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MULTIPLIERS ON WEIGHTED L_p-SPACES OVER CERTAIN TOTALLY DISCONNECTED GROUPS

BY

C. W. ONNEWEER

ABSTRACT. Let G be a locally compact totally disconnected group with a suitable sequence of open compact subgroups. We prove a multiplier theorem for certain weighted L_p-spaces over G, which is a generalization of a Hörmander-type multiplier theorem for L_p-spaces over a local field, due to Taibleson.

1. Introduction. For functions on $\mathbf{R}$ the multiplier theorem of Hörmander can be formulated most conveniently using the following notation; cf. [**7**]. Let $m \in L_\infty(\mathbf{R})$, $s \in \mathbf{R}$ and $l \in \mathbf{N}$; we say that $m \in M(s,l)$ if for all integers α with $0 \leq \alpha \leq l$ we have

$$\sup\left\{\left(R^{\alpha s-1}\int_{R<|x|<2R}|D^\alpha m(x)|^s\,dx\right)^{1/s};\ R>0\right\}<\infty.$$

In [**4**, Theorem 2.5] Hörmander proved the following theorem, stated here for functions on $\mathbf{R}$ instead of on $\mathbf{R}^n$.

THEOREM H. *Let $m \in M(2,l)$ for some $l \geq 1$ and let $1 < p < \infty$. Then there exists a constant $C > 0$ so that $\|Tf\|_p \leq C\|f\|_p$, where Tf is defined by $(Tf)^\wedge = m\hat{f}$ for suitable $f \in L_p(\mathbf{R})$.*

In 1979 Kurtz and Wheeden extended this theorem from L_p-spaces to weighted L_p-spaces (over $\mathbf{R}^n$), where the weight functions either satisfied the Muckenhoupt A_p condition or else were suitable powers of $|x|$. Of particular interest to us is the following result [**7**, Theorem 3], again formulated for functions on $\mathbf{R}$ instead of on $\mathbf{R}^n$.

THEOREM KW. *Let $1 < s \leq 2$ and $m \in M(s,1)$. If $1 < p < \infty$ and $-1 < \alpha < p-1$, then there exists a $C > 0$ so that*

$$\|Tf\|_{p,|x|^\alpha} \leq C\|f\|_{p,|x|^\alpha}.$$

Recently several attempts have been made, among others by the author in [**9**], to develop a theory for differentiability of functions on certain groups, including the Vilenkin groups and the local fields. Attempts to obtain a Hörmander-type multiplier theorem on such groups, using as an assumption the condition $M(s,l)$ but with the new differentiation concept, have so far been unsuccessful. However,

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for functions on a local field Taibleson [**17**, Theorem 1] formulated a condition that implied a multiplier theorem which in many respects resembles Hörmander's theorem for functions on $\mathbf{R}^n$. Taibleson used his multiplier theorem to prove an M. Riesz-type theorem for functions on the ring of integers of a local field [**17**, Corollary 1]: if $S_n f$ denotes the nth partial sum of the Fourier series of such a function then $\|S_n f\|_p \leq C\|f\|_p$, provided $1 < p < \infty$. This and a subsequent result [**17**, Corollary 4] were used by Hunt and Taibleson in their proof of the a.e. convergence of the Fourier series of an L_p-function on the ring of integers of a local field, $1 < p < \infty$; cf. [**5**].

In the present paper we have extended Taibleson's multiplier theorem in several directions. First of all, we consider functions defined on a class of totally disconnected groups that include the additive group of a local field as a special case. We also have obtained a larger class of multipliers than Taibleson did in [**17**]; cf. Corollary 2(ii). Moreover, our main result yields multipliers for certain weighted L_p-spaces, where the weight functions are similar to the powers of $|x|$ considered by Kurtz and Wheeden; cf. Corollaries 1 and 2(i).

We mention here that our results can be used to give an alternative proof to Gosselin's proof of an M. Riesz-type inequality (called Paley's theorem in [**2**]) for functions on order-bounded Vilenkin groups. This proof closely resembles Taibleson's proof of Corollary 1 in [**17**] and we therefore omit it. Since the multiplier theorems in this paper allow certain weighted L_p-norms it is likely that the Riesz inequality can be extended to such weighted L_p-spaces. This will be considered elsewhere as will be applications of our multiplier results to certain singular integral operators on L_p-spaces with or without weight functions.

The proof of our main theorem is based on techniques first used by Hörmander in [**4**] and with modifications that, for functions on $\mathbf{R}$ or $\mathbf{R}^n$, were introduced by Hirschman [**3**], Igari [**6**] and Kurtz and Wheeden [**7**]. We conclude the Introduction with a brief outline of this paper. In §2 we present most of the definitions and notation needed. In §3 we prove some lemmas that will be used in §4, in which we prove our main theorem. In the final section we give some results that complement the main theorem.

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2. Definitions and notation. Throughout this paper G will denote a locally compact abelian topological group with a suitable collection of compact open subgroups in the sense of Edwards and Gaudry [**1**, §4.1]. This means that there exists a sequence $(G_n)_{-\infty}^{\infty}$ such that

(i) each G_n is an open compact subgroup of G,

(ii) $G_{n+1} \subsetneqq G_n$ and $\operatorname{order}(G_n/G_{n+1}) < \infty$,

(iii) $\bigcup_{-\infty}^{\infty} G_n = G$ and $\bigcap_{-\infty}^{\infty} G_n = \{0\}$.

Moreover, we shall assume that G is order-bounded, that is,

$$\sup\{\operatorname{order}(G_n/G_{n+1});\ n \in \mathbf{Z}\} < \infty.$$

Several examples of such groups are given in [**1**, §4.1.2]. We mention one additional example that is of special interest to us; namely, G is the additive group of a local field K, i.e., $G = (K, +)$. With the notation of [**15** or **18**], if we take $G_n = \mathfrak{p}^n$ then it is easy to show that conditions (i), (ii) and (iii) are satisfied.

Let Γ denote the dual group of G, and for each $n \in \mathbf{Z}$ let Γ_n denote the annihilator of G_n, that is,

$$\Gamma_n = \{\gamma \in \Gamma; \gamma(x) = 1 \text{ for all } x \in G_n\}.$$

Then we have (cf. [**1**, §4.1.4])

(i)* each Γ_n is an open compact subgroup of Γ,

(ii)* $\Gamma_n \subsetneqq \Gamma_{n+1}$ and $\operatorname{order}(\Gamma_{n+1}/\Gamma_n) = \operatorname{order}(G_n/G_{n+1})$,

(iii)* $\bigcup_{-\infty}^{\infty} \Gamma_n = \Gamma$ and $\bigcap_{-\infty}^{\infty} \Gamma_n = \{1\}$.

If we choose Haar measures μ on G and λ on Γ so that $\mu(G_0) = \lambda(\Gamma_0) = 1$, then $\mu(G_n) = (\lambda(\Gamma_n))^{-1} := (m_n)^{-1}$ for each $n \in \mathbf{Z}$. We mention here two simple inequalities for the m_n that will be used frequently. For each $\alpha > 0$ and $k \in \mathbf{Z}$ we have

$$\sum_{n=k}^{\infty} (m_n)^{-\alpha} \leq C(m_k)^{-\alpha}, \tag{1}$$

$$\sum_{n=-\infty}^{k} (m_n)^{\alpha} \leq C(m_k)^{\alpha}. \tag{2}$$

Here, as in the sequel, C denotes a constant that may change in value from one occurrence to the next. Inequality (1) follows from the fact that $m_{n+1} \geq 2m_n$ for each $n \in \mathbf{Z}$, whereas (2) is a consequence of the order-boundedness of G.

