DECOMPOSITIONS INTO CODIMENSION-TWO MANIFOLDS

BY

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ABSTRACT. Let M denote an orientable $(n + 2)$ -manifold and let G denote an upper semicontinuous decomposition of M into continua having the shape of closed, orientable n -manifolds. The main result establishes that the decomposition space M/G is a 2-manifold.

1. Introduction. Let M denote an orientable $(n + 2)$ -manifold and let G denote an upper semicontinuous decomposition of M into continua having the shape of closed, orientable n -manifolds. The main result here establishes that the decomposition space M/G is a 2-manifold.

This theorem lifts what is known about decompositions into codimension-two submanifolds to nearly the same level as what is known about decompositions into codimension-one submanifolds. Liem [L] proved that if G is a decomposition of an $(n + 1)$ -manifold N into compacta having the shape of S^n , then N/G is a 1-manifold, and for $n > 4$ the decomposition map $p: N \rightarrow N/G$ can be approximated by a locally trivial bundle map. Daverman [D] showed that if G is a decomposition of N into continua having the shape of arbitrary closed n -manifolds, then N/G is a 1-manifold (possibly with boundary if various orientability conditions are not all met).

In an earlier paper [DW1] we derived the conclusion that M/G is a 2-manifold in case M is an $(n + 2)$ -manifold and each g in G has the shape of S^n . Then the natural map $p: M \rightarrow M/G$ must be particularly nice, being an approximate fibration when $n > 1$ (the same is "almost" true when $n = 1$). The present work evolved from techniques similar to those of [DW1], which were inspired for the most part by methods of Coram and Duvall [CD1, CD2]. However, the strong conclusion that $p: M \rightarrow M/G$ is an approximate fibration cannot be attained in the more general situation at hand.

In the fourth section we determine the structure of a certain sheaf $\mathcal{H}^n$ on M/G induced by p . The properties of this sheaf can be used to recover the conclusions in [DW1] just mentioned, in the special situation where the elements of G have the shape of a n -sphere.

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The manifold features present in such decomposition spaces M/G do not persist through successively larger codimensions. If G is a decomposition of a manifold M into codimension-three submanifolds, M/G need not be a manifold nor need it even be a generalized manifold. It seems that the fundamental question to address is whether M/G , if finite dimensional, is an ANR.

It should be pointed out that, in our terminology, a *manifold* is separable, metric and boundaryless. We will speak of a *manifold with boundary* as the need to permit a boundary occurs.

The symbol $\approx$ will be used to mean “is homeomorphic to” and the symbol $\cong$ will be used to mean “is isomorphic to.” Unless expressly stated otherwise, homology and cohomology groups are determined with integral ($\mathbf{Z}$) coefficients.

2. The winding function and its continuity set. Throughout the remainder of the paper we shall employ the following notation: M for an orientable $(n + 2)$ -manifold, $n \geq 1$, G for a usc (the standard abbreviation of “upper semicontinuous”) decomposition of M into compact connected elements each having the shape of a closed orientable n -manifold, B for the decomposition space M/G , and p for the decomposition map $M \rightarrow B$.

We begin with a variation on the Coram-Duvall definition of local winding number function. Temporarily fix $b_0 \in B$. Since $g_0 = p^{-1}b_0$ is movable, there exist connected neighborhoods U and U_0 of b_0 in B such that $b_0 \in U_0 \subset U$ and (the inclusion induced)

$$\phi_0: \check{H}_n(p^{-1}b_0) \rightarrow H_n(p^{-1}U)$$

is an isomorphism onto the image of

$$\phi: H_n(p^{-1}U_0) \rightarrow H_n(p^{-1}U).$$

For any $b \in U_0$, the image of

$$\phi_b: \check{H}_n(p^{-1}b) \rightarrow H_n(p^{-1}U)$$

is contained in $\text{Im } \phi$. Hence,

$$\phi_0^{-1}\phi_b: \check{H}_n(p^{-1}b) \rightarrow \check{H}_n(p^{-1}b_0)$$

is a well-defined homomorphism between copies of $\mathbf{Z}$, meaning that it amounts to multiplication by some integer $q_b \geq 0$. Define the winding function $\alpha: U_0 \rightarrow \mathbf{Z}$ by $\alpha(b) = q_b$.

Let K denote the set of all points b_0 in B such that every neighborhood W of b_0 (with $W \subset U_0$) contains some point w for which $\alpha(w) = 0$. As in previous analyses in [CD1, CD2, DW1], the set K is a troublesome spot. Clearly, K is closed in B .

LEMMA 2.1. *K is nowhere dense in B .*

PROOF. See [CD2, Lemma 3.1].

Define B_+ as $B - K$. In the rest of this section we outline an argument showing that the complement in B_+ of a certain countable closed subset is a 2-manifold. Although operating under less limiting hypotheses, we are able to extract enough from arguments in [CD1 and DW1, §3] to satisfy our needs.

We shall continue to work locally throughout most of this section. Assume that $b_0 \in B$ mentioned near the beginning lies in B_+ and that the neighborhood U_0 mentioned there satisfies $U_0 \subset B_+$ and

$$\operatorname{Im}\{\check{H}_1(p^{-1}b_0) \rightarrow H_1(p^{-1}U)\} = \operatorname{Im}\{H_1(p^{-1}U_0) \rightarrow H_1(p^{-1}U)\}$$

as well. The three facts stated below are direct analogues of Lemmas 1–3 in [CD1].

(1) If $b \in U_0$, then there is a neighborhood V of b in U_0 such that to each $b' \in V$ there corresponds a positive integer k such that $\alpha(b') = k\alpha(b)$.

(2) α is lower semicontinuous.

(3) The continuity set $C = \{b \in U_0; \alpha \text{ is continuous at } b\}$ is open and dense in U_0 ; its complement $D = U_0 - C$ can be written as $D_1 \cup D_2$ where D_2 is countable, D_1 is closed, and each neighborhood of $d \in D_1$ contains uncountably many points of D_1 .

(Here $D_2 = \{d \in D; \text{some neighborhood of } d \text{ in } D \text{ is countable}\}$, and $D_1 = D - D_2$.)

LEMMA 2.2. *Let A be an arc in C . Then, using either $\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z}$ -coefficients,*

- (i) $\check{H}^{n+1}(p^{-1}A) \cong 0$;
- (ii) for $x \in A$, $i^*: \check{H}^n(p^{-1}A) \rightarrow \check{H}^n(p^{-1}x)$ is surjective; and
- (iii) for $x, y \in A$, $\ker\{\check{H}^n(p^{-1}A) \rightarrow \check{H}^n(p^{-1}x)\} = \ker\{\check{H}^n(p^{-1}A) \rightarrow \check{H}^n(p^{-1}y)\}$.

PROOF. We assume the coefficient group is $\mathbf{Z}$, the argument being the same for other coefficient groups. For each $x \in A$, there is a closed subarc A_x , a neighborhood of x in A , for which there is a shape strong deformation retraction of $p^{-1}A_x$ to $p^{-1}x$, the deformation occurring in a neighborhood of $p^{-1}x$. Consequently, the diagram

$$\begin{array}{ccc} & \check{H}^n(p^{-1}A_x) & \\ i^* \swarrow & & \searrow i^* \\ \check{H}^n(p^{-1}y) & \begin{array}{c} \xleftarrow{\cong} \\ \psi \end{array} & \check{H}^n(p^{-1}x) \end{array}$$

commutes for each $y \in A_x$, where ψ is induced by the retraction and is an isomorphism as we are working in the continuity set. It follows easily that the arc A_x satisfies conclusions (ii) and (iii).

