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ON INFINITE DEFICIENCY IN R^∞-MANIFOLDS

BY

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ABSTRACT. Using the notion of inductive proper q-1-LCC introduced in this note, we will prove the following theorems.

THEOREM 1. *Let M be an R^∞-manifold and let $H: X \times I \to M$ be a homotopy such that H_0 and H_1 are R^∞-deficient embeddings. Then, there is a homeomorphism F of M such that $F \circ H_0 = H_1$. Moreover, if H is limited by an open cover α of M and is stationary on a closed subset X_0 of X and W_0 is an open neighborhood of*

$$H[(X - X_0) \times I] \quad \text{in } M,$$

then we can choose F to also be $\mathrm{St}^4(\alpha)$-close to the identity and to be the identity on $\dot{X}_0 \cup (M - W_0)$.

THEOREM 2. *Every closed, locally $R^\infty(Q^\infty)$-deficient subset of an $R^\infty(Q^\infty)$-manifold M is $R^\infty(Q^\infty)$-deficient in M. Consequently, every closed, locally compact subset of M is $R^\infty(Q^\infty)$-deficient in M.*

0. Introduction and definitions. In this note, we introduce a criterion for R^∞-deficiency in R^∞-manifolds, inductive proper q-1-LCC, and prove some properties similar to those in Hilbert-cube manifolds and Hilbert-space manifolds: (1) The controlled version of the unknotting theorem for R^∞-deficient embeddings in R^∞-manifolds (Theorem 2.1); (2) local $R^\infty(Q^\infty)$-deficiency implying global $R^\infty(Q^\infty)$-deficiency in $R^\infty(Q^\infty)$-manifolds (Theorems 5.1 and 5.3); and (3) consequently, every closed locally compact subset of an $R^\infty(Q^\infty)$-manifold being $R^\infty(Q^\infty)$-deficient.

Throughout this note, let R^n denote the n-Euclidean space, I the unit interval $[0,1]$, Q the Hilbert cube $\prod_1^\infty I_n$, R^∞ the direct limit space $\varinjlim\{R^n\}$, and Q^∞ the direct limit space $\varinjlim\{Q^n\}$ where Q^n is the product of n copies of Q. By *$R^\infty(Q^\infty)$-manifolds*, we mean paracompact spaces that are locally homeomorphic to $R^\infty(Q^\infty)$. A closed subset X of an $R^\infty(Q^\infty)$-manifold M is said to be $R^\infty(Q^\infty)$-*deficient* if there is a homeomorphism $h: M \to M \times R^\infty$ $(M \times Q^\infty)$ such that $h(X) \subset M \times \{0\}$, where $0 = (0,0,\ldots)$ (or $h: M \to M \times I$, as we already observed in $[\mathbf{L_1}]$ and Proposition 1 in $[\mathbf{L_2}]$). An embedding $f: X \to M$ is said to be $R^\infty(Q^\infty)$-*deficient* if $f(X)$ is. For basic notions and results in Q-manifold theory as Z-set, unknotting theorem,..., we refer to **[Ch]**.

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Given an open cover α of a topological space M, a homotopy $H\colon X \times I \to M$ is said to be *α-limited* if for each $x \in X$, there is an open $U \in \alpha$ such that $H(\{x\} \times I) \subset U$. Two maps f and $g\colon X \to M$ are α-close if for each $x \in X$ there is an open set $U \in \alpha$ such that $f(x),\ g(x) \in U$. For a subset A of X, let $\mathrm{St}(A, \alpha)$ denote $\bigcup\{U \in \alpha | U \cap A \neq \varnothing\}$. Let $\mathrm{St}(\alpha)$ denote the open cover $\{\mathrm{St}(U, \alpha) | U \in \alpha\}$ of M; and inductively let $\mathrm{St}^n(\alpha)$ denote $\{\mathrm{St}(V, \alpha) | V \in \mathrm{St}^{n-1}(\alpha)\}$. Let (X, d) be a metric space. Given a subset A of X and a $\delta > 0$, we write $N_\delta(A) = \{x \in X | d(x, A) < \delta\}$, the δ-neighborhood of A in X.

For basic notions and results in piecewise-linear (PL) topology as PL homeomorphism, PL embedding, PL collar, collapsing ($\searrow$), etc., we refer to [**Hd**]. For a PL manifold M, let ∂M and $\operatorname{Int} M$ denote its boundary and its interior. Let aB^d denote the PL d-ball $[-a, a] \times \cdots \times [-a, a]$ and B^d the unit PL d-ball.

Let M be an n-manifold. A closed subset X of M is *locally simply co-connected* (1-LCC) if for every $x \in X$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that every loop in $N_\delta(x) - X$ is null-homotopic in $N_\varepsilon(x) - X$. A closed embedding $f\colon X \to M$ is 1-LCC if $f(X)$ is 1-LCC in M. The complement $M - X$ is *uniformly simply-connected* (1-ULC) (as in [**B**]) if given an $\varepsilon > 0$ there is a $\delta > 0$ such that every δ-loop in $M - X$ is null-homotopic in an ε-subset of $M - X$. It is observed that if X is a compact subset of M, where M is a compact manifold or the interior of a compact manifold, then X is 1-LCC in M iff $M - X$ is 1-ULC (by use of the local contractibility of M and $M - X$, and the local compactness of M).

An embedding f of a polyhedron P into a PL manifold M is said to be *tame* (see [**R**, p. 51]) if there is a homeomorphism h of M such that hf is PL. If f is a 1-LCC embedding of a compact polyhedron into $\operatorname{Int} M^n$ ($n \geqslant 6$) with $2 \dim P + 2 \leqslant n$, then there is a PL embedding $g\colon P \to \operatorname{Int} M^n$ which is arbitrarily close to f (Corollary 1.6.6 in [**R**]); hence, it follows from [**B**] that f is *locally tame* (see definition in [**R**, p. 120]), and from Theorem 3.8.1 in [**R**] that f is ε-tame for any given $\varepsilon > 0$.

Now, we restate some lemmas already observed in [$\mathbf{L_1}$] that we will need in the sequel.

LEMMA A. *A space M is an R^∞-manifold if and only if M is homeomorphic to $\varinjlim M_n$ where, for each n, M_n is a compact finite-dimensional manifold and it is a* 1-*LCC subset of the interior of M_{n+1} with* $2 \dim M_n + 2 \leqslant \dim M_{n+1}$. $\square$

LEMMA B. *Let $X = \varinjlim \{X_n\}$ where X_n is a metric subspace of X_{n+1} for each n. If K is a compact subset of X, then there is an integer n_0 such that K is contained in X_{n_0}.* $\square$

COROLLARY 0. *Every compact subset of an R^∞-manifold is R^∞-deficient; therefore, by Theorem* 5.3 *in* [$\mathbf{L_1}$], *strongly negligible.* $\square$

LEMMA C. *If Y is an R^∞-deficient subset of an R^∞-manifold M, then M can be written as $\varinjlim M_n$, where M_n is a compact PL manifold such that, for each* $n = 1, 2, \ldots,$

(1) $M_n \subset \partial M_{n+1}$,

(2) $Y \cap M_n \subset \partial M_n$, $2 \dim(Y \cap \partial M_n) + 3 \leqslant \dim \partial M_n$, *and*

(3) *each* $M_n \cong$ *a PL submanifold of R^q with* $q = \dim M_n$.

PROOF. Let $M = \varinjlim N_n$ as in Lemma A and let $R^\infty \cong \varinjlim I^n$. From the Open Embedding Theorem [$\mathbf{H}_5$], we can assume that each N_n is homeomorphic to a PL submanifold of $R^{\dim N_n}$. Now, we identify M with $M \times R^\infty$ such that $Y \subset M \times \{0\}$ and define $M_n = N_n \times I^{d_n}$ with $\dim N_n + 4 \leqslant d_n < d_{n+1}$. Then, by Corollary III.1 in [$\mathbf{H}_4$], it follows that $M \times R^\infty \cong \varinjlim M_n$ which satsifies (1) and (2) since $I^{d_n} \equiv I^{d_n} \times \{0\} \subset \partial I^{d_{n+1}}$. $\square$

A closed subset X of a space N is *collared* (*bicollared*) in N if there is an open embedding ϕ: $X \times [0,1) \to N$ (ϕ: $X \times (-1,1) \to N$) such that $\phi(x,0) = x$ for all $x \in X$. A collar ϕ: $X \times [0,1) \to N$ is *normal* if $\phi(X \times [0,s])$ is closed in N for each $s \in [0,1)$, and $\phi|(X \times [0,s])$ of a normal collar ϕ is called a *closed collar* of X in N. *Normal bicollars* and *closed bicollars* are defined similarly.

Observe that if N is paracompact and X is a collared subset in N, then there is a normal collar of X in N. Since N is paracompact, it follows that there is an open neighborhood U of X in N such that $(\bar{U} \subset \phi(X \times [0,1)))$. Then, $\phi^{-1}(U)$ is an open neighborhood of $X \times \{0\}$ in $X \times [0,1)$. Since X is paracompact, there is a map α: $X \to (0,1]$ such that for each $s \in [0,1)$, the set $C_s \equiv \{(x,t) | x \in X,\ t \in [0, s\alpha(x)]\}$ is contained in $\phi^{-1}(U)$. Define ψ: $X \times [0,1) \to N$ by $\psi(x,t) = \phi(x, t\alpha(x))$. Then, ψ is a normal collar of X in N. For, if $s \in [0,1)$, $\psi(X \times [0,s]) = \phi(C_s)$ is closed in $\operatorname{Im}(\phi)$ and $\overline{\psi(X \times [0,s])} \subset \bar{U} \subset \operatorname{Im}(\phi)$; hence, $\psi(X \times [0,s])$ is closed in N.

Therefore, without loss of generality, we will use normal collars (bicollars) if the involved spaces are paracompact.

The layout of this note is as follows. The inductive proper q-1-LCC, which is defined in §1, is proved to be equivalent to the R^∞-deficiency. Thereby, the union of two R^∞-deficient subsets is shown to be R^∞-deficient. Then, the unknotting theorem is proved in §2. In §3, it is shown that a collared submanifold is $R^\infty(Q^\infty)$-deficient. Next, some technical lemmas, which will be used to prove local deficiency implying global deficiency in §5, are proved in §4.

1. Characterization of R^∞-deficiency. In this section, we will establish a criterion, the inductive proper q-1-LCC, that determines the R^∞-deficiency in R^∞-manifolds. The result is similar to Proposition 1 in [$\mathbf{L}_2$]. Then, we prove that the union of a finite family of R^∞-deficient subsets is R^∞-deficient in Proposition 1.5. Finally, we show that an inductive trivial boundary subset is also R^∞-deficient (Theorem 1.6).

A codimension-4 closed subset X of a compact manifold P is *properly* 1-LCC if $X \cap \operatorname{Int} P$ and $X \cap \partial P$ are 1-LCC subsets of $\operatorname{Int} P$ and ∂P, respectively. Given an integer q, a closed subset X of a compact manifold N^n is *properly* q-1-LCC in N if X is properly 1-LCC in N, $2\dim X + q \leqslant n$ and $2\dim(X \cap \partial N) + q \leqslant n - 1$. A closed subset X of an R^∞-manifold N is *inductively properly* q-1-LCC if N can be written as $\varinjlim N_n$ as in Lemma A such that $X \cap N_n$ is properly q-1-LCC in N_n for each n.

In the following observations, N, K and B denote compact PL manifolds of dimension greater than 5, and all spaces X, Y, Z and W are compact.