For p with $1 \leq p \leq \infty$ we shall denote its conjugate by p'; thus $1/p + 1/p' = 1$. For an arbitrary set A we denote its characteristic function by ξ_A. The symbols $^\wedge$ and $^\vee$ will be used to denote the Fourier and inverse Fourier transform, respectively. It is easy to see that for each $n \in \mathbf{Z}$ we have

$$(\xi_{G_n})^\wedge = (\lambda(\Gamma_n))^{-1}\xi_{\Gamma_n} := F_n.$$

We now define the weighted L_p-spaces that are of interest to us. For $\alpha \in \mathbf{R}$ we define the functions v_α: $G \to \mathbf{R}$ and w_α: $\Gamma \to \mathbf{R}$ by

$$v_\alpha(x) = \begin{cases} (m_n)^{-\alpha} & \text{if } x \in G_n \backslash G_{n+1} \qquad (n \in \mathbf{Z}), \\ 0 & \text{if } x = 0, \end{cases}$$

and

$$w_\alpha(\gamma) = \begin{cases} (m_n)^{\alpha} & \text{if } \gamma \in \Gamma_{n+1} \backslash \Gamma_n \qquad (n \in \mathbf{Z}), \\ 0 & \text{if } \gamma = 1. \end{cases}$$

If G is the additive group of a local field, $v_1(x)$ is equal to the nonarchimedian norm $|x|$ of $x \in G$ and, hence, $v_\alpha(x) = |x|^\alpha$ for all $x \in G$. The same is true for the functions w_α. We shall denote the L_p-spaces with respect to the measures $\mu_\alpha = v_\alpha \, d\mu$ on G and $\lambda_\alpha = w_\alpha \, d\lambda$ on Γ by $L_{p,\alpha}(G)$ and $L_{p,\alpha}(\Gamma)$, respectively. Also, for f: $G \to \mathbf{C}$ or g: $\Gamma \to \mathbf{C}$ and $1 \leq p < \infty$ we set

$$\|f\|_{p,\alpha} = \left(\int_G |f(x)|^p v_\alpha(x) \, d\mu \right)^{1/p},$$

$$\|g\|_{p,\alpha} = \left(\int_\Gamma |g(\gamma)|^p \, w_\alpha(\gamma) \, d\lambda \right)^{1/p}.$$

We shall use the notation $\|\cdot\|_{p,\alpha}$ for both norms, because it will always be clear from the context which of the norms is meant.

In order to give the definition of a (Fourier) multiplier we first introduce the space $S(G)$ of so-called test functions on G. A function $\phi\colon G \to \mathbf{C}$ belongs to $S(G)$ if ϕ has compact support and if ϕ is constant on the cosets of some subgroup G_n (n depending on ϕ) of G. Then the Fourier transform maps $S(G)$ one-to-one onto $S(\Gamma)$, with $S(\Gamma)$ defined like $S(G)$; cf. [**18**, p. 37]. Also, a simple computation, like in the proof of Lemma 1(a), shows that for $\alpha > -1$ each $\xi_{G_n} \in L_{p,\alpha}(G)$ and, hence, $S(G) \subset L_{p,\alpha}(G)$. Moreover, it is easy to see that for $\alpha > -1$ and $1 \le p < \infty$, $S(G)$ is a dense subset of $L_{p,\alpha}(G)$.

DEFINITION 1. Let $\alpha > -1$ and $1 \le p < \infty$. For $f \in L_\infty(\Gamma)$ and $\phi \in S(G)$ define $T\phi$ by $(T\phi)^\wedge = f\hat{\phi}$. The function f is a multiplier on $L_{p,\alpha}(G)$ if there exists a constant $C > 0$ so that for all $\phi \in S(G)$ we have $\|T\phi\|_{p,\alpha} \le C\|\phi\|_{p,\alpha}$.

REMARK 1. For $f \in L_\infty(\Gamma)$ and $k \in \mathbf{Z}$ let $f_k = f\xi_{\Gamma_k}$ and for $\phi \in S(G)$ let $T_k\phi$ be defined by $(T_k\phi)^\wedge = f_k\hat{\phi}$. Since each $\hat{\phi} \in S(\Gamma)$ has compact support, $T\phi = T_k\phi$ for k sufficiently large. Thus to prove that f is a multiplier on $L_{p,\alpha}(G)$ is equivalent to proving that the operators T_k are all bounded on $S(G)$ with uniformly bounded operator norms. Also, note that $T_k\phi = (f_k)^\vee * \phi$, the convolution of two functions on G.

We now state Taibleson's multiplier theorem referred to in §1, using the notation introduced here; see [**17**, Theorem 1, or **18**, p. 218, Theorem (1.1)].

THEOREM T. *Let $G = (K, +)$ be the additive group of a local field K, let $f \in L_\infty(\Gamma)$, and assume there exist constants B, $\varepsilon > 0$ so that for all $l \in \mathbf{Z}$ we have*

$$\int_{\Gamma_l}\int_{\Gamma_{l+1}\setminus\Gamma_l} |f(\xi+\eta) - f(\xi)|^2\, d\lambda(\xi) w_{-(2+\varepsilon)}(\eta)\, d\lambda(\eta) \le B^2(m_l)^{-\varepsilon}. \tag{3}$$

Then f is a multiplier on $L_p(G)$ for $1 < p < \infty$.

One of the crucial steps in the proof of Theorem T consists in showing that inequality (3) implies an inequality for the inverse Fourier transform of the functions $f_k = f\xi_{\Gamma_k}$; see [**17**, Lemma 5, or **18**, p. 221, Lemma (1.8)]. We state this result here in the form of a lemma.

LEMMA T1. *Let $G = (K, +)$ and let $f \in L_\infty(\Gamma)$ satisfy the hypothesis of Theorem* T. *Then there exists a $C > 0$ so that for all k, $l \in \mathbf{Z}$ we have*

$$\sup\left\{\int_{G\setminus G_l} |(f_k)^\vee(x+y) - (f_k)^\vee(x)|\, d\mu(x);\ y \in G_l\right\} \le C(B + \|f\|_\infty). \tag{4}$$

The other important step in the proof of Theorem T consists in proving the following

LEMMA T2. *Let $G = (K, +)$; if $f \in L_\infty(\Gamma)$ satisfies* (4) *then f is a multiplier on $L_p(G)$ for $1 < p < \infty$.*

The main result of this paper is a far-reaching generalization of Lemma T2. We state it here as Theorem 1; a proof will be given in §4. To simplify its formulation we shall say that a function $f \in L_\infty(\Gamma)$ satisfies condition $C(k,r)$ for some $k \in \mathbf{Z}$ and $r \in [1,\infty)$ if there exist $C,\ \varepsilon > 0$ so that for all $l, n \in \mathbf{Z}$ with $n < l$ we have

$$\sup\left\{\left(\int_{G_n\setminus G_{n+1}} |(f_k)^\vee(x-y) - (f_k)^\vee(x)|^r\, d\mu(x)\right)^{1/r};\ y \in G_l\right\} \leq C(m_n)^{\varepsilon+1/r'}(m_l)^{-\varepsilon} \quad \text{if } 1 < r < \infty, \tag{5}$$

and there exists $C > 0$ so that for all $l \in \mathbf{Z}$ we have

$$\sup\left\{\int_{G\setminus G_l} |(f_k)^\vee(x-y) - (f_k)^\vee(x)|\, d\mu(x);\ y \in G_l\right\} \leq C \quad \text{if } r = 1. \tag{6}$$

