TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 288, Number 1, March 1985

REGULARIZATION OF L^2 NORMS OF LAGRANGIAN DISTRIBUTIONS

BY
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ABSTRACT. Let X be a compact smooth manifold, $\dim X = n$. Let Λ be a fixed Lagrangian submanifold of T^*X. The space of Lagrangian distributions $I^k(X,\Lambda)$ is contained in $L^2(X)$ if $k < -n/4$. When $k = n/4$, $I^{-n/4}(X,\Lambda)$ just misses $L^2(X)$. A new inner product $\langle u,v\rangle_R$ is defined on $I^{-n/4}(X,\Lambda)/I^{-n/4-1}(X,\Lambda)$ in terms of symbols. This inner product contains "L^2 information" in the following sense: Slight regularizations of the Lagrangian distributions are taken, putting them in $L^2(X)$. The asymptotic behavior of the L^2 inner product is examined as the regularizations approach the identity. Three different regularization schemes are presented and, in each case, $\langle u,v\rangle_R$ is found to regulate the growth of the ordinary L^2 inner product.

0. Let X be a compact smooth manifold, $\dim X = n$. Let $\Lambda \subset T^*X\backslash\{0\}$ be a closed homogeneous Lagrangian submanifold and M a line bundle over Λ. We denote by $S^k(\Lambda, M)$ the space of smooth homogeneous sections of M, with degree of homogeneity k. By $S^k(\Lambda)$ we mean the space of smooth degree k homogeneous functions on Λ. Ω_Λ (resp. $\Omega_\Lambda^{1/2}$) will denote the line bundle of densities (resp. half-densities) on Λ. When no confusion will arise, the subscript Λ will be omitted. $I^k(X,\Lambda)$ will denote the space of classical, order k, Lagrangian distributions on X associated with Λ. This implies that if $u \in I^k(X,\Lambda)$, then u is a generalized half-density on X, and the wavefront set of u, denoted $\mathrm{WF}(u)$, is contained in Λ. The symbol of u is denoted $\sigma(u)$ and is an element of $S^{k+n/4}(\Lambda, \Omega^{1/2}\otimes L)$, where L is the Maslov bundle of Λ.

If $u \in I^k(X,\Lambda)$, where $k+n/4<0$, then $u\in L^2(X)$ by a theorem of Duistermaat and Hörmander [**DHo**]. The purpose of this paper is to examine the critical case $k+n/4=0$, when u just misses $L^2(X)$, to see what "L^2 information" can be extracted. The critical case $I^{-n/4}(X,\Lambda)$ will be denoted $I_{\mathrm{cr}}(X,\Lambda)$.

The L^2 inner product breaks down for $I_{\mathrm{cr}}(X,\Lambda)$, but we can make

$$I_{\mathrm{cr}}(X,\Lambda)/I^{-n/4-1}(X,\Lambda)$$

into an inner product space in a natural way by taking integrals of the symbols of the equivalence classes of distributions in $I_{\mathrm{cr}}(X,\Lambda)/I^{-n/4-1}(X,\Lambda)$. Since the symbol map σ factors through $I_{\mathrm{cr}}(X,\Lambda)/I^{-n/4-1}(X,\Lambda)$, when no confusion arises the distinction between an element of $I_{\mathrm{cr}}(X,\Lambda)$ and its image in $I_{\mathrm{cr}}(X,\Lambda)/I^{-n/4-1}(X,\Lambda)$ will not be made. Also, the term "distribution", unless otherwise indicated, refers to a generalized half-density. The inner product on $I_{\mathrm{cr}}(X,\Lambda)/I^{-n/4-1}(X,\Lambda)$ will be denoted $\langle\ ,\ \rangle_R$ and will be called the regularized inner product.

Received by the editors April 16, 1984.
1980 *Mathematics Subject Classification*. Primary 58G15, 46F99.

We demonstrate that this inner product contains "L^2 information". The general idea is the following: We start by slightly regularizing $u, v \in I_{\mathrm{cr}}(X, \Lambda)$, so as to put them in $L^2(X)$. Then the L^2 inner product is taken of the regularizations u_s and v_s of u and v.

The asymptotic behavior of $\langle u_s, v_s \rangle$ is examined as $u_s \to u$ and $v_s \to v$. This procedure is done three different ways, and each time $\langle u, v \rangle_R$ is found to be the coefficient of the leading term in an asymptotic expansion in the regularization parameter s.

This leads us to believe that the information contained in $\langle u, v \rangle_R$ is, in some canonical sense, L^2 information.

In §1, first the regularized inner product of $u, v \in I_{\mathrm{cr}}(X, \Lambda)$ is defined in terms of the symbols of u and v. Fortunately, the Maslov factors in the symbols conveniently cancel out. Then the relationship between $\langle u, v \rangle_R$ and $\langle Fu, Fv \rangle_R$ is determined, where F is an order-0 Fourier integral operator. The calculus of composition of Lagrangian distributions is used in calculating the above relationship and is needed again in a later section, so some results from the calculus are stated here.

In §2, $\langle u, v \rangle_R$ is exhibited as the coefficient of the singularity at $s = 0$ of the zeta function $Z_{P^{-1},u,v}(s) = \langle P^{s/2}u, P^{s/2}v \rangle$, where $P \in \Psi^{-1}(X)$ is positive, selfadjoint and elliptic. Here, and for the rest of this paper, $\Psi^k(X)$ denotes the space of order k classical pseudodifferential operators on X.

The result is obtained by a variation of a technique called the "algorithm of the '70s" by Fefferman [**Fe**]: The result is demonstrated first for the case $\Lambda = N^*\{0\} \subset T^*\mathbf{R}^n$, the conormal bundle of the origin. The result is transferred to a more general Λ (but with small microsupport) by certain Fourier integral operators examined in §1. Then the global result is pieced together from the microlocal result by a microlocal partition of unity.

The section finishes with a calculation of the constant term in the expansion $Z_{P^{-1},u,v}$ about $s = 0$.

In §3, the asymptotic growth as $\lambda \to \infty$ of $\langle E_\lambda, u, v \rangle$ is examined, where $\{E_\lambda\}$ is the spectral resolution of a positive, elliptic selfadjoint $Q \in \Psi^1(X)$. This is done by examining the singularity at 0 of the Fourier transform of $\langle dE_\lambda u/d\lambda, u \rangle$ using the calculus of composition of Lagrangian distributions and then applying a Tauberian argument to obtain that $\langle E_\lambda u, v \rangle \sim \langle u, v \rangle_R \log \lambda$, as $\lambda \to \infty$. The results of §2 are strengthened, showing that $Z_{P^{-1},u,v}(s)$ can be extended meromorphically to $\mathbf{C}$ with poles only at the nonpositive integers and with the residue at zero $= \langle u, v \rangle_R$.

