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BEST APPROXIMATION AND QUASITRIANGULAR ALGEBRAS[1]

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ABSTRACT. If $\mathscr{P}$ is a linearly ordered set of projections on a Hilbert space and $\mathscr{K}$ is the ideal of compact operators, then $\operatorname{Alg}\mathscr{P} + \mathscr{K}$ is the quasitriangular algebra associated with $\mathscr{P}$. We study the problem of finding best approximants in a given quasitriangular algebra to a given operator: given T and $\mathscr{P}$, is there an A in $\operatorname{Alg}\mathscr{P} + \mathscr{K}$ such that $\|T - A\| = \inf\{\|T - S\|: S \in \operatorname{Alg}\mathscr{P} + \mathscr{K}\}$? We prove that if $\mathscr{A}$ is an operator subalgebra which is closed in the weak operator topology and satisfies a certain condition Δ, then every operator T has a best approximant in $\mathscr{A} + \mathscr{K}$. We also show that if $\mathscr{E}$ is an increasing sequence of finite rank projections converging strongly to the identity then $\operatorname{Alg}\mathscr{E}$ satisfies the condition Δ. Also, we show that if T is not in $\operatorname{Alg}\mathscr{E} + \mathscr{K}$ then the best approximants in $\operatorname{Alg}\mathscr{E} + \mathscr{K}$ to T are never unique.

1. Introduction. The concept of quasitriangular operators on a Hilbert space was introduced by Halmos in **[5]**, where an operator T is said to be quasitriangular if there is a sequence $\{E_n\}$ of finite rank projections strongly converging to the identity such that $\|(1 - E_n)TE_n\| \to 0$.

For a fixed increasing sequence $\{P_n\}$ of finite rank projections strongly converging to the identity, Arveson **[2]** defined the quasitriangular algebra $QT(\{P_n\})$ to be the set of all operators T for which $\|(1 - P_n)TP_n\| \to 0$. He proved a distance formula for $QT(\{P_n\})$ and showed that $QT(\{P_n\}) = \operatorname{Alg}\{P_n\} + \mathscr{K}$, where $\operatorname{Alg}\{P_n\} = \{T: (1 - P_n)TP_n = 0 \text{ for all } n\}$ is the triangular algebra associated with $\{P_n\}$ and $\mathscr{K}$ is the ideal of compact operators.

For any linearly ordered set $\mathscr{P}$ of projections which is closed in the strong operator topology and contains 0 and 1, Fall, Arveson, and Muhly **[4]** showed that the algebra $\operatorname{Alg}\mathscr{P} + \mathscr{K}$ is norm closed, where $\operatorname{Alg}\mathscr{P}$ is the triangular algebra associated with $\mathscr{P}$, namely $\operatorname{Alg}\mathscr{P} = \{T: (1 - P)TP = 0, \text{ all } P \in \mathscr{P}\}$. They also gave a characterization of $\operatorname{Alg}\mathscr{P} + \mathscr{K}$ as a generalized quasitriangular algebra.

In this paper we study the problem of finding best quasitriangular approximants to a given operator: given an operator T does there exist an operator A in $\operatorname{Alg}\mathscr{P} + \mathscr{K}$ for which $\|T - A\| = \inf\{\|T - S\|: S \in \operatorname{Alg}\mathscr{P} + \mathscr{K}\}$? We prove that if $\mathscr{A}$ is an operator subalgebra which is closed in the weak operator topology and satisfies a certain condition $\Delta(\mathscr{A})$, then every operator T has a best approximant in $\mathscr{A} + \mathscr{K}$.

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We also show that if $\{P_n\}$ is an increasing sequence of finite rank projections strongly converging to 1, then $\mathrm{Alg}\{P_n\}$ satisfies the condition $\Delta(\mathrm{Alg}\{P_n\})$. Hence, best approximants in $\mathrm{Alg}\{P_n\} + \mathscr{K}$ exist for every operator T. Moreover, we show that if $T \notin \mathrm{Alg}\{P_n\} + \mathscr{K}$, then such best approximants are never unique.

Some of our results are reminiscent of those proved in **[3]** by Axler, Berg, Jewell, and Shields, where it is shown, for example, that every L^∞ function on the unit circle has a best approximant in the algebra $H^\infty + C$. In fact, the general approach to proving our main result is inspired by that paper.

2. Preliminaries. In what follows, H will be a separable infinite-dimensional Hilbert space with $\mathscr{L}(H)$ denoting the algebra of all bounded linear operators on H and $\mathscr{K}(H)$, or simply $\mathscr{K}$, denoting the ideal of compact operators in $\mathscr{L}(H)$. All subspaces of H are assumed to be closed and all projections are selfadjoint. For a projection P let $P^\perp = 1 - P$.