REMARK 2. If inequality (5) holds for some $k \in \mathbf{Z}$ and $r \in [1,\infty)$ then for each $l \in \mathbf{Z}$ and $y \in G_l$ we have

$$\left(\int_{G\setminus G_l} |(f_k)^\vee(x-y) - (f_k)^\vee(x)|^r\, d\mu(x)\right)^{1/r} \leq C(m_l)^{1/r'}.$$

To prove this, observe that for $y \in G_l$

$$\begin{aligned}
&\left(\int_{G\setminus G_l} |(f_k)^\vee(x-y) - (f_k)^\vee(x)|^r\, d\mu(x)\right)^{1/r} \\
&\quad = \left(\sum_{n=-\infty}^{l-1} \int_{G_n\setminus G_{n+1}} |(f_k)^\vee(x-y) - (f_k)^\vee(x)|^r\, d\mu(x)\right)^{1/r} \\
&\quad \leq C\left(\sum_{n=-\infty}^{l-1} ((m_n)^{\varepsilon+1/r'}(m_l)^{-\varepsilon})^r\right)^{1/r} \\
&\quad \leq C(m_l)^{-\varepsilon}(m_l)^{\varepsilon+1/r'} = C(m_l)^{1/r'},
\end{aligned}$$

by inequality (2). Also, it follows immediately from Hölder's inequality that if condition $C(k,r)$ holds for some $r \in [1,\infty)$ then $C(k,\tilde{r})$ holds for all $\tilde{r}$ with $1 \leq \tilde{r} \leq r$.

THEOREM 1. (i) *Let $f \in L_\infty(\Gamma)$ and assume that condition $C(k,r)$ holds for all $k \in \mathbf{Z}$, for some r with $1 < r < \infty$, and with constants C and ε independent of $k \in \mathbf{Z}$. If f is a multiplier on $L_{2,\alpha_0}(G)$ for some α_0 with $-1/r' < \alpha_0 < 1/r'$, then f is a multiplier on $L_{p,\alpha}(G)$ for all p, α such that $1 < p < \infty$ and $-|\alpha_0| \leq \alpha \leq (p-1)|\alpha_0|$.*

(ii) *If $f \in L_\infty(\Gamma)$ and if $C(k,1)$ holds for all $k \in \mathbf{Z}$ and with C independent of $k \in \mathbf{Z}$, then f is a multiplier on $L_p(G)$ for $1 < p < \infty$.*

3. Preliminary results. For future reference we give here a lemma in which some useful properties of the measures $\mu_\alpha = v_\alpha\, d\mu$ have been collected.

LEMMA 1. *Let $\alpha > -1$, $x \in G$ and $k \in \mathbf{Z}$.*

(a) $\mu_\alpha(G_k) \approx (m_k)^{-(1+\alpha)}$.

(b) *If $\alpha \leq 0$ and $(x+G_k)^* := (x+G_k)\setminus\{0\}$, then*

$$\mu_\alpha(x+G_k) \leq C(m_k)^{-1} \inf\{v_\alpha(y);\ y \in (x+G_k)^*\}.$$

(c) $\mu_\alpha(x+G_k) \le C\mu_\alpha(x+G_{k+1})$.
(d) $\mu_\alpha(G_k) \le C\mu_\alpha(G_k\backslash G_{k+1})$.

PROOF. (a) We clearly have

$$\mu_\alpha(G_k) \ge \mu_\alpha(G_k\backslash G_{k+1}) = (m_k)^{-\alpha}\mu(G_k\backslash G_{k+1}) \ge \tfrac{1}{2}(m_k)^{-(1+\alpha)}.$$

Also

$$\begin{aligned}\mu_\alpha(G_k) &= \int_{G_k} v_\alpha(t)\,d\mu = \sum_{j=k}^{\infty}\int_{G_j\backslash G_{j+1}} v_\alpha(t)\,d\mu \\ &\le \sum_{j=k}^{\infty}(m_j)^{-\alpha}(m_j)^{-1} \le C(m_k)^{-(1+\alpha)},\end{aligned}$$

by inequality (1).

(b) If $x+G_k = G_k$ then $v_\alpha(y) \ge (m_k)^{-\alpha}$ for all $y \in (G_k)^*$. In this case (b) follows immediately from (a). If $x+G_k \ne G_k$, then $x+G_k \subset G_l\backslash G_{l+1}$ for some $l<k$ and, hence, $v_\alpha(y) = (m_l)^{-\alpha}$ for all $y \in x+G_k$. In this case we have

$$\mu_\alpha(x+G_k) = \int_{x+G_k} v_\alpha(t)\,d\mu = (m_l)^{-\alpha}(m_k)^{-1} = (m_k)^{-\alpha}v_\alpha(y),$$

whenever $y \in x+G_k$.

(c) If $x \in G_{k+1}$ then $x+G_{k+1} = G_{k+1}$ and $x+G_k = G_k$. Then (c) follows from (a) and the order-boundedness of G. If $x \in G_k\backslash G_{k+1}$ we have

$$\mu_\alpha(x+G_{k+1}) = (m_k)^{-\alpha}\mu(x+G_{k+1}) \ge C(m_k)^{-(1+\alpha)} \ge C\mu_\alpha(G_k),$$

according to (a). Finally, if $x \in G_l\backslash G_{l+1}$ for some $l<k$ then $x+G_k$ and $x+G_{k+1}$ are both subsets of $G_l\backslash G_{l+1}$ and we have

$$\mu_\alpha(x+G_k) = (m_l)^{-\alpha}\mu(x+G_k) \le C(m_l)^{-\alpha}\mu(x+G_{k+1}) = C\mu_\alpha(x+G_{k+1}).$$

This completes the proof of (c).

(d) According to (a) we have

$$\mu_\alpha(G_k) \le C(m_k)^{-(1+\alpha)} \le 2C\mu_\alpha(G_k\backslash G_{k+1}).$$

The next lemma is a Calderón-Zygmund-type decomposition theorem for functions in $L_{1,\alpha}(G)$. A local field version of this theorem for the unweighted case, i.e., $\alpha = 0$, was proved by Phillips in [**12**]; see also [**18**, p. 148]. In [**6**] Igari proved a result similar to our Lemma 2 for functions defined on, among others, $\mathbf{R}$ instead of G. Since the proof of Lemma 2 is by and large the same as Igari's proof of Lemma 4 in [**6**], we shall only give the definition of the components in which a given $\phi \in L_{1,\alpha}(G)$ is decomposed and the proof for one inequality, (10), which corresponds to inequality (3.22) in [**6**].

LEMMA 2. *Let $-1<\alpha\le 0$, let $\phi \in L_{1,\alpha}(G)$, and let $\sigma>0$ be given. Then there exist functions $(\phi_j)_0^\infty$ such that*
(i) $\phi = \sum_{j=0}^\infty \phi_j$,
(ii) $\phi_j \in L_{1,\alpha}(G)$ *for each* $j \ge 0$,
(iii) $\sum_{j=0}^\infty \|\phi_j\|_{1,\alpha} \le C\|\phi\|_{1,\alpha}$,
(iv) $|\phi_0(x)| \le C\sigma$ *for a.e.* $x \in G$,
(v) *there exist disjoint sets* $I_j = x_j + G_{m(j)}$ *such that* $\mathrm{supp}(\phi_j) \subset I_j$ *for* $j \in \mathbf{N}$,

(vi) $\sum_{j=1}^{\infty} \mu_\alpha(I_j) \leq \sigma^{-1}\|\phi\|_{1,\alpha}$,
(vii) $\int_{I_j} \phi_j(x)\, d\mu = 0$ *for* $j \in \mathbf{N}$.