If a closed subarc A' of A satisfies conclusion (ii), then obviously so does each closed subarc A'' ; moreover, inclusion induces a surjection

$$\check{H}^n(p^{-1}A') \rightarrow \check{H}^n(p^{-1}A'').$$

To see why this is so, express $A' = E_1 \cup A'' \cup E_2$ as the union of closed subarcs (one of E_1 or E_2 being empty, should A'' contain an endpoint of A'), where $E_1 \cap A''$ and $E_2 \cap A''$ are endpoints of A'' . Given an element $z \in \check{H}^n(p^{-1}A'')$, use conclusion (ii) to find $z_j \in \check{H}^n(p^{-1}E_j)$ for $j \in \{1, 2\}$ such that

$$\operatorname{Im}\{z_j \rightarrow \check{H}^n(p^{-1}(A'' \cap E_j))\} = \operatorname{Im}\{z \rightarrow \check{H}^n(p^{-1}(A'' \cap E_j))\}.$$

From the Mayer-Vietoris sequence of $(p^{-1}(A'' \cup E_1), p^{-1}A'', p^{-1}E_1)$,

$$\check{H}^n(p^{-1}(A'' \cup E_2)) \rightarrow \check{H}^n(p^{-1}A'') \oplus \check{H}^n(p^{-1}E_1) \xrightarrow{\Delta} \check{H}^n(p^{-1}(A'' \cap E_1)),$$

we see that $\Delta(z \oplus z_1) = 0$, which gives $z'' \in \check{H}^n(p^{-1}(A'' \cup E_1))$, whose restrictions (= inclusion-induced images) in $\check{H}^n(p^{-1}A'')$ and $\check{H}^n(p_1^{-1})$ are z and z_1 , respectively. Similarly, starting from part of the Mayer-Vietoris sequence for

$$(p^{-1}(E_1 \cup A'' \cup E_2), p^{-1}(E_1 \cup A''), p^{-1}E_2),$$

we obtain $z' \in H^n(p^{-1}A')$ whose restriction to $H^n(p^{-1}A'')$ is z . It now follows easily that if, in addition, A satisfies conclusion (iii), then so does each closed subarc.

Let A_1 and A_2 denote closed subarcs of A such that $A_1 \cap A_2 \neq \emptyset$. By the preceding argument, if A_1 and A_2 satisfy conclusion (ii), so does their union $A_1 \cup A_2$. If both A_1 and A_2 satisfy conclusion (iii), we can verify that the same is true of A by supposing $y \in A_1 \cap A_2$ and $x \in A_1$, say, and considering the commutative diagram:

$$\begin{array}{ccccc} & & \check{H}^n(p^{-1}A) & & \\ & \swarrow & \downarrow & \searrow & \\ \check{H}^n(p^{-1}x) & \leftarrow & \check{H}^n(p^{-1}A_1) & \rightarrow & \check{H}^n(p^{-1}y) \end{array}$$

Let A' be a maximal subarc of A such that conclusions (ii) and (iii) are valid for each proper closed subarc of A' . Each endpoint of A' is contained in A' , else the arc A_x , where x is an endpoint of A' , together with the conclusions of the previous paragraph would lead to a contradiction of the maximality of A' . For the same reason, $A' = A$.

Finally, conclusion (i) is a consequence of conclusion (ii) since repeated applications of the Mayer-Vietoris sequence reveal

$$\check{H}^{n+1}(p^{-1}A) \cong \check{H}^{n+1}(p^{-1}A_1) \oplus \cdots \oplus \check{H}^{n+1}(p^{-1}A_k)$$

for any "partition" $A_1, \dots, A_k$ of A into subarcs. The nontriviality of $\check{H}^{n+1}(p^{-1}A)$ would ultimately detect the nontriviality of $\check{H}^{n+1}(p^{-1}x)$ for some $x \in A$, but this is not possible.

LEMMA 2.3. *Neither a point nor an arc in C separates C , but every simple closed curve there does separate C .*

PROOF. Let A denote an arc in C and V the component of C containing A . From the exact sequence of the pair $(p^{-1}V, p^{-1}V - p^{-1}A)$, we obtain

$$H_1(p^{-1}V, p^{-1}V - p^{-1}A) \rightarrow \tilde{H}_0(p^{-1}V - p^{-1}A) \rightarrow \tilde{H}_0(p^{-1}A) \cong 0$$

and, by duality [S, p. 296] and Lemma 2.2,

$$H_1(p^{-1}V, p^{-1}V - p^{-1}A) \cong \check{H}^{n+1}(p^{-1}A) \cong 0.$$

Thus $p^{-1}V - p^{-1}A$ and $V - A$ are connected.

Similarly, no point separates C .

Now, let J denote a simple closed curve in some component V of C . Express J as the union of two arcs A_1 and A_2 whose intersection is the boundary $\{a_1, a_2\}$ of each. From the Mayer-Vietoris sequence for $p^{-1}J = p^{-1}A_1 \cup p^{-1}A_2$, we analyse with $\mathbf{Z}/2\mathbf{Z}$ -coefficients

$$\begin{aligned} \cdots \rightarrow \check{H}^n(p^{-1}A_1) \oplus \check{H}^n(p^{-1}A_2) &\xrightarrow{\beta} \check{H}^n(p^{-1}\{a_1, a_2\}) \rightarrow \check{H}^{n+1}(p^{-1}J) \rightarrow 0 \\ &\cong \downarrow \phi \\ \check{H}^n(p^{-1}a_1) \oplus \check{H}^n(p^{-1}a_2) &\cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \end{aligned}$$

(where the zero at the right comes from Lemma 2.2). Write $\phi \circ \beta = \delta_1 \oplus \delta_2$, where, for $i = 1, 2$, $\delta_i: \check{H}^n(p^{-1}A_1) \oplus \check{H}^n(p^{-1}A_2) \rightarrow \check{H}^n(p^{-1}a_i)$. The conclusion in part (iii) of Lemma 2.2 reveals that $\delta_1(z, 0) = 0$ if and only if $\delta_2(z, 0) = 0$ and, likewise, that $\delta_1(0, z) = 0$ if and only if $\delta_2(0, z) = 0$. It follows that the image of $\phi\beta$ is contained in the diagonal subgroup $\{(0, 0), (1, 1)\}$ and, in light of part (ii) of Lemma 2.2, is equal to this subgroup. Consequently,

$$\check{H}^{n+1}(p^{-1}J; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}.$$

Continuing throughout the remainder of the proof to use $\mathbf{Z}/2\mathbf{Z}$ -coefficients, we establish that $p^{-1}J$ separates $p^{-1}V$ by showing that the homomorphism γ in the sequence below is trivial:

$$\begin{array}{ccccccc} H_1(p^{-1}V) & \xrightarrow{\gamma} & H_1(p^{-1}V, p^{-1}V - p^{-1}J) & \rightarrow & \check{H}_0(p^{-1}V - p^{-1}J) & \rightarrow & 0 \\ & & & & \downarrow \cong & & \\ & & & & \check{H}^{n+1}(p^{-1}J) \cong \mathbf{Z}/2\mathbf{Z} & & \end{array}$$

A simple chase through the diagram

$$\begin{array}{ccccccc} H_1(p^{-1}V - p^{-1}J) & \rightarrow & H_1(p^{-1}V) & \xrightarrow{\gamma} & H_1(p^{-1}V, p^{-1}V - p^{-1}J) & & \\ \downarrow & & \downarrow & & \downarrow \cong & & \\ H_1(p^{-1}U_0 - p^{-1}J) & \rightarrow & H_1(p^{-1}U_0) & \rightarrow & H_1(p^{-1}U_0, p^{-1}U_0 - p^{-1}J) & & \\ \downarrow & & j \downarrow & & \downarrow \cong & & \\ H_1(p^{-1}U - p^{-1}J) & \xrightarrow{\eta} & H_1(p^{-1}U) & \xrightarrow{\gamma'} & H_1(p^{-1}U, p^{-1}U - p^{-1}J) & & \end{array}$$

establishes the triviality of γ provided $\text{Im } \eta \supset \text{Im } j$. One can detect the last containment by choosing $c_0 \in U_0 - J$ so close to b_0 that c_0 lies in a neighborhood W of b_0 for which there is a shape deformation retraction in $p^{-1}U_0$, say $r: p^{-1}W \rightarrow p^{-1}b_0$. Since the restriction of r induces an isomorphism $\check{H}_n(p^{-1}c_0) \rightarrow \check{H}_n(p^{-1}b_0)$ with $\mathbf{Z}$ and, hence, with $\mathbf{Z}/2\mathbf{Z}$ -coefficients (as we are working in the continuity set), r induces a surjection $\check{H}_1(p^{-1}c_0) \rightarrow \check{H}_1(p^{-1}b_0)$. It follows that

$$\text{Im}\{\check{H}_1(p^{-1}c_0) \rightarrow H_1(p^{-1}U)\} = \text{Im}\{\check{H}_1(p^{-1}b_0) \rightarrow H_1(p^{-1}U)\}.$$

We stipulated earlier (for $\mathbf{Z}$ and, hence, $\mathbf{Z}/2\mathbf{Z}$ -coefficients) that

$$\text{Im}\{\check{H}_1(p^{-1}b_0) \rightarrow H_1(p^{-1}U)\} = \text{Im}\{H_1(p^{-1}U_0) \rightarrow H_1(p^{-1}U)\},$$

and, as

$$\text{Im } \eta \supset \text{Im}\{\check{H}_1(p^{-1}c_0) \rightarrow H_1(p^{-1}U)\},$$

we conclude that $\text{Im } \eta \supset \text{Im } j$.