OBSERVATIONS. (α) If X is a subset of N, then for each PL ball B^q, $2\dim X + 2 \leqslant q$, there is an embedding f: $X \to B^q$ such that $f^{-1}(\partial B^q) = X \cap \partial N$ ($\equiv Y$). First, from [**H-W**], there is an embedding of Y into ∂B^q which, then, extends to a map g: $X \to B^q$ such that $g(X - Y) \subset \operatorname{Int} B^q$; finally, from Corollary 5 in [**H-T**], there is an embedding f approximating g (rel. Y) as we desired.

(α') In (α) if Z is a subset of X with $Y \cap Z = \varnothing$, and 1-LCC embedding h: $Z \to \operatorname{Int} B^q$ is given, then the embedding f in (α) can be chosen to be an extension of h [**H-T**, Corollary 5], since $h(Z)$ is a Z^k-set in $\operatorname{Int} B^q$ by Lemma 1 of [**B**] where $k \leqslant q - \dim X - 1$.

(β) If W is a 1-LCC subset of $\operatorname{Int} K^q$ and f: $X \to \operatorname{Int} K$ is a map such that $f | X_0$ is a 1-LCC embedding, where X_0 is a compact subset of X and where $2 \dim X + 2$, $2 \dim W + 2 \leqslant q$, then there is a 1-LCC embedding g: $X \to \operatorname{Int} K$ approximating f (rel X_0) such that $g(X - X_0) \cap W = \varnothing$. For, let Z (Z_0, resp.) denote $(X \cup W)/\sim$ ($(X_0 \cup W)/\sim$, resp.) where $f(x) \sim x$ if $x \in X_0$ and $f(x) \in W$. Then, the map f and the inclusion $W \subset K$ define a map $\bar{f}$: $Z \to \operatorname{Int} K$ such that $\bar{f}|Z_0$ is a 1-LCC embedding (by Lemma 1.2 in [$\mathbf{L_1}$]). Then, g will be the restriction on X of an approximation to $\bar{f}$ (rel. Z_0) given by Corollary 5 in [**H-T**].

(γ) If $X \subset R^{q-1} \subset R^q$ is a closed subset with $\dim X + 3 \leqslant q$, then X does not separate R^{q-1} locally; consequently, by use of the natural bicollar of R^{q-1} in R^q, it is straightforward to show that X is 1-LCC in R^q.

LEMMA 1.1. *Let X be a compact subset of a compact PL manifold N and assume that* $2 \dim X + 3$, $2 \dim(X \cap \partial N) + 4 \leqslant k$. *Let f: $X \to aB^{k-1} \subset aB^k$ be an embedding given by Observation* (α). *Then, each neighborhood V of $f(X)$ in aB^k contains a compact-PL-manifold neighborhood K of $f(X)$ such that*

(1) $f^{-1}(\partial K) = \partial N \cap X$, *and*

(2) $f(X)$ *is a properly* 1-*LCC subset of K.*

PROOF. It follows from (γ) that $f(X)$ is properly 1-LCC in aB^k. Let L be a polyhedron neighborhood of $f(X)$ in V. Define K to be a second derived neighborhood of L in V, then K meets $\partial(aB^k)$ regularly [**R**, p. 23]; hence, it satisfies both properties (1) and (2). □

PROPOSITION. 1.2. *A subset X of an R^∞-manifold N is R^∞-deficient in N if and only if it is an inductively properly q-1-LCC subset of N for* $q \geqslant 3$.

PROOF. Let X be an R^∞-deficient subset of $N \cong N \times R^\infty$. We can assume that $X \subset N \times \{0\} \subset N \times R^\infty$. Let $N = \varinjlim N'_k$ as in Lemma A, and let $N_k = N'_k \times kB^{\dim N'_k + q}$; then, it is clear that $X \cap N_k \subset N'_k \times \{0\}$ is properly q-1-LCC in N_k and $N \times R^\infty = \varinjlim N_n$.

(ii) Let X be an inductively properly 3-1-LCC subset in N. We will show that there is a homeomorphism f: $N \times R^\infty \to N$ such that $f^{-1}(X) \subset N \times \{0\}$. Let $N = \varinjlim N_n$ and $X_n = X \cap N_n$ as in the definition of the inductive proper 3-1-LCC of X in N. In the sequel, we will use two families of PL balls: $\{B_1^k\}_{k=1}^\infty$ and $\{B_2^k\}_{k=1}^\infty$.

First, from Lemma 1.1, if $m_1 = \dim X_1 + 1$, there is a properly 1-LCC embedding g_1: $X_1 \to K_1 \subset B_1^{2m_1+3}$ where K_1 is a compact-PL-manifold neighborhood of $g_1(X_1)$ in $B_1^{2m_1+3}$ which can be chosen so small that $g_1^{-1}|g_1(X_1)$ has an extension h' over K_1

into Int N_2. Identify K_1 with $K_1 \times \{0\} \subset K_1 \times B_2^1$ and define $h = h' \circ \mathrm{proj}\colon K_1 \times B_2^1 \to K_1 \to \mathrm{Int}\, N_2$. Without loss of generality, we can assume that $4m_1 + 10$, $2 \dim N_1 + 2 \leqslant \dim N_2$. Observe that

(a) $2 \dim X_2 + 3 \leqslant \dim N_2$ from the definition of inductive proper 3-1-LCC,

(b) $2 \dim(K_1 \times B_2^1) + 2 = 4m_1 + 10 \leqslant \dim N_2$, and

(c) $h(g_1(X_1)) = g_1^{-1}(g_1(X_1)) = X_1$ is 1-LCC in Int N_2. Therefore, from Observation (β), it follows that there is a 1-LCC embedding $f_1\colon K_1 \times B_2^1 \to \mathrm{Int}\, N_2$ such that

$$(1.1) \qquad f_1|g_1(X_1) = g_1^{-1}|g_1(X_1),$$

and

$$(1.2) \qquad (N_1 \cup X_2) \cap \mathrm{Im}(f_1) = X_1 \quad (\text{since } X_1 = X_2 \cap N_1 \subset N_1 \cup X_2).$$

Define $M_1 = \mathrm{Im}(f_1)$.

Second, in a similar manner, we define K_2, f_2 and M_2. Write $X_2' = X_2 \cup N_1 \cup M_1$ and let $\phi\colon K_1 \times B_2^1 \to K_1 \times [-1,1]$ be the natural PL homeomorphism. Let $m_2 = \dim X_2' + 1 = \max\{\dim X_2, \dim N_1, \dim M_1\} + 1$. We have the following natural inclusions: $K_1 \times [-1,1] \subset B_1^{2m_1+3} \times [-1,1] = B_1^{2m_1+4} \subset B_1^{2m_2+3} \subset 2B_1^{2m_2+3}$. So, it follows from Observations (α') and (γ) that there is a 1-LCC embedding g_2: $X_2' \to 2B_1^{2m_2+3}$ that is an extension of ϕf_1^{-1} with $g_2^{-1}(\partial(2B_1^{2m_2+3})) = X_2' \cap \partial N_2$. Now, we assume similarly that $2\dim N_2 + 2$, $4m_2 + 10 \leqslant \dim N_3$; then, there is a compact-PL-manifold neighborhood K_2 of $g_2(X_2')$ in $2B_1^{2m_2+3}$ such that by use of Observation (β) with $g_2(X_2')$ and $N_2 \cup (X_3 \cap \mathrm{Int}\, N_3)$ playing the role of X_0 and W respectively, we can define a 1-LCC embedding $f_2'\colon K_2 \times 2B_2^2 \to \mathrm{Int}\, N_3$ (recalling that $X_2' \subset N_2$ is 1-LCC in Int N_3) such that

$$(2.1) \qquad f_2'|g_2(X_2') = g_2^{-1}|g_2(X_2'),$$

especially, if $x \in K_1 \times [-1,1] \subset g_2(X_2')$, then $f_2'(x) = g_2^{-1}(x) = (\phi f_1^{-1})^{-1}(x) = f_1\phi^{-1}(x)$, and

$$(2.2) \qquad (N_2 \cup X_3) \cap \mathrm{Im}(f_2') = X_2'.$$

Now, observe that the PL embedding $j\phi\colon K_1 \times B_2^1 \to K_2 \times 2B_2^2$, where j is the inclusion $K_1 \times [-1,1] \subset K_2 \times \{0\} \subset K_2 \times 2B_2^2$, is homotopic to the inclusion $i_1 \times i_2\colon K_1 \times B_2^1 \to K_2 \times 2B_2^2$ through a homotopy H, fixing $K_1 \times \{0\}$, such that $H(x, s, t) \notin g_2(X_2)$ for $x \in K_1$, $s \in B_2^1 - \{0\}$, $t \in I$. (Recall that $2 \dim X_2 + 3 \leqslant \dim K_2$ and that $g_2(X_2)$ is properly 1-LCC in $K_2 \times 2B_2^2$.) Hence, there is, by Theorem 1.5 [**L₁**], an isotopy h_t^2 of $K_2 \times 2B_2^2$ such that

$$(*) \qquad h_0^2 = \mathrm{id},$$

$$(**) \qquad h_t^2(x) = x \quad \text{if } x \in (K_1 \times \{0\}) \cup g_2(X_2),$$

and

$$(***) \qquad h_1^2(x) = j\phi(x) = \phi(x) \quad \text{if } x \in K_1 \times B_2^1.$$

Then, we define $f_2 = f_2'h_1^2$ and $M_2 = \mathrm{Im}(f_2) = \mathrm{Im}(f_2')$. Observe that the following diagram is commutative,

$$\begin{array}{ccc} K_1 \times B_2^1 & \xrightarrow{f_1} & M_1 \\ \downarrow i_1 \times i_2 & & \cap \\ K_2 \times 2B_2^2 & \xrightarrow{f_2} & M_2 \end{array}$$

since for each $x \in K_1 \times B_2^1, f_2(x) = f_2'h_1^2(x) = f_1\phi^{-1}\phi(x) = f_1(x)$ by (2.1).

In a similar manner, we can continue to define X_3', g_3, K_3, f_3', $h_t^3, f_3 = f_3'h_1^3$, $M_3, \ldots$ to extend the above commutative diagram.

Finally, let $f = \varinjlim f_n$: $\varinjlim (K_n \times nB_2^n) \to \varinjlim M_n$, then f is a homeomorphism. Moreover, since M_{n+1} contains X_n' that contains N_n for each n, $\varinjlim M_n = N$; and since K_n and nB_2^n are compact, it can be shown that $\varinjlim (K_n \times nB_2^n)$ is homemorphic to $(\varinjlim K_n) \times (\varinjlim nB_2^n) = N \times R^\infty$; and $f^{-1}(X) \subset N \times \{0\}$ since

$$\begin{aligned} f^{-1}(X_n) = f_n^{-1}(X_n) &= (f_n'h_1^n)^{-1}(X_n) = (h_1^n)^{-1}(f_n')^{-1}(X_n) \\ &= (h_1^n)^{-1}(g_n(X_n)) \quad \text{by } (n.1) \\ &= g_n(X_n) \subset K_n \equiv K_n \times \{0\} \quad \text{by } (**). \end{aligned}$$

Hence, the proof is complete. □

REMARKS. 1.a. The closed subset $R^\infty \times \{0\}$ is not R^∞-deficient in $R^\infty \times R^4$. Therefore, the dimension condition in the definition of inductive proper q-1-LCC cannot be dropped.