I would like to thank my advisor Victor Guillemin, for his help on this paper, which is based on my Ph.D. Thesis [**Iz**].

1. Let Y be a principal fiber bundle over a compact manifold B, with fiber $\mathbf{R}^+$. The following sequence is exact:

$$0 \to T_y\mathbf{R}^+ \to T_yY \xrightarrow{d\pi} T_bB \to 0,$$

where $\pi\colon Y \to B$ is the fiber projection. Therefore, for all $y \in Y$, $|T_yY| \cong |T_bB| \otimes |\mathbf{R}|$, where $\pi(y) = b$.

Choose a section s of Y, $s\colon B \to Y$. Let $|dt|$ be the standard density on $\mathbf{R}$. Then to each section $v \in |TY|$ we can associate a section $v_s^\# \in |TB|$ given by

$$v_s^\#(b) = v(b, s(b)) \otimes |dt|^{-1}.$$

Now suppose $u \in |TY|$ is a homogeneous density, homogeneous of degree 0. Then for any two sections s_1, s_2 of Y and $b \in B$, $u(b, s_1(b)) = u(b, s_2(b))$.

Thus we can canonically associate to u a density $u^{\#} \in |TB|$, $u^{\#} = u^{\#}_s$ for any section s of Y, since $u^{\#}_s$ will be independent of the section s chosen.

DEFINITION. $\operatorname{res}_Y u = \int_B u^{\#}$.

Let X be a compact manifold with $\dim X = n$. Let Λ be a closed homogeneous Lagrangian submanifold of $T^*X\backslash\{0\}$ and let $\Lambda^{\#} = \Lambda/\mathbf{R}^+$.

PROPOSITION 1.1. *Let L be the Maslov bundle of Λ and $a, b \in S^0(\Lambda, \Omega^{1/2} \otimes L)$. Then $a\bar{b} \in S^0(\Lambda, \Omega)$ canonically.*

PROOF. Choose a set $\{M_j\}$ of conic neighborhoods in $T^*X\backslash\{0\}$ such that L is constant above $\Lambda_j = M_j \cap \Lambda$ for each j. Let $\{\tau_j\}$ be a trivialization of L above $\{\Lambda_j\}$. For all j, $\tau_j^4 = 1$. Above Λ_j, $a' = \tau_j a$, $b' = \tau_j b$, where a', b' are the sections belonging to $S^0(\Lambda, \Omega^{1/2})$ obtained when trivializing L with $\{\tau_j\}$. Note that a' and b' are by no means canonical. However, $a'\bar{b}' \in S^0(\Lambda, \Omega)$ is canonically defined since, for each $\lambda \in \Lambda_j$,

$$a'\bar{b}'(\lambda) = (\tau_j a)(\overline{\tau_j b})(\lambda) = |\tau_j|^2 a\bar{b}(\lambda) = a\bar{b}(\lambda). \tag{1.1}$$

The canonical sections above the Λ_j's patch together to give a global section in $S^0(\Lambda, \Omega)$ which is also canonically defined, since all the transition functions for $a\bar{b}$ are identically one, independent of the trivialization chosen for L.

Let $\alpha\colon S^0(\Lambda, \Omega^{1/2} \otimes L) \times S^0(\Lambda, \Omega^{1/2} \otimes L) \to \mathbf{C}$, $\alpha(a, b) \to \operatorname{res}_\Lambda a\bar{b}$.

PROPOSITION 1.2. *$S^0(\Lambda, \Omega^{1/2} \otimes L)$ is an inner product space with inner product α.*

PROOF. The proof is clear.

The symbol map

$$\sigma\colon I_{\mathrm{cr}}(X, \Lambda)/I^{-n/4-1}(X, \Lambda) \to S^0(\Lambda, \Omega^{1/2} \otimes L)$$

is an isomorphism, so we automatically get an inner product space structure on $I_{\mathrm{cr}}(X, \Lambda)/I^{-n/4-1}(X, \Lambda)$. Let $u, v \in I_{\mathrm{cr}}(X, \Lambda)/I^{-n/4-1}(X, \Lambda)$.

DEFINITION. $\langle u, v\rangle_R = (2\pi)^{-n}\alpha(\sigma(u), \sigma(v)) = (2\pi)^{-n} \operatorname{res}_\Lambda \sigma(u)\overline{\sigma(v)}$. The reason for the dimensional constant $(2\pi)^{-n}$ will be apparent later. Define $\|u\|_R = (\langle u, u\rangle_R)^{1/2}$.

It would be interesting and useful to know some of the functorial properties of the inner product $\langle\ ,\ \rangle_R$. In order to develop these properties, it is necessary to use the calculus of composition of Lagrangian distributions. We list some of the main results here. (See [**DGu**] for the proofs.)

Let X, Y be compact manifolds, Γ a closed Lagrangian submanifold of $T^*(X \times Y)\backslash\{0\}$, and Λ a closed Lagrangian submanifold of $T^*Y\backslash\{0\}$.

DEFINITION. $\Gamma \circ \Lambda = \{(x, \xi) \in T^*X | \exists (y, \eta) \in \Lambda \text{ with } (x, \xi, y, \eta) \in \Gamma\}$.

In the following discussion it is assumed that:

(1) There are no points of the form $(x, 0) \in \Gamma \circ \Lambda$. That is, $\Gamma \circ \Lambda \subset T^*X\backslash\{0\}$.

(2) There are no points of the form $(x, \xi, y, 0) \in \Gamma$. That is, there are no zero covectors in $\pi(\Gamma)$, where π is the projection of Γ on T^*Y.

LEMMA 1.1. *If π and the inclusion map i in the fiber product diagram*

$$\begin{array}{ccc} \Gamma & \leftarrow & F \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ T^*Y & \stackrel{i}{\leftarrow} & \Lambda \end{array} \tag{1.2}$$

intersect cleanly, then:

(1) *$\Gamma \circ \Lambda$ is an immersed Lagrangian submanifold of $T^*X\backslash\{0\}$, and*

(2) *the projection $\beta\colon F \to \Gamma \to \Gamma \circ \Lambda$ is a fiber mapping with compact fiber.*

LEMMA 1.2. *There is an isomorphism*

$$|\Lambda|^{1/2} \otimes |\Gamma|^{1/2} \cong |\Gamma \circ \Lambda|^{1/2} \otimes |\operatorname{Ker} \beta| \otimes |T^*Y|^{1/2}.$$

However, T^*Y is naturally a symplectic manifold, hence it has a canonical positive, nonvanishing half-density $\omega^{1/2}$. Thus we get the natural isomorphism

$$|\Lambda|^{1/2} \otimes |\Gamma|^{1/2} \cong |\Gamma \circ \Lambda|^{1/2} \otimes |\operatorname{Ker} \beta|.$$

Let $\sigma \in |\Gamma|^{1/2}$ and $\tau \in |\Lambda|^{1/2}$. By the above isomorphism, $\sigma \otimes \tau \otimes \omega^{-1/2}$ is a half-density on $\Gamma \circ \Lambda$ times a density in the fiber direction. Let $v \in \Gamma \circ \Lambda$.