ADDENDUM. Having established that $p^{-1}J$ separates, we can conclude that $\check{H}^{n+1}(p^{-1}J; \mathbf{Z}) \cong \mathbf{Z}$. The only point in the preceding argument that made essential use of $\mathbf{Z}/2\mathbf{Z}$ -coefficients arose in the conclusion that the image of β equalled a diagonal subgroup. For $\mathbf{Z}$ -coefficients, we can draw the analogous conclusion provided we know $\check{H}^{n+1}(p^{-1}J; \mathbf{Z}) \neq 0$, which holds because $p^{-1}J$ separates V (and since the homomorphism γ analysed in the proof is trivial for $\mathbf{Z}$ -coefficients as well).

THEOREM 2.4. *The space C is a 2-manifold.*

PROOF. This is a consequence of Lemma 2.3 and [Wi, p. 95].

LEMMA 2.5. *The set D is countable.*

PROOF. By fact (3), $D = D_1 \cup D_2$ where D_1 is closed in D and dense-in-itself, and where D_2 is countable. We shall prove that $D_1 = \emptyset$.

Suppose $D_1 \neq \emptyset$. Since $\alpha|_{D_1}$ is lower semicontinuous, the set where $\alpha|_{D_1}$ is continuous forms a dense open subset of D_1 . Thus, we can find $b \in D_1$ and some neighborhood W of b in U_0 such that $\alpha|_{D_1 \cap W}$ is constant. Accordingly, as in fact (1), W can be restricted so that for all $w \in W$ there exists an integer $k(w) > 0$ such that $\alpha(w) = k(w) \cdot \alpha(b)$. Since b is a discontinuity of α , there is some $w \in W - D_1$ for which $k(w) > 1$. This yields $k(w') \geq k(w) > 1$ for all w' in some neighborhood of w , so we can assume $w \in W - D$.

Using the countability of D_2 , we build an arc A in W with endpoints c, d in D_1 and with $A - \{c, d\} \subset W - D$ such that, for $x \in A - \{c, d\}$,

$$\alpha(x) = k\alpha(c) = k\alpha(d) \quad (\text{some } k > 1).$$

It suffices to compute that $\check{H}^{n+1}(p^{-1}A) \cong \mathbf{Z}/k\mathbf{Z}$, for one can verify that this is impossible by inspecting the sequence

$$\begin{aligned} H_1(p^{-1}W) &\xrightarrow{\gamma} H_1(p^{-1}W, p^{-1}W - p^{-1}A) \rightarrow \check{H}_0(p^{-1}W - p^{-1}A) \rightarrow 0 \\ &\quad \downarrow \cong \\ &\check{H}^{n+1}(p^{-1}A) \cong \mathbf{Z}/k\mathbf{Z} \end{aligned}$$

while recalling the argument in Lemma 2.3 that γ is the trivial homomorphism.

Express A as the union of two subarcs A_1, A_2 having a common endpoint e . It suffices to determine that, for $i = 1, 2$, $\check{H}^{n+1}(p^{-1}A_i) \cong 0$ and that

$$\text{Im}\{\check{H}^n(p^{-1}A_i) \rightarrow \check{H}^n(p^{-1}e)\} = k\mathbf{Z}.$$

Then the tail of a Mayer-Vietoris sequence

$$\check{H}^n(p^{-1}A_1) \oplus \check{H}^n(p^{-1}A_2) \rightarrow \check{H}^n(p^{-1}e) \rightarrow \check{H}^{n+1}(p^{-1}A) \rightarrow 0$$

detects that $\check{H}^{n+1}(p^{-1}A) \cong \mathbf{Z}/k\mathbf{Z}$.

SUBLEMMA 2.6. *Let W and D be as above and let A be an arc with one endpoint $c \in D$, with $A \setminus \{c\} \subset W$, and with $\alpha(x) = k\alpha(c)$ for $x \in A - \{c\}$ (some $k > 1$). Then*

- (i) $\check{H}^{n+1}(p^{-1}A) = 0$, and
- (ii) $\text{Im}\{\check{H}^n(p^{-1}A) \rightarrow \check{H}^n(p^{-1}e)\} = k\mathbf{Z}$, where e is the endpoint of $A - \{c\}$.

PROOF. Conclusion (i) follows from conclusion (ii) of Lemma 2.2, just as conclusion (i) of Lemma 2.2 followed. (The key observation is that, expressing $A = A_1 \cup A_2$ as the union of subarcs having just an endpoint in common,

$$\check{H}^n(p^{-1}A_1) \oplus \check{H}^n(p^{-1}A_2) \rightarrow \check{H}^n(p^{-1}(A_1 \cap A_2))$$

is surjective, since one of A_1 and A_2 is contained in the continuity set.)

Take an arbitrary neighborhood N of $p^{-1}A$. Let A_1 be a proper subarc of A containing c for which there exists a shape strong deformation of $p^{-1}A_1$ to $p^{-1}c$, the deformation taking place in N . Let d denote the other endpoint of A_1 and $A' = \text{Cl}(A - A_1)$. It follows that

$$\text{Im}\{\check{H}^n(p^{-1}A_1) \rightarrow \check{H}^n(p^{-1}d)\} = k\mathbf{Z}$$

because $\alpha(d) = k\alpha(c)$. From the cohomology ladder

$$\begin{array}{ccccccc} \check{H}^n(p^{-1}A) & \rightarrow & \check{H}^n(p^{-1}A') & \rightarrow & \check{H}^{n+1}(p^{-1}A, p^{-1}A') & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \cong & & \\ \check{H}^n(p^{-1}A_1) & \rightarrow & \check{H}^n(p^{-1}d) & \rightarrow & \check{H}^{n+1}(p^{-1}A_1, p^{-1}d) & \rightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \\ & & \mathbf{Z} & & \mathbf{Z}/k\mathbf{Z} & & \end{array}$$

we see that $\text{Im}\{\check{H}^n(p^{-1}A) \rightarrow \check{H}^n(p^{-1}d)\} = k\mathbf{Z}$. The desired conclusion that

$$\text{Im}\{\check{H}^n(p^{-1}A) \rightarrow \check{H}^n(p^{-1}e)\} = k\mathbf{Z}$$

follows from the following diagram and the conclusion of Lemma 2.2 that the surjections i_e^* and i_d^* have equal kernels:

$$\begin{array}{ccccc} & & \check{H}^n(p^{-1}A) & & \\ & \swarrow & \downarrow & \searrow & \\ \check{H}^n(p^{-1}e) & \xleftarrow{i_e^*} & \check{H}^n(p^{-1}A') & \xrightarrow{i_d^*} & \check{H}^n(p^{-1}d) \end{array}$$

This completes the proof that D is countable.

Now it is appropriate to let the point b_0 , identified shortly after the statement of Lemma 2.1, take on different values in $B_+ = B - K$. For each $b \in B_+$ we have a neighborhood $U_b \subset B_+$, a closed and countable subset D_b of U_b , a dense open subset $C_b = U_b - D_b$ of U_b , and a winding function $\alpha_b: U_b \rightarrow \mathbf{Z}$ with the properties described at the outset of this section. In case $c \in U_{b(1)} \cap U_{b(2)}$, then $c \in C_{b(1)}$ iff $c \in C_{b(2)}$, since continuity of $\alpha_{b(i)}$ at c depends only on the behavior of point inverses near $p^{-1}c$. Select a countable subcollection $\{U_{b(i)}\}$ of $\{U_b\}$ covering B_+ and define C_+ as $\bigcup_i C_{b(i)}$. From Theorem 2.4 and Lemma 2.5 we obtain

THEOREM 2.7. *The set C_+ is open and dense in B and $D_+ = B_+ - C_+$ is a countable, relatively closed subset of B_+ . Furthermore, C_+ is a 2-manifold.*

3. The structure of the decomposition space. At this point we launch an investigation of the subset $K = B - B_+$, with an eye toward ultimately concluding it is locally finite. The crucial step is detecting that K is totally disconnected, for we have

LEMMA 3.1. *If K is totally disconnected, then B is a 2-manifold.*

SKETCH OF THE PROOF. The argument in Lemma 2.3 shows each $y_0 \in K \cup D_+$ to have a connected neighborhood U_0 in B such that every simple closed curve in $U_0 \cap C_+$ separates $U_0 \cap C_+$. Thus, as argued in [DW1, Lemma 4.1], $U_0 \cap C_+$ is homeomorphic to the complement in S^2 of some closed 0-dimensional subset. The

points of $(K \cup D_+) \cap U_0$ can be considered to be (some of the) ends of U_0 and the homeomorphism of $U_0 \cap C_+$ into S^2 can be extended, in an end-preserving way, to one from U_0 onto an open subset of S^2 .

