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A REFLEXIVITY THEOREM FOR WEAKLY CLOSED SUBSPACES OF OPERATORS

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ABSTRACT. It was proved in [4] that the ultraweakly closed algebras generated by certain contractions on Hilbert space have a remarkable property. This property, in conjunction with the fact that these algebras are isomorphic to H^∞, was used in [3] to show that such ultraweakly closed algebras are reflexive. In the present paper we prove an analogous result that does not require isomorphism with H^∞, and applies even to linear spaces of operators. Our result contains the reflexivity theorems of [3, 2 and 9] as particular cases.

Let $\mathscr{L}(\mathscr{H})$ denote the algebra of (linear, bounded) operators acting on the Hilbert space $\mathscr{H}$, and let $\mathscr{M}$ denote a linear subspace of $\mathscr{L}(\mathscr{H})$. Then $\mathscr{M}$ is endowed with the weak and ultraweak topologies that it inherits from $\mathscr{L}(\mathscr{H})$ (cf. [6, §15]). For two arbitrary vectors $x, y \in \mathscr{H}$ we can define the (ultra) weakly continuous functional $[x \otimes y]$ on $\mathscr{M}$ by

$$[x \otimes y](A) = \langle Ax, y\rangle, \qquad A \in \mathscr{M},$$

where $\langle \cdot , \cdot \rangle$ stands for the scalar product in $\mathscr{H}$.

DEFINITION 1. Let n be a natural number, $n \geqslant 1$. The subspace $\mathscr{M}$ has property $(\tilde{\mathbf{B}}_n)$ [respectively $(\tilde{\mathbf{A}}_n)$] if for every positive number ε there exists a positive number $\delta = \delta(\varepsilon, n)$ such that for every system $\{\phi_{ij}: 1 \leqslant i, j \leqslant n\}$ of weakly [respectively ultraweakly] continuous functionals on $\mathscr{M}$ and every system $\{x_i, y_j: 1 \leqslant i, j \leqslant n\}$ of vectors in $\mathscr{H}$ satisfying the inequalities $\|\phi_{ij} - [x_i \otimes y_j]\| < \delta$ there exist vectors $\{x_i', y_j': 1 \leqslant i, j \leqslant n\}$ in $\mathscr{H}$ such that

$$\phi_{ij} = \left[x_i' \otimes y_j'\right], \qquad 1 \leqslant i, j \leqslant n,$$

and

$$\left\|x_i - x_i'\right\| < \varepsilon, \quad \left\|y_j - y_j'\right\| < \varepsilon, \qquad 1 \leqslant i, j \leqslant n.$$

Since every weakly continuous functional on $\mathscr{M}$ is also ultraweakly continuous, property $(\tilde{\mathbf{B}}_n)$ is weaker than $(\tilde{\mathbf{A}}_n)$. (ADDED IN PROOF. It was pointed out by C. Apostol that $(\tilde{\mathbf{B}}_n)$ and $(\tilde{\mathbf{A}}_n)$ are in fact equivalent. This fact is not used below.)

We recall now from [8] that a linear subspace $\mathscr{M}$ of $\mathscr{L}(\mathscr{H})$ is said to be *reflexive* if it contains every operator $T \in \mathscr{L}(\mathscr{H})$ with the property that $Tx \in (\mathscr{M}x)^-$ for every

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$x \in \mathscr{H}$. Of course, reflexive subspaces are weakly closed. This definition coincides with the usual definition ($\mathscr{M} = \operatorname{Alg\,Lat} \mathscr{M}$) if $\mathscr{M}$ is a subalgebra of $\mathscr{L}(\mathscr{H})$.

We state now the main result of this paper.

THEOREM 2. *Let $\mathscr{M}$ be a weakly closed subspace of $\mathscr{L}(\mathscr{H})$. If $\mathscr{M}$ has property $(\mathbf{B}_n^{\sim})$ for every natural number n, then $\mathscr{M}$ is reflexive. Moreover, every weakly closed subspace of $\mathscr{M}$ is also reflexive.*

Before going into the proof, we relate this result with the reflexivity theorem from [**3**]. It was proved in [**4**] that, if T is a (BCP)-operator, the ultraweakly closed algebra A_T generated by T has property $(\mathbf{A}_n^{\sim})$ for every $n = 1, 2, \ldots$. The reflexivity of A_T follows then from Theorem 2 and the following lemma.