1.b. From part (i) of the proof, up to a homeomorphism, we can assume as in Lemma C that each manifold N_k in the definition of inductive proper q-1-LCC is a PL submanifold of R^d, where $d = \dim N_k$.

In the following, we will use the same notation to indicate an upper semicontinuous decomposition and the collection of its nondegenerate elements. By a *pseudo-isotopy* g of a space X, we mean a surjective, level-preserving map g: $X \times I \to X \times I$ such that $g_0 = \mathrm{id}_X$ and g_t: $X \to X$ is a homeomorphism for each $t \in [0, 1)$.

LEMMA 1.3. *Let N^n be a manifold without boundary and p: $E \to N \times [0,2)$ an R^k-fiber bundle. Assume that E_1 is a closed subspace of $p^{-1}(N \equiv N \times \{0\})$ such that $p_1 = p|E_1$: $E_1 \to N$ is a B^k-fiber subbundle of $p|p^{-1}(N)$. Let Y be a closed subset of N and $G = \{p_1^{-1}(y) | y \in Y\}$. Then, there is a homeomorphism f: $E \to E/G$ such that*

(i) *$f(x) = q(x)$ for all $x \in K \equiv i(N \times [0,2)) \cup (E - W) \cup p^{-1}(N \times [1,2))$ where q: $E \to E/G$ is the quotient map, i is the zero-section of p, and W is an open neighborhood of $i(N \times [0,2)) \cup E_1$ in E, and*

(ii) *$p_G f = p$ where p_G: $E/G \to N \times [0,2)$ is the natural map induced from p.*

PROOF. Let $\{C_i | i = 1, 2, \ldots\}$ and $\{D_i | i = 1, 2, \ldots\}$ be nbd-finite covers **[D]** of N each of whose members is a closed n-ball such that D_i is a neighborhood of C_i for each $i = 1, 2, \ldots$. Let $F_i = C_1 \cup \cdots \cup C_i$, $Y_i = Y \cap F_i$, $G_i = \{p_1^{-1}(y) | y \in Y_i\}$ for each $i = 1, 2, \ldots$, and $G_0 = \varnothing$. For each i, let q_i: $E \to E/G_i$, $q_{i,j}$: $E/G_i \to E/G_j$ if

$i < j$, $q_{i,\infty}$: $E/G_i \to E/G$ be the quotient maps, and p_{G_i}: $E/G_i \to N \times [0,2)$ the natural map induced from p. In the sequel, we will use the same notation for any restriction of a map if no confusion occurs.

We will prove by induction the following statement: For each $i = 0, 1, \dots,$ there is a homeomorphism h_i: $E \to E/G_i$ such that $p_{G_i} h_i = p$ and that if $f_i = q_{i,\infty} h_i$: $E \to E/G$, then

(1) $p_G f_i = p$,

(2) $f_i | p^{-1}(F_i \times [0,2))$ is a homeomorphism,

(3) $f_i = f_{i-1}$ on $p^{-1}((F_i - D_i) \times [0,2))$, and

(4) $f_i(z) = q(z)$ if $z \in K$.

Then, by (2), (3) and the nbd-finiteness of $\{D_i | i = 1, 2, \dots\}$, it follows that $f = \lim_{i \to \infty} f_i$ is a well-defined homeomorphism that satisfies the lemma by (1) and (4).

For $i = 0$, let $h_0 = \mathrm{id}_E$ and $f_0 = q$; then, there is nothing to prove. Now, assume by induction that h_n and f_n have been defined. Consider the restrictions of h_n and f_n on $p^{-1}(D_{n+1} \times [0,2)) \cong D_{n+1} \times [0,2) \times R^k$, and G', G'_n, G'_{n+1} the upper semicontinuous decompositions of $D_{n+1} \times [0,2) \times R^k$ induced from G, G_n, G_{n+1}, respectively. Without loss of generality, we can assume that $D_{n+1} \times [0,2) \times B^k \subset W$. Let ϕ: $D_{n+1} \times [0,2) \to [0,1]$ be a map such that

(a) $\phi(z) = 1$ if $z \in D_{n+1} \times [1,2)$;

(b) $\phi(x, 0) = 0$ if $x \in Y_n \cap D_{n+1}$, and

(c) if we write $P_z = \{su | u \in \partial B^k,\ 0 \leqslant s \leqslant 1 + \phi(z)\}$ for each $z \in Z \equiv (D_{n+1} \times [0,2)) - (Y_n \times \{0\})$, then $\{z\} \times P_z \subset W$ and $\mathrm{diam}(h_n^{-1} q_n(\{z\} \times P_z)) \leqslant \mathrm{diam}(h_n^{-1} q_n(\{z\} \times B^k)) + \mathrm{dist}(z, Y_n \times \{0\})$.

For each $z \in Z$, define a pseudo-isotopy $g_{z,t}$: $R^k \to R^k$ $(t \in I)$ by $g_{z,t}(su) = (\lambda_{1+\phi(z),t}(s))u$ for each $u \in \partial B^k$ and $s \in [0, \infty)$, where for each $a > 1$, λ_a is the natural pseudo-isotopy of $[0, \infty)$ defined by: first, $\lambda_{a,t}(0) = 0$, $\lambda_{a,t}(s) = s$ if $s \geqslant a$, $\lambda_{a,t}(1) = 1 - t$; then, extends linearly over each interval [0, 1] and [1, a].

Observe that $q_n|(Z \times R^k)$ is a fiber-preserving (f.p.) homeomorphism. Therefore, we can define a f.p. pseudo-isotopy F of $Z \times R^k$ by

$$F_{z,t}(v) = \left(h_n^{-1} q_n\right)_z \left(g_{z, t\alpha(\zeta)}\right) \left(h_n^{-1} q_n\right)_z^{-1} (v)$$

where α: $D_{n+1} \times [0,2) \to [0,1]$ is a map such that $\alpha^{-1}(1) = (Y \cap C_{n+1}) \times \{0\}$ and $\alpha^{-1}(0)$ contains $Z_0 \equiv (\partial D_{n+1} \times [0,2)) \cup (D_{n+1} \times [1,2))$. Observe that

$$(*) \qquad F_t = \mathrm{id} \quad \text{on } \left(Z_0 \times R^k\right) \cup \left(Z \times R^k - h_n^{-1} q_n\left(\bigcup\{\{z\} \times P_z | z \in Z\}\right)\right).$$

From (c) and (*), it is straightforward to verify that F_t can extend via the identity to a f.p. pseudo-isotopy $\bar{F}_t$ of $D_{n+1} \times [0,2) \times R^k$ that only shrinks the members of $\{h_n^{-1} q_n(\{z\} \times B^k) | z \in Y_{n+1} - Y_n\}$ to points. Consequently, $\bar{F}_1$ induces a homeomorphism

$$\bar{f}: \left(D_{n+1} \times [0,2) \times R^k\right) / G''_{n+1} \to D_{n+1} \times [0,2) \times R^k,$$

where $G''_{n+1} = \{h_n^{-1} q_n(A) | A \in G'_{n+1}\}$, such that $\bar{f} \bar{q} = \bar{F}_1$ where $\bar{q}$ is the quotient map (refer to the diagram below).

In brief, we obtain the following diagram, where the nondefined maps are the natural ones,

$$\begin{tikzcd}
 & (D_{n+1}\times[0,2)\times R^k)/G'_n \arrow[dl, "q_{n,\infty}"'] \arrow[dd, bend right=60, "q_{n,n+1}"'] & D_{n+1}\times[0,2)\times R^k \arrow[l, dashed, "h_n"'] \arrow[d, "h_n^{-1}", "\cong"'] \\
(D_{n+1}\times[0,2)\times R^k)/G' & D_{n+1}\times[0,2)\times R^k \arrow[l] \arrow[u, "q_n"'] \arrow[r, "h_n^{-1}q_n"] \arrow[d, "q_{n+1}"] \arrow[dr, "\bar q h_n^{-1} q_n"] & D_{n+1}\times[0,2)\times R^k \arrow[u, "\bar F_1"'] \arrow[d, "\bar q"] \\
 & (D_{n+1}\times[0,2)\times R^k)/G'_{n+1} \arrow[ul, "q_{n+1,\infty}"] \arrow[r, "\psi", "\cong"'] & (D_{n+1}\times[0,2)\times R^k)/G''_{n+1} \arrow[uu, bend right=60, "\bar f"', "\cong"]
\end{tikzcd}$$

that has the following properties:

(α) if the dotted map h_n is deleted, then the diagram is commutative,

(β) if the whole diagram is restricted to the subspaces corresponding to $Z_0 \times R^k$, then we will obtain a commutative diagram since $\bar F_1 = \mathrm{id}$ on $Z_0 \times R^k$, and

(γ) every map in the diagram is f.p. with respect to the natural maps induced from p.

Now, define $h_{n+1} = \psi \bar f^{-1}$. Then, by ($\beta$), $h_{n+1} = q_{n,n+1}h_n$ on $Z_0 \times R^k$. Therefore, we can extend h_{n+1} over E via $q_{n,n+1}h_n$ and define $f_{n+1} = q_{n+1,\infty}h_{n+1}$. Then, h_{n+1} and f_{n+1} satisfy the inductive hypothesis and (1)–(4). So, the lemma is proved. □

We will only use the trivial-bundle version of the following lemma in the proof of Proposition 1.5. Note that the proof of the trivial-bundle version is much shorter.

LEMMA 1.4. *Let $p\colon E \to M^m$ be a B^k-fiber bundle over a manifold M, Y a closed subset of ∂M and $G = \{p^{-1}(y) \mid y \in Y\}$. Then, there is a homeomorphism $f\colon (E, \partial E) \to (E/G, \partial E/G)$ such that $f(i(x)) = q(i(x))$ where i is the zero-section of p and q is the quotient map.*

PROOF. Identify $\partial M \times [0,2)$ with an open collar of ∂M in M such that $\partial M \times [0,1]$ is closed in M. Since there is a homeomorphism $\phi\colon [0,2) \times B^k \to [0,2) \times R^k$ such that $\phi(0,x) = (0,x)$ and $\phi(t,0) = (t,0)$ for each $x \in B^k$ and $t \in [0,2)$, it follows that there is an R^k-fiber bundle structure $\bar p\colon p^{-1}(\partial M \times [0,2)) \to \partial M \times [0,2)$ such that $\bar p^{-1}(\{x\} \times [0,2)) = p^{-1}(\{x\} \times [0,2))$. From Lemma 1.3, there is a homeomorphism $\bar f\colon p^{-1}(\partial M \times [0,2)) \to p^{-1}(\partial M \times [0,2))/G$ such that $\bar f(x) = q(x)$ if $x \in i(\partial M \times [0,2)) \cup (p^{-1}(\partial M \times [0,2)) - W) \cup \bar p^{-1}(\partial M \times [1,2))$, where W is an open neighborhood of $p^{-1}(\partial M \times \{0\}) \cup i(\partial M \times [0,2))$ such that $\overline{W} \cap \bar p^{-1}(A)$ is compact if A is a compact subset of $\partial M \times [0,2)$. Hence, since $\partial M \times [0,1]$ is closed in M, the extension f of $\bar f$ over E via q is a well-defined homeomorphism that we wanted. □.