Suppose now that $K_1 \subset K$ is a closed subset such that $K - K_1$ is totally disconnected (equivalently, 0-dimensional, as $K - K_1$ is locally compact). Using [DH, Theorem 2.10], we name a point $b_0 \in K_1$, a neighborhood U_0 of b_0 as above, and then another neighborhood W of b_0 in U_0 for which there exists a shape retraction $\underline{r}: p^{-1}W \rightarrow p^{-1}b_0$ such that the restriction of $\underline{r}$ to $p^{-1}b$ provides a shape equivalence from $p^{-1}b$ to $p^{-1}b_0$ for each $b \in K_1 \cap W$.

PROPOSITION 3.2. *There is a nonempty relatively open subset $\mathcal{O}$ of $K_1 \cap W$ that is totally disconnected (equivalently, in this setting, 0-dimensional).*

Before embarking on the proof, which occupies most of the remainder of this section, we derive the important

COROLLARY 3.3. *K is totally disconnected.*

PROOF. Specify K_1 by requiring $K - K_1$ to be a maximal relatively open subset of K that is 0-dimensional. The separability of K and the Sum Theorem for 0-dimensional sets [HW, p. 18] ensure that such a subset exists. Then $K_1 = \emptyset$, for otherwise a nonempty, 0-dimensional, relatively open subset $\mathcal{O}$ of K_1 would exist, and $(K - K_1) \cup \mathcal{O}$ would be larger than $K - K_1$. It would be obvious that $(K - K_1) \cup \mathcal{O}$ is relatively open in K and the Sum Theorem of [HW] again would reveal that $(K - K_1) \cup \mathcal{O}$ is 0-dimensional. Thus, K itself is 0-dimensional.

LEMMA 3.4. *There is a connected neighborhood $W_1 \subset W$ of b_0 and a component V of $W_1 - K_1$ such that every arc A with $\partial A \subset K_1$ and $A - \partial A \subset V \cap C_+$ separates B and one component of $B - A$ is contained in U_0 .*

PROOF. Choose $W_1 \subset W$ so that $H_{n+1}(p^{-1}W_1) \rightarrow H_{n+1}(p^{-1}U_0)$ is trivial. Since $b_0 \in K$, there exists $w \in W_1$ such that $\check{H}_n(p^{-1}w) \rightarrow H_n(p^{-1}W_1)$ is trivial. Because $\alpha|_{K_1 \cap W}$ is nonzero, $w \notin K_1$. It follows from the 0-dimensionality of $[(K - K_1) \cup D_+] \cap W_1$ (and from the fact that $\check{H}_n(p^{-1}w') \rightarrow H_n(p^{-1}W_1)$ is trivial for all w' in some neighborhood of w) that we can choose $w \in W_1 \cap C_+$.

Let V denote the component of $W_1 - K_1$ containing w . Observe that $V \cap K \subset K - K_1$ is a relatively closed 0-dimensional subset of V and, hence, $V \cap C_+ = V - K$ is a dense, connected, open subset of V .

Let A be an arc with $\partial A \subset K_1$ and $A - \partial A \subset V \cap C_+$. Split A into subarcs A_1, A_2 having just an endpoint e in common. Then $e \in V \cap C_+$. An argument like that in Lemma 2.5 verifies that $\check{H}^{n+1}(p^{-1}A; \mathbf{Z}) \cong \mathbf{Z}$. The diagram

$$\begin{array}{c} H_1(p^{-1}W_1) \xrightarrow{\gamma} H_1(p^{-1}W_1, p^{-1}W_1 - p^{-1}A) \rightarrow \check{H}_0(p^{-1}W_1 - p^{-1}A) \rightarrow 0 \\ \downarrow \cong \\ \check{H}^{n+1}(p^{-1}A) \cong \mathbf{Z} \end{array}$$

indicates that $p^{-1}A$ separates $p^{-1}W_1$ (recall the previous argument in the proof of Lemma 2.3 that γ is trivial). Similarly, $p^{-1}A$ separates $p^{-1}U_0$. Hence, A separates both W_1 and U_0 .

Our initial specification that $H_{n+1}(p^{-1}W_1) \rightarrow H_{n+1}(p^{-1}U_0)$ be trivial starts the chase through the diagram

$$\begin{array}{ccc}
 H_{n+1}(p^{-1}U_0) & \xrightarrow{\cong} & H^1(\text{Cl } p^{-1}U_0, \text{Fr } p^{-1}U_0) \\
 \uparrow & & \nwarrow j \\
 & & H^1(\text{Cl } p^{-1}U_0, \text{Cl } p^{-1}U_0 - p^{-1}W_1) \\
 & & \swarrow \cong \\
 H_{n+1}(p^{-1}W_1) & \xrightarrow{\cong} & H^1(\text{Cl } p^{-1}W_1, \text{Fr } p^{-1}W_1)
 \end{array}$$

leading to the detection that j is trivial. (The isomorphisms on the left arise from Lefschetz Duality [S, p. 297] while that on the right is an excision.) Since j is trivial, the diagram

$$\begin{array}{ccccc}
 0 & \rightarrow & \check{H}^0(\text{Fr } p^{-1}U_0) & \rightarrow & \check{H}^1(\text{Cl } p^{-1}U_0, \text{Fr } p^{-1}U_0) \\
 & & \uparrow i & & \uparrow j \\
 0 & \rightarrow & \check{H}^0(\text{Cl } p^{-1}U_0 - p^{-1}W_1) & \rightarrow & H^1(\text{Cl } p^{-1}U_0, \text{Cl } p^{-1}U_0 - p^{-1}W_1)
 \end{array}$$

can be used to detect that i is trivial and, consequently, $\text{Fr } p^{-1}U_0$ is contained in a single component of $(\text{Cl } p^{-1}U_0) - p^{-1}W_1$. Thus, $\text{Fr } p^{-1}U_0$ is contained in a single component of $(\text{Cl } p^{-1}U_0) - p^{-1}A$. This shows that one component of $(\text{Cl } p^{-1}U_0) - p^{-1}A$ is also a component of $M - p^{-1}A$, as well as that one component of $B - A$ is a subset of U_0 . This completes the proof of Lemma 3.4.

It remains to analyze the set $W_1 \cap \text{Fr } V$ with the aim of determining that it is totally disconnected. Thereby we shall discover that $V = W_1$ and that $W_1 \cap K_1$ is a nonempty ($b_0 \in W_1 \cap K_1$), totally disconnected subset of K_1 .

LEMMA 3.5. $W_1 \cap \text{Fr } V$ contains no arc.

PROOF. Suppose that $A \subset W_1 \cap \text{Fr } V$ is an arc. Recall that the choice of W and b_0 was based on the stipulation that the shape retraction $r: p^{-1}W \rightarrow p^{-1}b_0$ restricts to a shape equivalence from $p^{-1}b$ to $p^{-1}b_0$, for each $b \in K_1 \cap W$. It follows that, for each subarc A' of A and $x \in A'$, the inclusion $p^{-1}x \rightarrow p^{-1}A'$ is a shape equivalence. Consequently, we can choose connected open sets P and N , with $P \subset N \subset W_1$, such that

$$N \cap A = A - \partial A = P \cap A,$$

the inclusion induces an injection of $\pi_1(p^{-1}(A - \partial A))$ to $\pi_1(p^{-1}N)$, and

$$\text{Im}\{\pi_1(p^{-1}(A - \partial A)) \rightarrow \pi_1(p^{-1}N)\} = \text{Im}\{\pi_1(p^{-1}P) \rightarrow \pi_1(p^{-1}N)\}.$$

CLAIM. The set $p^{-1}(A - \partial A)$ separates $p^{-1}P$ into precisely two connected pieces. Furthermore, if A' is a subarc of $A - \partial A$ and if $P' \subset N' \subset P$ are connected open sets chosen so that $N' \cap A' = A' - \partial A' = P' \cap A'$, inclusion induces an injection of $\pi_1(p^{-1}(A' - \partial A'))$ to $\pi_1(p^{-1}N')$, and

$$\text{Im}\{\pi_1(p^{-1}(A' - \partial A')) \rightarrow \pi_1(p^{-1}N')\} = \text{Im}\{\pi_1(p^{-1}P') \rightarrow \pi_1(p^{-1}N')\},$$

then distinct components of $p^{-1}P' - p^{-1}(A' - \partial A')$ are contained in distinct components of $p^{-1}P - p^{-1}(A - \partial A)$.