DEFINITION. $\sigma \circ \tau(v) = \int_{\beta^{-1}(v)} \sigma \otimes \tau \otimes \omega^{-1/2}$. The integral is well defined since $\beta^{-1}(v)$ is compact. $\sigma \circ \tau \in |\Gamma \circ \Lambda|^{1/2}$ is called the composition of σ and τ.

PROPOSITION 1.3. *Suppose that σ and τ are homgeneous half-densities. Then $\sigma \circ \tau$ is homogeneous with* $\operatorname{degree} \sigma \circ \tau = \operatorname{degree} \sigma + \operatorname{degree} \tau - (\dim Y)/2 + e/2$, *where e is the excess of diagram* (1.2).

Let Γ be a Lagrangian submanifold of $T^*(X \times Y)\backslash\{0\}$ with the usual symplectic structure.

DEFINITION. $\Gamma' = \{(x, \xi, y, \eta) \in T^*(X \times Y)\backslash\{0\} | (x, \xi, y, -\eta) \in \Gamma\}$. Note that Γ' is Lagrangian on $T^*(X \times Y^-)\backslash\{0\}$, where $T^*(X \times Y^-)\backslash\{0\}$ is $T^*(X \times Y)\backslash\{0\}$ with the twisted symplectic form $\rho^*\omega_{T^*X} - \pi^*\omega_{T^*Y}$. Here ρ is the projection of $T^*(X \times Y)\backslash\{0\}$ onto T^*X.

THEOREM 1.1. *Let k be a generalized half-density on $X \times Y$, and K the operator with Schwartz kernel k. Let Γ, Λ be as above. If $k \in I^m(X \times Y; \Gamma)$ then:*

(1) *K maps $I^s(Y, \Lambda) \to I^{m+s+e/2}(X, \Gamma \circ \Lambda)$,*

(2) *if $u \in I^s(Y, \Lambda)$, then $\sigma(ku) = (2\pi i)^{-e/2} \sigma(k) \circ \sigma(u)$, modulo Maslov factors.*

The notation $K \in I^m(X, Y; \Gamma')$ denotes that K is the Fourier integral operator associated with $k \in I^m(X \times Y; \Gamma)$. Theorem 1.1 enables us to get the functorial properties we need.

Let $\dim X = \dim Y = n$. Let $\Lambda \subset T^*Y\backslash\{0\}$ and $\Gamma \subset T^*(X \times Y)\backslash\{0\}$ be closed homogeneous Lagrangian submanifolds. Let Γ intersect Λ transversally. Let $H = \Gamma \circ \Lambda$. Suppose there are conic open sets $N \subset T^*X\backslash\{0\}$ and $M \subset T^*Y\backslash\{0\}$ and a symplectic map $\phi\colon N \to M$ such that $\operatorname{graph} \phi \subset \Gamma$. Set $\Lambda_M = \Lambda \cap M$ and $H_N = H \cap N$ $(= \Gamma \circ \Lambda_M = \phi^{-1}(\Lambda_M))$. Let $u \in I_{\mathrm{cr}}(Y, \Lambda_M)$, $F \in I^0(X, Y; \Gamma')$, $a = \sigma(u)$, $c = \sigma(Fu)$ and $f = \sigma(F)$. By Theorem 1.1, we have that $Fu \in I_{\mathrm{cr}}(X, H_N)$.

PROPOSITION 1.4. *Fix $F \in I^0(X, Y; \Gamma')$. Then for all $u \in I_{\mathrm{cr}}(Y, \Lambda_M)$, $\|Fu\|_R^2 \leq \|\tilde{f}\| \, \|u\|_R^2$, where $\|\tilde{f}\|^2$ is defined below. Hence F, which is bounded as an operator from $L^2(Y)$ to $L^2(X)$, is also bounded as an operator from $I_{\mathrm{cr}}(Y, \Lambda_M)$ to $I_{\mathrm{cr}}(X, H_N)$.*

PROOF. By Theorem 1.1, modulo Maslov factors $c = f \circ a = f \otimes a \otimes \omega^{-1/2}$, where ω is the standard density on T^*Y. So $|c|^2 = |f|^2 \otimes |a|^2 \otimes \omega^{-1}$. Here all the Maslov factors have cancelled. The above identification is done pointwise. That is, for all $p \in H_N$,

$$|c|^2(p) = |f|^2(p, \phi(p)) \otimes |a|^2(\phi(p)) \otimes \omega^{-1}(\phi(p)).$$

Let us look at this even closer. Let $\psi = \phi|_{H_N}$. Then $\phi_N \colon H_N \to \operatorname{graph} \psi$, $p \to (p, \phi(p))$ is a diffeomorphism, as is $\phi_M \colon \operatorname{graph} \psi \to \Lambda_M$, and $(p, \phi(p)) \to \phi(p)$, since $\psi = \phi_M \circ \phi_N$ is a diffeomorphism. The idea is to transport densities above Λ to densities above $\operatorname{graph} \psi$ by ϕ_M^*, and then transport those densities down to H_N by ϕ_N^*. By Lemma 1.2 the density factors in all directions except the H_N direction will cancel, leaving us with a density on H_N.

$|f|^2$ is a density on Γ which is homogeneous of degree n. If $|f|^2$ is restricted to $\operatorname{graph} \psi$ we get a density on Γ above $\operatorname{graph} \psi$. By this we mean that for all $\gamma \in \operatorname{graph} \psi$, $\Gamma(\gamma)$ is a density on $T_\gamma \Gamma$. Hence, by Lemma 1.2, $|f|^2 \otimes \phi_M^* |a|^2 \otimes \phi_M^* \omega^{-1}$ is a density on H_N above $\operatorname{graph} \psi$, and $|c|^2 = \phi_N^*(|f|^2 \otimes \phi_M^* |a|^2 \otimes \phi_M^* \omega^{-1})$ is a density on H_N. Therefore,

$$|c|^2 = \phi_N^* |f|^2 \otimes \psi^* |a|^2 \otimes \psi^* \omega^{-1}$$

and

$$\begin{aligned} \operatorname{res}_{H_N} |c|^2 &= \int_{H_N^\#} (|c|^2)^\# = \int_{H_N^\#} (\phi_N^* |f|^2 \otimes \psi^* |a|^2 \otimes \psi^* \omega^{-1})^\# \\ &= \int_{\Lambda_M^\#} ((\phi_M^*)^{-1} |f|^2 \otimes |a|^2 \otimes \omega^{-1})^\#. \end{aligned}$$