Given an R^k-vector bundle $p\colon \mathscr{E} \to M$, there is a Euclidean metric $\mu\colon E \to [0,\infty)$ on $\mathscr{E}$ [**M-S**, p. 23]. Given a map $\lambda\colon M \to [0,1]$, and letting Y denote $\lambda^{-1}(0)$, we define a *pinched k-tube neighborhood* (*pinched bicollar* for the trivial R^1-bundle p) at Y of M in $\mathscr{E}$ to be $N = \{x \in \mathscr{E} \mid \mu(x) \leqslant \lambda(p(x))\}$. If E is the corresponding closed B^k-fiber subbundle of $\mathscr{E}$, $\{x \in \mathscr{E} | \mu(x) \leqslant 1\}$, and if $Y \subset \partial M$, then by Lemma 1.4, $E \cong E/G \cong N$ by homeomorphisms that leaves the zero-section fixed.

PROPOSITION 1.5. *If X and Y are two R^∞-deficient subsets of an R^∞-manifold M, then $X \cup Y$ is R^∞-deficient in M.*

PROOF. By taking subsequences, we can choose $M = \varinjlim M_i$ and $M = \varinjlim M_j'$ such that the following hold:

(1) $X_i = X \cap M_i$ is a properly 4-1-LCC compact subset of M_i, $\dim M_1 \geqslant 6$ and M_i is a 1-LCC compact subset of Int M_{i+1}, for $i = 1, 2, \ldots$ (from Proposition 1.2),

(2) $Y_j = Y \cap M_j' \subset \partial M_j'$, $2 \dim Y_j + 3 \leqslant \dim \partial M_j'$, and M_j' is a compact PL submanifold of $\partial M_{j+1}'$ for each $j = 1, 2, \ldots$ (from Lemma C),

(3) $M_i \subset \partial M_i'$ for $i = 1, 2, \ldots$ (from Lemmas C and B),

(4) $M_j' \subset \operatorname{Int} M_{j+1}$, $2 \dim M_j' + 4 \leqslant \dim M_{j+1}$ for $j = 1, 2, \ldots$ (from (1) and Lemma B), and

(5) each of M_i's and M_j''s are PL submanifolds of $R^{\dim M_i}$ and $R^{\dim M_j'}$, respectively (from Remark 1.b and Lemma C).

We will construct a sequence of manifolds $\{N_j \subset \operatorname{Int} M_{j+1} | j = 1, 2, \ldots\}$ each of which is homeomorphic to a PL manifold such that

(i) $(X \cup Y) \cap \partial N_j$ is 3-1-LCC in ∂N_j,

(ii) $\partial M_j'$ is 1-LCC in ∂N_j, and

(iii) $\dim N_j = \dim M_{j+1}$.

Consequently, $\partial M_j'$ is tame in ∂N_j by (4), (iii) and Theorem 3.8.1 in [**R**]; hence, $\partial N_j \subset \partial N_{j+1}$ is 2-1-LCC from (3), (4), (iii) and Observation (γ). Then the proposition will follow from Proposition 1.2 since $X \cup Y$ will be inductively properly 3-1-LCC in $\varinjlim \partial N_j = M$ by (i).

For fixing our idea, let $j = 1$. We can assume that M_1' is a 1-LCC compact subset of Int M_2; hence, it is ε-tame in Int M_2 (see the Introduction). Then, without loss of generality, we can assume that M_1' is a PL submanifold of M_2 and that (see [**R**, p. 24]) its second derived neighborhood V (rel. $\partial M_1'$) in Int M_2 is a PL manifold which is, by (5) and Theorem 1.6.6 in [**R**], homeomorphic to $M_1' \times B^k$, where $k = \dim M_2 - \dim M_1'$. Let W be a pinched k-tube neighborhood of M_1' at Y_1 contained in $(\operatorname{Int} V - Y) \cup Y_1$. Since Y_1 is contained in $\partial M_1'$, from Lemma 1.4, there is a homeomorphism θ: $V \to W$ such that $\theta(x) = x$ for all $x \in M_1'$. Hence, Y_1 and $\partial M_1'$ are 1-LCC subsets of ∂W. Let $G(a) = \{\{y\} \times [-a, a] | y \in Y_1\}$ and $\Gamma(a) = (\partial W \times [-a, a])/G(a)$.

Now, let c: $\Gamma(1) \to (\operatorname{Int} M_2 - Y) \cup Y_1$ be a closed embedding that defines a pinched bicollar of ∂W at Y_1 (refer to [**R**, p. 41]). Let $W' = \operatorname{Im}(c) \cup W$, then W' is homeomorphic to W. Now, observe that $X \cap \partial M_1' \subset \partial M_1' \subset \partial W$ is a 1-LCC subset of ∂W; hence, there is a 1-LCC embedding f: $X \cap \partial W \to \partial W$ approximating the inclusion (rel. $X \cap \partial M_1')^{(\#)}$. Observe that the restriction $f|$: $X \cap (\partial W - Y_1) \to$ Int W' is a proper 1-LCC embedding. Therefore, from 1.3 of [$\mathbf{L}_1$], if f is chosen to be close to the inclusion, then there is a proper, 1-LCC embedding g: $X \cap \operatorname{Int} W' \to$ Int W' such that

(a) $g(x) = f(x)$ if $x \in X \cap (\partial W - Y_1)$,

(b) $g(x) = x$ if $x \notin c(\Gamma(1/2))$,

(c) $g \simeq_p i\colon X \cap \operatorname{Int} W' \subset \operatorname{Int} W'$ (rel. $X \cap [\{\operatorname{Int} W' - c(\Gamma(1/2))\} \cup M_1'])$ say by a proper homotopy H, and

(d) $g^{-1}(\partial W) = X \cap (\partial W - Y_1)$ (by use of a bicollar of $\partial W - Y_1$ in $\operatorname{Int} W'$).

Extend H_t and g via the identity over $\operatorname{Int} M_1'$ to obtain a proper homotopy $\overline{H}_t$: $\overline{G} \simeq_p i\colon (X \cup M_1') \cap \operatorname{Int} W' \subset \operatorname{Int} W'$ (rel. $(X \cap \{\operatorname{Int} W' - c(\Gamma(1/2))\}) \cup \operatorname{Int} M_1')$. Then, since $2\dim\{(X \cup M_1') \cap \operatorname{Int} W'\} + 4 \leqslant \dim W' = \dim M_2 \geqslant 6$ by (1) and (4), it follows from Corollary 1.6 of [$\mathbf{L_1}$] that there is a homeomorphism $\bar{h}$ of $\operatorname{Int} W'$ that extends $\bar{g}$ and equals the identity on the complement of $c(\Gamma(1/2))$. So, $\bar{h}$ can extend via the identity to a homeomorphism h of W'. Observe that $h = \mathrm{id}$ on $M_1' \cup \partial W'^{(*)}$. Define $N_1 = h^{-1}(W)$.

First, since $Y_1 \subset \partial M_1'$ is 1-LCC in ∂W, $Y_1 = h^{-1}(Y_1)$ is a 1-LCC subset of $h^{-1}(\partial W) = \partial N_1$. From (a) and (#), observe that h is an extension of f; hence, $h(X \cap W') \cap \partial W = h(X \cap \partial W) = f(X \cap \partial W)$ is 1-LCC in ∂W, where the former equality follows from (d) and (*). So,

$$X \cap \partial N_1 = (X \cap W') \cap \partial N_1 = h^{-1}(h(X \cap W') \cap \partial W) = h^{-1}(f(X \cap \partial W))$$

is 1-LCC in $h^{-1}(\partial W) = \partial N_1$. Therefore,

$$(X \cup Y) \cap \partial N_1 = (X \cap \partial N_1) \cup (Y \cap \partial N_1) = (X \cap \partial N_1) \cup Y_1$$

is 1-LCC in ∂N_1 by [$\mathbf{L_1}$, Lemma 1.2]. Second, $2\dim Y_1 + 3 \leqslant \dim \partial N_1$ from (2) and (4), and $2\dim(X \cap \partial N_1) + 3 \leqslant \dim M_2 - 1 = \dim \partial N_1$ by (1). Therefore, (i) follows. On the other hand, since $h^{-1}(\partial M_1') = \partial M_1'$, it follows that $\partial M_1'$ is a 1-LCC subset of ∂N_1; hence, (ii) is verified. Moreover, (iii) is trivially satisfied.

Similarly, by use of $M_j' \subset M_{j+1} \subset \partial M_{j+1}'$, $j > 1$, we can construct N_j such that $\partial N_j \subset \partial M_{j+1}'$ and $(X \cup Y) \cap \partial N_j$ is 3-1-LCC in ∂N_j. Hence, $\varinjlim \partial N_j = \varinjlim \partial M_j' = M$ by (2) and (ii). Therefore, $X \cup Y$ is an inductively properly 3-1-LCC subset of M with respect to $\{\partial N_j\}$; hence, it is R^∞-deficient in M by Proposition 1.2, and the proof is complete. □

Observe from the proof of the above proposition that we only need that Y satisfy (2). Therefore, if we call such a subset Y of M an *inductive trivial boundary subset* of M, combining with Lemma C, we can state the following result. (Note that Y_j is not required to be 1-LCC in ∂M_j in (2).)

THEOREM 1.6. *A closed subset Y of an R^∞-manifold M is R^∞-deficient in M if and only if Y is an inductive trivial boundary subset of M.* □

2. Unknotting theorem. In this section, we will prove the controlled relative version of the unknotting theorem for R^∞-deficient embeddings in R^∞-manifolds, which is similar to Theorem 19.4 in [**Ch**] for Q-manifolds, and Main Theorem in [$\mathbf{L_2}$] for Q^∞-manifolds. Besides that, as in the l_2-manifold theory, we also show that every locally compact, closed subset of an R^∞ (or Q^∞)-manifold is R^∞ (or Q^∞)-deficient.

THEOREM 2.1 (UNKNOTTING THEOREM). *Let X be an R^∞-deficient subset of an R^∞-manifold M, and let $f\colon X \to M$ be an R^∞-deficient embedding homotopic to the* *inclusion* i: $X \subset M$. *Then, there is a homeomorphism* F *of* M *onto itself such that the following hold.*

(1) $F|X = f$.

(2) *Given an open cover* α *of* M, *if* H: $i \simeq f$ *is limited by* α, *then* F *can be chosen to be* $\mathrm{St}^4(\alpha)$-*close to the identity.*

(3) *Moreover, if* $H_t|X_0 =$ *inclusion for each* $t \in I$, *where* X_0 *is a closed subset of* X, *and if* W_0 *is an open neighborhood of* $H[(X - X_0) \times I]$ *in* M, *then* F *can be chosen to be the identity on* $X_0 \cup (M - W_0)$.

PROOF. Since $X \cup F(X)$ is R^∞-deficient in M by Proposition 1.4, we can think of M as $M \times R^\infty$ and $X \cup f(X) \subset M \times \{0\}$. The theorem then follows from Lemma 3.1 in [$\mathbf{L_1}$] and its addendum. □

We now prove an application of the unknotting theorem for later use. A family $\{X_n\}$ of closed subsets of a space M is said to be *discrete* if the union of any subfamily is closed in M. Recall that each X_n is paracompact if M is an R^∞-manifold [$\mathbf{H_1}$, Proposition III].

LEMMA 2.2. *Let* $\{X_n | n = 1, 2, \ldots\}$ *be a discrete sequence of pairwise disjoint* R^∞-*deficient subsets of an* R^∞-*manifold* M. *Then,* $X = \bigcup_1^\infty X_n$ *is* R^∞-*deficient.*

PROOF. Let Ω be an open cover of M such that $\mathrm{St}(X_m, \Omega) \cap \mathrm{St}(X_n, \Omega) = \varnothing$ if $m \neq n$. The existence of such an open cover is from the paracompactness of each X_n and the discreteness of the sequence $\{X_n\}$ as follows. First, there is inductively a locally neighborhood-finite family $\{V_{n,\alpha} | \alpha \in A_n\}$ of open sets in M covering X_n such that the closure $\tilde{X}_n$ of $\bigcup\{V_{n,\alpha} | \alpha \in A_n\}$ is disjoint from $\tilde{X}_1 \cup \cdots \cup \tilde{X}_{n-1} \cup (\bigcup\{X_j | j > n\})$. Then, choose Ω to be a refinement of $\{V_{n,\alpha} | \alpha \in \bigcup_1^\infty A_n\} \cup \{M - X\}$.