LEMMA 3. *Let $\mathscr{M}$ be a linear subspace of $\mathscr{L}(\mathscr{H})$ having property $(\mathbf{A}_1^{\sim})$. Then the weak and ultraweak closures of $\mathscr{M}$ coincide, and the weak and ultraweak topologies coincide on the weak closure of $\mathscr{M}$.*

PROOF. Since every ultraweakly continuous functional on $\mathscr{M}$ extends continuously to the ultraweak closure of $\mathscr{M}$, there is no loss of generality in assuming that $\mathscr{M}$ is ultraweakly closed. Let $\delta = \delta(1,1)$ be as in Definition 1, and let ϕ be an arbitrary ultraweakly continuous functional on $\mathscr{M}$. Then $\|\delta\phi/(2\|\phi\|) - [0 \otimes 0]\| < \delta$ so that we can find vectors x' and y' such that $\|x'\| < 1$, $\|y'\| < 1$ and $\delta\phi/(2\|\phi\|) = [x' \otimes y']$ or, equivalently, $\phi = [x \otimes y]$ with $x = (2\|\phi\|/\delta)^{1/2}x'$, $y = (2\|\phi\|/\delta)^{1/2}y'$. Thus we can write ϕ as $[x \otimes y]$ with $\|x\| < (2/\delta)^{1/2}\|\phi\|^{1/2}$, $\|y\| < (2/\delta)^{1/2}\|\phi\|^{1/2}$. We can now apply, e.g., the proof of [**3**, Theorem 1] to conclude that $\mathscr{M}$ is weakly closed and the weak and ultraweak topologies coincide on $\mathscr{M}$.

We have therefore the following consequence of Theorem 2, which also implies the reflexivity results of [**2** and **9**].

COROLLARY 4. *Let $\mathscr{M}$ be an ultraweakly closed subspace of $\mathscr{L}(\mathscr{H})$. If $\mathscr{M}$ has property $(\mathbf{A}_n^{\sim})$ for every natural number n, then $\mathscr{M}$ is weakly closed and reflexive. Moreover, every weakly closed subspace of $\mathscr{M}$ is also reflexive.*

For the proof of Theorem 2, we need two lemmas. The first was proved in [**3**] for the case in which $\mathscr{M}$ is a weakly closed algebra. The proof for linear subspaces of $\mathscr{L}(\mathscr{H})$ is identical (and easy) so we content ourselves with the statement.

LEMMA 5. *Let $\mathscr{M}$ be a linear subspace of $\mathscr{L}(\mathscr{H})$. An operator $T \in \mathscr{L}(\mathscr{H})$ is in the weak closure of $\mathscr{M}$ if and only if for every natural n and every system $\{x_i, y_i\colon 1 \leqslant i \leqslant n\}$ of vectors in $\mathscr{H}$ such that $\sum_{i=1}^n [x_i \otimes y_i] = 0$, we have $\sum_{i=1}^n \langle Tx_i, y_i\rangle = 0$.*

LEMMA 6. *Let $\mathscr{M}$ be a linear subspace of $\mathscr{L}(\mathscr{H})$. Assume that $\mathscr{M}$ has property $(\mathbf{B}_n^{\sim})$ for every natural number n. Then for every natural number n, every system $\{x_i, y_i\colon 1 \leqslant i \leqslant n\}$ of vectors in $\mathscr{H}$ and every $\varepsilon > 0$, there exist vectors $\{x_{ij}, y_{ij}\colon 1 \leqslant i, j \leqslant n\}$ such that $[x_{ij} \otimes y_{kl}] = \delta_{jl}[x_i \otimes y_k]$, $1 \leqslant i, j, k, l \leqslant n$, and $\|x_i - x_{i1}\| < \varepsilon$, $\|y_j - y_{j1}\| < \varepsilon$, $1 \leqslant i, j \leqslant n$. (Here δ_{jl} denotes, as usual, Kronecker's symbol.)*