PROOF. It follows from Lemma 1(a) that $\mu_\alpha(G_k) \to \infty$ as $k \to -\infty$. Thus there exists a $k_0 \in \mathbf{Z}$ such that

$$\min\{1, (2C_0)^{-1}\}\mu_\alpha(G_0) \geq \sigma^{-1}\|\phi\|_{1,\alpha}, \tag{7}$$

where C_0 is chosen so that $\mu_\alpha(G_k) \leq C_0(m_k)^{-(1+\alpha)}$ for all $k \in \mathbf{Z}$; cf. Lemma 1(a). Then for $l \leq k_0$ we have

$$\begin{aligned} \mu_\alpha(G_l \backslash G_{l+1}) &\geq \tfrac{1}{2}(m_l)^{-(1+\alpha)} \geq \tfrac{1}{2}(m_{k_0})^{-(1+\alpha)} \\ &\geq (2C_0)^{-1}\mu_\alpha(G_{k_0}) \geq \sigma^{-1}\|\phi\|_{1,\alpha}. \end{aligned} \tag{8}$$

Now consider those cosets $x + G_{k_0+1}$ in G_{k_0} for which

$$(\mu_\alpha(x + G_{k_0+1}))^{-1} \int_{x+G_{k_0+1}} |\phi(t)|\, d\mu_\alpha \geq \sigma, \tag{9}$$

and for each $l < k_0$ consider those cosets $x + G_{l+1}$ in $G_l \backslash G_{l+1}$ for which

$$(\mu_\alpha(x + G_{l+1}))^{-1} \int_{x+G_{l+1}} |\phi(t)|\, d\mu_\alpha \geq \sigma. \tag{10}$$

Call these cosets $I_{1,j}$. Then we have

$$\sigma\mu_\alpha(I_{1,j}) \leq \int_{I_{1,j}} |\phi(t)|\, d\mu_\alpha \leq \|\phi\|_{1,\alpha} \leq C\sigma\mu_\alpha(I_{1,j}),$$

with the last inequality following from (7) and (8) and Lemma 1(c). For the sets $x + G_{k_0+1}$ in G_{k_0} for which (9) does not hold we consider subsets that are cosets of G_{k_0+2}; similarly for the sets $x + G_{l+1}$ in $G_l \backslash G_{l+1}$ for which (10) does not hold we consider subsets that are cosets of G_{l+2}. From among these sets we select those that satisfy an inequality analogous to (9) or (10). Call these sets $I_{2,j}$. Continuing this process we obtain a countable collection of disjoint sets, say $(I_j)_1^\infty$ with $I_j = x_j + G_{m(j)}$, and so that for each $j \in \mathbf{N}$ we have

$$\sigma \leq (\mu_\alpha(I_j))^{-1} \int_{I_j} |\phi(t)|\, d\mu_\alpha \leq C\sigma. \tag{11}$$

Inequality (11) corresponds to (3.19) in [**6**]. We now prove the analogue of (3.22) in [**6**]: for each $j \in \mathbf{N}$ we have

$$(\mu(I_j))^{-1} \int_{I_j} |\phi(t)|\, d\mu \leq C(\mu_\alpha(I_j))^{-1} \int_{I_j} |\phi(t)|\, d\mu_\alpha. \tag{12}$$

We distinguish two cases.

(i) If $I_j = G_n$ for some $n \in \mathbf{Z}$ then, according to Lemma 1(a),

$$\begin{aligned} (\mu(G_n))^{-1} \int_{G_n} |\phi(t)|\, d\mu &\leq C(\mu_\alpha(G_n))^{-1}(m_n)^{-\alpha} \int_{G_n} |\phi(t)|\, d\mu \\ &\leq C(\mu_\alpha(G_n))^{-1} \int_{G_n} |\phi(t)| v_\alpha(t)\, d\mu, \end{aligned}$$

because for $t \in G_n \backslash \{0\}$ and $\alpha \leq 0$ we have $(m_n)^{-\alpha} \leq v_\alpha(t)$.

(ii) If $I_j = x + G_n$ for some $n \in \mathbf{Z}$ and $x \in G \backslash G_n$ then $x + G_n \subset G_l \backslash G_{l+1}$ for some $l < n$. In this case we have

$$(\mu(I_j))^{-1} \int_{I_j} |\phi(t)|\, d\mu = (\mu_\alpha(I_j))^{-1}(m_l)^{-\alpha} \int_{I_j} |\phi(t)|\, d\mu$$
$$= (\mu_\alpha(I_j))^{-1} \int_{I_j} |\phi(t)| v_\alpha(t)\, d\mu.$$

This completes the proof of (12). We now define the functions $(\phi_j)_0^\infty$ as follows:

$$\phi_0(x) = \begin{cases} (\mu(I_j))^{-1} \int_{I_j} \phi(t)\, d\mu & \text{if } x \in I_j,\ j \geq 1, \\ \phi(x) & \text{if } x \notin D_\sigma, \end{cases}$$

where $D_\sigma = \bigcup_1^\infty I_j$, and for $j \geq 1$ we set

$$\phi_j(x) = \begin{cases} \phi(x) - \phi_0(x) & \text{if } x \in I_j, \\ 0 & \text{if } x \notin I_j. \end{cases}$$

As we already mentioned, the proof that the functions $(\phi_j)_0^\infty$ satisfy the conditions of our lemma is virtually the same as the proof of Lemma 4 in [**6**] and will be omitted.

4. Multipliers on $L_{p,\alpha}(G)$. The first proposition of this section is a duality theorem for multipliers on $L_{p,\alpha}(G)$-spaces.

PROPOSITION 1. *Let $f \in L_\infty(\Gamma)$, $1 < p < \infty$, $-1 < \alpha < p - 1$ and $k \in \mathbf{Z}$. Assume that there exists a $C > 0$ so that for all $\phi \in S(G)$ we have*

$$\|T_k \phi\|_{p,\alpha} \leq C \|\phi\|_{p,\alpha}. \tag{13}$$

Let $q = p'$ and $\beta = (1-q)\alpha$. Then we have for all $\phi \in S(G)$ and with the same constant C as in (13),

$$\|T_k \phi\|_{q,\beta} \leq C \|\phi\|_{q,\beta}.$$

Thus f is a multiplier on $L_{p,\alpha}(G)$ if and only if f is a multiplier on $L_{q,\beta}(G)$.

PROOF. Since $S(G)$ is dense in $L_{p,\alpha}(G)$ we can define $T_k g$ for all $g \in L_{p,\alpha}(G)$ and so that (13) remains valid with ϕ replaced by g. For $\phi \in S(G)$ and with $\delta = \beta/q$ we have

$$\|T_k\phi\|_{q,\beta} = \|T_k \phi v_\delta\|_q = \sup \left\{ \left| \int_G T_k\phi(x) v_\delta(x) \psi(x)\, d\mu \right| ; \psi \in S(G),\ \|\psi\|_p = 1 \right\}.$$

A straightforward computation shows that

$$\int_G T_k \phi(x) v_\delta(x) \psi(x)\, d\mu = \int_G \phi(x) T_k g(x)\, d\mu,$$

where $g(x) = v_\delta(x)\psi(-x) \in L_{p,\alpha}(G)$. Thus, Hölder's inequality implies that

$$\left| \int_G T_k\phi(x) v_\delta(x)\psi(x)\, d\mu \right| \leq \left(\int_G |\psi(x) v_\delta(x)|^q\, d\mu \right)^{1/q} \left(\int_G |T_k g(x) v_{-\delta}(x)|^p\, d\mu \right)^{1/p}$$
$$= \|\phi\|_{q,\beta} \|T_k g\|_{p,\alpha} \leq C \|\phi\|_{q,\beta} \|g\|_{p,\alpha} = C \|\phi\|_{q,\beta} \|\psi\|_p.$$

Therefore, we may conclude that

$$\|T_k \phi\|_{q,\beta} \leq C \|\phi\|_{q,\beta},$$

which completes the proof of Proposition 1.