Let W_n denote $\mathrm{St}(X_n, \Omega)$ and let f: $X \to M$ be an R^∞-deficient embedding which is Ω-homotopic to the inclusion i: $X \subset M$ (from [$\mathbf{L_1}$, Theorem 2.3]). From the Unknotting Theorem 2.1, there is a sequence of homeomorphisms $\{f_n\}$ of M such that, for each n,

(1) $f_n | X_n = f | X_n$,

(2) $f_n(x) = x$ if $x \notin W_n$.

Now, if we let F: $M \to M$ be defined by

$$F(x) = \begin{cases} f_n(x) & \text{if } x \in W_n, \\ x & \text{if } x \notin \bigcup_1^\infty W_n, \end{cases}$$

then F is a homeomorphism of M such that $F|X = f$; consequently, X is R^∞-deficient in M.

To conclude this section, we prove the following theorems that provide nontrivial examples of $R^\infty(Q^\infty)$-deficient subsets in $R^\infty(Q^\infty)$-manifolds. Although they are also corollaries to Theorems 5.1 and 5.2 below, we take the liberty to state them here because of the simplicity of their proofs.

THEOREM 2.3. *Every locally compact, closed subset* X *of an* R^∞-*manifold* M *is* R^∞-*deficient.*

PROOF. Let $g: X \to [0, \infty)$ be a proper map, and define $X_n = g^{-1}([n-1, n])$. Let $X^e = \{X_n | n \text{ is even}\}$ and $X^o = \{X_n | n \text{ is odd}\}$. Then, X^e and X^o are discrete families in M. Moreover, each X_n is R^∞-deficient in M since it is compact. Therefore, by Lemma 2.2, the union of each family X^e and X^o is R^∞-deficient; so, X is R^∞-deficient by Proposition 1.4. □

REMARKS. 2.a. *Let $X_0 \subset X$ be Q^∞-deficient subsets of a Q^∞-manifold M, and let H: $F \simeq i$ be a homotopy (rel. X_0) where f: $X \to M$ is Q^∞-deficient embedding. Let W be an open neighborhood of $H[(X - X_0) \times I]$ in M. Then, the ambient isotopy F_t ($t \in I$) of M with $F_1 | X = f$ in Main Theorem* [**L$_2$**] *can be chosen to be identity on $X_0 \cup (M - W)$.* As in the proof of Main Theorem [**L$_2$**], without loss of generality, we can assume that $H[(X - X_0) \times I) \cap X_0 = \varnothing$. Also from that proof, there is a sequence of compact Q-manifolds $\{N_k | k = 1, 2, \dots\}$ such that $M = \varinjlim N_k$ and that for each k, (a) N_k is a Z-set of N_{k+1}, (b) $H_t | X_k$: $X_k \to N_k$ is a Z-set embedding for each $t \in I$ where $X_k = X \cap N_k$, and (c) $H[(X - X_k) \times I] \cap N_k = \varnothing$. On the other hand, observe that the fiber version of the Z-sets unknotting theorem [**F**] can be restated as follows: "*Let $X_0 \subset X$ be Z-set compacta in a Q-manifold N, and let H_t: $X \to N$ ($t \in I$) be a continuous family of Z-set embeddings such that $H_0(x) = x$ if $x \in X$ and that $H_t(x) = x$ if $x \in X_0$. If Ω is an open neighborhood of $H[(X - X_0) \times I]$ in N, then H extends to an isotopy $\overline{H}$ of N with $\overline{H}_0 = \mathrm{id}$ and $\overline{H}_t = \mathrm{id}$ on $X_0 \cup (N - \Omega)$.*" Now, let W_1 be a closed neighborhood of $H[(X_1 - X_0) \times I]$ in $W \cap N_1$ such that $W_1 \cup X_1$ is compact. Define inductively W_k to be a closed neighborhood of $H[(X_k - X_0) \times I)$ in $W \cap N_k$ such that $W_k \cup X_k$ is compact and $W_{k-1} \subset \mathrm{Int}_{N_k}(W_k) \equiv \mathring{W}_k$. Then, from the above fiber version of the Z-sets unknotting theorem, we can construct inductively a sequence $\{\overline{H}^k | k = 1, 2, \dots\}$ where $\overline{H}^k$ is an isotopy of N, such that $\overline{H}^k_t = \mathrm{id}$ on $(X_0 \cap N_k) \cup (N_k - \mathring{W}_k)$ and $\overline{H}^{k+1}_t | N_k = \overline{H}^k_t$ ($t \in I$). Therefore, the map F: $M \times I \to M$ defined by $F_t | N_k = \overline{H}^k_t$, $k = 1, 2, \dots,$ is an ambient isotopy that we wanted.

2.b. A result similar to Lemma 2.2 for Q^∞-manifolds also holds true by use of Theorem 3.3 in [**L$_3$**], Proposition 2 [**L$_2$**] and the above version of the Main Theorem in [**L$_2$**].

Consequently, we also have the following.

THEOREM 2.4. *Every locally compact closed subset of a Q^∞-manifold is Q^∞-deficient.* □

The author thanks Dr. M. Hale for reminding him about the question of whether Theorem 2.3 above holds true.

3. Deficiency of closed collared submanifolds. We now detect a special class of R^∞ (or Q^∞)-deficient subsets of R^∞ (or Q^∞)-manifolds, the class of closed collared submanifolds. This is a special case of Theorem 5.1; however, it is an ingredient of the proof of the latter.

LEMMA 3.1. *Let $(N; M_0, M_1)$ be a triad of R^∞-manifolds such that M_0 and M_1 are disjoint and collared in N, and that the inclusion $M_i \to N$ is a homotopy equivalence* (*h.e.*) *for each* $i = 0, 1$. *If* N *has the homotopy type of a finite complex* K, *then*

$$(N; M_0, M_1) \cong (M_0 \times I; M_0 \times \{0\}, M_0 \times \{1\}).$$

PROOF. (1) From [$\mathbf{H_2}$, Theorem 3], there are homeomorphisms μ_0: $K \times R^\infty \cong M_0$ and μ_1: $K \times R^\infty \cong M_1$ such that $\mu_0 \simeq \mu_1$ in N. Let μ: $\mu_0|(K \times \{0\}) \simeq \mu_1|(K \times \{0\})$ be a homotopy. By [$\mathbf{L_1}$, Theorem 2.3], we can assume that μ is an embedding since the compact subspace $\mu(K \times \{0\} \times \partial I)$ is R^∞-deficient in N (Corollary 0). Moreover, by use of collars of $M_0 \cup M_1$ in N, we can assume that $\mu(K \times \{0\} \times (0,1)) \cap (M_0 \cup M_1) = \varnothing$. Now, by use of μ_0, $\mu_1$and μ, we can define an embedding

$$f_0: (K \times R^\infty \times \partial I) \cup (K \times \{0\} \times I) \to N$$

such that $f_0(K \times R^\infty \times \{i\}) = M_i$, $i = 0, 1$.

In the rest of the proof, let $N = \varinjlim N_k$ where $N_k = K \times I^{n_k}$ with $n_1 < n_2 < n_3 < \cdots$.

(2) By Lemma B, we can assume that $f_0(K \times \{0\} \times I) \subset N_1$. We will define a subset Q_1 of N such that $Q_1 \cong N_1$, $f_0(K \times \{0\} \times I) \subset Q_1$, and there is an embedding h_1: $M_0 \cup M_1 \cup Q_1 \to K \times R^\infty \times I$ that extends f_0^{-1} and

$$h_1(Q_1) \cap (K \times R^\infty \times \partial I) = K \times \{0\} \times \partial I^{(*)}.$$

First, by use of pinched collars of $M_0 \cup M_1$ at $f_0(K \times \{0\} \times \partial I)$ in N that miss $f_0(K \times \{0\} \times (0,1))$, we can obtain a subset Q_1 of N homeomorphic to N_1 such that $Q_1 \cap (M_0 \cup M_1) = f_0(K \times \{0\} \times \partial I)$ and $Q_1 \supset f_0(K \times \{0\} \times I)$. Then, since $f_0|(K \times R^\infty \times \{0\})$: $K \times R^\infty \times \{0\} \to N$ is a h.e., it follows that the inclusion $f_0(K \times \{0\} \times I) \subset N_1$ is a h.e.; hence, so is the inclusion $f_0(K \times \{0\} \times I) \subset Q_1$. Moreover, since $(Q_1, f_0(K \times \{0\} \times I))$ is a pair of compact ANR's, it has the absolute homotopy extension property [**Hu**, p. 31]; consequently, $f_0(K \times \{0\} \times I)$ is a strong deformation retract of Q_1 [**Sp**, Corollary 1.4.10]. So, there is a retraction r: $M_0 \cup M_1 \cup Q_1 \to M_0 \cup M_1 \cup f_0(K \times \{0\} \times I)$. Now, from [$\mathbf{L_1}$, Theorem 2.3], there is an embedding $\tilde{h}_1$: $M_0 \cup M_1 \cup Q_1 \to N$ that approximates $f_0^{-1}r$ (rel. $M_0 \cup M_1 \cup f_0(K \times \{0\} \times I)$). Then, by use of a pinched collar of $K \times R^\infty \times \partial I$ at $K \times \{0\} \times \partial I$ in $K \times R^\infty \times I$, we can obtain from $\tilde{h}_1$ an embedding h_1 as we wanted.

(3) For some large integer d_1, let C_1 denote the ball $d_1 B^{d_1}$. We will constsruct an embedding f_1: $(K \times R^\infty \times \partial I) \cup (K \times C_1 \times I) \to N$ which is an extension of h_1^{-1} (also of f_0). Assume the compact ANR subset $h_1(Q_1)$ of $K \times R^\infty \times I$ is contained in $K \times C_1 \times I$. Then, the inclusion $h_1(Q_1) \subset K \times C_1 \times I$ is a h.e. since $C_1 \searrow 0$ and since $K \times \{0\} \times I \subset h_1(Q_1)$ is a h.e. Moreover, since $C_1 \searrow 0$ and since $h_1(Q_1) \cap (K \times C_1 \times \partial I) = K \times \{0\} \times \partial I$ it follows that

$$h_1(Q_1) \cup (K \times C_1 \times \partial I) \searrow h_1(Q_1) \cup (K \times \{0\} \times \partial I) = h_1(Q_1) \quad \text{by } (*).$$

So, the inclusion $h_1(Q_1) \cup (K \times C_1 \times \partial I) \subset K \times C_1 \times I$ is a h.e. As in (2), there is a retraction from $(K \times C_1 \times I) \cup (K \times R^\infty \times \partial I)$ onto $h_1(Q_1) \cup (K \times R^\infty \times \partial I)$, then an embedding

$$f_1: (K \times C_1 \times I) \cup (K \times R^\infty \times \partial I) \to N$$

extending $h_1^{-1}|[h_1(Q_1) \cup (K \times R^\infty \times \partial I)]$.