If $\mathscr{S}$ is any subset of $\mathscr{L}(H)$ and $T \in \mathscr{L}(H)$, then the distance of T from $\mathscr{S}$ is given by $d(T, \mathscr{S}) = \inf\{\|T - S\|: S \in \mathscr{S}\}$. Also, Lat $\mathscr{S}$ will denote the set of all projections P for which $PSP = SP$ whenever $S \in \mathscr{S}$. If $\mathscr{P}$ is a set of projections in $\mathscr{L}(H)$, then Alg $\mathscr{P}$ denotes the set of all operators T in $\mathscr{L}(H)$ for which $PTP = TP$ whenever $P \in \mathscr{P}$. A subalgebra $\mathscr{S} \subset \mathscr{L}(H)$ is said to be reflexive if Alg Lat $\mathscr{S} = \mathscr{S}$.

A nest is a family of projections which is linearly ordered by range inclusion, contains 0 and 1, and is closed in the strong operator topology (SOT). A nest algebra is a subalgebra $\mathscr{A}$ of $\mathscr{L}(H)$ for which $\mathscr{A} =$ Alg $\mathscr{P}$ for some nest $\mathscr{P}$. Equivalently, it is not hard to see that a nest algebra is a reflexive algebra $\mathscr{A}$ such that Lat $\mathscr{A}$ is linearly ordered (cf. **[9]**).

In **[2]** Arveson established the following distance formula for a nest algebra $\mathscr{A}$.

$$(2.1) \qquad d(T, \mathscr{A}) = \sup\{\|P^\perp TP\|: P \in \mathrm{Lat}\,\mathscr{A}\} \quad \text{for } T \in \mathscr{L}(H).$$

For a nest $\mathscr{P}$ define the quasitriangular algebra associated with $\mathscr{P}$ by $QT(\mathscr{P}) = \mathrm{Alg}\,\mathscr{P} + \mathscr{K}(H)$. In **[4]** Fall, Arveson, and Muhly showed that $QT(\mathscr{P})$ is a norm closed algebra and that

$$\begin{aligned} QT(\mathscr{P}) = \{T \in \mathscr{L}(H) \colon &\text{(i)}\ \ P^\perp TP \in \mathscr{K}(H), \text{ for all } P \in \mathscr{P},\\ &\qquad \text{the map } P \mapsto P^\perp TP \text{ is continuous}\\ &\text{(ii) with respect to the SOT on } \mathscr{P} \text{ and}\\ &\qquad\qquad \text{the norm topology on } \mathscr{K}(H)\}. \end{aligned}$$

In the case when $\mathscr{P} = \{P_n\}$ is an increasing sequence of finite rank projections converging strongly to 1, this yields the definition of $QT(\{P_n\})$ given by Arveson in **[2]**. For this special case Arveson has established the following distance formula.

$$(2.2) \qquad d(T, QT(\{P_n\})) = \overline{\lim}\, \|P_n^\perp TP_n\|, \qquad n \to \infty, \text{ for } T \in \mathscr{L}(H).$$

In this case (2.1) can be written as

$$(2.1') \qquad d(T, \mathrm{Alg}\{P_n\}) = \sup\left\{\|P_n^\perp TP_n\|: \text{all } n\right\} \quad \text{for } T \in \mathscr{L}(H).$$

We also need the following known result.

LEMMA 2.3. *If $\mathscr{A} \subset \mathscr{L}(H)$ is closed in the weak operator topology (WOT), then every T in $\mathscr{L}(H)$ has a best approximant in $\mathscr{A}$.*

PROOF. The proof is a standard argument using the compactness, in the weak operator topology, of the closed unit ball in $\mathscr{L}(H)$. □

Finally, we observe that if $\mathscr{P}$ is a nest then Alg $\mathscr{P}$ is closed in the WOT. Indeed, if $\{A_\lambda\} \subset$ Alg $\mathscr{P}$ is a net of operators such that $A_\lambda \to A$ (WOT), then, for each $P \in \mathscr{P}$, $0 = P^\perp A_\lambda P \to P^\perp AP$ (WOT), which implies that $A \in$ Alg $\mathscr{P}$.