By Lemma 1.2, above $\operatorname{graph} \phi$, $|f|^2$ is the product of a density f_1 in the H_N direction with a density f_2 in the $\pi(\Gamma)$ direction. Note that f_1 and f_2 cannot be chosen canonically, but the product $f_1 \otimes f_2$ is, of course, independent of the choice made. Let $f_N = \phi_N^* f_1$. f_N is a density on H_N. Let $f_M = (\psi^*)^{-1} f_N = (\phi_M^*)^{-1} f_1$. f_M is a density on Λ_M since ψ gives an isomorphism of $|H_N|$ with $|\Lambda_M|$. Therefore, $(\phi_M^*)^{-1} |f|^2 = f_M \otimes (\phi_M^*)^{-1} f_2$ is a canonically defined nonnegative density on T^*Y above Λ_M because Γ intersects Λ transversally. Since the above operations preserve homgeneity, $(\phi_M^*)^{-1} |f|^2$ is still homogeneous of degree n. So, dividing by the standard positive density ω, which is also homogeneous of degree n, leaves a canonically defined nonnegative function on Λ_M, which is homogeneous of degree 0. Denote this function by

$$|\tilde{f}|^2 = (\phi_M^*)^{-1} |f|^2 \otimes \omega^{-1} = f_M \otimes (\phi_M^*)^{-1} f_2 \otimes \omega^{-1}.$$

Hence,

$$\operatorname{res}_{H_N} |c|^2 = \int_{\Lambda_M^\#} (|\tilde{f}|^2 |a|^2)^\#. \tag{1.3}$$

Let $\|\tilde{f}\|^2 = \sup_{\Lambda_M} |\tilde{f}|^2$. Then

$$\operatorname{res}_{H_N} |c|^2 \leq \|\tilde{f}\|^2 \int_{\Lambda_M^\#} (|a|^2)^\# = \|\tilde{f}\|^2 \operatorname{res}_{\Lambda_M} |a|^2.$$

But $\mathrm{res}_H|c|^2 = \mathrm{res}_{H_N}|c|^2$ since $\mathrm{supp}(|c|^2) \subset H_N$, and $\mathrm{res}_\Lambda|a|^2 = \mathrm{res}_{\Lambda_M}|a|^2$ since $\mathrm{supp}(|a|^2) \subset \Lambda_M$. Thus,

$$\mathrm{res}_H|c|^2 \leq \|\tilde{f}\|^2 \mathrm{res}_\Lambda|a|^2.$$

Multiplying by $(2\pi)^{-n}$ completes the proof.

PROPOSITION 1.5. *Proposition 1.4 gives a sharp estimate.*

PROOF. Suppose Proposition 1.4 is not sharp. Then there is a constant g, $0 < g < \|\tilde{f}\|^2$, such that for all $u \in I_{\mathrm{cr}}(Y, \Lambda_M)$

$$\mathrm{res}_H|c|^2 \leq g\,\mathrm{res}_\Lambda|a|^2.$$

Let $\varepsilon = \|\tilde{f}\|^2 - g$. By assumption $\varepsilon > 0$. Let

$$E = \{\lambda \in \Lambda_M \mid |\tilde{f}|^2(\lambda) > (g + \varepsilon/2)\}.$$

E is nonempty since $|\tilde{f}|^2$ is continuous. Choose $u_0 \in I_{\mathrm{cr}}(Y, \Lambda_M)$ such that $\mathrm{supp}(|a_0|^2) \subset E$ and $\mathrm{res}|a_0|^2 \neq 0$, where $a_0 = \sigma(u_0)$. Let $c_0 = \sigma(Fu_0)$. Then (1.3) gives

$$\mathrm{res}_H|c_0|^2 \geq (g + \varepsilon/2) \int_{\Lambda^\#} (|a_0|^2)^\# = (g + \varepsilon/2)\mathrm{res}_\Lambda|a_0|^2.$$

So $g\,\mathrm{res}_\Lambda|a_0|^2 \geq \mathrm{res}_H|c_0|^2 \geq (g+\varepsilon/2)\,\mathrm{res}_\Lambda|a_0|^2$, which is a contradiction since both $\mathrm{res}_\Lambda|a_0|^2$ and ε are positive. Thus Proposition 1.4 is sharp.

Without any additional information about $F \in I^0(X, Y; \Gamma')$, the estimates of (1.3) and Proposition 1.4 are the best that can be obtained. However, for a certain class of F's, we can go a bit further.

DEFINITION. An operator $F \in I^0(X, Y; \Gamma')$ is said to be unitary on M, a conic neighborhood in $T^*Y\backslash\{0\}$ if $X = Y$, and $F^*F - I$ is smoothing on M.

REMARK. The definition can easily be extended to any conic open set in $T^*Y\backslash\{0\}$.

If F is unitary on M in Proposition 1.4, then it follows from [**Tr**] that $|f|^2 = 1$ on M. Therefore, (1.3) becomes

$$\mathrm{res}_H|c|^2 = \int_{\Lambda^\#} (|a|^2)^\# = \mathrm{res}_\Lambda|a|^2.$$

Multiplying by $(2\pi)^{-n}$ and taking square roots proves

PROPOSITION 1.6. *Let F in Proposition 1.4 be unitary on M. Then $\|Fu\|_R = \|u\|_R$.*

COROLLARY 1.1. *Let u, v be as in Proposition 1.4. Let F in Proposition 1.4 be unitary on M. Then $\langle Fu, Fv\rangle_R = \langle u, v\rangle_R$.*

PROOF. Polarize:

$$\begin{aligned}\langle Fu, Fv\rangle_R &= \frac{\|F(u+iv)\|_R^2 + \|F(u+v)\|_R^2}{2} - \|Fu\|_R^2 - \|Fv\|_R^2 \\ &= \frac{\|u+iv\|_R^2 + \|u+v\|_R^2}{2} - \|u\|_R^2 - \|v\|_R^2 = \langle u, v\rangle_R.\end{aligned}$$

2. In this section the first regularization scheme is presented. Let $u, v \in I_{\mathrm{cr}}(X, \Lambda)$. Let $P \in \Psi^{-1}(X)$ be positive, selfadjoint and elliptic. Since X is compact, for $s > 0$ we have $P^{s/2} \in \Psi^{s/2}(X)$, with $\sigma(P^{s/2}) = [\sigma(P)]^{s/2}$. Hence $P^{s/2}u, P^{s/2}v \in I^{-n/4-s/2}(X, \Lambda)$. For positive s let $Z_{P^{-1},u,v}(s)$ be the L^2 inner product $\langle P^{s/2}u, P^{s/2}v\rangle$. The subscript P^{-1} is used instead of P so as to keep the notation consistent with that of §3 when $Z_{P^{-1},u,v}$ is extended to complex s. As $S \to 0^+$, $Z_{P^{-1},u,v}$ will usually approach ∞. The main theorem of this section is

THEOREM 2.1. *Let u, v, P be as above. Then*

$$\lim_{s \to 0^+} sZ_{P^{-1},u,v}(s) = \langle u, v\rangle_R.$$

Note that the right-hand side is independent of the choice of the operator P.