(4) By Lemma B, without loss of generality, we can assume that $N_1 \cap (M_0 \cup M_1) \subset f_1(K \times C_1 \times \partial I) \subset f_1(K \times C_1 \times I) \subset N_2$. Then, following the constructions of Q_1 and h_1 in (2), we can similarly construct a compact ANR subset Q_2 of N such that

(a) $N_2 \cong Q_2 \supset N_1$,

(b) $Q_2 \cap (M_0 \cup M_1) = f_1(K \times C_1 \times \partial I)$,

(c) $f_1(K \times C_1 \times I) \subset Q_2$ is a h.e., and

(d) there is an embedding h_2: $M_0 \cup M_1 \cup Q_2 \to K \times R^\infty \times I$ extending f_1^{-1} (consequently, h_2 extends h_1) such that $h_2(Q_2) \cap (K \times R^\infty \times \partial I) = K \times C_1 \times \partial I$.

(5) Finally, in a similar manner, we construct inductively $C_2 = d_2 B^{d_2}$ (for a large d_2), f_2, Q_3, $h_3, \ldots$ such that the following diagram is commutative,

$$\begin{tikzcd}
(K \times R^\infty \times \partial I) \cup (K \times 0 \times I) \arrow[r, "f_0"] \arrow[d] & M_0 \cup M_1 \cup Q_1 \arrow[d] \arrow[dl, "h_1"'] \\
(K \times R^\infty \times \partial I) \cup (K \times C_1 \times I) \arrow[r, "f_1"] \arrow[d] & M_0 \cup M_1 \cup Q_2 \arrow[d] \arrow[dl, "h_2"'] \\
(K \times R^\infty \times \partial I) \cup (K \times C_2 \times I) \arrow[r, "f_2"] \arrow[d] & M_0 \cup M_1 \cup Q_3 \arrow[d] \arrow[dl, "h_3"'] \\
\vdots & \vdots
\end{tikzcd}$$

where the vertical maps are inclusions. It is easy to check that $f = \varinjlim f_n$ and $h = \varinjlim h_n$ are inverses of each other. Also, it is clear that

$$\varinjlim \left[(K \times R^\infty \times \partial I) \cup (K \times C_n \times I) \right] = K \times R^\infty \times I \cong M_0 \times I.$$

Moreover, $\varinjlim (M_0 \cup M_1 \cup Q_j) = N$ since $N_j \subset Q_{j+1}$. Therefore, f is a homeomorphism that we desired; and the proof is now complete. □

In the process of proving Theorem 3.3 below, we need the following special case.

LEMMA 3.2. *Let M be a collared R^∞-submanifold of an R^∞-manifold N. If $M \cong K \times R^\infty$, K a finite complex, then M is R^∞-deficient in N.*

PROOF. Let ϕ: $M \times [0, 2) \to N$ be a normal collar of M in N, and let M_t and $M_{[a,b]}$ denote $\phi(M \times t)$ and $\phi(M \times [a, b])$, respectively. From $[\mathbf{H_2}]$ there is a homeomorphism g: $N \times I \to N$ which is so close to the projection p_N that (i) $M \cap g(M_1 \times I) = \varnothing$, and (ii) $g|M \times \{0\}$ is homotopic to the inclusion $M \times \{0\} \subset g(M_{[0,1]} \times I)$ in $g(M_{[0,1]} \times I)$.

Now, it can be shown that the triad $(g(M_{[0,1]} \times I);\ M,\ g(M_1 \times I))$ satisfies Lemma 3.1 (recall that $g(M_1 \times I)$ is bicollared in N); so, it is homeomorphic to the triad $(M \times [0, 1];\ M \times \{0\},\ M \times \{1\})$. Therefore, $g(M_{[0,1]} \times I)$ is a closed collar of M in N whose frontier is $g(M_1 \times I)$. Then, by use of a closed collar of $g(M_1 \times I)$ in $N - g(M_{[0,1]} \times I)$, we can construct a homeomorphism h: $N \to N - g(M_{[0,1)} \times I)$ such that

$$h(M) = g(M_1 \times I).$$

Moreover, the latter is R^∞-deficient in $N - g(M_{[0,1)} \times I)$ since $M_1 \times I$ is R^∞-deficient in $(N - M_{[0,1)}) \times I$ by Lemma 4.4 in [$\mathbf{L_1}$]. Consequently, M is R^∞-deficient in N. □

THEOREM 3.3. *Let M be a closed R^∞-submanifold of an R^∞-manifold N. Then, M is R^∞-deficient in N if and only if M is collared in N.*

PROOF. By Theorem 4.2 in [$\mathbf{L_1}$], we only have to show the "if" part. Let K be a countable locally finite simplicial complex such that $M \cong K \times R^\infty$, and let ϕ: $K \to [0, \infty)$ be a PL proper map. Then, $K = \bigcup_{n=0}^\infty K_n$ where $K_n = \phi^{-1}([n, n+1])$. It is clear that $K_n \cap K_m = \phi$ if $|n - m| \geqslant 2$, and each K_n is a finite complex. Moreover, $M^e = \bigcup\{K_n \times R^\infty | n \text{ is even}\}$, $M^o = \bigcup\{K_n \times R^\infty | n \text{ is odd}\}$ are discrete families of closed subsets of M. Therefore, by Lemma 3.2, Lemma 2.2, and Proposition 1.4, the theorem will follow if $K_n \times R^\infty$ is collared in N for each $n = 1, 2, \ldots$.

Fix an n, and let $\varepsilon > 0$ be so small that K_n is a strong deformation retract of $N_\varepsilon(K_n)$; in particular, the inclusion $K_i \subset N_\varepsilon(K_i)$ is a h.e. Now, from Addendum to Lemma 4.1 in [$\mathbf{L_1}$], $K_n \times R^\infty \times \{0\}$ is R^∞-deficient in both $K_n \times R^\infty \times [0,1)$ and $N_\varepsilon(K_n) \times R^\infty \times [0,1)$; hence, the inclusion $K_n \times R^\infty \times [0,1) \subset N_\varepsilon(K_n) \times R^\infty \times [0,1)$ is homotopic to a homeomorphism h (by [$\mathbf{H_2}$, Theorem 3]) such that $h(x,0) = (x,0)$ for all $x \in K_n \times R^\infty$ (by Theorem 2.1 above). Finally, the open embedding ϕh: $K_n \times R^\infty \times [0,1) \to N$, where ϕ is a given collar on M in N, will be a collar on $K_n \times R^\infty$ in N as we desired; and the proof of the theorem is now complete. □

REMARKS. 3.a. In the proof of Lemma 3.1, if we substitute the cell C_n by $Q^n = Q_1 \times \cdots \times Q_n$, then we obtain a result similar to Lemma 3.1 for Q^∞-manifolds.

3.b. In the proof of Lemma 3.2, we use nothing but Lemma 3.1, Lemma 4.4 in [$\mathbf{L_1}$], the projection p_N: $N \times I \to N$ being a near homeomorphism, and the unknotting theorem. Recall that the Q^∞-manifold version of Lemma 4.4 in [$\mathbf{L_1}$] is Lemma 2.6 in [$\mathbf{L_3}$]. Moreover, for Q^∞-manifolds, results similar to the last two theorems have been proved in [$\mathbf{H_3}$ and $\mathbf{L_2}$]. Therefore, a result similar to Lemma 3.2 for Q^∞-manifolds also holds true; and we can prove the following.

THEOREM 3.4. *Let M be a closed Q^∞-submanifold of a Q^∞-manifold N. Then, M is Q^∞-deficient in N if and only if M is collared in N.*

PROOF. Similar to the proof of Theorem 3.3, we will use the Q^∞-manifold version of Lemma 2.2 (see Remark 2.b) to prove the "if" part. The "only if" part is Theorem 2.3 in [$\mathbf{L_3}$]. □

4. Some primary properties of $R^\infty(Q^\infty)$-deficient subsets. Given a subset A of a topological space Y, let $\mathrm{Fr}(A, Y)$ denote the topological frontier of A in Y, and $\bar{A}$ the closure of A in Y. A closed subset Z of a paracompact space M is said to be *clean* in M if there is a normal bicollar ϕ: $\mathrm{Fr}(Z, M) \times (-2, 2) \to M$. In the following, we will use a closed bicollar $\phi|(\mathrm{Fr}(Z, M) \times [-1, 1])$ with $\phi(\mathrm{Fr}(Z, M) \times [-1, 0]) \subset Z$ when we say Z is clean in M.

We now begin this section by proving a technical lemma.

LEMMA 4.1. *Let X be a proper closed subset of a connected R^∞-manifold M. Given an open neighborhood V of X in M, then X has a clean R^∞-manifold neighborhood W in V (also in M) whose frontier is an R^∞-manifold.*

PROOF. As in §1, from Theorem 2 in [**H**$_2$], we can think of M as an open subset of R^∞. Consequently, M is the union of a sequence of compact PL manifolds $\{M_n\}$ such that, for each n,

(a) M_n is a compact PL submanifold for $R^{d_n} \cap M$ where d_n is defined to be $2^{n+2} - 2$,

(b) $\dim M_n = d_n$, and $M_n \subset \operatorname{Int} M_{n+1}$.

For each n, $X_n = X \cap M_n$ is a compact subset of the d_n-manifold $V_n = V \cap M_n$, an open subset of M_n. Observe that $V_n \subset \operatorname{Int} V_{n+1}$. We will construct inductively a sequence of compact PL manifolds $\{W_n\}$ enjoying the following properties:

(1) $X_n \subset W_n \subset W_{n+1}$,

(2) W_n is a clean PL submanifold of V_n with a bicollar ϕ_n: $\operatorname{Fr}(W_n, V_n) \times (-2, 2) \to V_n$,

(3) $W_{n+1} \cap V_n = W_n$, and

(4) ϕ_{n+1} is an extension ϕ_n.

Then, $W = \varinjlim W_n$ will be a clean neighborhood of X with a bicollar $\phi = \varinjlim \phi_n$ that we desired.

First, to construct W_1, ϕ_1, let us assume $X_1 \neq V_1$. Let W_1' be a PL-manifold neighborhood of X_1 in V_1, and let W_1 be a second derived neighborhood of W_1' in V_1 ([**Hd**, p. 57 or **R**, p. 23]) which meets ∂V_1 regularily [**R**, Theorem 1.6.5(1), p. 24]. Hence, there is a PL bicollar ϕ_1 on $\operatorname{Fr}(W_1, V_1)$,

$$\phi_1 \colon \operatorname{Fr}(W_1, V_1) \times [-1, 1] \to V_1,$$

such that $\operatorname{Im}(\phi_1) \cap X = \varnothing$ (since $X \subset W_1'$).