PROOF. Let $\delta = \delta(\varepsilon, n^2)$ be as in Definition 1. Set

$$\eta = \min\{\delta/(2\|[x_i \otimes y_k]\|) : [x_i \otimes y_k] \neq 0\}$$

and define $\phi_{ij,kl} = 0$ if $j \neq l$, $\phi_{i1,k1} = [x_i \otimes y_k]$, $\phi_{ij,kj} = \eta[x_i \otimes y_k]$, $j \geqslant 2$. The vectors $\{x_{ij}^0, y_{ij}^0 : 1 \leqslant i, j \leqslant n\}$ defined by $x_{ij}^0 = \delta_{j1}x_i$, $y_{ij}^0 = \delta_{j1}y_i$, $1 \leqslant i, j \leqslant n$, obviously satisfy the inequalities

$$\left\|\phi_{ij,kl} - \left[x_{ij}^0 \otimes y_{kl}^0\right]\right\| < \delta, \qquad 0 \leqslant i, j, k, l \leqslant n,$$

and therefore, by property ($\mathbf{B}_{n^2}^{\sim}$), we can find vectors $\{x'_{ij}, y'_{ij} : 1 \leqslant i, j \leqslant n\}$ in $\mathscr{H}$ such that

$$\phi_{ij,kl} = \left[x'_{ij} \otimes y'_{kl}\right], \qquad 0 \leqslant i, j, k, l \leqslant n,$$

and $\|x_{ij}^0 - x'_{ij}\| < \varepsilon$, $\|y_{ij}^0 - y'_{ij}\| < \varepsilon$, $0 \leqslant i, j \leqslant n$. Then the vectors $\{x_{ij}, y_{ij} : 1 \leqslant i, j \leqslant n\}$ defined by $x_{i1} = x'_{i1}$, $y_{i1} = y'_{i1}$, $x_{ij} = \eta^{-1/2}x'_{ij}$, $y_{ij} = \eta^{-1/2}y'_{ij}$, $1 \leqslant i \leqslant n$, $2 \leqslant j \leqslant n$, satisfy the requirements of the lemma.

PROOF OF THEOREM 2. Let $T \in \mathscr{L}(\mathscr{H})$ satisfy the property that $Tx \in (\mathscr{M}x)^-$ for every $x \in \mathscr{H}$. We first note that the equality $[x \otimes y] = 0$, $x, y \in \mathscr{H}$, means that y is orthogonal to $(\mathscr{M}x)^-$, and hence it implies $\langle Tx, y\rangle = 0$.

In order to show that $T \in \mathscr{M}$, we must prove, according to Lemma 5, that the equality $\sum_{i=1}^n [x_i \otimes y_i] = 0$, $x_i, y_i \in \mathscr{H}$, $1 \leqslant i \leqslant n$, implies $\sum_{i=1}^n \langle Tx_i, y_i\rangle = 0$. By what has just been said, this property is satisfied for $n = 1$. Assume therefore that $n \geqslant 2$, $x_i, y_i \in \mathscr{H}$, $1 \leqslant i \leqslant n$, and $\sum_{i=1}^n [x_i \otimes y_i] = 0$. For every $\varepsilon > 0$ we can find, using Lemma 6, vectors $x_{ij} = x_{ij}(\varepsilon)$, $y_{ij} = y_{ij}(\varepsilon)$, $0 \leqslant i, j \leqslant n$, satisfying,

$$\left[x_{ij} \otimes y_{kl}\right] = \delta_{jl}[x_i \otimes y_k], \qquad 1 \leqslant i, j, k, l \leqslant n, \tag{1}$$

and

$$\begin{aligned} &\|x_i - x_{i1}\| = \|x_i - x_{i1}(\varepsilon)\| < \varepsilon, \\ &\|y_j - y_{j1}\| = \|y_j - y_{j1}(\varepsilon)\| < \varepsilon, \qquad 1 \leqslant i, j \leqslant n. \end{aligned} \tag{2}$$

We now remark that by (1)

$$\begin{aligned} \left[\sum_{i=1}^n x_{ii} \otimes \sum_{i=1}^n y_{ii}\right] &= \sum_{i=1}^n [x_{ii} \otimes y_{ii}] + \sum_{i \neq j} [x_{ii} \otimes y_{jj}] \\ &= \sum_{i=1}^n [x_i \otimes y_i] = 0 \end{aligned}$$

and therefore

$$\left\langle T\left(\sum_{i=1}^n x_{ii}\right), \sum_{i=1}^n y_{ii}\right\rangle = 0. \tag{3}$$