This theorem will be proven in three steps. First it will be verified for $X = \mathbf{R}^n$ and $\Lambda = N^*\{0\}$, the conormal bundle to the origin.

Note that although in this case X is not compact, the definition of $\langle u, v\rangle_R$ given in §1 still makes sense because the integration is over $\Lambda^\# \cong S^{n-1}$, which is compact. For this case the theorem is just Parseval's formula and a switch to polar coordinates.

The second step is to extend the result of Step 1 to distributions with small microsupport in $T^*X\backslash\{0\}$. This is done by using the results of §1 to construct a microlocally unitary Fourier integral operator which takes distributions with small microsupport back to the conormal distributions of Step 1.

Finally, the global result is obtained by patching together with a microlocal partition of unity.

PROOF. First consider the case where $u_1, u_2 \in I_{\mathrm{cr}}(\mathbf{R}^n, N^*\{0\})$. Let $x \in \mathbf{R}^n$, $\xi \in (\mathbf{R}^n)^*$,

$$u_j = (2\pi)^{-n} \int_{\mathbf{R}^n_\xi} e^{i\langle x,\xi\rangle} a_j(\xi)|d\xi|^{1/2}|dx|^{1/2}, \qquad j = 1, 2,$$

where $a_j(\xi) = \widehat{u_j}$, the Fourier transform of u_j.

Since we are on $\mathbf{R}^n$, $a_j(\xi)$ is also the total symbol of u_j, and therefore is an asymptotic sum of half-densities of decreasing homogeneity. It will be more convenient to work with functions, so let $b_j(\xi) = a_j(\xi)|d\xi|^{-1/2}$, where $|d\xi|$ is the standard density on $(\mathbf{R}^n)^*$. Let $a_{j_0} = \sigma(u_j)$ and $b_{j_0} = a_{j_0}|d\xi|^{-1/2}$. a_{j_0} and b_{j_0} are just the leading terms in the asymptotic expansions of a_j and b_j, respectively. Let P be a constant coefficient, selfadjoint, positive elliptic operator belonging to $\Psi^{-1}(\mathbf{R}^n)$. Let $p(\xi)|d\xi|^{1/2}$ denote the total symbol and $p_0(\xi)|d\xi|^{1/2}$ the principal symbol of P. Then

$$P^{s/2}u_j = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} p^{s/2}(\xi) b_j(\xi)|d\xi|\,|dx|^{1/2}.$$

By Parseval's formula

$$Z_{P^{-1},u_1,u_2} = (2\pi)^{-n} \int_{\mathbf{R}^n} p^s(\xi) b_1(\xi)\bar{b}_2(\xi)|d\xi|.$$

Convert to polar coordinates. Let $\xi = \omega t$, where $\omega \in S^{n-1}$. Then

$$Z_{P^{-1},u_1,u_2}(s) = (2\pi)^{-n} \int_{S^{n-1}} \int_0^\infty p^s(\omega t) b_1(\omega t)\bar{b}_2(\omega t) t^{n-1}|dt\, d\omega|.$$

Now $b_{1_0}\bar{b}_{2_0} \in S^{-n}(N^*\{0\})$, and $p_0^s \in S^{-s}(N^*\{0\})$. Hence as $s \to 0^+$

$$\begin{aligned} Z_{P^{-1},u_1,u_2}(s) &\sim (2\pi)^{-n} \int_{S^{n-1}} \int_1^\infty p_0^s(\omega) b_{1_0}(\omega)\overline{b_{2_0}}(\omega) t^{-s-1} |dt\, d\omega| \\ &= (2\pi)^{-n} \int_{S^{n-1}} p_0^s(\omega) b_{1_0}(\omega)\overline{b_{2_0}}(\omega)|d\omega| \int_1^\infty t^{-s-1}\, dt \qquad (2.1)\\ &= \frac{(2\pi)^{-n}}{s} \int_{S^{n-1}} e^s(\omega), \quad \text{where } e^s(\omega) = P_0^s(\omega) b_{1_0}(\omega)\overline{b_{2_0}}(\omega)|d\omega|. \end{aligned}$$

The lower order terms in the asymptotic expansion of $Z_{P^{-1},u_1,u_2}(s)$ are finite as $s \to 0^+$, hence,

$$\begin{aligned} \lim_{s\to 0^+} s Z_{P^{-1},u_1,u_2}(s) &= \lim_{s\to 0^+} (2\pi)^{-n} \int_{S^{n-1}} e^s(\omega) \\ &= (2\pi)^{-n} \int_{S^{n-1}} b_{1_0}(\omega)\overline{b_{2_0}}(\omega)|d\omega| \qquad (2.2)\\ &= (2\pi)^{-n} \int_{S^{n-1}} (a_{1_0}(\omega)\bar{a}_{2_0}(\omega))^{\#} \\ &= (2\pi)^{-n} \mathrm{res}_{N^*\{0\}} a_{1_0}\bar{a}_{2_0} = \langle u_1, u_2\rangle_R. \end{aligned}$$

We claim (2.2) is also valid when P is a variable coefficient operator.

PROOF OF CLAIM. We need the following

LEMMA 2.1. *Let $u \in I^k(X,\Lambda)$, $P \in \Psi(X)$, $p_0 = \sigma(P)$ and $a_0 = \sigma(u)$. Then*

$$\sigma(Pu) = p_0|_\Lambda a_0. \qquad (2.3)$$

PROOF. This is a consequence of equation (4.10) in Chapter 6 of [**Tr**].

Apply Lemma 2.1 to the conormal case, $\Lambda = N^*\{0\}$, $X = \mathbf{R}^n$. So

$$\sigma(P^s u) = p_0^s(x,\xi)|_{N^*\{0\}} a_0(\xi) = p_0^s(0,\xi)a_0(\xi).$$

Let $P \in \Psi^{-1}(X)$ be positive, elliptic and selfadjoint as above except that now variable coefficients are permitted. Then $P^{s/2}u_j \in I^{-n/4-s/2}(\mathbf{R}^n, N^*\{0\})$ and define $q_{j,s}(\xi)$ by

$$q_{j,s}(\xi)|d\xi|^{1/2} = (P^{s/2}u_j)\widehat{\ }, \qquad j = 1,2.$$

The above lemma tells us that the leading term of $q_{j,s}(\xi)$ is $p_0^{s/2}(0,\xi)b_{j_0}(\xi) = \sigma(P^{s/2}u_j)$.