PROPOSITION 2. *Let $f \in L_\infty(\Gamma)$ and assume that f satisfies condition $C(k,r)$ for some $k \in \mathbf{Z}$ and with $r \in (1,\infty)$. If T_k is of type $(2,2)$ on $L_{2,\alpha}(G)$ for some α with $-1/r' < \alpha \leq 0$ then T_k is of weak type $(1,1)$ on $L_{1,\alpha}(G)$. Moreover, if $C(k,1)$ holds then T_k is of weak type $(1,1)$ on $L_1(G)$.*

PROOF. Take any $\phi \in L_{1,\alpha}(G)$. Fix $\sigma > 0$ and apply the Calderón-Zygmund decomposition, as given in Lemma 2, to ϕ. Using the notation of that lemma we obtain

$$\phi = \phi_0 + \psi \quad \text{with } \psi = \sum_{j=1}^{\infty} \phi_j.$$

Then

$$\begin{aligned}&\{x \in G;\ |T_k\phi(x)| > \sigma\}\\ &\qquad \subset \{x \in G;\ |T_k\phi_0(x)| > \sigma/2\} \cup \{x \in G;\ |T_k\psi(x)| > \sigma/2\} := E_\sigma \cup F_\sigma.\end{aligned}$$

For the set E_σ we have

$$\begin{aligned}\mu_\alpha(E_\sigma) &= \mu_\alpha(\{x \in G; 4\sigma^{-2}|T_k\phi_0(x)| > 1\})\\ &\leq 4\sigma^{-2}\|T_k\phi_0\|_{2,\alpha}^2 \leq C\sigma^{-2}\|\phi_0\|_{2,\alpha}^2.\end{aligned}$$

Hence, it follows from Lemma 2(iii) and (iv) that

$$\mu_\alpha(E_\sigma) \leq C\sigma^{-1}\int_G \phi_0(x)\,d\mu_\alpha \leq C\sigma^{-1}\|\phi\|_{1,\alpha}.$$

Next we observe that

$$\begin{aligned}\mu_\alpha(F_\sigma) &= \mu_\alpha(F_\sigma \cap D_\sigma) + \mu_\alpha(F_\sigma \backslash D_\sigma)\\ &\leq \mu_\alpha(D_\sigma) + 2\sigma^{-1}\int_{G\backslash D_\sigma} |T_k\psi(x)|\,d\mu_\alpha\\ &\leq \sigma^{-1}\|\phi\|_{1,\alpha} + 2\sigma^{-1}\int_{G\backslash D_\sigma} |(f_k)^\vee * \psi(x)|\,d\mu_\alpha,\end{aligned}$$

by Lemma 2(vi). We now derive a suitable inequality for the last integral that we shall denote by J. Lemma 2(v) and (vii) imply that

$$\begin{aligned}J &= \int_{G\backslash D_\sigma}\left|\sum_{j=1}^{\infty}\int_{I_j}(f_k)^\vee(x-y)\phi_j(y)\,d\mu(y)\right| d\mu_\alpha(x)\\ &\leq \sum_{j=1}^{\infty}\int_{G\backslash D_\sigma}\int_{I_j}|(f_k)^\vee(x-y)-(f_k)^\vee(x-x_j)|\,|\phi_j(y)|\,d\mu(y)\,d\mu_\alpha(x)\\ &\leq \sum_{j=1}^{\infty}\int_{I_j}|\phi_j(y)|\int_{G\backslash I_j}|(f_k)^\vee(x-y)-(f_k)^\vee(x-x_j)|v_\alpha(x)\,d\mu(x)\,d\mu(y)\\ &\leq \sum_{j=1}^{\infty}\int_{G_{m(j)}}|\phi_j(y+x_j)|\int_{G\backslash G_{m(j)}}|(f_k)^\vee(x-y)-(f_k)^\vee(x)|\\ &\qquad\qquad\qquad\qquad\qquad\qquad \cdot v_\alpha(x+x_j)\,d\mu(x)\,d\mu(y).\end{aligned}$$

We now consider the inner integrals in the above sum and denote them by $K_{j,\alpha}$. It follows from Hölder's inequality that if $r \in (1,\infty)$ then

$$\begin{aligned}
K_{j,\alpha} &= \sum_{n=-\infty}^{m(j)-1} \int_{G_n \backslash G_{n+1}} |(f_k)^\vee(x-y) - (f_k)^\vee(x)| v_\alpha(x+x_j)\, d\mu(x) \\
&\le \sum_{n=-\infty}^{m(j)-1} \left(\int_{G_n \backslash G_{n+1}} |(f_k)^\vee(x-y) - (f_k)^\vee(x)|^r\, d\mu(x) \right)^{1/r} \\
&\qquad\qquad \cdot \left(\int_{G_n \backslash G_{n+1}} (v_\alpha(x+x_j))^{r'}\, d\mu(x) \right)^{1/r'} \\
&\le \sum_{n=-\infty}^{m(j)-1} \left(\int_{G_n \backslash G_{n+1}} |(f_k)^\vee(x-y) - (f_k)^\vee(x)|^r\, d\mu(x) \right)^{1/r} (\mu_{\alpha r'}(x_j + G_n))^{1/r'}.
\end{aligned}$$

Applying inequality (5) and Lemma 1(b) we see that for $y \in G_{m(j)}$,

$$\begin{aligned}
K_{j,\alpha} &\le C \sum_{n=-\infty}^{m(j)-1} (m_n)^{\varepsilon + 1/r'} (m_{m(j)})^{-\varepsilon} ((m_n)^{-1} \inf\{v_{\alpha r'}(t); t \in (x_j + G_n)^*\})^{1/r'} \\
&\le C (m_{m(j)})^{-\varepsilon} \sum_{n=-\infty}^{m(j)-1} (m_n)^{\varepsilon} \inf\{v_\alpha(t); t \in (x_j + G_{m(j)})^*\} \\
&\le C \inf\{v_\alpha(t); t \in (x_j + G_{m(j)})^*\}.
\end{aligned}$$

Substituting this inequality for $K_{j,\alpha}$ into the inequality derived earlier for J we see that

$$\begin{aligned}
J &\le C \sum_{j=1}^{\infty} \int_{G_{m(j)}} |\phi_j(y+x_j)| v_\alpha(y+x_j)\, d\mu(y) \\
&\le C \sum_{j=1}^{\infty} \int_{I_j} |\phi_j(t)|\, d\mu_\alpha(t) \\
&= C \sum_{j=1}^{\infty} \|\phi_j\|_{1,\alpha} \le C \|\phi\|_{1,\alpha},
\end{aligned}$$

according to Lemma 2(iii). Thus we may conclude that $\mu_\alpha(F_\sigma) \le C\sigma^{-1} \|\phi\|_{1,\alpha}$, and, consequently,

$$\mu_\alpha(\{x \in G; |T_k \phi(x)| > \sigma\}) \le C\sigma^{-1} \|\phi\|_{1,\alpha},$$

that is, T_k is of weak type $(1,1)$ on $L_{1,\alpha}(G)$.

In case $r = 1$ and $\alpha = 0$ we do not have to apply Hölder's inequality and Lemma 1(b) to show that $K_{j,0} \le C$, since this follows immediately from inequality (6). The remainder of the proof remains the same and we may conclude that T_k is of weak type $(1,1)$ on $L_1(G)$.