Since $[x_{ii} \otimes y_{jj}] = 0$ for $i \neq j$, we also have $\langle Tx_{ii}, y_{jj}\rangle = 0$ for $i \neq j$ so that (3) can be rewritten as

$$\sum_{i=1}^n \langle Tx_{ii}, y_{ii}\rangle = 0. \tag{4}$$

Assume now that $i \neq 1$. We have by (1)

$$\begin{aligned}[(x_{ii} - x_{i1}) \otimes (y_{ii} + y_{i1})] &= [x_{ii} \otimes y_{ii}] - [x_{i1} \otimes y_{i1}] + [x_{ii} \otimes y_{i1}] - [x_{i1} \otimes y_{ii}] \\ &= [x_i \otimes y_i] - [x_i \otimes y_i] = 0\end{aligned}$$

and therefore

$$\begin{aligned}0 &= \langle T(x_{ii} - x_{i1}), y_{ii} + y_{i1} \rangle \\ &= \langle Tx_{ii}, y_{ii} \rangle - \langle Tx_{i1}, y_{i1} \rangle + \langle Tx_{ii}, y_{i1} \rangle - \langle Tx_{i1}, y_{ii} \rangle.\end{aligned}$$

The last two terms are zero because $[x_{ii} \otimes y_{i1}] = [x_{i1} \otimes y_{ii}] = 0$ and we conclude that $\langle Tx_{ii}, y_{ii} \rangle = \langle Tx_{i1}, y_{i1} \rangle$. Therefore (4) can now be written as $\sum_{i=1}^{n} \langle Tx_{i1}, y_{i1} \rangle = 0$. We now let ε approach zero. We have $\lim_{\varepsilon \to 0} x_{i1}(\varepsilon) = x_i$, $\lim_{\varepsilon \to 0} y_{i1}(\varepsilon) = y_i$ so that

$$\sum_{i=1}^{n} \langle Tx_i, y_i \rangle = \lim_{\varepsilon \to 0} \sum_{i=1}^{n} \langle Tx_{i1}(\varepsilon), y_{i1}(\varepsilon) \rangle = 0$$

and the reflexivity of $\mathscr{M}$ is proved by Lemma 5. The last statement of the theorem follows from [**8**, Theorem 2.3] (cf. also [**7**]).

We conclude with a condition implying property $(\mathbf{A}_n^{\sim})$ and which is sometimes easier to verify. For an arbitrary linear subspace $\mathscr{M}$ of $\mathscr{L}(\mathscr{H})$ we will denote by $\mathscr{M}_*$ the Banach space of all ultraweakly continuous functionals on $\mathscr{M}$. It is well known that the dual space of $\mathscr{M}_*$ coincides with the ultraweak closure of $\mathscr{M}$; we will not use this fact here. The following two definitions were given in [**1**] for ultraweakly closed algebras $\mathscr{M}$ (cf. [**1**, Definitions 1.4 and 1.5]).

DEFINITION 7. Suppose $\mathscr{M} \subset \mathscr{L}(\mathscr{H})$ is a linear subspace and $0 \leqslant \theta < +\infty$. We denote by $X_\theta(\mathscr{M})$ the set of all ϕ in $\mathscr{M}_*$ such that there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ satisfying the following conditions:

$$\|x_i\| \leqslant 1, \quad \|y_i\| \leqslant 1, \qquad 1 \leqslant i \leqslant \infty,$$

$$\limsup_{i \to +\infty} \|\phi - [x_i \otimes y_i]\| \leqslant \theta,$$

and

$$\lim_{i \to \infty} \left(\|[x_i \otimes z]\| + \|[z \otimes x_i]\| + \|[y_i \otimes z]\| + \|[z \otimes y_i]\| \right) = 0, \qquad z \in \mathscr{H}.$$

DEFINITION 8. Suppose $\mathscr{M} \subset \mathscr{L}(\mathscr{H})$ is a linear subspace and $0 \leqslant \theta < \gamma < +\infty$. We say that $\mathscr{M}$ has property $X_{\theta,\gamma}$ if the closed absolutely convex hull of the set $X_\theta(\mathscr{M})$ contains the closed ball of radius γ centered at the origin in $\mathscr{M}_*$:

$$\overline{\text{aco}}\, X_\theta(\mathscr{M}) \supset \{\phi \in \mathscr{M}_* : \|\phi\| \leqslant \gamma\}.$$

The following result coincides with [**1**, Theorem 1.9] if $\mathscr{M}$ is an ultraweakly closed algebra. However, neither the algebra structure, nor the ultraweak closedness of $\mathscr{M}$ has been used in the proof of that theorem, so that we refer to [**1**] for the proof.