Before proving this, we use it to arrive at the contradiction that establishes Lemma 3.5. Transferring the conclusions via the map p , we have that $P - A = Z_1 \cup Z_2$, where Z_1 and Z_2 are distinct components. An easy consequence of the “furthermore” in the Claim is that every point of $A - \partial A$ is arcwise accessible from each of Z_1 and Z_2 . As one of Z_1 or Z_2 meets V , say $Z_1 \cap V \neq \emptyset$, there is a 4-od T (i.e., T is the cone on a four-point set) with $T \subset (Z_1 \cap C_+) \cup (A - \partial A)$, with the cone point in V and with $T \cap (A - \partial A)$ equal to the four endpoints of T . Let T' denote the subset of T that is the “sub-4-od” whose endpoints lie in $\text{Fr } V$ but is otherwise contained in $V \cap C_+$. Name the endpoints of T to be a_1, a_2, a_3, a_4 and the corresponding endpoints of T' to be a'_1, a'_2, a'_3, a'_4 . Let $\langle x, y \rangle$ denote the subarc of T connecting any given pair of points $x, y \in T$. Lemma 3.4 implies that for some pair, say a'_1, a'_2 , $p^{-1}(\langle a'_1, a'_2 \rangle)$ separates $p^{-1}a'_3$ from $p^{-1}a'_4$ and, therefore, that $\langle a'_1, a'_2 \rangle$ separates a'_3 from a'_4 . This is an impossibility, since $\langle a'_3, a'_3 \rangle \cup Z_2 \cup \langle a'_4, a'_4 \rangle$ is a connected set missing $\langle a'_1, a'_2 \rangle$.

PROOF OF THE CLAIM. Since $p^{-1}x$ and $p^{-1}y$ are “connected” by a shape equivalence for $x, y \in A$, an argument like the one given for Lemma 2.2 shows that inclusion induces an isomorphism $\check{H}^n(p^{-1}A') \rightarrow \check{H}^n(p^{-1}a)$ and that $\check{H}^{n+1}(p^{-1}A') = 0$ for each subarc A' of A and each $a \in A'$. The exact cohomology sequence for the pair $(p^{-1}A', p^{-1}\partial A')$ and properties of cohomology with compact supports [S, p. 321] give

$$\check{H}_c^{n+1}(p^{-1}(A' - \partial A')) \cong \check{H}^{n+1}(p^{-1}A', p^{-1}\partial A') = 0,$$

and a comparison of two such sequences reveals that inclusion induces an isomorphism

$$H_c^{n+1}(p^{-1}(A' - \partial A')) \rightarrow H_c^{n+1}(p^{-1}(A'' - \partial A''))$$

for subarcs $A'' \subset A'$ of A . The diagram

$$\begin{array}{ccccccc} H_1(p^{-1}N) & \xrightarrow{\beta} & H_1(p^{-1}N, p^{-1}(N - A)) & & & & \\ \uparrow \phi & & \uparrow \cong & & & & \\ H_1(p^{-1}P) & \xrightarrow{\gamma} & H_1(p^{-1}P, p^{-1}(P - A)) & \rightarrow & \check{H}_0(p^{-1}(P - A)) & \rightarrow & 0 \\ & & \uparrow \cong & & & & \\ & & H_c^{n+1}(p^{-1}(A - \partial A)) & \cong & Z & & \end{array}$$

can be used to establish the Claim, provided γ is the trivial homomorphism. [The “furthermore” follows by comparing the diagrams for $A \subset P \subset N$ and $A' \subset P' \subset N'$ and by using the inclusion-induced isomorphism

$$H_c^{n+1}(p^{-1}(A - \partial A)) \rightarrow H_c^{n+1}(p^{-1}(A' - \partial A')).]$$

The triviality of γ is immediate if $\pi_1(p^{-1}a) = 0$ for some (hence, any) $a \in A$, as the homomorphism ϕ is then trivial. An outline of the argument that γ is trivial in

general will follow; this will be done by showing $\beta(2\phi(e)) = 0$ for each $e \in H_1(p^{-1}P)$ and, hence, as β is a homomorphism to $\mathbf{Z}$, $\beta\phi$ is trivial. Of course, it suffices to show that $2\phi(e)$ is an element of

$$\text{Im}\{H_1(p^{-1}(N - A)) \rightarrow H_1(p^{-1}N)\}.$$

In fact, we show that in the diagram

$$\begin{array}{ccc} \pi_1(p^{-1}(N - A), x_0) & \xrightarrow{\delta} & \pi_1(p^{-1}N, x_0) \\ & \downarrow \eta & \\ & \pi_1(p^{-1}P, x_0) & \end{array}$$

$2\eta(e) \in \text{Im } \delta$ for each $e \in \pi_1(p^{-1}P, x_0)$, where the basepoint x_0 lies in $p^{-1}(P - A)$.

Let $f: \tilde{N} \rightarrow p^{-1}N$ denote the universal covering space. Choose $y_0 \in f^{-1}(x_0)$. Let $\tilde{P}_0$ denote the component of $f^{-1}(P)$ containing y_0 and let

$$\tilde{A}_0 = \tilde{P}_0 \cap f^{-1}p^{-1}(A - \partial A).$$

It follows from the π_1 conditions stipulated prior to the statement of the Claim that $\tilde{A}_0$ is the only component of $f^{-1}p^{-1}(A - \partial A)$ in $\tilde{P}_0$ and that $\pi_1(\tilde{A}_0)$ is trivial. A computation outlined below establishes $H_c^{n+1}(\tilde{A}) \cong \mathbf{Z}$, and then the simply-connected case of this result applies to reveal $\tilde{P}_0 - \tilde{A}_0 = C_1 \cup C_2$, where C_1 and C_2 are distinct components. For an arbitrary representative $e: (I, \{0, 1\}) \rightarrow (p^{-1}P, x_0)$ of an element (also called) e of $\pi_1(p^{-1}P, x_0)$, if there is a lift $\tilde{e}: I \rightarrow \tilde{P}_0$ of e with $\tilde{e}(0)$ and $\tilde{e}(1)$ contained in the same C_i , then the loop obtained by connecting $\tilde{e}(0)$ and $\tilde{e}(1)$ via a path in that C_i is necessarily null homotopic in $\tilde{N}$ and, consequently, e is homotopic rel $\{0, 1\}$ in $p^{-1}N$ to $e': (I, \{0, 1\}) \rightarrow (f(C_i), x_0)$. As $f(C_i) \cap p^{-1}A = \emptyset$, the element $\eta(e) = \eta(e')$ is in $\text{Im } \delta$. If no lift $\tilde{e}$ to $\tilde{P}_0$ has $\tilde{e}(0)$ and $\tilde{e}(1)$ contained in the same C_i , then each lift of $2e$ to $\tilde{P}_0$ must have this property. Thus, in either case, $\eta(2e) \in \text{Im } \delta$.