3. Main results.

DEFINITION 3.1. A subalgebra $\mathscr{A}$ of $\mathscr{L}(H)$ satisfies condition $\Delta(\mathscr{A})$ provided that, for each $T \in \mathscr{L}(H)$, for each sequence of operators $\{A_n\} \subset \mathscr{L}(H)$ satisfying $A_n \to 0$ (SOT), and for each $\varepsilon > 0$, there exists an N such that

$$d(T + A_N, \mathscr{A}) \leqslant \varepsilon + \max\{d(T, \mathscr{A}), d(T, \mathscr{A} + \mathscr{K}) + d(A_N, \mathscr{A})\}.$$

Two remarks are in order. First, if condition $\Delta(\mathscr{A})$ holds and T, $\{A_n\}$, and ε are chosen as indicated, then there exists an N such that

$$d(T + \beta A_N, \mathscr{A}) \leqslant \varepsilon + \max\{d(T, \mathscr{A}), d(T, \mathscr{A} + \mathscr{K}) + d(\beta A_N, \mathscr{A})\}$$

for all $\beta \in [0,1]$. Otherwise, for each n, take $\beta_n \in [0,1]$ such that the inequality fails for $\beta_n A_n$. The sequence $\{\beta_n A_n\}$ satisfies $\beta_n A_n \to 0$ (SOT), so the assumption that condition $\Delta(\mathscr{A})$ holds yields a contradiction. Secondly, for any fixed M, N can be chosen so that $N \geqslant M$ by restricting attention to the sequence $\{A_n : n \geqslant M\}$.

The next result enables us to reduce the problem of finding best approximants in $\mathscr{A} + \mathscr{K}(H)$ to that of finding best approximants in $\mathscr{A}$.

THEOREM 3.2. *Let $\mathscr{A} \subset \mathscr{L}(H)$ be a subalgebra satisfying condition $\Delta(\mathscr{A})$. Choose $T \in \mathscr{L}(H) \setminus \mathscr{A} + \mathscr{K}$ and suppose the sequence $\{T_n\} \subset \mathscr{A} + \mathscr{K}$ satisfies $T_n \to T$ (SOT). Then there is a sequence $\{a_n\}$ of nonnegative real numbers satisfying $\Sigma a_n = 1$ and such that, if $K = \Sigma a_n T_n$, then $d(T - K, \mathscr{A}) = d(T, \mathscr{A} + \mathscr{K})$.*

PROOF. Let $A_n = T - T_n$ so that $A_n \to 0$ (SOT). For convenience let

$$r = d(T, \mathscr{A} + \mathscr{K}).$$

CLAIM. There exists an increasing sequence of positive integers $\{n(k)\}$ and a sequence $\{\alpha_k\}$ of positive real numbers such that $\Sigma \alpha_k = 1$ and such that, for all $N = 1, 2, \ldots,$

$$d\left(\sum_{k=1}^{N} \alpha_k A_{n(k)}, \mathscr{A}\right) = r - \varepsilon_N, \quad \text{where } \varepsilon_N = r/3^N.$$

PROOF OF CLAIM. Choose $n(1) = 1$. Since $A_1 \notin \mathscr{A} + \mathscr{K}$, it follows that $d(A_1, \mathscr{A}) \neq 0$. Choose α_1 such that $\alpha_1 \cdot d(A_1, \mathscr{A}) = r - \varepsilon_1$. Since $\alpha_1 \cdot d(A_1, \mathscr{A}) = d(\alpha_1 A_1, \mathscr{A})$, it follows that $d(\alpha_1 A_1, \mathscr{A}) = r - \varepsilon_1$. The relations

$$d(A_1, \mathscr{A}) = d(T, \mathscr{A} + T_1) \geqslant d(T, \mathscr{A} + \mathscr{K})$$

imply $0 < \alpha_1 < 1$.

Suppose $n(1), \ldots, n(N)$ and $\alpha_1, \ldots, \alpha_N$ have been chosen as required. Applying condition $\Delta(\mathscr{A})$ to the operator $\Sigma_{k=1}^{N} \alpha_k A_{n(k)}$, the sequence $\{A_n\}$, and ε_{N+1}, choose

$n(N+1) > n(N)$ such that

$$(3.3)\quad d\left(\sum_{k=1}^{N}\alpha_k A_{n(k)} + \beta A_{n(N+1)}, \mathscr{A}\right)$$
$$\leqslant \varepsilon_{N+1} + \max\left\{ d\left(\sum_{k=1}^{N}\alpha_k A_{n(k)}, \mathscr{A}\right), d\left(\sum_{k=1}^{N}\alpha_k A_{n(k)}, \mathscr{A}+\mathscr{K}\right)\right.$$
$$\left. + d\left(\beta A_{n(N+1)}, \mathscr{A}\right)\right\}$$
$$\text{for all } \beta \in [0,1].$$