PROOF OF THEOREM 1. First, we consider the case where $1 < r < \infty$. Since each $f \in L_\infty(\Gamma)$ is a multiplier on $L_2(G) = L_{2,0}(G)$ it follows from Stein's interpolation theorem for weighted L_p-spaces [**13**, Theorem 2] and Proposition 1 that f is a multiplier on $L_{2,\alpha}(G)$ for all α with $-|\alpha_0| \le \alpha \le |\alpha_0|$. Now consider p and α with $1 < p < 2$ and $-|\alpha_0| \le \alpha \le 0$. Since each T_k, $k \in \mathbf{Z}$, is of type $(2,2)$ on $L_{2,\alpha}(G)$, it follows from Proposition 2 that T_k is also of weak type $(1,1)$ on

$L_{1,\alpha}(G)$. Therefore, the Stein-Weiss interpolation theorem for weighted L_p-spaces [**14**, Theorem (2.9)] implies that T_k is of type (p,p) on $L_{p,\alpha}(G)$. In view of Remark 1, the assumption that the constants ε and C in inequality (5) are independent of k implies that f is a multiplier on $L_{p,\alpha}(G)$. Next, consider p, α with $1 < p < 2$ and $0 < \alpha \leq (p-1)|\alpha_0|$. Choose p_0 such that $1 < p_0 < (p|\alpha_0| - 2\alpha)(|\alpha_0| - \alpha)^{-1}$. Then $1 < p_0 < p < 2$, so that each T_k is of type (p_0, p_0) on $L_{p_0}(G)$. Also, choose $\alpha_1 = (2\alpha - p_0\alpha)(p - p_0)^{-1}$; then $0 < \alpha_1 < |\alpha_0|$ and, hence, each T_k is of type $(2,2)$ on $L_{2,\alpha_1}(G)$. Since $1/p = (1-\theta)/p_0 + \theta/2$ and $\alpha/p = \theta\alpha_1/2$, where $\theta = (2p - 2p_0)(2p - pp_0)^{-1}$, it follows from Stein's interpolation theorem for weighted L_p-spaces that each T_k is of type (p,p) on $L_{p,\alpha}(G)$. As before, we may conclude that f is a multiplier on $L_{p,\alpha}(G)$. Finally, Proposition 1 enables us to extend this conclusion to all p and α such that $1 < p < \infty$ and $-|\alpha_0| < \alpha < (p-1)|\alpha_0|$. If $C(k,1)$ holds then the theorem follows immediately from Proposition 2 and the Marcinkiewicz interpolation theorem.

We conclude this section by stating explicitly the most interesting case of Theorem 1.

COROLLARY 1. *Let $f \in L_\infty(\Gamma)$. Assume $C(k,r)$ holds for all $k \in \mathbf{Z}$ and all $r \geq 1$ and with ε and C independent of k. If f is a multiplier on $L_{2,\alpha}(G)$ for all α with $-1 < \alpha < 1$, then f is a multiplier on $L_{p,\alpha}(G)$ for all p, α with $1 < p < \infty$ and $-1 < \alpha < p - 1$.*

5. Complimentary results. In view of the assumption in Theorem 1 that the given function $f \in L_\infty(\Gamma)$ must be a multiplier on $L_{2,\alpha}(G)$ for some value of α it is clearly of interest to find conditions for f that imply this assumption. One such result will be given in Proposition 3. However, we first prove a lemma that is the analogue on G of a result of Hirschman for functions on $\mathbf{R}$; see [**3**, Theorem 3a].

LEMMA 3. (a) *For $\alpha > 0$ and $x \in G$ with $x \neq 0$ we have*

$$v_\alpha(x) \approx \int_\Gamma |1 - \sigma(x)|^2 w_{-(1+\alpha)}(\sigma)\, d\lambda(\sigma) := I_\alpha(x).$$

(b) *For $\alpha > 0$ and $\phi \in L_2(G)$ we have*

$$\|\phi\|_{2,\alpha}^2 \approx \int_\Gamma \int_\Gamma |\hat{\phi}(\rho) - \hat{\phi}(\rho - \sigma)|^2 w_{-(1+\alpha)}(\sigma)\, d\lambda(\rho)\, d\lambda(\sigma).$$

PROOF. (a) Since $x \neq 0$, we have $x \in G_k \backslash G_{k+1}$ for some $k \in \mathbf{Z}$ and $v_\alpha(x) = (m_k)^{-\alpha}$. Also, for $x \in G_k$ and $\sigma \in \Gamma_k$ we have $\sigma(x) = 1$. Therefore,

$$\begin{aligned} I_\alpha(x) &= \sum_{l=k}^{\infty} \int_{\Gamma_{l+1} \backslash \Gamma_l} |1 - \sigma(x)|^2 w_{-(1+\alpha)}(\sigma)\, d\lambda \\ &\leq 4 \sum_{l=k}^{\infty} (m_l)^{-(1+\alpha)} \lambda(\Gamma_{l+1} \backslash \Gamma_l) \\ &\leq C \sum_{l=k}^{\infty} (m_l)^{-\alpha} = C(m_k)^{-\alpha}, \end{aligned}$$

by (1). On the other hand we obviously have

$$I_\alpha(x) \geq \int_{\Gamma_{k+1}\setminus\Gamma_k} |1-\sigma(x)|^2 w_{-(1+\alpha)}(\sigma)\, d\lambda$$
$$= (m_k)^{-(1+\alpha)} \int_\Gamma (\xi_{\Gamma_{k+1}} - \xi_{\Gamma_k})(\sigma)(2-\sigma(x)-\overline{\sigma(x)})\, d\lambda.$$

Since $(\xi_{\Gamma_n})^\vee(x) = m_n \xi_{G_n}(x)$ for each $n \in \mathbf{Z}$ we see that for $x \in G_k\setminus G_{k+1}$,

$$I_\alpha(x) \geq (m_k)^{-(1+\alpha)}(2(m_{k+1}-m_k)+m_k+m_k) \geq C(m_k)^{-\alpha},$$

which completes the proof of (a).

(b) Since for each $\sigma \in \Gamma$, $\phi(x)(1-\sigma(x)) = (\hat{\phi} - \tau_\sigma\hat{\phi})^\vee(x)$ a.e. on G, where $\tau_\sigma\hat{\phi}(\rho) = \hat{\phi}(\rho-\sigma)$, Plancherel's equality implies that

$$\int_\Gamma |\hat{\phi}(\rho) - \hat{\phi}(\rho-\sigma)|^2\, d\lambda(\rho) = \int_G |\phi(x)(1-\sigma(x))|^2\, d\mu.$$

Thus, after an interchange in the order of integration, it follows from (a) that

$$\int_\Gamma\int_G |\phi(x)(1-\sigma(x))|^2\, d\mu\; w_{-(1+\alpha)}(\sigma)\, d\lambda$$
$$= \int_G |\phi(x)|^2 \int_\Gamma |1-\sigma(x)|^2 w_{-(1+\alpha)}(\sigma)\, d\lambda\, d\mu$$
$$\approx \int_G |\phi(x)|^2 v_\alpha(x)\, d\mu = \|\phi\|_{2,\alpha}^2.$$

Therefore, (b) holds.

PROPOSITION 3. *Let $f \in L_\infty(\Gamma)$ and assume that for all $\rho, \tau \in \Gamma$ such that $w_1(\rho) \leq w_1(\tau)$ we have*

$$|f(\rho) - f(\tau)| \leq C w_1(\rho-\tau)(w_1(\tau))^{-1}.$$

If $-1 < \alpha < 1$ and $k \in \mathbf{Z}$ then for all $\phi \in S(G)$ we have

$$\|T_k\phi\|_{2,\alpha} \leq C\|\phi\|_{2,\alpha},$$

with C independent of k, that is, f is a multiplier on $L_{2,\alpha}(G)$.