THEOREM 9. *Suppose $\mathscr{M} \subset \mathscr{L}(\mathscr{H})$ is a linear subspace with property $X_{\theta,\gamma}$ for some $\gamma > \theta \geqslant 0$. Then for every $\phi \in \mathscr{M}_*$ there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ such that*

$$\phi = [x_i \otimes y_i], \qquad 1 \leqslant i < \infty,$$

$$\limsup_{i \to \infty} \|x_i\| \leqslant (\gamma - \theta)^{-1/2} \|\phi\|^{1/2}, \quad \limsup_{i \to \infty} \|y_i\| \leqslant (\gamma - \theta)^{-1/2} \|\phi\|^{1/2},$$

and

$$\lim_{i\to\infty}\left(\|[x_i\otimes z]\|+\|[z\otimes x_i]\|+\|[y_i\otimes z]\|+\|[z\otimes y_i]\|\right)=0,\qquad z\in\mathscr{H}.$$

It was seen in [1] that this theorem implies that $\mathscr{M}$ has property $(\mathbf{A}_n)$ for each n; we recall that property $(\mathbf{A}_n)$ requires the solvability for x_i and y_i of arbitrary systems of the form $[x_i\otimes y_j]=\phi_{ij}$, $\phi_{ij}\in\mathscr{M}_*$, $1\leqslant i,\ j\leqslant n$. In order to prove the stronger property $(\tilde{\mathbf{A}}_n)$ we need the following lemma, whose proof is reminiscent of the techniques of Robel [9].

LEMMA 10. *Suppose $\mathscr{M}\subset\mathscr{L}(\mathscr{H})$ is a linear subspace with property $X_{\theta,\gamma}$ for some $\gamma>\theta\geqslant 0$. If n is a natural number, $a>0$, $\varepsilon>0$, and $\phi_{ij}\in\mathscr{M}_*$, $x_i,\ y_j\in\mathscr{H}$, $1\leqslant i,\ j\leqslant n$, are such that*

$$\left\|\phi_{ij}-\left[x_i\otimes y_j\right]\right\|<a,\qquad 1\leqslant i,\ j\leqslant n,$$

then there exist $\{x_i',\ y_j'\colon 1\leqslant i,\ j\leqslant n\}$ in $\mathscr{H}$ such that

$$\left\|\phi_{ij}-\left[x_i'\otimes y_j'\right]\right\|<\varepsilon,\qquad 1\leqslant i,\ j\leqslant n,$$

and

$$\|x_i-x_i'\|<n(\gamma-\theta)^{-1/2}a^{1/2},\quad \|y_j-y_j'\|<n(\gamma-\theta)^{-1/2}a^{1/2},\qquad 1\leqslant i,\ j\leqslant n.$$

PROOF. Let $\delta>0$ be such that $(n^2+2n-1)\delta<\varepsilon$. An application of Theorem 9 to $\phi=\phi_{ij}-[x_i\otimes y_j]$ yields sequences $\{\xi_{ij}(k)\}_{k=1}^{\infty}$, $\{\eta_{ij}(k)\}_{k=1}^{\infty}$ such that

$$\phi_{ij}-\left[x_i\otimes y_j\right]=\left[\xi_{ij}(k)\otimes\eta_{ij}(k)\right],\qquad 1\leqslant k<\infty, \tag{6}$$

$$\|\xi_{ij}(k)\|<(\gamma-\theta)^{-1/2}a^{1/2},\quad \|\eta_{ij}(k)\|<(\gamma-\theta)^{-1/2}a^{1/2},\qquad 1\leqslant k<\infty, \tag{7}$$

and

$$\lim_{k\to\infty}\left(\left\|\left[\xi_{ij}(k)\otimes z\right]\right\|+\left\|\left[z\otimes\eta_{ij}(k)\right]\right\|\right)=0,\qquad z\in\mathscr{H}. \tag{8}$$