Consider the quantity $d(\sum_{k=1}^{N}\alpha_k A_{n(k)} + \alpha A_{n(N+1)}, \mathscr{A})$ as a function of α. When $\alpha = 0$ this quantity equals $r - \varepsilon_N$. Note that $r - \varepsilon_N < r - \varepsilon_{N+1}$. As $\alpha \to \infty$ this quantity also approaches ∞. (Here we use the fact that A_k does not belong to $\mathscr{A}$ for any k.) Thus, there exists some value of α, call it α_{N+1}, for which

$$d\left(\sum_{k=1}^{N}\alpha_k A_{n(k)} + \alpha_{N+1}\cdot A_{n(N+1)}, \mathscr{A}\right) = r - \varepsilon_{N+1}.$$

Note that

$$r - \varepsilon_{N+1} = d\left(\sum_{k=1}^{N+1}\alpha_k A_{n(k)}, \mathscr{A}\right) = d\left(\sum_{k=1}^{N+1}\alpha_k\left(T - T_{n(k)}\right), \mathscr{A}\right)$$
$$\geqslant d\left(\sum_{k=1}^{N+1}\alpha_k T, \mathscr{A}+\mathscr{K}\right) \quad \text{since } T_{n(k)} \in \mathscr{A}+\mathscr{K}$$
$$= \left(\sum_{k=1}^{N+1}\alpha_k\right)\cdot d(T, \mathscr{A}+\mathscr{K}) = \left(\sum_{k=1}^{N+1}\alpha_k\right)\cdot r$$

and, hence, $\sum_{k=1}^{N+1}\alpha_k < 1$. It remains to show that $\sum\alpha_k = 1$.

Referring to inequality (3.3), with α_{N+1} in place of β, suppose that

$$d\left(\sum_{k=1}^{N+1}\alpha_k A_{n(k)}, \mathscr{A}\right) \leqslant \varepsilon_{N+1} + d\left(\sum_{k=1}^{N}\alpha_k A_{n(k)}, \mathscr{A}\right).$$

Then $r - \varepsilon_{N+1} \leqslant \varepsilon_{N+1} + (r - \varepsilon_N)$, which implies that $\varepsilon_N \leqslant 2\varepsilon_{N+1}$, a contradiction of the definition of $\{\varepsilon_n\}$. It follows that

$$r - \varepsilon_{N+1} = d\left(\sum_{k=1}^{N+1}\alpha_k A_{n(k)}, \mathscr{A}\right)$$
$$\leqslant \varepsilon_{N+1} + d\left(\sum_{k=1}^{N}\alpha_k A_{n(k)}, \mathscr{A}+\mathscr{K}\right) + d\left(\alpha_{N+1}A_{n(N+1)}, \mathscr{A}\right)$$
$$= \varepsilon_{N+1} + \left(\sum_{k=1}^{N}\alpha_k\right)\cdot r + \alpha_{N+1}\cdot d\left(A_{n(N+1)}, \mathscr{A}\right).$$

If $N \to \infty$ then $\varepsilon_{N+1} \to 0$ and, since $\sum\alpha_k \leqslant 1$, it follows that $\alpha_{N+1} \to 0$. Since $A_n \to 0$ (SOT) we see that $\{\|A_n\|\}$, and hence $\{d(A_n, \mathscr{A})\}$, is a bounded set. Thus, letting $N \to \infty$ in the above yields $r = d(\Sigma\alpha_k A_{n(k)}, \mathscr{A}) \leqslant (\Sigma\alpha_k) \cdot r$, which implies that $\Sigma\alpha_k \geqslant 1$. This completes the proof of the claim.

To complete the proof of the theorem, define the sequence $\{a_n\}$ by $a_{n(k)} = \alpha_k$ and $a_j = 0$ if j is not of the form $n(k)$ for any k. Also, let $K = \Sigma a_n T_n = \Sigma\alpha_k T_{n(k)} = T - \Sigma\alpha_k A_{n(k)}$. This sum converges since $\Sigma\alpha_k = 1$ and since $\{\|A_n\|\}$ is a bounded set. It follows from the foregoing discussion that $d(T - K, \mathscr{A}) = d(\Sigma\alpha_k A_{n(k)}, \mathscr{A}) = r = d(T, \mathscr{A} + \mathscr{K})$, which completes the proof. □

Note that if $\mathscr{A} + \mathscr{K}(H)$ is norm closed then $K \in \mathscr{A} + \mathscr{K}(H)$. Also, if $\{T_n\}$ is taken to be a sequence of *compact* operators converging to T (SOT), then $K \in \mathscr{K}(H)$, since $\mathscr{K}(H)$ is norm closed.