Apply Parseval's formula again:

$$\begin{aligned} Z_{P^{-1},u_1,u_2}(s) &= (2\pi)^{-n} \int_{\mathbf{R}^n} q_{1,s}(\xi)\overline{q_{2,s}}(\xi)|d\xi| \\ &= (2\pi)^{-n} \int_{S^{n-1}} \int_0^\infty q_{1,s}(\omega t)\overline{q_{2,s}}(\omega t) t^{n-1}|dt\, d\omega| \\ &\sim (2\pi)^{-n} \int_{S^{n-1}} \int_1^\infty p_0^s(0,\omega) b_{1_0}(\omega)\bar{b}_{2_0}(\omega) t^{-s-1}|dt\, d\omega| \quad \text{for } s \text{ near } 0. \end{aligned}$$

This is just (2.1), so the claim follows.

Now consider X compact and a general $\Lambda \subset T^*X\backslash\{0\}$. Let $u, v \in I_{\mathrm{cr}}(X,\Lambda)$ with $\mathrm{WF}(u)$ and $\mathrm{WF}(v)$ contained in a small conic open neighborhood M of $(x_0,\xi_0) \in \Lambda$.

By a suitable choice of coordinates on M, we can regard M as a conic neighborhood in $T^*\mathbf{R}^n_X\backslash\{0\}$, with $\Lambda_M = \Lambda \cap M$ defined by $x = dH(\xi)$. Then the homogeneous canonical transformation $\chi\colon T^*\mathbf{R}^n_X\backslash\{0\} \to T^*\mathbf{R}^n_Y\backslash\{0\}$, $\chi(x,\xi) = (x - dH(\xi), \xi)$, maps Λ_M into $N^*\{0\} \subset T^*\mathbf{R}^n_Y$. Let C be the graph of χ in $T^*(\mathbf{R}^n_X \times \mathbf{R}^n_Y)\backslash\{0\}$. Next we construct a Fourier integral operator $F \in I^0(\mathbf{R}^n_X, \mathbf{R}^n_Y; C')$ which is unitary on M. Let $f(x,y,\theta)$ be the leading term of an amplitude of F and let $\phi(x,y,\theta)$ be the corresponding phase function.

In addition to ensuring F is unitary on M, we can make F smoothing off a slightly larger conic neighborhood $M' \supset M$.

By Theorem 1.1, $Fu \in I_{\mathrm{cr}}(\mathbf{R}^n, N^*\{0\})$. Let $Q, P \in \Psi^{-1}(\mathbf{R}^n)$ be elliptic, positive and selfadjoint. From [**Tr**, Chapter 6], the leading terms of the amplitudes of the compositions PF and FQ are, respectively,

$$f(x,y,\theta)p_0(x,\phi_x(x,y,\theta))$$

and

$$f(x,y,\theta)q_0(y,-\phi_y(x,y,\theta)).$$

Thus, if we are given a Q as above, we can construct a positive, selfajoint, elliptic $P \in \Psi^{-1}(X)$ such that $PF - FQ$ is of degree -2 and smoothing outside of M' by first choosing an operator P' with principal symbol p_0 such that

$$p_0(x,\phi_x(x,y,\theta)) = q_0(y,-\phi_y(x,y,\theta)).$$

Then we get a selfadjoint P by setting $P = (P' + P'^*)/2$. P now has the required properties. Consider the operators P^sF and FQ^s. By construction P^sF and FQ^s have the same phase function as F, and the leading amplitude of P^sF is

$$p_0^s(x,\phi_x(x,y,\theta))f(x,y,\theta) = q_0^s(x,-\phi_y(x,y,\theta))f(x,y,\theta),$$

which is the leading amplitude of FQ^s. Thus

$$P^sF - FQ^s \in I^{-s-1}(\mathbf{R}^n_X, \mathbf{R}^n_Y; C),$$

and is smoothing off of M'. Let $\sigma(u) = a$, $\sigma(v) = b$. Then

$$\begin{aligned}
\lim_{s\to 0^+} sZ_{Q^{-1},u,v}(s) &= \lim_{s\to 0^+} s\langle Q^{s/2}u, Q^{s/2}v\rangle \\
&= \lim_{s\to 0^+} s\langle FQ^{s/2}u, FQ^{s/2}v\rangle, \quad \text{since } F \text{ is unitary on } M, \\
&= \lim_{s\to 0^+} s\langle P^{s/2}Fu, P^{s/2}Fv\rangle = (2\pi)^{-n}\mathrm{res}_{N^*\{0\}}\sigma(Fu)\overline{\sigma(Fv)} \\
&= \langle Fu, Fv\rangle_R = \langle u, v\rangle_R, \quad \text{by Corollary 1.1.}
\end{aligned}$$

This proves Theorem 2.1 for u, v with small wavefront sets.

The microlocal version of Theorem 2.1 will now be globalized. Let $u, v \in I_{\mathrm{cr}}(X,\Lambda)$. Choose a microlocal partition of unity $\{\phi_j\}$ on the conic open sets $\{M_j\}$ which cover Λ. Since X is compact, only a finite number, l, of open sets M_j are needed. For all j, $\phi_j \in \Psi^0(X)$ and $\mathrm{WF}'(\phi_j) \subset M_j$. Let $u_j = \phi_j u, v_j = \phi_j v$ and let $\Lambda \cap M_j = \Lambda_j$. Then $u_j, v_j \in I_{\mathrm{cr}}(X, \Lambda_j)$. Since $\sum_{j=1}^l \phi_j =$ Identity modulo smoothing operators, we have $u = \sum_{j=1}^l u_j$ and $v = \sum_{j=1}^l v_j$ modulo smooth half-densities. The $\{M_j\}$'s can be chosen small enough so that the microlocal version of the theorem applies to $M_j \cup M_k$ for all pairs j, k when $M_j \cap M_k \neq \varnothing$. Of course, if the intersection is empty, then $\mathrm{res}_\Lambda \sigma(u_j)\overline{\sigma(v_k)} = 0$,

$$\lim_{s\to 0^+} s\langle P^{s/2}u_j, P^{s/2}v_k\rangle = (2\pi)^{-n}\mathrm{res}_\Lambda \sigma(u_j)\overline{\sigma(v_k)},$$

$$\begin{aligned}
\lim_{s\to 0^+} sZ_{P^{-1},u,v}(s) &= \lim_{s\to 0^+} s\langle P^{s/2}u, P^{s/2}v\rangle \\
&= \lim_{s\to 0^+} s\left\langle P^{s/2}\sum_{j=1}^{l} u_j, P^{s/2}\sum_{k=1}^{l} v_k\right\rangle \\
&= \sum_{j,k=1}^{l} \lim_{s\to 0^+} s\langle P^{s/2}u_j, P^{s/2}v_k\rangle \\
&= \sum_{j,k=1}^{l} (2\pi)^{-n}\mathrm{res}_\Lambda \sigma(u_j)\overline{\sigma(v_k)} \\
&= (2\pi)^{-n}\mathrm{res}_\Lambda \sum_{j,k=1}^{l} \sigma(u_j)\overline{\sigma(v_k)} \\
&= (2\pi)^{-n}\mathrm{res}_\Lambda \sigma(u)\overline{\sigma(v)} = \langle u, v\rangle_R.
\end{aligned}$$