Second, to construct W_2, let W_2' be a compact-PL-submanifold neighborhood of X_2 in V_2 such that $W_2' \cap \operatorname{Im}(\phi_1) = \varnothing$ and $W_2' \cap V_1 \subset W_1$. Let W_2 be a second-derived neighborhood of $W_2' \cup W_1(\operatorname{mod} \operatorname{Fr}(W_1, V_1))$ in V_2 such that $W_2 \cap V_1 = W_1$. Observe that $W_2' \cup W_1$ is link-collapsible on $\operatorname{Fr}(W_1, V_1)$ [**R**, Example 1.6.3(c), p. 22], and $\operatorname{Fr}(W_1, V_1) \cap \partial V_2 = \varnothing$ since $V_1 \subset \operatorname{Int} V_2$. Therefore, W_2 is a compact PL submanifold of V_2 meeting ∂V_2 regularly [**R**, Theorem 1.6.5(1), p. 24]. Hence, $\operatorname{Fr}(W_2, V_2)$ is bicollared in V_2, say a bicollar ψ_2: $\operatorname{Fr}(W_2, V_2) \times [-1, 1] \to V_2$ such that $\operatorname{Im}(\psi_2) \cap X = \varnothing$. Now, observe that

$$\begin{aligned} \phi_1 \psi_2^{-1} | \psi_2(\operatorname{Fr}(W_1, V_1) \times [0,1]) \colon \psi_2(\operatorname{Fr}(W_1, V_1) \times [0,1]) \\ \to \phi_1(\operatorname{Fr}(W_1, V_1) \times [0,1]) \end{aligned}$$

is a PL homeomorphism, which is homotopic (rel. $\operatorname{Fr}(W_1, V_1)$) in $\overline{V_2 - W_2}$ to the inclusion

$$\psi_2(\operatorname{Fr}(W_1, V_1) \times [0,1]) \subset \overline{V_2 - W_2}.$$

Moreover, since

$$\psi_2(\operatorname{Fr}(W_1, V_1) \times (0,1]) \cup \phi_1(\operatorname{Fr}(W_1, V_1) \times (0,1]) \subset \operatorname{Int}(V_2 - W_2)$$

and

$$2\dim(\mathrm{Fr}(W_1, V_1) \times [0,1]) + 2 = \dim \overline{V_2 - W_2},$$

there is a PL homeomorphism $h_{2,1}$ of $\overline{V_2 - W_2}$ fixing $\partial(\overline{V_2 - W_2})$ such that

$$h_{2,1}|\psi_2(\mathrm{Fr}(W_1, V_1) \times [0,1]) = \phi_1\psi_2^{-1}|\psi_2(\mathrm{Fr}(W_1, V_1) \times [0,1]),$$

from Theorem 10.1 in **[Hd]** (Theorem 9.8 will play the role of Theorem 9.2 in the proof of Theorem 10.1). Similarly, there is a PL homeomorphism $h_{2,-1}$ of W_2 fixing $\mathrm{Fr}(W_2, V_2)$ such that

$$h_{2,-1}\phi_2|\mathrm{Fr}(W_1, V_1) \times [-1,0] = \phi_1|\mathrm{Fr}(W_1, V_1) \times [-1,0].$$

Combining $h_{2,1}$ and $h_{2,-1}$, we obtain a homeomorphism h_2 of V_2 such that $\phi_2 = h_2\psi_2$ is a PL bicollar extending ϕ_1, i.e. the following diagram is commutative,

$$\begin{array}{ccc}
\mathrm{Fr}(W_1, V_1) \times [-1,1] & \xrightarrow{\phi_1} & V_1 \\
\downarrow i & & \downarrow i \\
\mathrm{Fr}(W_2, V_2) \times [-1,1] & \xrightarrow{\phi_2} & V_2.
\end{array}$$

Finally, in a similar manner, we can continue to construct inductively $W_3, \phi_3, \ldots$ to extend the above commutative diagram. Then, by (1) and (3), observe that $V_k - \bigcup_1^\infty W_n = V_k - W_k$ for each k. So,

$$V - \bigcup_1^\infty W_n = \bigcup_1^\infty \left(V_k - \bigcup_1^\infty W_n \right) = \bigcup_1^\infty (V_k - W_k).$$

Hence,

$$\overline{V - \bigcup_1^\infty W_n} = \overline{\bigcup_1^\infty (V_n - W_n)} = \bigcup_1^\infty \overline{V_n - W_n}$$

since the latter is closed in V. Therefore, $\bigcup_1^\infty \mathrm{Fr}(W_n, V_n) = \mathrm{Fr}(W, V)(= \mathrm{Fr}(W, M)$ since $W = \bigcup_1^\infty W_n$ is closed in M). By (a), (3) and Lemma A, we observe that $\mathrm{Fr}(W, V)$ is an R^∞-manifold. Now, if we define $\phi\colon \mathrm{Fr}\, W \times]-1,1[\to V$ by $\phi(x,t) = \phi_n(x,t)$ if $x \in \mathrm{Fr}(W_n, V_n)$, then ϕ is an open embedding which defines a bicollar that we desired. □

To prove a result similar to Lemma 4.1 for Q^∞-manifolds, we need the following lemma.

LEMMA 4.2. *Let V_1 and X be compact subsets of a Q-manifold V_2. Assume that V_1 is a Q-manifold Z-set in V_2. If W_1 is a clean neighborhood of $X \cap V_1$ in V_1, then there is a clean neighborhood W_2 of X in V_2 such that $W_2 \cap V_1 = W_1$.*

PROOF. Since $\overline{V_1 - W_1}$ is a compact Q-manifold Z-set in V_2, it is collared in V_2. Let U_1 be a pinched collar (refer to **[R** or **L₂]**) on $\overline{V_1 - W_1}$ at $\mathrm{Fr}(W_1, V_1)$ in V_2 such that $U_1 \cap X = \varnothing$. Then, we can show that U_1 is a Q-manifold homeomorphic to $\overline{V_1 - W_1}$, and $\mathrm{Fr}(U_1, V_2) \cong (V_1 - W_1) \cup \mathrm{Fr}(W_1, V_1) = \overline{V_1 - W_1}$ since $V_1 - W_1$ is open in V_1. By use of Theorem 3.1(3) in **[Ch]**, we can show that $\mathrm{Fr}(U_1, V_2)$ is a Z-set in both U_1 and $\overline{V_2 - U_1}$. Therefore, $\mathrm{Fr}(U_1, V_2)$ is bicollared in V_2; consequently, $W_2 = \overline{V_2 - U_1}$ is a clean neighborhood of X in V_2 that we wanted. □

LEMMA 4.3. *Let X be a proper closed subset of a Q^∞-manifold M. Given an open neighborhood V of X in M, then X has a clean Q^∞-manifold neighborhood W in V (also in M) whose frontier is a Q^∞-manifold.*

PROOF. We will go along with the proof of Lemma 4.1 and use similar notations. Now, M_n is a Q-manifold Z-subset of M_{n+1} as in Lemma A in [**L₂**]. In $V \cap M_n$, we take a compact Q-manifold V_n (inductively) such that $X_n \cup V_{n-1} \subset V_n$ and $\bigcup_1^\infty V_n = V$ (assume $V_0 = \varnothing$). Then, the existence of clean neighborhoods $\{W_n\}$ in the inductive construction is from Lemma 4.2, and matching up these bicollars will follow from the relative version of the unknotting theorem for Z-embeddings [**Ch**, Theorem 19.4]. The rest of the proof is the same. □

Following are a few lemmas that we need in §5.

LEMMA 4.4. *Let X be an R^∞-deficient subset of an R^∞-manifold V and let W be an R^∞-manifold neighborhood of X in V, then X is R^∞-deficient in $\mathring{W}$ (the interior of W in V) and W.*

PROOF. Let $g\colon V \to V \times [0,1]$ be a homeomorphism such that $g(x) = (x, 0)$ if $x \in X$. Let $U = g^{-1}(V \times \{0\})$, then $\phi = g^{-1}((g|U) \times \mathrm{id}_I)\colon U \times I \to V$ is a homeomorphism. For convenience, let U also denote $U \times \{0\}$. Define $W' = \phi^{-1}(W)$ and $\mathring{W}' = \phi^{-1}(\mathring{W})$.

Observe that $U \cap \mathring{W}'$ is a neighborhood of X in U, then let Z be a clean neighborhood of X in $U \cap \mathring{W}'$ with a bicollar $\lambda\colon \mathrm{Fr}(Z, U \cap \mathring{W}') \times [-1,1] \to U \cap \mathring{W}'$ from Lemma 4.1. Since X is a closed subset of Z, the proof of Lemma 4.4 will be complete by Theorem 3.3 if we can show that Z is collared in $\mathring{W}'$. Let $\psi\colon U \cap \mathring{W}' \to (0,1)$ be a map such that

$$\bigcup\{\{x\} \times [0, \psi(x)] \,|\, x \in U \cap \mathring{W}'\} \subset \mathring{W}'.$$

Let

$$Z' = Z \cup \lambda(\mathrm{Fr}(Z, U \cap \mathring{W}') \times [0,1)) \quad \text{and} \quad \Omega = \{\{z\} \times [0, \psi(z)) \,|\, z \in Z'\}.$$

Then, Ω is an open subset of $\mathring{W}'$ and there is a natural homeomorphism $\omega\colon Z' \times [0,1) \to \Omega$. Now, by use of a homeomorphism from $[-1,0] \times [0,1)$ onto $[-1,1) \times [0,1)$ fixing $([-1,0] \times \{0\}) \cup (\{-1\} \times [0,1))$, and the bicollar λ, we can construct a homeomorphism $\theta\colon Z \times [0,1) \to Z' \times [0,1)$ such that $\theta(x, 0) = (x, 0)$, $x \in Z$, then an open embedding $h = \omega\theta\colon Z \times [0,1) \to \Omega \subset \mathring{W}'$ such that $h(z, 0) = z$ for each $z \in Z$. Therefore, h is a collar on Z in $\mathring{W}'$ and W'. The proof of the lemma is now complete. □

Using Lemma 4.3 and Theorem 3.4, we can similarly prove the following.

LEMMA 4.5. *Let X be a Q^∞-deficient subset of a Q^∞-manifold V and let W be a Q^∞-manifold neighborhood of X in V, then X is Q^∞-deficient in both W and $\mathring{W}$.* □

LEMMA 4.6. *Let X be a closed subset of an R^∞-manifold M. If X has an open neighborhood V in M such that X is R^∞-deficient in V, then X is R^∞-deficient in M.*

PROOF. By use of a clean neighborhood W of X in V and in M, from Lemma 4.1, we will construct a homeomorphism $f\colon M \times I \to M$ such that $f(x,0) = x$ for all $x \in X$. Recall, that by Lemma 4.1, the R^∞-manifold $\mathrm{Fr}(W, M)$ is bicollared in M, say by ϕ. For the sake of simplicity, let $\mathrm{Fr}\, W$ and $\mathring{W}$ denote the topological frontier and the interior of W in M, respectively. Then, $\mathrm{Fr}\, W$ is an R^∞-deficient subset of both W and $M - \mathring{W}$ by Theorem 3.3 and so is $\mathrm{Fr}\, W \times I$ in $W \times I$ and $(M - \mathring{W}) \times I$.

We will, first, obtain special homeomorphisms $h\colon W \times I \to W$ and $h_0\colon (M - \mathring{W}) \times I \to M - \mathring{W}$, near the corresponding projections, such that $h|(\mathrm{Fr}\, W \times I) = h_0|(\mathrm{Fr}\, W \times I)$. For the existence of h, using the stability theorem in $[\mathbf{H}_2]$, we have homeomorphisms $h\colon W \times I \to W$ and $h'\colon \mathrm{Fr}\, W \times I \to \mathrm{Fr}\, W$, all of which are so near the projection that we can assume that $h|(\mathrm{Fr}\, W \times I)$ and h' are homotopic through a homotopy missing X. By use of the unknotting theorem (Theorem 2.1), we can assume that $h|(\mathrm{Fr}\, W \times I) = h'$ (so, $h(\mathrm{Fr}\, W \times I) = \mathrm{Fr}\, W$). Similarly, we can have a homeomorphism $\bar{h}_0\colon (M - \mathring{W}) \times I \to M - \mathring{W}$, near the projection, such that $\bar{h}_0(\mathrm{Fr}\, W \times I) = \mathrm{Fr}\, W$. Furthermore, we can assume that $h\bar{h}_0^{-1}|\mathrm{Fr}\, W\colon \mathrm{Fr}\, W \to \mathrm{Fr}\, W$ is homotopic to the identity. By Theorem 2.1 again, we have a homeomorphism k of $M - \mathring{W}$ which extends $h\bar{h}_0^{-1}|\mathrm{Fr}\, W$. Define $h_0 = k\bar{h}_0$. Then, h and h_0 will agree on $\mathrm{Fr}\, W \times I$.