We are now in a position to prove one of our main results on the existence of best approximants.

THEOREM 3.4. *Let $\mathscr{A} \subset \mathscr{L}(H)$ be a subalgebra which is WOT-closed and satisfies condition $\Delta(\mathscr{A})$, and suppose $T \in \mathscr{L}(H)$. Then there exists $B \in \mathscr{A} + \mathscr{K}(H)$ such that $\|T - B\| = d(T, \mathscr{A} + \mathscr{K}(H))$.*

PROOF. Assume $T \in \mathscr{L}(H) \backslash \mathscr{A} + \mathscr{K}$, since otherwise the result is obvious. Let $\{e_j : j \geqslant 0\}$ be an orthonormal basis for H and define E_n to be the projection onto the subspace spanned by $\{e_j : j \leqslant n\}$. Each E_n has finite rank and $E_n \to 1$ (SOT). Set $T_n = E_n T E_n$. Each T_n is compact and $T_n \to T$ (SOT).

By Theorem 3.2 there is a sequence $\{a_n\}$ of nonnegative real numbers satisfying $\Sigma a_n = 1$ and such that, if $K = \Sigma a_n T_n$, $d(T - K, \mathscr{A}) = d(T, \mathscr{A} + \mathscr{K})$. Note that $K \in \mathscr{K}$. By Lemma 2.3 there exists $A \in \mathscr{A}$ such that $\|(T - K) - A\| = d(T - K, \mathscr{A})$. Therefore, the operator $B = A + K$ is in $\mathscr{A} + \mathscr{K}$ and satisfies

$$\|T - B\| = d(T, \mathscr{A} + \mathscr{K}).$$

In other words, B is a best approximant to T in $\mathscr{A} + \mathscr{K}$. □

We remarked earlier that every nest algebra is WOT-closed, so Theorem 3.4 applies, in particular, to any nest algebra $\mathscr{A}$ which satisfies condition $\Delta(\mathscr{A})$.

The following corollary shows that if $\mathscr{A} + \mathscr{K}$ is norm closed, then the operator K in the conclusion of Theorem 3.2 is not unique.

COROLLARY 3.5. *Let $\mathscr{A}$, T, and $\{T_n\}$ be as in the statement of Theorem 3.2, and also suppose that $\mathscr{A} + \mathscr{K}$ is norm closed. Then there exist two sequences $\{a_n\}$ and $\{b_n\}$ of nonnegative real numbers such that $\Sigma a_n = \Sigma b_n = 1$ and such that, if $K = \Sigma a_n T_n$ and $K_1 = \Sigma b_n T_n$, then $K \neq K_1$ and $d(T - K, \mathscr{A}) = d(T - K_1, \mathscr{A}) = d(T, \mathscr{A} + \mathscr{K})$.*

PROOF. Let $\{a_n\}$ and $K = \Sigma a_n T_n$ be as in the conclusion of Theorem 3.2. Then $(T_n - K) \to (T - K)$ (SOT). Let $\mathscr{O}$ be a convex neighborhood of $T - K$ in the strong operator topology whose closure does not contain 0. Deleting a finite number of terms if necessary, assume that $T_n - K \in \mathscr{O}$ for all n.

Since $\mathscr{A} + \mathscr{K}$ is norm closed, we see that $K \in \mathscr{A} + \mathscr{K}$ and, hence, $(T_n - K) \in \mathscr{A} + \mathscr{K}$ for all n. Thus by Theorem 3.2 we can construct a sequence $\{b_n\}$ such that $\Sigma b_n = 1$ and such that if $K' = \Sigma b_n(T_n - K)$, then

$$d((T - K) - K', \mathscr{A}) = d(T - K, \mathscr{A} + \mathscr{K}) = d(T, \mathscr{A} + \mathscr{K}).$$

Thus, the operator $K_1 = K + K'$ satisfies $d(T - K_1, \mathscr{A}) = d(T, \mathscr{A} + \mathscr{K})$. Since K' is a convex combination of elements of $\mathscr{O}$, it follows that $K' \neq 0$ and, hence, $K_1 \neq K$. This proves the corollary. $\square$

We noted earlier that $\mathscr{A} + \mathscr{K}$ is norm closed whenever $\mathscr{A}$ is a nest algebra, so Corollary 3.5 applies, in particular, to any nest algebra satisfying condition $\Delta(\mathscr{A})$. Also note that if $\{T_n\}$ is taken to be a sequence of compact operators, then K is compact as well and the requirement that $\mathscr{A} + \mathscr{K}$ be norm closed is superfluous.