This completes the proof of Theorem 2.1. Thus $\langle u, v\rangle_R$ represents the singular part of $Z_{P^{-1},u,v}$ as $s \to 0^+$ and is giving a measure on how the integral $u\bar{v}$ diverges. If the singular part of $Z_{P^{-1},u,v}$ is subtracted off, the resulting function will be finite as $s \to 0^+$.

DEFINITION.

$$\langle u, v\rangle_{R,P} = \lim_{s\to 0^+}\left(\langle P^{s/2}u, P^{s/2}v\rangle - \frac{\langle u, v\rangle_R}{s}\right).$$

Similarly,

$$\|u\|^2_{R,P} = \lim_{s\to 0^+}\left(\|P^{s/2}u\|^2 - \frac{\|u\|^2_R}{s}\right).$$

Unfortunately, $\langle u, v\rangle_{R,P}$ is not independent of P, but the following relates $\langle u, v\rangle_{R,P_0}$ to $\langle u, v\rangle_{R,P_1}$:

THEOREM 2.2. *Let $u, v \in I_{\mathrm{cr}}(X, \Lambda)$, $\sigma(u) = a$, $\sigma(v) = b$. Let $P_0, P_1 \in \Psi^{-1}(X)$ be positive, selfadjoint and elliptic. Let $q = \sigma(P_0)/\sigma(P_1)$. Then*

$$\langle u, v\rangle_{R,P_0} - \langle u, v\rangle_{R,P_1} = (2\pi)^{-n}\mathrm{res}_\Lambda(\log q)a\bar{b}.$$

PROOF. Let $Q = P_0 P_1^{-1}$. Then

$$Q \in \Psi^0(X) \quad \text{and} \quad \sigma(Q) = \sigma(P_0)/\sigma(P_1) = q;$$

q is real since both P_0 and P_1 are selfadjoint. Let $T(s) = P_0^s P_1^{-s} \in \Psi^0(X)$. Then $\sigma(T(s)) = q^s = e^{s\log q}$, $s \geq 0$. Since T is real analytic in s, we can write $T(s) = I + sR + O(s^2)$, where R is an operator $\Psi^0(X)$ with $\sigma(R) = \log q$. Then

$$\begin{aligned}
\langle P_0^{s/2}u, P_0^{s/2}v\rangle &= \langle P_0^s u, v\rangle = \langle T(s)P_1^s u, v\rangle \\
&= \langle P_1^S u, v\rangle + s\langle RP_1^s u, v\rangle + O(s^2).
\end{aligned}$$

Therefore,

$$\langle P_0^{s/2}u, P_0^{s/2}v\rangle - \langle P_1^{s/2}u, P_1^{s/2}v\rangle = s\langle P_1^{s/2}u, P_1^{s/2}R^*v\rangle + O(s^2).$$

Taking limits as $s \to 0^+$ gives

$$\langle u, v\rangle_{R,P_0} - \langle u, v\rangle_{R,P_1} = \langle u, R^*v\rangle_R = (2\pi)^{-n}\mathrm{res}_\Lambda(\log q)a\bar{b}.$$

3. Let Q be a positive, elliptic, selfadjoint operator in $\Psi^1(X)$, where as before, X is compact and $\dim X = n$. Then the spectrum of Q is positive and discrete. Denote the symbol of Q by $q(x,\xi)$. Let $u, v \in I(X,\Lambda)$, $\sigma(u) = a$, $\sigma(v) = b$.

Consider the family of spectral projections $\{E_\lambda\}$ of Q. We have $I = \int_0^\infty dE_\lambda$. The Schwartz kernel of E_λ is $\sum_{\lambda_j \in \mathrm{spec}(Q), \lambda_j \le \lambda} e_j(x)\overline{e_j(y)}$, where $e_j(x)$ is the normalized eigenfunction associated with the eigenvalue λ_j of Q. Note that $Q^s = \int_0^\infty \lambda^s\, dE_\lambda$, with Schwartz kernel $\sum_{\lambda_j\in\mathrm{spec}(Q)} \lambda_j^s e_j(x)\overline{e_j(y)}$. Let $u_j = \langle u, e_j\rangle$, the coefficient of e_j in the expansion of u about the orthonormal set $\{e_j\}$. Let

$$g_{u,v}(\lambda) = \sum_{\lambda_j\in\mathrm{spec}(Q)} u_j\bar{v}_j = \langle E_\lambda u, v\rangle.$$

If $u, v \in L^2(x)$, then $\lim_{\lambda\to\infty} g_{u,v}(\lambda) = \langle u, v\rangle$.

THEOREM 3.1. *Let $u, v \in I_{\mathrm{cr}}(X,\Lambda)$. Then as $\lambda \to \infty$, $g_{u,v}(\lambda) \sim \langle u, v\rangle_R \log\lambda + C + O(\lambda^{-1})$.*

PROOF. By an easy polarization argument similar to the one in the proof of Corollary 1.1, it suffices to prove Theorem 3.1 for the case when $u = v$.

Define $g_u = g_{u,u}$. Let $\rho \in C_0^\infty(\mathbf{R})$, $\rho(0) = 1$, with $\mathrm{supp}\,\rho \subset (-\varepsilon, \varepsilon)$, where $\varepsilon < |T_1|$ and T_1 is the smallest nonzero period of the Hamiltonian flow Φ^t of the Hamiltonian vector field H_q. Let $\check{\rho}$ be the inverse Fourier transform of ρ. Choose ρ so that $\check{\rho} \ge 0$ on $\mathbf{R}$ and $\check{\rho} \ge 1$ on $[0,1]$. For the construction of a suitable ρ see [**Gu1**].

Claim 3.1. Suppose

$$\check{\rho} * \frac{dg_u}{d\lambda} = \|u\|_R^2\lambda^{-1} + O(\lambda^{-2}) \quad \text{as } \lambda\to\infty. \tag{3.1}$$

Then $g_u = \|u\|_R^2 \log\lambda + C + O(\lambda^{-1})$.

