a basis for this space as a module over the enveloping V of $\mathfrak{s}/\mathfrak{r}$. Since $\mathfrak{s}/\mathfrak{r}$ and $L/\mathfrak{s}$ are free Lie algebras over **Z**, and hence free **Z**-modules, it follows that $L/\mathfrak{r}$ is a free **Z**-module. If we let K be the subalgebra of $L/\mathfrak{r}$ generated by the images of $\xi_2, \ldots, \xi_N$, the canonical projection of K onto $L/\mathfrak{s}$ is an isomorphism; we use this isomorphism to identify K with $L/\mathfrak{s}$, which can be in turn identified with the free Lie algebra on $\xi_2, \ldots, \xi_N$. Hence $L/\mathfrak{r}$ is the direct sum of $\mathfrak{s}/\mathfrak{r}$ and K as **Z**-modules. If W is the enveloping algebra of K, then V and W are subalgebras of U and the canonical map of $V \otimes W$ into U defined by $v \otimes w \mapsto vw$ is an isomorphism of **Z**-modules (cf. [**2**, §2, No. 7, Corollary 6]). Hence

$$U = \bigoplus V\xi_{i_1}\xi_{i_2} \cdots \xi_{i_n} \qquad (2 \leqslant i_1, i_2, \ldots, i_n \leqslant N, n \geqslant 0).$$

Let $\bar{\rho}_i$ be the image of ρ_i in $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$. Since $\xi_i \cdot \bar{\rho}_j$ is the image of $\mathrm{ad}(\xi_i)(\rho_j)$, it follows that condition (ii) is satisfied.

To treat Example 1, take $\mathfrak{s}$ to be the ideal of L generated by the elements $\xi_2, \xi_4, \ldots, \xi_{2m}$, where $2m$ is the largest even integer $\leqslant N$, and proceed as above.

We now give an example to show that the theorem is not true under the hypotheses suggested in [**12**]. Let $N = 4$, $t = 3$, and let $r_1 = [x_1, x_2]$, $r_2 = [x_2, x_3]$, $r_3 = [x_3, x_1][x_2, [x_2, x_4]]^2$. Then $\rho_1 = [\xi_1, \xi_2]$, $\rho_2 = [\xi_2, \xi_3]$, $\rho_3 = [\xi_3, \xi_1]$ and we see that ρ_1, ρ_2, ρ_3 are part of a basis for $\mathrm{gr}_2(F)$. Since

$$r = [x_3, r_1][x_1, r_2][x_2, r_3] \equiv [x_2, [x_2, [x_2, x_4]]]^2 u \pmod{F_5},$$

where the image of u in F_4 is in the ideal $\mathfrak{a}$ of L generated by ξ_1 and ξ_3, it follows that r is an element of R whose initial form is not in $\mathfrak{a}$. Since $\mathfrak{r} \subseteq \mathfrak{a}$, it follows that $\mathfrak{r} \neq \mathrm{gr}(R)$. Here condition (ii) is not satisfied since $[\xi_3, \rho_1] + [\xi_1, \rho_2] + [\xi_3, \rho_3] = 0$.

5. p-filtrations. In this section we extend the above results to p-filtrations of G (p a fixed prime); for example, the filtration (G_n) of G defined by $G_1 = G$, $G_{n+1} = G_n^p[G, G_n]$. In general, a p-filtration of a group G is a filtration (G_n) of G which satisfies the additional condition

(d) $G_n^p \subseteq G_{n+1}$.

However, it is useful to extend the notion of a p-filtration as in Lazard [**10**] (for one application, see [**8**]); the general reference for this section is [**10**].