4. More main results. Throughout this section let $\mathscr{P} = \{P_n\}$ be a fixed increasing sequence of finite rank projections such that $P_n \to 1$ (SOT). Let

$$\mathscr{A} = \operatorname{Alg}\{P_n\} = \left\{T \in \mathscr{L}(H) \colon P_n^\perp T P_n = 0 \text{ for all } n\right\}$$

and let

$$QT = QT(\{P_n\}) = \left\{T \in \mathscr{L}(H) \colon \left\|P_n^\perp T P_n\right\| \to 0, n \to \infty\right\}.$$

The following result establishes the validity of condition $\Delta(\mathscr{A})$ in this special case. It then follows from Theorem 3.4 that best approximants in QT exist for every operator in $\mathscr{L}(H)$.

PROPOSITION 4.1. *The algebra* $\mathscr{A} = \operatorname{Alg}\{P_n\}$ *satisfies condition* $\Delta(\mathscr{A})$.

PROOF. Choose $T \in \mathscr{L}(H)$ and let $\{A_n\} \subset \mathscr{L}(H)$ satisfy $A_n \to 0$ (SOT). Fix $\varepsilon > 0$. If condition $\Delta(\mathscr{A})$ is not satisfied, then by the distance formulas (2.2) and (2.1′) there is a sequence $\{m_n\}$ of nonnegative integers such that $\|P_{m_n}^\perp(T + A_n)P_{m_n}\| > \varepsilon + \alpha_n$, where

$$\alpha_n = \max\left(\sup_{j \geqslant 0}\left\|P_j^\perp T P_j\right\|, \ \varlimsup_k \left\|P_k^\perp T P_k\right\| + \sup_{j \geqslant 0}\left\|P_j^\perp A_n P_j\right\|\right).$$

Consider two cases.

Case 1. Suppose no nonnegative integer appears infinitely often in the sequence $\{m_n\}$. Passing to a subsequence if necessary, assume that $\{m_n\}$ is an increasing sequence. By the definition of lim sup, there exists some N such that $n \geqslant N$ implies that $\|P_{m_n}^\perp T P_{m_n}\| \leqslant \varlimsup_k \|P_k^\perp T P_k\| + \varepsilon/2$. Thus, for $n \geqslant N$, we have

$$\begin{aligned}\left\|P_{m_n}^\perp (T + A_n) P_{m_n}\right\| &\leqslant \left\|P_{m_n}^\perp T P_{m_n}\right\| + \left\|P_{m_n}^\perp A_n P_{m_n}\right\| \\ &\leqslant \varlimsup_k \left\|P_k^\perp T P_k\right\| + \varepsilon/2 + \sup_j \left\|P_j^\perp A_n P_j\right\| \\ &\leqslant \alpha_n + \varepsilon/2.\end{aligned}$$

This contradicts the definition of the sequence $\{m_n\}$.

Case 2. Suppose some nonnegative integer, call it M, appears infinitely often in the sequence $\{m_n\}$. Passing to a subsequence if necessary, assume that

$$\left\|P_M^\perp (T + A_n) P_M\right\| > \varepsilon + \alpha_n \quad \text{for all } n.$$

Since P_M is compact and $A_n \to 0$ (SOT), it follows that $\|A_n P_M\| \to 0$. Choose N such that $\|A_N P_M\| < \varepsilon/2$. We then have

$$\begin{aligned}\left\|P_M^\perp (T + A_N) P_M\right\| &\leqslant \left\|P_M^\perp T P_M\right\| + \left\|P_M^\perp A_N P_M\right\| \\ &\leqslant \sup_j \left\|P_j^\perp T P_j\right\| + \varepsilon/2 \leqslant \alpha_N + \varepsilon/2.\end{aligned}$$

This yields a contradiction to the definition of the sequence $\{m_n\}$ and completes the proof of the proposition. $\square$

We now show that best approximants in QT are never unique for operators not in QT.

PROPOSITION 4.2. *For each $T \in \mathscr{L}(H)\setminus QT$ there exist operators B and B_1 in QT such that $B \neq B_1$ and $\|T - B\| = \|T - B_1\| = d(T, QT)$.*

PROOF. Consider two cases.