Moreover, we can assume that $h(x,0) = x$ for all $x \in X$ as follows. Since we can choose h to be as close to the projection as we wish, we assume that $h|X \times \{0\}$ is homotopic to the inclusion $i\colon X \times \{0\} \to W \times \{0\}$ in $W \times \{0\} \equiv W$. Furthermore, without loss of generality, we also can assume that both X and $h(X \times \{0\})$ miss the collar $\phi(\mathrm{Fr}\, W \times [-1,0])$ in W, and so does the homotopy from $h|X \times \{0\}$ to i; i.e. the homotopy is limited by a family $\mathscr{W}$ of open sets of $\mathring{W}$ missing $\phi(\mathrm{Fr}\, W \times [-1,0])$. Observe that $h(X \times \{0\})$ is R^∞-deficient in W since $X \times \{0\}$ is R^∞-deficient in $W \times I$. On the other hand, X is R^∞-deficient in W by Lemma 4.4 above. Again, by the unknotting theorem (Theorem 2.1), there is a homeomorphism g of W such that (1) $gh(x,0) = x$ if $x \in X$, and (2) $g(x) = x$ if $x \notin \bigcup\{\omega | \omega \in \mathscr{W}\}$. Let $\bar{g}$ denote the extension of g over W by the identity. Then, $\bar{g}h\colon W \times I \to W$ is a homeomorphism such that

(1) $\bar{g}h(x,0) = x$ if $x \in X$, and

(2) $\bar{g}h(x,t) = h_0(x,t)$ if $x \in \mathrm{Fr}\, W$, $t \in I$.

Now, if we define $f\colon M \times I \to M$ by

$$f(x,t) = \begin{cases} \bar{g}h(x,t) & \text{if } x \in W, \\ h_0(x,t) & \text{if } x \in M - \mathring{W}, \end{cases}$$

then f is a homeomorphism (near the projection p_M) such that $f(x,0) = x$ for $x \in X$. Therefore, X is R^∞-deficient in M, and the proof is complete. □

Finally, if we use Lemma 4.3, Theorem 3.4 above, and the stronger version of the unknotting theorem in $[\mathbf{L}_2]$ as observed in Remark 2.a, we can prove a similar result for Q^∞-manifolds.

LEMMA 4.7. *Let X be a closed subset of a Q^∞-manifold M. If X has an open neighborhood V such that X is Q^∞-deficient in V, then X is Q^∞-deficient in M.* □

5. Local $R^\infty(Q^\infty)$-deficiency. A closed subset X of an R^∞-manifold M is said to be *locally R^∞-deficient* if each $x \in X$ has an open neighborhood V in M such that $X \cap V$ in R^∞-deficient in V. It is clear that an R^∞-deficient subset is locally R^∞-deficient. We define similarly the notion of local Q^∞-deficiency, and obtain a similar observation in Q^∞-manifolds.

THEOREM 5.1. *A locally R^∞-deficient subset X of an R^∞-manifold M is R^∞-deficient.*

To have the proof to be more readable, we include an intermediate lemma. For the notion of nbd-finite family, we refer to [**D**].

LEMMA 5.2. *There is a countable nbd-finite family $\{W_s\}$ of open subsets of M covering X and a countable nbd-finite family $\{X_s\}$ of closed subsets of X covering X such that*

(1) *$X_s \subset W_s$, and*

(2) *X_s is R^∞-deficient in M.*

PROOF. Let $\{V_a | a \in A\}$ be a family of open subsets of M covering X such that $X \cap V_a$ is R^∞-deficient in V_a. Since X is closed and M is paracompact, there is a nbd-finite family of open sets $\{W_s\}$ of M covering X which is a refinement of $\{V_a\}$. Then, since X is Lindelöf [**D**, p. 175], we can assume that $\{W_s\}$ is countable. Next, since X is paracompact [**D**, p. 165], there is a nbd-finite open covering $\{X'_s\}$ of X such that the closure X_s of X'_s in X is a subset of $X \cap W_s$ for each s. Finally, since X_s is contained in $X \cap V_a$ for some a, X_s is R^∞-deficient in V_a for some a; so, it is R^∞-deficient in M by Lemma 4.6. The proof of the lemma is complete. □

PROOF OF THEOREM 5.1. We will construct a closed embedding $f\colon Z \to M$ extending the inclusion $X \subset M$, where Z is a closed R^∞-manifold neighborhood of X in M such that $f(Z)$ is collared in M. Then, the theorem will follow from Theorem 3.3. Let us consider a countable family $\{(W_s, X_s)\}$ from the above lemma, and for each s let V_{a_s} denote a member of $\{V_a\}$ containing W_s.

First, let W'_1 be a clean neighborhood of X_1 (in M) contained in W_1 (see Lemma 4.1), and let U_1 be a clean neighborhood of X_1 (in M) contained in $\mathring{W}'_1$. Observe that $X \cap W'_1$ is R^∞-deficient in V_{a_1}; so, it is R^∞-deficient in M by Lemma 4.6. From the relative approximation theorem [$\mathbf{L_1}$, Theorem 2.3], there is an R^∞-deficient embedding $f'_1\colon U_1 \cup (X \cap W'_1) \to M$ enjoying the following properties:

(1) f'_1 is so close to the inclusion that $f'_1(U_1) \subset \mathring{W}'_1$, and

(2) $f'_1(x) = x$ if $x \in X \cap W'_1$.

Extend f'_1 by identity to a closed embedding $f_1\colon X \cup U_1 \to M$.

Next, let W'_2 be a clean neighborhood of X_2 (in M) contained in W_2, then $f_1^{-1}(\mathring{W}'_2)$ is an open subset of $X \cup U_1$. Let Y be an open subset of M such that $f_1^{-1}(\mathring{W}'_2) = Y \cap (X \cup U_1)$, then Y is also a neighborhood of X_2. Let U_2 be a clean neighborhood of X_2 (in M) contained in $Y \cap \mathring{W}'_2$ such that f_1 can extend to a map $\bar{f}_1\colon X \cup U_1 \cup U_2 \to M$ with $\bar{f}_1(U_2) \subset \mathring{W}'_2$ (Proposition III.3 of [$\mathbf{H_1}$]), and let X'_2 denote $(X \cap W'_2) \cup (U_2 \cap U_1)$. Observe that $f_1 | X'_2\colon X'_2 \to M$ is an R^∞-deficient embedding by Proposition 1.4, and $f_1(U_2 \cap U_1) \subset \mathring{W}'_2$. Again, by Theorem 2.3 in [$\mathbf{L_1}$] (rel. X'_2), there is an R^∞-deficient embedding $f'_2\colon U_2 \cup (X \cap W'_2) \to M$ such that

(1) $f'_2(x) = f_1(x)$ if $x \in X'_2$, and

(2) f_2' is so close to $\bar{f}_1$ that $f_2'(U_2) \subset \mathring{W}_2'$.
Then, define f_2: $X \cup U_1 \cup U_2 \to M$ by

$$f_2(x) = \begin{cases} f_2'(x) & \text{if } x \in U_2 \cup (X \cap W_2'), \\ f_1(x) & \text{otherwise.} \end{cases}$$

Observe that $f_2|U_1 \cup U_2 \cup [X \cap (W_1' \cup W_2')]$ is an R^∞-deficient embedding (Proposition 1.4).

Then, let us construct f_n after f_{n-1} has been defined with $f_{n-1}|$: $(\bigcup_1^{n-1} U_s) \cup [X \cap (\bigcup_1^{n-1} W_s')] \to M$ being an R^∞-deficient embedding. Let W_n' be a clean neighborhood of X_n in W_n as above. Then, $f_{n-1}^{-1}(\mathring{W}_n')$ is an open neighborhood of X_n in $X \cup U_1 \cup \cdots \cup U_{n-1}$, the domain of f_{n-1}. Let Y be an open neighborhood of X_n in M whose intersection with the domain of f_{n-1} is $f_{n-1}^{-1}(\mathring{W}_n')$, and let U_n be a small clean neighborhood of X_n contained in $Y \cap \mathring{W}_n'$. Then, $f_{n-1}(U_n \cap U_s) \subset \mathring{W}_n'$ for each $s = 1, \ldots, n-1$. In a similar manner, we can obtain a closed embedding f_n: $X \cup U_1 \cup \cdots \cup U_n \to M$ extending f_{n-1} such that f_n is R^∞-deficient on $[X \cap (W_1' \cup \cdots \cup W_n')] \cup (U_1 \cup \cdots \cup U_n)$ and $f_n(U_n) \subset \mathring{W}_n'$. Therefore, the inductive construction is complete.

Now, let us define $U = \bigcup_1^\infty U_s$ and f: $U \to M$ by $f(x) = f_s(x)$ if $x \in U_s$. Then, f is a well-defined and continuous injection since the family $\{U_s\}$ is nbd-finite [**D**, p. 83]. Moreover, since the family $\{f(U_s) | s = 1, 2, \ldots\}$ of closed subsets of M, a refinement of $\{W_s\}$, is also nbd-finite, $f(U) = \bigcup_1^\infty f(U_s)$ is closed in M [**D**, p. 82]; and f^{-1}: $f(U) \to U$, which is continuous on each $f(U_s)$, is continuous. Therefore, f is a closed embedding.

Now, it is clear that U is a neighborhood of X in M. We will show that $f(\mathring{U})$ is locally collared in M. For a given $x \in \mathring{U}$ has a clean neighborhood ω_x in $\mathring{U}$ meeting only finitely many U_s's, say $U_{s_1}, \ldots, U_{s_m}$. So, being contained in $f(U_{s_1} \cup \cdots \cup U_{s_m})$, $f(\omega_x)$ is R^∞-deficient; hence, it is collared in M [$\mathbf{L}_1$, Theorem 4.2]; therefore, $f(\mathring{U})$ is locally collared in M.

Let A be an open subset of M such that $A \cap f(U) = f(\mathring{U})(f(\mathring{U})$ being a relatively open set in the subspace $f(U))$. Then, $f(\mathring{U})$ is relatively closed in A since $f(U)$ is closed in M. Therefore, $f(\mathring{U})$ is collared in A [**Br**, Theorem 4.3, p. 228] since $f(\mathring{U})$ is also locally collared in A. Now, let Z be a clean neighborhood of X in $\mathring{U}$. Then, $f(Z)$ is collared in A (consequently, in M) as in the last part of the proof of Lemma 4.4. So, it follows from Theorem 3.3 that $f(Z)$ is R^∞-deficient in M; therefore, so is $X = f(X)$ in M. The proof now is complete. □

We conclude this note with the following theorem in the Q^∞-manifold theory.

THEOREM 5.3. *A locally Q^∞-deficient subset X of a Q^∞-manifold M is Q^∞-deficient.*

PROOF. The proof is similar to that of Theorem 5.1. Here, we use the paracompactness of X and M, Lemmas 4.7 and 4.3, Theorem 3.4, and the relative approximation theorem and Proposition 2 in [$\mathbf{L}_3$]. □

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