As in §2, we imbed $\Lambda = \mathbf{Z}[F]$ in the Magnus algebra A. If $\tau_1, \ldots, \tau_N$ are real numbers > 0, we define a valuation w of A by setting

$$W\left(\sum a_{i_1,\ldots,i_k} X_{i_1} \cdots X_{i_k}\right) = \operatorname{Inf}\left\{\tau_{i_1} + \cdots + \tau_{i_k} + v(a_{i_1,\ldots,i_k}: a_{i_1,\ldots,i_k} \neq 0\right\},$$

where v is the p-adic valuation on $\mathbf{Z}$ ($v(p) = 1$). For each real number $\alpha > 0$ let $A_\alpha = \{u \in A: w(u) \geqslant \alpha\}$. Then $A_0 = A$, $A_\alpha \subseteq A_\beta$ if $\beta \leqslant \alpha$, and $A_\alpha \cdot A_\beta \subseteq A_{\alpha+\beta}$, which implies that A_α is an ideal of A. Since $p \in A_1$, we have $pA_\alpha \subseteq A_{\alpha+1}$. Let $A_{\alpha+} = \bigcup_{\beta > \alpha} A_\beta$, let $\mathrm{gr}_\alpha(A) = A_\alpha / A_{\alpha+}$, and let $\mathrm{gr}(A) = \bigoplus_{\alpha \geqslant 0} \mathrm{gr}_\alpha(A)$. Then $\mathrm{gr}(A)$ is a graded vector space over $\mathbf{F}_p[\pi]$, where $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ and π is the image of p in $\mathrm{gr}_1(A)$. Moreover, π is transcendental over $\mathbf{F}_p$ and $\mathrm{gr}(A)$ is the free associative algebra over $\mathbf{F}_p[\pi]$ on the elements $\xi_1, \ldots, \xi_N$, where ξ_i is the image of X_i in $\mathrm{gr}_\alpha(A)$, $\alpha = \tau_i$.

For any real number $\alpha > 0$, let $F_\alpha = F \cap (1 + A_\alpha)$ and let ω be the corresponding weight function on F. The F_α are subgroups of F with

(a′) $\bigcup_{\alpha > 0} F_\alpha = F$;

(b′) $F_\alpha \subseteq F_\beta$ if $\beta \leqslant \alpha$;

(c′) $[F_\alpha, F_\beta] \subseteq F_{\alpha+\beta}$;

(d′) $F_\alpha^p \subseteq F_{\phi(\alpha)}$, where $\phi(\alpha) = \min(p\alpha, \alpha + 1)$.

Then $F_{\alpha+} = \bigcup_{\beta > \alpha} F_\beta$ is a normal subgroup of F with $F_\alpha / F_{\alpha+}$ abelian. Let $\mathrm{gr}_\alpha(F)$ be the abelian group $F_\alpha / F_{\alpha+}$, written additively, and let i_α: $F_\alpha \to \mathrm{gr}_\alpha(F)$ be the canonical surjection. If $\xi = i_\alpha(x)$ and $\eta = i_\beta(y)$ then $[\xi, \eta] = i_{\alpha+\beta}([x, y])$ uniquely defines an element $[\xi, \eta] \in \mathrm{gr}_{\alpha+\beta}(F)$, and this bracket operation yields a Lie algebra structure (over $\mathbf{F}_p$) on the graded vector space $\mathrm{gr}(F) = \bigoplus_{\alpha > 0} \mathrm{gr}_\alpha(F)$. Now the mapping θ: $\mathrm{gr}(F) \to \mathrm{gr}(A)$ defined by $x \pmod{F_{\alpha+}} \mapsto x - 1 \pmod{A_{\alpha+}}$ is an injective Lie algebra homomorphism of $\mathrm{gr}(F)$ in the underlying Lie algebra of $\mathrm{gr}(A)$ (over $\mathbf{F}_p$). We use this map to identify $\mathrm{gr}(F)$ with its image in $\mathrm{gr}(A)$.