Case 1. Suppose there is a subsequence $\{n_k\}$ such that $(P_{n_{k+1}} - P_{n_k})TP_{n_0} \neq 0$ for all $k \geqslant 0$. Set $E_k = P_{n_k}$ and let $T_k = E_kTE_k$. Now, let $K = \Sigma a_kT_k$ and $K_1 = \Sigma b_kT_k$ be as in the conclusion of Corollary 3.5. Note that K and K_1 are compact. By Lemma 2.3 we can find operators A and A_1 in $\mathscr{A}$ such that $\|T - K - A\| = d(T - K, \mathscr{A})$ and $\|T - K_1 - A_1\| = d(T - K_1, \mathscr{A})$. Thus, $B = A + K$ and $B_1 = A_1 + K_1$ are best approximants in QT to T. To show that $B \neq B_1$, it suffices to show that $K - K_1 \notin \mathscr{A}$.

Suppose, to the contrary, that $K - K_1 \in \mathscr{A}$. Then it follows, in particular, that

$$\begin{aligned} 0 = E_0^{\perp}(K - K_1)E_0 &= \sum_{k\geqslant 0} (a_k - b_k)E_0^{\perp}E_kTE_kE_0 \\ &= \sum_{k\geqslant 1} (a_k - b_k)(E_k - E_0)TE_0. \end{aligned}$$

Letting $C_k = \Sigma_{j\geqslant k}(a_j - b_j)$, a summation by parts shows that

$$\sum_{k=1}^{N} C_k(E_k - E_{k-1})TE_0 = \sum_{k=1}^{N-1} (a_k - b_k)E_kTE_0 + C_NE_NTE_0 - C_1E_0TE_0.$$

As $N \to \infty$, $|C_N| \to 0$, so $\|C_NE_NTE_0\| \to 0$. Thus,

$$\sum_{k=1}^{\infty} C_k(E_k - E_{k-1})TE_0 = \sum_{k=1}^{\infty} (a_k - b_k)E_kTE_0 - C_1E_0TE_0.$$

We thus have that

$$\begin{aligned} &\sum_{k=1}^{\infty}\Big(\sum_{j\geqslant k}(a_j - b_j)\Big)(E_k - E_{k-1})TE_0 \\ &\qquad = \sum_{k=1}^{\infty} (a_k - b_k)E_kTE_0 - \sum_{k=1}^{\infty} (a_k - b_k)E_0TE_0 \\ &\qquad = \sum_{k=1}^{\infty} (a_k - b_k)(E_k - E_0)TE_0 = 0. \end{aligned}$$

Since the range of $(E_l - E_{l-1})$ is orthogonal to that of $(E_j - E_{j-1})$ whenever $l \neq j$, and since, by assumption, $(E_k - E_{k-1})TE_0 \neq 0$ for $k \geqslant 1$, it follows that

$$\sum_{j\geqslant k} (a_j - b_j) = 0 \quad \text{for } k \geqslant 1.$$

The fact that $\Sigma a_n = \Sigma b_n = 1$ implies that $\Sigma_{j\geqslant 0}(a_j - b_j) = 0$ as well. Hence, $a_j = b_j$ for all $j \geqslant 0$, which contradicts the assumption that $K \neq K_1$. Thus, $K - K_1 \notin \mathscr{A}$ and, consequently, $B \neq B_1$.

Case 2. Suppose there is no subsequence $\{n_k\}$ for which $(P_{n_{k+1}} - P_{n_k})TP_{n_0} \neq 0$ for all k. Then for each k there is a smallest integer $m(k)$ such that $P_{m(k)}^{\perp}TP_k = 0$. We claim that $m(k) \geqslant k + 1$ for infinitely many k. Indeed, were this not so then there would exist N such that $m(k) \leqslant k$ for $k \geqslant N$. Hence, $P_k^{\perp} TP_k = 0$ for $k \geqslant N$, which implies that $d(T, QT) = 0$, contradicting the assumption that $T \notin QT$.

We make the following remarks.

(a) If $m(k) \geqslant k + 1$, then $(P_{m(k)} - P_k)TP_k \neq 0$. This follows from the choice of $m(k)$ as the *smallest* integer such that $P_{m(k)}^{\perp}TP_k = 0$.

(b) It is clear that if $(P_{m(k)} - P_k)TP_k \neq 0$, then $(P_j - P_k)TP_k \neq 0$ for $j \geqslant m(k)$.

Now, choose k_0 such that $m(k_0) \geqslant k_0 + 1$ and $TP_{k_0} \neq 0$. For $j \geqslant 1$ inductively choose k_j such that $m(k_j) \geqslant k_j + 1$ and $k_j > m(k_{j-1})$. Set $E_j = P_{k_j}$ and let $T_j = E_jTE_j$. From this we get $K = \Sigma a_n T_n$ and $K_1 = \Sigma b_n T_n$, as in the conclusion of Corollary 3.5. To complete the proof it suffices, as in the previous case, to show that $K - K_1 \notin \mathscr{A}$.