Finally, to finish off the Claim, we describe the computation of $H_c^{n+1}(\tilde{A}_0)$. Set $\tilde{E}$ equal to that component of $f^{-1}p^{-1}E$ contained in $\tilde{A}_0$, where E is a closed subarc of A . Given any point $a_0 \in E$, we specify a shape equivalence s from $p^{-1}a_0$ to a closed orientable n -manifold g_0 and a shape deformation retraction $r': p^{-1}E \rightarrow p^{-1}a_0$. Denote by $\tilde{g}_0$ the universal cover of g_0 and by $\tilde{s}: \tilde{E} \cap f^{-1}p^{-1}a_0 \rightarrow \tilde{g}_0$ and $\tilde{r}': \tilde{E} \rightarrow \tilde{E} \cap f^{-1}p^{-1}a_0$ lifts of s and r' , respectively. Computing with $\tilde{s}$ and $\tilde{r}'$ establishes that $H_c^n(E \cap f^{-1}p^{-1}a_0) \cong \mathbf{Z}$, that $\check{H}_c^{n+1}(\tilde{E}) = 0$, and that inclusion induces an isomorphism

$$\check{H}_c^n(\tilde{E}) \rightarrow \check{H}_c^n(\tilde{E} \cap f^{-1}p^{-1}a_0).$$

The exact cohomology sequence (with compact supports) of the pair $(\tilde{E}, \tilde{E} \cap f^{-1}p^{-1}\partial E)$ displays that

$$\check{H}_c^{n+1}(\tilde{E}, \tilde{E} \cap f^{-1}p^{-1}\partial E) \cong \check{H}_c^{n+1}(\tilde{E} - f^{-1}p^{-1}\partial E) \cong \mathbf{Z}.$$

For a pair of closed subarcs $E \subset F$ of A , a comparison of the computation just described for each reveals that inclusion induces an isomorphism

$$\check{H}_c^{n+1}(\tilde{E} - f^{-1}p^{-1}\partial E) \cong \check{H}_c^{n+1}(\tilde{F} - f^{-1}p^{-1}\partial F).$$

(Specifically, compare the pair $(\tilde{F}, \tilde{F} \cap f^{-1}p^{-1}\partial F)$ and the pair $(\tilde{F}, \tilde{F} \cap f^{-1}p^{-1}\text{Cl}(F - E))$ to determine that

$$\check{H}_c^{n+1}(\tilde{F}, \tilde{F} \cap f^{-1}p^{-1}\partial F) \cong \check{H}_c^{n+1}(\tilde{F}, \tilde{F} \cap f^{-1}p^{-1}\text{Cl}(F - E))$$

and, then, use the excision axiom to deduce that the latter group is isomorphic to

$$\check{H}_c^{n+1}(\tilde{E} - f^{-1}p^{-1}\partial E).$$

Finally, $\check{H}_c^{n+1}(\tilde{A}_0)$ is the direct limit of the groups $\check{H}_c^{n+1}(\tilde{E} - f^{-1}p^{-1}\partial E)$ where E ranges over closed subarcs of A and, hence, $\check{H}_c^{n+1}(\tilde{A}_0) \cong \mathbf{Z}$.

PROOF OF PROPOSITION 3.2. Since $W_1 \cap \text{Fr } V$ contains no arc, no compact connected subset of it, other than a singleton, can be locally connected. Consequently, we can show $W_1 \cap \text{Fr } V$ to be totally disconnected by proving it contains no sequence $Z_1, Z_2, \dots$ of pairwise disjoint, compact connected subsets, each with at least two points, that converges in the Hausdorff metric to another compact connected subset Z , also with at least two points. The possibility that such a sequence might exist is eliminated by considering the following two cases.

First, Z contains four points a_1, a_2, a_3, a_4 which are arcwise accessible from V . Then there exists a 4-od T in $(V \cap C_+) \cup Z$, with $T \cap Z = \{a_1, a_2, a_3, a_4\}$. Lemma 3.4 implies that for some pair, say $\{a_1, a_2\}$, the subarc $\langle a_1, a_2 \rangle$ of T separates a_3 from a_4 in V . This is not possible because the connected sets $Z_1, Z_2, \dots$ miss T but have both a_3 and a_4 as limit points.

Second, the set of points in Z that are arcwise accessible from V is not dense in Z . In this case, there exists a 4-od T in $(V \cap C_+) \cup (W_1 \cap \text{Fr } V)$, having its endpoints $a_1, \dots, a_4$ in Z and containing a “sub-4-od” T' , which, in turn, has its endpoints $a'_1, \dots, a'_4$ in $W_1 \cap \text{Fr } V$, disjoint from $\{a_1, \dots, a_4\}$, and satisfying $T' - \{a'_1, \dots, a'_4\} \subset V \cap C_+$. Lemma 3.4 again implies that for some pair, say $\{a'_1, a'_2\}$, the subarc $\langle a'_1, a'_2 \rangle$ of T' separates a'_3 from a'_4 . This is not possible because $\langle a'_3, a'_3 \rangle \cup Z \cup \langle a'_4, a'_4 \rangle$ is a connected subset of B missing $\langle a'_1, a'_2 \rangle$ but containing both a'_3 and a'_4 .

Lemma 3.1 and Corollary 3.3 combine to form our main result.

THEOREM 3.6. *If M is an orientable $(n + 2)$ -manifold and G is a usc decomposition of M into continua having the shape of closed, orientable n -manifolds, then $B = M/G$ is a 2-manifold (without boundary).*

Orientability of the elements of G is a fundamental hypothesis in Theorem 3.6; orientability of the source manifold M is less crucial, as the next result demonstrates.

THEOREM 3.7. *If G is a usc decomposition of an $(n + 2)$ -manifold M into continua having the shape of closed, orientable n -manifolds, then $B = M/G$ is a 2-manifold possibly with boundary.*

PROOF. Assume M to be nonorientable. Consider the orientable double covering $\theta: \tilde{M} \rightarrow M$ and usc decomposition $\tilde{G}$ into the components of $\theta^{-1}(g)$, $g \in G$. The

nontrivial covering homeomorphism $h: \tilde{M} \rightarrow \tilde{M}$, which respects elements of $\tilde{G}$, induces an involution u on $\tilde{M}/\tilde{G}$, and the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\theta} & M \\ \tilde{p} \downarrow & & p \downarrow \\ \tilde{M}/\tilde{G} & \xrightarrow{T} & M/G \end{array}$$

induces a map $T: \tilde{M}/\tilde{G} \rightarrow M/G$ behaving just like the map of $\tilde{M}/\tilde{G}$ to the orbit space of u . It is well known [Wh] that the orbit space of every involution on a 2-manifold is a 2-manifold with boundary (the boundary may be empty).

Before closing down our analysis of the troublesome spots D and K , the discontinuity and degeneracy subsets of B , we provide one last analysis.

PROPOSITION 3.8. *Assume M (orientable) satisfies $H_1(M) = 0$ and $K \neq \emptyset$. Then $K \cup D$ consists of exactly one point.*

PROOF. Find an arc A in B with $\partial A \subset K \cup D$, $\partial A \cap K \neq \emptyset$, and $A - \partial A \subset C_+$. If $\partial A \subset K$, then as before $\check{H}^{n+1}(p^{-1}A; \mathbf{Z}) \cong \mathbf{Z}$; if $d \in \partial A \cap D$, then

$$\check{H}^{n+1}(p^{-1}A; \mathbf{Z}/k\mathbf{Z}) \cong \mathbf{Z}/k\mathbf{Z},$$

where $\alpha_d(x) = k$ for all $x \neq d$ sufficiently close to d . As a result, with the appropriate coefficients the diagram

$$\begin{array}{c} 0 = H_1(M) \rightarrow H_1(M, M - p^{-1}A) \rightarrow \check{H}_0(M - p^{-1}A) \rightarrow \check{H}_0(M) = 0 \\ \downarrow \cong \\ \check{H}^{n+1}(p^{-1}A) \end{array}$$

reveals $\check{H}_0(M - p^{-1}A)$ to be nontrivial, implying that the arc A separates the 2-manifold B , an impossibility.

As applications of Theorem 3.6, we mention some structural limitations upon the manifolds M admitting decompositions into codimension-two submanifolds.

PROPOSITION 3.9. *Suppose M is a connected $(n + 2)$ -manifold with at least two ends and with $H_1(M; \mathbf{Z}/2\mathbf{Z}) = 0$. Then M admits no usc decomposition G into continua having the shape of closed, orientable n -manifolds.*

PROOF. If it did, the (noncompact) decomposition space B would satisfy $H_1(B; \mathbf{Z}/2\mathbf{Z}) = 0$ [B], indicating that B is the plane. Here the key property is that B has one end. The desired contradiction stems from the fact that proper, monotone maps preserve the number of ends.

PROPOSITION 3.10. *Suppose M is a noncompact, connected $(n + 2)$ -manifold such that for each compact subset C of M there exists another compact set $C' \supset C$ such that*

$$H_1(M - C'; \mathbf{Z}/2\mathbf{Z}) \rightarrow H_1(M - C; \mathbf{Z}/2\mathbf{Z})$$

is trivial. Then M admits no usc decomposition G into continua having the shape of closed, orientable n -manifolds.