An easy induction using (8) shows that we can find natural numbers k_{ij}, $1\leqslant i,\ j\leqslant n$, such that the vectors $\xi_{ij}=\xi_{ij}(k_{ij})$ and $\eta_{ij}=\eta_{ij}(k_{ij})$ satisfy the inequalities

$$\begin{cases}\left\|\left[\xi_{ij}\otimes\eta_{kl}\right]\right\|<\delta & \text{if } (i,j)\neq(k,l),\\ \left\|\left[x_i\otimes\eta_{kl}\right]\right\|<\delta, & 1\leqslant i,k,l\leqslant n,\\ \left\|\left[\xi_{ij}\otimes y_k\right]\right\|<\delta, & 1\leqslant i,j,k\leqslant n.\end{cases} \tag{9}$$

We can now set

$$x_i'=x_i+\sum_{k=1}^{n}\xi_{ik},\quad y_j'=y_j+\sum_{l=1}^{n}\eta_{lj}$$

and note that we obviously have from (7)

$$\|x_i'-x_i\|\leqslant\sum_{k=1}^{n}\|\xi_{ik}\|<n(\gamma-\theta)^{-1/2}a^{1/2},\qquad 1\leqslant i\leqslant n,$$

and similarly

$$\|y_j' - y_j\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \qquad 1 \leqslant j \leqslant n.$$

Finally, we observe that

$$\phi_{ij} - [x_i' \otimes y_j'] = \phi_{ij} - [x_i \otimes y_j] - [\xi_{ij} \otimes \eta_{ij}] - \sum_{l=1}^{n} [x_i \otimes \eta_{lj}]$$
$$- \sum_{k=1}^{n} [\xi_{ik} \otimes y_j] - \sum_{(l,k) \neq (i,j)} [\xi_{ik} \otimes \eta_{lj}]$$

and we obtain, using (6) and (9),

$$\|\phi_{ij} - [x_i' \otimes y_j']\| \leqslant n\delta + n\delta + (n^2 - 1)\delta < \varepsilon.$$

The lemma follows.

A routine argument shows now that Lemma 10 is self-improving to yield the following result.

THEOREM 11. *Suppose $\mathscr{M} \subset \mathscr{L}(\mathscr{H})$ is a linear subspace with property $X_{\gamma,\theta}$ for some $\gamma > \theta \geqslant 0$. If n is a natural number, $a > 0$ and $\phi_{ij} \in \mathscr{M}_*$, x_i, $y_j \in \mathscr{H}$, $1 \leqslant i, j \leqslant n$, are such that*

$$\|\phi_{ij} - [x_i \otimes y_j]\| < a, \qquad 1 \leqslant i, j \leqslant n,$$

then there exist $\{x_i', y_j' \colon 1 \leqslant i, j \leqslant n\}$ in $\mathscr{H}$ such that

$$\phi_{ij} = [x_i' \otimes y_j'], \qquad 1 \leqslant i, j \leqslant n,$$

and

$$\|x_i - x_i'\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \quad \|y_j - y_j'\| < n(\gamma - \theta)^{-1/2} a^{1/2}, \qquad 1 \leqslant i, j \leqslant n.$$

PROOF. Choose a positive number b such that

$$\|\phi_{ij} - [x_i \otimes y_j]\| < b < a, \qquad 1 \leqslant i, j \leqslant n,$$

and let ε be a positive number to be specified later (ε will only depend on a and b). By Lemma 10, we can find vectors $\{x_i^1, y_j^1 \colon 1 \leqslant i, j \leqslant n\}$ such that

$$\|\phi_{ij} - [x_i^1 \otimes y_j^1]\| < \varepsilon, \qquad 1 \leqslant i, j \leqslant n,$$

and

$$\|x_i^1 - x_i\| < n(\gamma - \theta)^{-1/2} b^{1/2}, \quad \|y_j^1 - y_j\| < n(\gamma - \theta)^{-1/2} b^{1/2}.$$