First observe that remarks (a) and (b) imply that $(E_n - E_l)TE_l \neq 0$ for $n \geqslant l + 1$. Also, by the construction of the sequence $\{E_n\}$, it follows that $(E_{j+1} - E_j)TE_l = 0$ for $j \geqslant l + 1$. Putting these together we see that, for $n \geqslant l + 1$,

$$(E_n - E_l)TE_l = \sum_{j=l}^{n-1} (E_{j+1} - E_j)TE_l = (E_{l+1} - E_l)TE_l \neq 0.$$

To see that $K - K_1 \notin \mathscr{A}$, suppose the contrary. Then, for $l \geqslant 0$, we must have

$$\begin{aligned} 0 = E_l^{\perp}(K - K_1)E_l &= \sum_{n \geqslant 0} (a_n - b_n) E_l^{\perp} E_n T E_n E_l \\ &= \sum_{n \geqslant l+1} (a_n - b_n)(E_n - E_l)TE_l \\ &= \sum_{n \geqslant l+1} (a_n - b_n)(E_{l+1} - E_l)TE_l \\ &= \left[\sum_{n \geqslant l+1} (a_n - b_n)\right](E_{l+1} - E_l)TE_l. \end{aligned}$$

Since $(E_{l+1} - E_l)TE_l \neq 0$, it follows that $\Sigma_{n \geqslant l+1}(a_n - b_n) = 0$ for $l \geqslant 0$. Since $\Sigma a_n = \Sigma b_n = 1$, it follows that $\Sigma_{n \geqslant l}(a_n - b_n) = 0$ for all $l \geqslant 0$ and, hence, $a_n = b_n$ for all n, contradicting the assumption that $K \neq K_1$. Hence, $K - K_1 \notin \mathscr{A}$ and the corollary is proved. □

5. Remarks. The obvious question is to ask which subalgebras $\mathscr{A}$ satisfy the condition $\Delta(\mathscr{A})$. Our proof of Proposition 4.1 and Arveson's proof of the distance formula (2.2) both use the finite dimensionality of the projections P_n in an important way. Some means of eliminating this dependence would apparently be needed to establish a broader validity of condition $\Delta(\mathscr{A})$. A generalization of Proposition 4.2 to the setting of §3 would also be useful.

A question related to Theorem 3.2 is the following. If the operators $\{T_n\}$ are taken to be compact, then the resulting K is also compact. It is possible that this K is a best compact approximant to T?

In [3] Axler, Berg, Jewell, and Shields employ what they call the "Basic Inequality" for a Banach space X. This inequality is similar to condition $\Delta(\mathscr{A})$ for $\mathscr{A} = \{0\}$, the zero operator. They show that the Basic Inequality is satisfied for $X = l^p$, $1 < p < \infty$. They also prove that the closed unit ball of $L^\infty/H^\infty + C$ has no extreme points. Two questions which arise are whether $\Delta(\mathscr{A})$ holds when $\mathscr{A}$ is the algebra of operators on l^p $(1 < p < \infty)$ with upper triangular matrix representations with respect to the standard basis, and whether the closed unit ball of $\mathscr{L}(H)/\mathscr{A} + \mathscr{K}(H)$ has any extreme points if $\mathscr{A}$ is a nest algebra satisfying condition $\Delta(\mathscr{A})$.

Another line of questioning is related to the theory of M-ideals, introduced in 1972 by Alfsen and Effros [1]. Luecking [8] showed that $H^\infty + C/H^\infty$ is an M-ideal in L^∞/H^∞, and it seems reasonable to ask if $\mathscr{A} + \mathscr{K}(H)/\mathscr{A}$ is an M-ideal in $\mathscr{L}(H)/\mathscr{A}$ for any nest algebra $\mathscr{A}$. An affirmative answer would imply, by a result of Holmes, Scranton, and Ward [7], that the collection $\mathscr{S}_T = \{A + \mathscr{A} \in \mathscr{A} + \mathscr{K}(H)/\mathscr{A}: d(T - A, \mathscr{A}) = d(T, \mathscr{A} + \mathscr{K}(H))\}$ would algebraically span $\mathscr{A} + \mathscr{K}(H)/\mathscr{A}$ for each $T \in \mathscr{L}(H) \setminus \mathscr{A} + \mathscr{K}(H)$.

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