PROOF. Suppose otherwise. Then $B = M/G$ is a noncompact 2-manifold for which $H_1(B; \mathbf{Z}/2\mathbf{Z})$ is trivial at infinity, by [B] and the hypothesis. This is impossible: every noncompact 2-manifold contains loops arbitrarily close to infinity that fail to be null homologous in the complement of a given compactum.

Our chief interest was in determining whether E^{n+2} admits a decomposition into codimension-two submanifolds, which now is settled.

COROLLARY 3.11. *There is no usc decomposition of E^{n+2} into continua having the shape of closed, orientable n -manifolds.*

REMARK. The upper semicontinuity of the decompositions G has been fundamental to the discussion throughout this paper. Reinforcing this point, a remarkably explicit partition of E^3 into (round) circles is given in [Sz].

Finally, in the compact case, we have

PROPOSITION 3.12. *Suppose M^{n+2} is a closed, orientable n -manifold for which $H_2(M; \mathbf{Z}/2\mathbf{Z}) = 0$. Then M admits no usc decomposition G into continua having the shape of closed n -manifolds with $\check{H}_1(g; \mathbf{Z}/2\mathbf{Z}) = 0$ for all $g \in G$.*

PROOF. If it did, B would be a closed 2-manifold, and $H_2(B; \mathbf{Z}/2\mathbf{Z}) \neq 0$. Under these hypotheses, the Vietoris-Begle Theorem [B] ensures that

$$p_*: H_2(M; \mathbf{Z}/2\mathbf{Z}) \rightarrow H_2(B; \mathbf{Z}/2\mathbf{Z})$$

is surjective, a contradiction.

4. Structure of the sheaf $\mathcal{H}^n$. We shall denote by $\mathcal{H}^n$ the presheaf on B determined by setting $\mathcal{H}^n(U) = H^n(p^{-1}U; \mathbf{Z})$ (and using inclusion induced homomorphisms) as well as the associated sheaf, whose stalk at a point $x \in B$ is $\check{H}^n(p^{-1}x; \mathbf{Z})$. The path leading to the conclusion that B is a 2-manifold also brought us to a determination that $\mathcal{H}^n$ is locally constant except at points of the sparse set $K \cup D_+$. A final refinement is

THEOREM 4.1. *Suppose M is an orientable $(n+2)$ -manifold and G is a usc decomposition of M into continua having the shape of closed, orientable n -manifolds. Then the decomposition space B is a 2-manifold and B contains a locally finite (closed) subset F such that $\mathcal{H}^n$ is locally constant at each point of $B - F$. Furthermore, if $\check{H}_1(g; \mathbf{Z}) \cong 0$ for each $g \in G$, then $F = \emptyset$.*

PROOF. Since it is already established that B is a 2-manifold and $\mathcal{H}^n$ is locally constant over $B \setminus K \cup D_+$, we need to show that $K \cup D_+$ is locally finite and, under the additional constraints of the “furthermore”, that it is empty.

The local finiteness of $K \cup D_+$ is detected by precluding the existence of three possible types of convergent sequences from $K \cup D_+$.

First, suppose there is a sequence $\{x_i\} \subset K$ with $x_i \rightarrow x \in K$. It had been previously observed that fact (1) detects that the local winding function at any point of K is zero at some point of the continuity set C_+ . As we currently know that $U \cap C_+$ is connected whenever U is an open connected set, each such local winding function is zero at every point of C_+ in its domain. Thus, the argument in Lemma

3.4 would show that an arc A connecting x_i and x_{i+1} , where i is chosen so that A is contained in an appropriately small neighborhood of x and $A - \partial A \subset C_+$, would separate a connected neighborhood of x . This clearly is not possible as B is a 2-manifold (without boundary).

Second, suppose that some point $x \in D_+$ is a limit point of $D_+ - x$. Let α denote the local winding function at x defined on a connected neighborhood U of x and, for $d \in D_+ \cap U$, let α_d denote the local winding function of d defined on a connected neighborhood $V_d \subset U$ of d . Since $U \cap C_+$ is connected, α is constant on $U \cap C_+$, say $\alpha(b) = r$. Similarly, α_d is constant on $V_d \cap C_+$, say $\alpha_d(b) = r_d$. According to fact (1), $r = k \cdot r_d$ for $b \in V_d \cap C_+$ and, consequently, the integers r_d are bounded (by r). Of course, $r_d > 1$ as $d \in D_+$. Thus, we can specify a sequence $\{x_i\} \subset D_+ - x$ with $x_i \rightarrow x$ and with the local winding functions at the x_i 's taking on the single value $q > 1$ at points in the continuity set C_+ . An argument as in Lemma 3.4 using $\mathbf{Z}/q\mathbf{Z}$ -coefficients would lead to the contradiction that an arc A , chosen as in the preceding paragraph, separates a connected neighborhood of x .

Third, suppose that there is a sequence $\{x_i\} \subset D_+$ with $x_i \rightarrow x \in K$. As before, we assume that the x_i 's are contained in a neighborhood U of x on which the local winding function α at x is defined. Specify connected neighborhoods $V_i \subset U$ of x_i on which the local winding function α_i at x_i is defined and set r_i equal to the constant value of α_i on $V_i \cap C_+$. In contrast with the previous situation, we cannot conclude that the r_i 's are bounded, as $\alpha(b) = 0$ for $b \in U_0 \cap C_+$. Consider an arc A connecting x and x_i where A is chosen in a small neighborhood of x containing x_i and where $A - \partial A \subset C$. An argument as in Lemma 3.4 using $\mathbf{Z}/r_i\mathbf{Z}$ -coefficients produces the contradiction that A separates a connected neighborhood of x provided the homomorphism $H_1(p^{-1}W_1) \rightarrow H_1(p^{-1}W_1, p^{-1}W_1 - p^{-1}A)$ is trivial (W_1 being chosen near the start of the proof at Lemma 3.4). In the preceding two paragraphs, the fact that $x \notin A$ facilitates the computation. In the current situation, we must insist that, after choices of neighborhoods $W_1 \subset U_0 \subset U$ of x are made, i is chosen with $p^{-1}A$ so close to $p^{-1}x$ that

$$\text{Im}\{\check{H}_1(p^{-1}x) \rightarrow H_1(p^{-1}U)\} = \text{Im}\{H_1(p^{-1}(U_0 - A)) \rightarrow H_1(p^{-1}U)\}.$$

This poses no difficulties since generators for $\text{Im}\{\check{H}_1(p^{-1}x) \rightarrow H_1(p^{-1}U)\}$ can be found in $H_1(p^{-1}(U_0 - x))$.

We now assume that $\check{H}_1(p^{-1}x; \mathbf{Z}) \cong 0$ for each $x \in B$. Suppose, contrary to the Claim of the "furthermore", that $b \in K \cup D_+$. Specify an open 2-cell $\Delta \subset B$ with $\Delta \cap (K \cup D_+) = \{b\}$ so that there is a shape strong deformation retraction $\underline{r}: p^{-1}\Delta \rightarrow p^{-1}b$, the deformation occurring in a neighborhood of $p^{-1}b$. Set $\Delta^* = \Delta - \{b\}$ and choose any $c \in \Delta^*$. Consider the diagram

$$\begin{array}{ccccc} H^n(p^{-1}\Delta) & \xrightarrow{\alpha} & H^n(p^{-1}\Delta^*) & \rightarrow & H^{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \\ \downarrow j^* & & \downarrow j^* & & \\ \check{H}^n(p^{-1}b) & \xrightarrow{\phi} & \check{H}^n(p^{-1}c) & & \end{array}$$

where the top row is part of the exact sequence of the pair $(p^{-1}\Delta, p^{-1}\Delta^*)$, ϕ is induced by the restriction of $\underline{r}$, and the remaining homomorphisms are induced by

inclusions. In particular, $\phi j^* = i^* \alpha$. The existence of r assures that j^* is into. The assumption that $\check{H}_1(p^{-1}b) \cong 0$ yields, by duality [S, p. 296] that $H^{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \cong 0$. Consequently, the contradiction that ϕ is onto (and hence an isomorphism) will be reached once i^* is shown to be onto.