PROOF. For $\alpha = 0$ the conclusion holds for each $f \in L_\infty(\Gamma)$. Assume $0 < \alpha < 1$. Since for $\phi \in S(G)$ and $k \in \mathbf{Z}$ we have $T_k\phi = (f_k)^\vee * \phi \in L_2(G)$, Lemma 3(b) implies that

$$\|T_k\phi\|_{2,\alpha} \leq C\left(\int_\Gamma\int_\Gamma |(f_k)(\rho)\hat{\phi}(\rho) - f_k(\rho-\sigma)\hat{\phi}(\rho-\sigma)|^2 \cdot w_{-(1+\alpha)}(\sigma)\, d\lambda(\rho)\, d\lambda(\sigma)\right)^{1/2}$$
$$\leq C(P^{1/2} + Q^{1/2}),$$

where

$$P = \int_\Gamma\int_\Gamma |f_k(\rho)|^2\, |\hat{\phi}(\rho) - \hat{\phi}(\rho-\sigma)|^2 w_{-(1+\alpha)}(\sigma)\, d\lambda(\rho)\, d\lambda(\sigma)$$

and

$$Q = \int_\Gamma \int_\Gamma |\hat{\phi}(\rho-\sigma)|^2 \, |f_k(\rho) - f_k(\rho-\sigma)|^2 w_{-(1+\alpha)}(\sigma)\, d\lambda(\rho)\, d\lambda(\sigma)$$
$$= \int_\Gamma |\hat{\phi}(\tau)|^2 \int_\Gamma |f_k(\rho) - f_k(\tau)|^2 w_{-(1+\alpha)}(\rho-\tau)\, d\lambda(\rho)\, d\lambda(\tau).$$

It follows immediately from Lemma 3(b) that

$$P \le C\|f\|_\infty^2 \|\phi\|_{2,\alpha}^2.$$

We now consider the inner integral in Q which we shall denote by $J(\tau)$. If $\tau \in \Gamma_{n+1}\backslash\Gamma_n$ for some $n \in \mathbf{Z}$ with $n \ge k$ then $f_k(\tau) = 0$ because $\operatorname{supp}(f_k) \subset \Gamma_k$. Thus in this case we have

$$J(\tau) \le \|f\|_\infty^2 \int_{\Gamma_k} w_{-(1+\alpha)}(\rho-\tau)\, d\lambda(\rho).$$

Furthermore, for $\rho \in \Gamma_k$ and $\tau \in \Gamma_{n+1}\backslash\Gamma_n$ with $n \ge k$ we have $\rho - \tau \in \Gamma_{n+1}\backslash\Gamma_n$; hence

$$J(\tau) \le \|f\|_\infty^2 (m_n)^{-(1+\alpha)} \lambda(\Gamma_k) \le \|f\|_\infty^2 (m_n)^{-(1+\alpha)} \lambda(\Gamma_n)$$
$$\le C(m_n)^{-\alpha} = Cw_{-\alpha}(\tau).$$

On the other hand, if $\tau \in \Gamma_{n+1}\backslash\Gamma_n$ with $n < k$ then

$$J(\tau) = \int_{\Gamma_{n+1}} + \int_{\Gamma\backslash\Gamma_{n+1}} |f_k(\rho) - f_k(\tau)|^2 w_{-(1+\alpha)}(\rho-\tau)\, d\lambda(\rho) := R + S.$$

Since for $\rho \in \Gamma_{n+1}$ and $\tau \in \Gamma_{n+1}\backslash\Gamma_n$ we have $w_1(\rho) \le w_1(\tau)$, the assumption of our proposition implies that

$$R \le C(m_n)^{-2} \int_{\Gamma_{n+1}} w_{1-\alpha}(\rho-\tau)\, d\lambda(\rho) = C(m_n)^{-2} \int_{\Gamma_{n+1}} w_{1-\alpha}(\sigma)\, d\lambda(\sigma)$$
$$\le C(m_n)^{-2}(m_n)^{1-\alpha}\lambda(\Gamma_{n+1}),$$

since $w_{1-\alpha}(\sigma) \le (m_n)^{1-\alpha}$ for $\sigma \in \Gamma_{n+1}\backslash\{0\}$ and $\alpha < 1$. Thus

$$R \le C(m_n)^{-\alpha} = Cw_{-\alpha}(\tau).$$

Furthermore, for S we have

$$S \le 4\|f\|_\infty^2 \sum_{l=n+1}^{\infty} \int_{\Gamma_{l+1}\backslash\Gamma_l} w_{-(1+\alpha)}(\rho-\tau)\, d\lambda(\rho)$$
$$\le 4\|f\|_\infty^2 \sum_{l=n+1}^{\infty} (m_l)^{-(1+\alpha)}\lambda(\Gamma_{l+1}) \le C(m_n)^{-\alpha} = Cw_{-\alpha}(\tau).$$

Thus, we have shown that $J(\tau) \le Cw_{-\alpha}(\tau)$. Substituting this in Q we see that

$$Q \le C\int_\Gamma |\hat{\phi}(\tau)|^2 w_{-\alpha}(\tau)\, d\lambda(\tau) = C\|\hat{\phi}\|_{2,-\alpha}^2 \le C\|\phi\|_{2,\alpha}^2,$$

by the Hausdorff-Young inequality for weighted L_p-spaces on G, which was proved in [**11**, Theorem 2]. Combining the inequalities obtained for P and Q we may conclude that Proposition 3 holds for $0 < \alpha < 1$. The extension to values of α with $-1 < \alpha < 0$ follows from Proposition 1.

In the final proposition of this paper we describe certain classes of functions $f \in L_\infty(\Gamma)$ with the property that the corresponding functions $(f_k)^\vee$ satisfy condition $C(k,r)$. The fact that condition $C(k,1)$ is somewhat more general than condition $C(k,r)$ for $r > 1$ (cf. Remark 2) is reflected in the fact that for $r = 1$ we obtain a somewhat larger class of functions than for $r > 1$. For all $r \in [1,\infty)$ the classes of functions are generalized Lipschitz spaces on Γ. These spaces were first defined in [**10**]; we repeat their definition here.

DEFINITION 2. Let $\alpha \in \mathbf{R}$, $1 \le p < \infty$ and $0 < q \le \infty$. A function $f\colon \Gamma \to \mathbf{C}$ belongs to the generalized Lipschitz space $\Lambda(\alpha,p,q;\Gamma)$ if

$$\|f\|_{\Lambda(\alpha,p,q;\Gamma)} := \|f\|_p + \left(\sum_{l=-\infty}^{\infty} \|(m_l)^{-\alpha}(F_{l+1} - F_l) * f\|_p^q\right)^{1/q} < \infty,$$

with the usual modification if $q = \infty$.

For some basic properties of these generalized Lipschitz spaces, see [**8** or **10**]. In particular, in [**8**] the equivalence of several norms on these Lipschitz spaces is proved when $\alpha > 0$, and one of these norms clearly shows the Lipschitz character of the Λ-spaces; see also [**18**, p. 80, Theorem (2.2)] for the case $G = (K,+)$, the additive group of a local field.