We can then use Lemma 10 to construct inductively sequences $\{x_i^k\}_{k=2}^{\infty}$, $\{y_j^k\}_{k=2}^{\infty}$, $1 \leqslant i, j \leqslant n$, such that

$$\|\phi_{ij} - [x_i^k \otimes y_j^k]\| < \varepsilon^k, \qquad 1 \leqslant i, j \leqslant n, 2 \leqslant k < \infty,$$

and

$$\|x_i^{k+1} - x_i^k\| < n(\gamma - \theta)^{-1/2} \varepsilon^{k/2}, \quad \|y_j^{k+1} - y_j^k\| < n(\gamma - \theta)^{-1/2} \varepsilon^{k/2},$$
$$1 \leqslant i, j \leqslant n, 1 \leqslant k < \infty.$$

It is obvious that the sequences $\{x_i^k\}_{k=1}^\infty$ and $\{y_j^k\}_{k=1}^\infty$, $1 \leqslant i, j \leqslant n$, are Cauchy and $\phi_{ij} = [x_i' \otimes y_j']$, $1 \leqslant i, j \leqslant n$, if

$$x_i' = \lim_{k\to\infty} x_i^k, \quad y_j' = \lim_{k\to\infty} y_j^k, \qquad 1 \leqslant i, j \leqslant n.$$

Finally,

$$\begin{aligned}\|x_i' - x_i\| &\leqslant \|x_i^1 - x_i\| + \sum_{k=1}^{\infty} \|x_i^{k+1} - x_i^k\| \\ &< n(\gamma - \theta)^{-1/2}\left(b^{1/2} + \sum_{k=1}^{\infty} \varepsilon^{k/2}\right) \\ &= n(\gamma - \theta)^{-1/2}\left(b^{1/2} + \varepsilon^{1/2}(1 - \varepsilon^{1/2})^{-1}\right), \qquad 1 \leqslant i \leqslant n,\end{aligned}$$

and analogously

$$\|y_j' - y_j\| < n(\gamma - \theta)^{-1/2}\left(b^{1/2} + \varepsilon^{1/2}(1 - \varepsilon^{1/2})^{-1}\right), \qquad 1 \leqslant j \leqslant n.$$

It suffices therefore to choose ε so small that $b^{1/2} + \varepsilon^{1/2}(1 - \varepsilon^{1/2})^{-1} < a^{1/2}$. The theorem is proved.

We are now able to prove the promised criterion.

COROLLARY 12. *Suppose $\mathscr{M} \subset \mathscr{L}(\mathscr{H})$ is a linear subspace with property $x_{\theta,\gamma}$ for some $\gamma > \theta \geqslant 0$. Then $\mathscr{M}$ has property $(\tilde{\mathbf{A}}_n)$ for every natural number n. In particular the ultraweak closure $\mathscr{M}^-$ of $\mathscr{M}$ is weakly closed and reflexive.*

PROOF. The last part of the statement follows from the first part, combined with Lemma 3 and Corollary 4. To prove the first part we only have to use Theorem 11. Observe that we can take $\delta(\varepsilon, n) = \varepsilon^2 n^{-2}(\gamma - \theta)$.

We finally note that one could give a definition analogous to Definition 8, in which the space $\mathscr{M}_*$ is replaced by the set $\mathscr{M}_\sim$ of all weakly continuous functionals on $\mathscr{M}$. The property thus defined would however be stronger than $X_{\theta,\gamma}$ since $\mathscr{M}_*$ coincides with the norm closure of $\mathscr{M}_\sim$; this is why we restricted ourselves to the space $\mathscr{M}_*$ and the properties $(\tilde{\mathbf{A}}_n)$. We do not know whether the weaker properties $(\mathbf{A}_n)$ imply reflexivity. Property $(\mathbf{A}_1)$ alone does not imply reflexivity. Indeed, the algebra $\mathscr{M}$ of 2×2 matrices defined as

$$\mathscr{M} = \left\{\begin{pmatrix} a & b \\ 0 & a\end{pmatrix} : a, b \in \mathbf{C}\right\}$$

is not reflexive, but it has property $(\mathbf{A}_1)$ (and even $(\tilde{\mathbf{A}}_1)$, as can be seen by an easy computation).

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