The universal coefficient theorem [S, p. 243] yields the short exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ext}(H_n(p^{-1}\Delta, p^{-1}\Delta^*), \mathbf{Z}) &\rightarrow H^{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \\ &\rightarrow \text{Hom}(H_{n+1}(p^{-1}\Delta, p^{-1}\Delta^*), \mathbf{Z}) \rightarrow 0 \end{aligned}$$

and duality [S, p. 296] produces

$$H_n(p^{-1}\Delta, p^{-1}\Delta^*) \cong \check{H}^2(p^{-1}b) \quad \text{and} \quad H_{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \cong \check{H}^1(p^{-1}b).$$

The same exact sequence applied to $\check{H}^1(p^{-1}b)$ reveals that $\check{H}^1(p^{-1}b) \cong 0$ and, then, to $\check{H}^2(p^{-1}b)$ reveals that $\check{H}^2(p^{-1}b)$ is free. It follows that $H^{n+1}(p^{-1}\Delta, p^{-1}\Delta^*) \cong 0$.

We begin the computation of i^* 's surjectiveness by expressing $\Delta - b = \Delta^*$ as the union of two open disks R_1 and R_2 whose intersection is a pair of open disks V_1 and V_2 with $c \in V_1$. Further, we specify a point $d \in V_2$ and arcs $A_1 \subset R_1$ and $A_2 \subset R_2$ connecting e and d . The strategy is to decompose i^* as

$$H^n(p^{-1}\Delta^*) \xrightarrow{\delta} H^n(p^{-1}(A_1 \cup A_2)) \xrightarrow{\beta} \check{H}^n(p^{-1}c),$$

and then to determine that both inclusion induced homomorphisms δ and β are surjective.

The Mayer-Vietoris sequence

$$\begin{array}{ccc} \check{H}^n(p^{-1}(A_1 \cap A_2)) & \rightarrow & \check{H}^{n+1}(p^{-1}(A_1 \cup A_2)) \rightarrow 0 \\ \downarrow \cong & & \downarrow \cong \\ \check{H}^n(p^{-1}c) \oplus \check{H}^n(p^{-1}d) & & \mathbf{Z} \\ \downarrow \cong & & \\ \mathbf{Z} \oplus \mathbf{Z} & & \end{array}$$

contains adequate information for discovering that β is surjective. The computation of $\check{H}^{n+1}(p^{-1}(A_1 \cup A_2))$ is essentially done in Lemma 2.3 while the zero at the right is a consequence of Lemma 2.2. A further consequence of the latter is that, for $i = 1, 2$, there is an element $a_i \in \check{H}^n(p^{-1}A_i)$ that restricts to a generator in both $\check{H}^n(p^{-1}c)$ and $\check{H}^n(p^{-1}d)$. Hence, essentially either $\gamma(a_1, a_2) = (0, 0)$ or $\gamma(a_1, a_2) = (2, 0)$. The latter would force $\check{H}^{n+1}(p^{-1}(A_1 \cup A_2))$ to be a torsion group, which it is not, while the former readily shows that $\check{H}^n(p^{-1}(A_1 \cup A_1)) \rightarrow \check{H}^n(p^{-1}A_1)$ is surjective, and, hence, β is surjective.

Detecting the surjectivity of

$$\check{H}^n(p^{-1}\Delta^*) \xrightarrow{\delta} \check{H}^n(p^{-1}(A_1 \cup A_2))$$

is accomplished by comparing the spectral sequences of $p|: p^{-1}\Delta^* \rightarrow \Delta^*$ and $p|: p^{-1}(A_1 \cup A_2) \rightarrow A_1 \cup A_2$. (The sequence is derived in [G] and is described adequately for our purpose in [DW2].) We only need to extract the commuting diagram

$$\begin{array}{ccccc} \check{H}^n(p^{-1}\Delta^*; \mathbf{Z}) & \rightarrow & H^0(\Delta^*; \mathcal{H}^n) & \rightarrow & 0 \\ \downarrow \delta & & \downarrow & & \\ \check{H}^n(p^{-1}A_1 \cup A_2; \mathbf{Z}) & \xrightarrow{\eta} & H^0(A_1 \cup A_2; \mathcal{H}^n) & \rightarrow & 0 \end{array}$$

where the right-hand vertical homomorphism is inclusion induced. Furthermore, the homomorphism is an isomorphism since the early computation that

$$\check{H}^{n+1}(p^{-1}(A_1 \cup A_2); \mathbf{Z}) \cong \mathbf{Z}$$

detects that the sheaf $\mathcal{H}^n$ is constant on Δ^* . The surjectivity of δ follows once we determine that the homomorphism η is an isomorphism. From the spectral sequence of

$$p|: p^{-1}(A_1 \cup A_2) \rightarrow A_1 \cup A_2$$

we determine that kernel $\eta \cong H^1(A_1 \cup A_2; \mathcal{H}^{n-1})$ but $\mathcal{H}^{n-1}$ is the zero-sheaf as each stalk $\check{H}^{n-1}(p^{-1}x; \mathbf{Z})$ is dually isomorphic to $\check{H}_1(p^{-1}x; \mathbf{Z}) \cong 0$ (by assumption).

5. Examples. As a summary, we want to compare what is known about usc decompositions G of orientable $(n+k)$ -manifolds M into connected, orientable n -manifolds in the two low-codimensional cases where $k=1$ and $k=2$. In either case $B=M/G$ is now known to be a boundaryless k -manifold.

Sometimes the decomposition map $p: M \rightarrow B$ can be employed to study the structure of M . This investigative tool is quite effective when p is an approximate fibration. Unfortunately, that is not always the prevailing situation. Example 3 of [D] provides a decomposition G of an $(n+1)$ -manifold M into homology n -spheres, some simply connected and others not; the decomposition map cannot be an approximate fibration because the decomposition elements have different shapes (different homotopy types). There is a similar decomposition of the $(n+2)$ -manifold $M \times E^1$ into homology n -spheres.

When $k=1$ the structure of M is exposed, to some extent, by one unifying feature: the elements of G are all pairwise homologically equivalent [D, Corollary 6.3]. Consequently, the shape (neighborhood) retractions $V \rightarrow g$ induce homology isomorphisms $H_j(g') \rightarrow H_j(g)$ for all g' in V sufficiently close to g . The first fact may appear to stem from the property that the continuity set C_+ of B coincides with B itself. However, when $k=2$ the elements of G need not all be homologically equivalent, not even if $C_+ = B$.

EXAMPLE. A usc decomposition G of a connected, orientable $(n+2)$ -manifold M ($n \geq 2$) into orientable n -manifolds such that (1) in the resulting decomposition space B , each point belongs to the continuity set C_+ , and (2) G contains a pair of homologically inequivalent elements.

The crux of the matter is manifested when $n=2$, so focus on that case. Start with a knot K in S^3 , like the trefoil, whose closed complement fibers over S^1 with fiber T , where T is a disk having a positive number of handles. Attach a (4-dimensional) 2-handle H to B^4 along $U(K)$, a tubular neighborhood of K in $S^3 = \partial B^4$, via the framing determined by the longitude ∂T of the knot space $\text{Cl}(S^3 - U(K))$. The interior M of the resulting 4-manifold with boundary represents the desired manifold.

The exceptional element g_0 of the decomposition G yet to be specified is the 2-sphere in M formed by the cone over K from the center of B^4 and the core of the

attached 2-handle, h . Topologically the other, more standard elements of G all are the union $\hat{T}$ of T and a 2-cell capping off ∂T . Each lies in M as a copy of $\hat{T}$ in

$$r \cdot (S^3 - U(K)) \subset B^4 \quad (\text{where } 0 < r < 1)$$

concentric with one of the given fibers of $S^3 - U(K)$, a copy of B^2 in

$$B^2 \times r \cdot \partial D^2 \subset B^2 \times D^2 = h,$$

and an annulus connecting the boundaries of the first two in the cone over $U(K)$ in B^4 , as suggested by Figure 1.

FIGURE 1

Away from the exceptional element g_0 , the decomposition map behaves like the projection of $\hat{T} \times (E^2 - 0)$ to $E^2 - 0$. One can see that the winding function α_b is locally constant at $b = p(g_0)$ by observing the existence of a retraction $r_0: M \rightarrow g_0$ such that $r_0(M \cap B^4) \subset g_0 \cap B^4$ and $r_0|h \cap M$ acts like the projection of $B^2 \times \text{Int } D^2$ to its core $B^2 \times 0 \subset g_0$. For geometric reasons, $r_0|g': g' \rightarrow g_0$ is a degree-one map, for each $g' \in G[E]$. Obviously, for each $g \neq g_0$, $H_1(g') = H_1(\hat{T}) \neq 0$.

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