For $f\colon \Gamma \to \mathbf{C}$ and $j \in \mathbf{Z}$ we define f^j by

$$f^j = f_{j+1} - f_j = f\xi_{\Gamma_{j+1}\setminus\Gamma_j}.$$

PROPOSITION 4. (i) *Let $f \in L_\infty(\Gamma)$ and $1 < s \le 2$, and assume that there exist $\varepsilon > 0$ and $C > 0$ so that each $f^j \in \Lambda(1/s+\varepsilon, s, \infty;\Gamma)$ with*

$$\|f^j\|_{\Lambda(1/s+\varepsilon,s,\infty;\Gamma)} \le C(m_j)^{-\varepsilon}. \tag{14}$$

Then the inequality in condition $C(k,r)$ holds for all $k \in \mathbf{Z}$ and r with $1 \le r \le s'$.

(ii) *Let $f \in L_\infty(\Gamma)$ and $1 < s \le 2$, and assume that there exist $\varepsilon > 0$ and $C > 0$ so that each $f^j \in \Lambda(1/s+\varepsilon, s, s';\Gamma)$ with*

$$\|f^j\|_{\Lambda(1/s+\varepsilon,s,s';\Gamma)} \le C(m_j)^{-\varepsilon}. \tag{15}$$

Then the inequality in condition $C(k,1)$ holds for all $k \in \mathbf{Z}$.

PROOF. (i) Fix $k \in \mathbf{Z}$. Choose $l, n \in \mathbf{Z}$ with $n < l$ and let $y \in G_l$. First assume $k \le l$. Since $\operatorname{supp}(f_k) \subset \Gamma_k$, $(f_k)^\vee$ is constant on the cosets of G_k and, hence, on the cosets of G_l. Therefore, if $x \in G$ and $y \in G_l$ we have $(f_k)^\vee(x-y) = (f_k)^\vee(x)$, so that for all $r \in (1,\infty)$

$$I(y) := \left(\int_{G_n\setminus G_{n+1}} |(f_k)^\vee(x-y) - (f_k)^\vee(x)|^r \, d\mu(x)\right)^{1/r} = 0.$$

If $k > l$ then we set $f_k = f_l + \sum_{j=l}^{k-1} f^j$. The foregoing argument and Minkowski's inequality imply that

$$\begin{aligned} I(y) \le \sum_{j=l}^{k-1} &\left(\int_{G_n\setminus G_{n+1}} |(f^j)^\vee(x-y)|^r \, d\mu(x)\right)^{1/r} \\ &+ \sum_{j=l}^{k-1} \left(\int_{G_n\setminus G_{n+1}} |(f^j)^\vee(x)|^r \, d\mu(x)\right)^{1/r}. \end{aligned}$$

Since for $y \in G_l$ and $n < l$ we have $x - y \in G_n \backslash G_{n+1}$ if and only if $x \in G_n \backslash G_{n+1}$, we see that $I(y) \leq 2 \sum_{j=l}^{k-1} (\int_{G_n \backslash G_{n+1}} |(f^j)^\vee(x)|^r \, d\mu(x))^{1/r}$. We now apply Hölder's inequality, the Hausdorff-Young inequality and (14), in this order, and we obtain, whenever $1 \leq r \leq s' < \infty$,

$$
\begin{aligned}
I(y) &\leq 2 \sum_{j=l}^{k-1} \left(\left(\int_{G_n \backslash G_{n+1}} |(f^j)^\vee(x)|^{s'} \, d\mu \right)^{r/s'} \left(\int_{G_n \backslash G_{n+1}} d\mu \right)^{1-r/s'} \right)^{1/r} \\
&\leq 2 \sum_{j=l}^{k-1} \left(\|(F_{n+1} - F_n) * f^j\|_s^r (m_n)^{-(1-r/s')} \right)^{1/r} \\
&\leq C \sum_{j=l}^{k-1} ((m_n)^{(1/s+\varepsilon)r} (m_j)^{-\varepsilon r} (m_n)^{r/s'-1})^{1/r} \\
&\leq C (m_n)^{1+\varepsilon-1/r} \sum_{j=l}^{\infty} (m_j)^{-\varepsilon} \leq C (m_n)^{\varepsilon+1/r'} (m_l)^{-\varepsilon},
\end{aligned}
$$

by inequality (1). This completes the proof of (i).

(ii) Fix $k, l \in \mathbf{Z}$ and let $y \in G_l$ and

$$
J(y) := \int_{G \backslash G_l} |(f_k)^\vee(x - y) - (f_k)^\vee(x)| \, d\mu(x).
$$

Like in the proof of part (i), if $k \leq l$ then $J(y) = 0$. Also, if $k > l$ then, compare again the proof of part (i):

$$
\begin{aligned}
J(y) &\leq 2 \sum_{j=l}^{k-1} \int_{G \backslash G_l} |(f^j)^\vee(x)| \, d\mu \\
&\leq 2 \sum_{j=l}^{k-1} \left\{ \left(\int_{G \backslash G_l} |(f^j)^\vee(x) v_{1/s+\varepsilon}(x)|^{s'} \, d\mu \right)^{1/s'} \left(\int_{G \backslash G_l} |v_{-(1/s+\varepsilon)}|^s \, d\mu \right)^{1/s} \right\} \\
&= 2 \sum_{j=l}^{k-1} \left\{ \left(\sum_{i=-\infty}^{l-1} \int_{G_i \backslash G_{i+1}} |(f^j)^\vee(x)|^{s'} v_{(1/s+\varepsilon)s'}(x) \, d\mu \right)^{1/s'} \right. \\
&\qquad \left. \cdot \left(\sum_{i=-\infty}^{l-1} \int_{G_i \backslash G_{i+1}} v_{-(1+\varepsilon s)}(x) \, d\mu \right)^{1/s} \right\} \\
&\leq 2 \sum_{j=l}^{k-1} \left\{ \left(\sum_{i=-\infty}^{\infty} (m_i)^{-(1/s+\varepsilon)s'} \|(F_{i+1} - F_i) * f^j\|_s^{s'} \right)^{1/s'} \right. \\
&\qquad \left. \cdot \left(\sum_{i=-\infty}^{l-1} (m_i)^{1+\varepsilon s} (m_i)^{-1} \right)^{1/s} \right\} \\
&\leq C \sum_{j=l}^{k-1} \|f^j\|_{\Lambda(1/s+\varepsilon,s,s';\Gamma)} (m_l)^{\varepsilon} \leq C (m_l)^{\varepsilon} \sum_{j=l}^{\infty} (m_j)^{-\varepsilon} = C,
\end{aligned}
$$

which proves that condition $C(k, 1)$ holds.

Combining Proposition 4 with Theorem 1 we obtain the following corollary. In case $G = (K, +)$, and $s = 2$ in part (ii), this result agrees with the multiplier theorem proved by Taibleson for functions on a local field; cf. [**18**, p. 238].

COROLLARY 2. (i) *Let $f \in L_\infty(\Gamma)$ and $1 < s \leq 2$. Assume there exist $\varepsilon > 0$ and $C > 0$ so that for all $j \in \mathbf{Z}$ the functions f^j satisfy inequality* (14). *If f is a multiplier on $L_{2,\alpha_0}(G)$ for some α_0 with $-1/s < \alpha_0 < 1/s$, then f is a multiplier on $L_{p,\alpha}(G)$ for all p, α with $1 < p < \infty$ and $-|\alpha_0| \leq \alpha \leq (p-1)|\alpha_0|$.*

(ii) *Let $f \in L_\infty(\Gamma)$ and $1 < s \leq 2$. Assume there exist $\varepsilon > 0$ and $C > 0$ so that for all $j \in \mathbf{Z}$ the functions f^j satisfy inequality* (15). *Then f is a multiplier on $L_p(G)$ for $1 < p < \infty$.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NEW MEXICO 87131