For $\alpha > 0$ let P_α: $\mathrm{gr}_\alpha(A) \to \mathrm{gr}_{\phi(\alpha)}(A)$ be the map defined by $P_\alpha(\eta) = \eta^p$ if $\alpha = 1/(p-1)$, $P_\alpha(\eta) = \eta^p + \pi\eta$ if $\alpha = 1/(p-1)$, and $P_\alpha(\eta) = \pi\eta$ if $\alpha > 1/(p-1)$. The operators $P = P_\alpha$ have the following properties:

(1) For $\alpha \leqslant 1/(p-1)$ and $\xi, \eta \in A_\alpha$, $P(\xi + \eta) = P\xi + P\eta + J(\xi, \eta)$, where $J(\xi, \eta)$ is the Lie polynomial $(\xi + \eta)^p - \xi^p - \eta^p$ in ξ, η with coefficients in $\mathbf{F}_p$;

(2) For $\alpha > 1/(p-1)$ and $\xi, \eta \in A_\alpha$, $P(\xi + \eta) = P\xi + P\eta$;

(3) For $\alpha < 1/(p-1)$ and $\xi \in A_\alpha$, $\eta \in A_\beta$, $[P\xi, \eta] = \mathrm{ad}(\xi)^p(\eta)$;

(4) For $\alpha = 1/(p-1)$ and $\xi \in A_\alpha$, $\eta \in A_\beta$, $[P\xi, \eta] = \mathrm{ad}(\xi)^p(\eta) + P[\xi, \eta]$;

(5) For $\alpha > 1/(p-1)$ and $\xi \in A_\alpha$, $\eta \in A_\beta$, $[P\xi, \eta] = P[\xi, \eta]$.

If $\xi = i_\alpha(x) \in \mathrm{gr}_\alpha(F)$, then $P_\alpha(x) = x^p \pmod{F_{\alpha+}}$. Thus $\mathrm{gr}(F)$ is stable under the operators P_α and hence is a mixed Lie algebra in the terminology of Lazard [**10**]. It is the mixed Lie subalgebra of $\mathrm{gr}(A)^+ = \bigoplus_{\alpha > 0} \mathrm{gr}_\alpha(A)$ generated by the elements $\xi_1, \ldots, \xi_N$ and is free by the Poincaré-Birkhoff-Witt theorem for mixed Lie algebras.

If the τ_i are all integers, then $F_\alpha = F_n$ if $n - 1 < \alpha \leqslant n$ and (F_n) is a p-filtration. If $\tau_i = 1$ for all i, then (F_n) is the lower p-central series of F. We thus see that, for

$p \neq 2$, the Lie algebra associated to the lower central series of F is a free $\mathbf{F}_p[\pi]$-Lie algebra.

Let $\rho_1, \ldots, \rho_t$ be the initial forms of the elements $r_1, \ldots, r_t \in F$ with respect to the (x, τ, p)-filtration. Let R be the normal subgroup of F generated by $r_1, \ldots, r_t$. Let $\mathrm{gr}(R) = \oplus_{\alpha > 0} \mathrm{gr}_\alpha(R)$ where $\mathrm{gr}_\alpha(R) = R_\alpha / R_{\alpha+}$ and $R_\alpha = R \cap F_\alpha$. Let L be the $\mathbf{F}_p[\pi]$-Lie subalgebra of $\mathrm{gr}(A)$ generated by $\xi_1, \ldots, \xi_N$. Then L is the free Lie algebra over $\mathbf{F}_p[\pi]$ on $\xi_1, \ldots, \xi_N$.

THEOREM 3. *Suppose that $\omega(r_i) > 1/(p-1)$ for all i, and that the elements $\rho_1, \ldots, \rho_t$ lie in L. Let $\mathfrak{r}$ be the ideal of L generated by $\rho_1, \ldots, \rho_N$ and let U be the enveloping algebra of $L/\mathfrak{r}$. If* (i) *$L/\mathfrak{r}$ is a free $\mathbf{F}_p[\pi]$-module and* (ii) *$\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a free U-module on the images of $\rho_1, \ldots, \rho_t$, then $\mathfrak{r} = \mathrm{gr}(R)$.*

The proof of this theorem is entirely analogous to the proof of Theorem 1; the same argument yields the corresponding result for pro-p-groups (cf. **[6]**).

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