PROOF. We can integrate (3.1) to obtain

$$\check{\rho} * g_u = \|u\|_R^2\log\lambda + C + O(\lambda^{-1})$$

as $\lambda\to\infty$. We will show that $\check{\rho} * g_u - g_u = O(\lambda^{-1})$. From (3.1) we have that

$$\check{\rho} * \frac{dg_u}{d\lambda} = \int_{-\infty}^{\infty} \frac{dg_u}{d\lambda}(\lambda-\mu)\check{\rho}(\mu)\,d\mu = \frac{\|u\|_R^2}{\lambda} + O(\lambda^{-2}).$$

But $\check{\rho} \le 1$ on $[0,1]$ and $dg_u/d\lambda \ge 0$, since g_u is increasing. Hence

$$\int_0^1 \frac{dg_u}{d\lambda}(\lambda-\mu)\,d\mu \le \frac{d}{\lambda},$$

where d is some constant larger than $\|u\|_R^2$. Integrating gives

$$g_u(\lambda) - g_u(\lambda-1) \le d/\lambda.$$

This immediately gives a crude estimate for $g_u(\lambda)$:

$$g_u(\lambda) = O(\lambda) \quad \text{as } \lambda \to \infty.$$

In the following discussion, assume $\lambda > 1$. $\check{\rho} * g_u - g_u$ can be written as the sum of the three integrals

$$\left(\int_{-\infty}^{0} + \int_{0}^{\lambda/2} + \int_{\lambda/2}^{\infty}\right) [g_u(\lambda - \mu) - g_u(\lambda)]\check{\rho}(\mu)\, d\mu.$$

Suppose first $\mu \geq \lambda/2$. Then

$$0 \leq g_u(\lambda) - g_u(\lambda - \mu) \leq g_u(\lambda) = O(\lambda) \quad \text{as } \lambda \to \infty.$$

Therefore, if $I_1 = \int_{\lambda/2}^{\infty}[g_u(\lambda - \mu) - g_u(\lambda)]\check{\rho}(\mu)\, d\mu$, then

$$|I_1| \leq g_u(\lambda)\left|\int_{\lambda/2}^{\infty} \check{\rho}(\mu)\, d\mu\right| = O(\lambda^{-N}) \quad \text{for all } N \in \mathbf{R}$$

since $\check{\rho}(\mu)$ is a Schwartz function.

Suppose next that $\mu \leq 0$. Then

$$0 \leq g_u(\lambda - \mu) - g_u(\lambda) \leq g_u(\lambda + [-\mu] + 1) - g_u(\lambda),$$

where $[\mu]$ is the greatest integer $\leq \mu$ and

$$\begin{aligned} g_u(\lambda + [-\mu] + 1) - g_u(\lambda) &= \sum_{n=0}^{[-\mu]} g_u(\lambda + n + 1) - g_u(\lambda + n) \\ &\leq d \sum_{n=0}^{[-\mu]} \frac{1}{\lambda + n + 1} \leq d \int_{\lambda}^{\lambda + [-\mu] + 1} \lambda^{-1}\, d\lambda, \end{aligned}$$

since the sum is a lower Riemann sum for the integral.

$$d \int_{\lambda}^{\lambda + [-\mu] + 1} \lambda^{-1}\, d\lambda = d \log\left(1 + \frac{[-\mu] + 1}{\lambda}\right) \leq d\frac{|\mu - 1|}{\lambda}.$$

Therefore,

$$I_2 = \int_{-\infty}^{0} [g_u(\lambda - \mu) - g_u(\lambda)]\check{\rho}(\mu)\, d\mu \leq \frac{d}{\lambda}\int_{-\infty}^{0} |\mu - 1|\check{\rho}(\mu)\, d\mu$$

so $I_2 = O(\lambda^{-1})$. Finally, if $0 \leq \mu \leq \lambda/2$, then

$$\begin{aligned} 0 &\leq g_u(\lambda) - g_u(\lambda - \mu) \leq g_u(\lambda) - g_u(\lambda - [\mu]) + [g_u(\lambda - [\mu]) - g_u(\lambda - [\mu] - 1)] \\ &= g_u(\lambda - [\mu]) - g_u(\lambda - [\mu] - 1) + \sum_{n=1}^{[\mu]} g_u(\lambda - n + 1) - g_u(\lambda - n) \\ &\leq \frac{d}{\lambda - [\mu]} + d\sum_{n=1}^{[\mu]} \frac{1}{\lambda - n + 1} \leq \frac{d}{\lambda - \mu} + d\int_{\lambda - [\mu]}^{\lambda} \frac{d\lambda}{\lambda} \\ &\leq \frac{d}{\lambda\left(1 - \frac{\mu}{\lambda}\right)} + d\log\frac{\lambda}{\lambda - \mu} \leq \frac{2d}{\lambda} + \frac{2d\mu}{\lambda} = \frac{2d}{\lambda}(1 + \mu) \end{aligned}$$

since $0 \leq u/\lambda \leq 1/2$. So if $I_3 = \int_0^{\lambda/2}[g_u(\lambda-\mu) - g_u(\lambda)]\check{\rho}(\mu)\,d\mu$, then $|I_3| \leq (2d/\lambda)|\int_0^{\lambda/2}(1+\mu)\check{\rho}(\mu)\,d\mu|$ so $I_3 = O(\lambda^{-1})$. Hence $|\check{\rho} * g - g| \leq |I_1| + |I_2| + |I_3| = O(\lambda^{-1})$, proving the claim.

Now all that remains to complete the proof of Theorem 3.1 is to show that (3.1) holds.

We will get the growth of $\check{\rho} * dg_u/d\lambda$ by examining the singularities of its Fourier transform in λ. For technical reasons g_u will be identified with the generalized half-density $g_u|d\lambda^{1/2}|$, where $(d\lambda)^{1/2}$ is the standard half-density on $\mathbf{R}$. We have

$$\check{\rho} * \frac{dg_u}{d\lambda} = \check{\rho} * \left\langle \frac{dE_\lambda}{d\lambda} u, u \right\rangle.$$

Therefore,

$$\left(\check{\rho} * \left\langle \frac{dE_\lambda}{d\lambda} u, u \right\rangle\right)^{\widehat{}} = \rho \left\langle \frac{\widehat{dE_\lambda}}{d\lambda} u, u \right\rangle.$$

The Schwartz kernel of $\widehat{dE_\lambda}/d\lambda$ is

$$\sum_{\lambda_j \in \operatorname{spec}(Q)} e^{-it\lambda_j} e_j(x)\overline{e_j(y)}.$$

Denote by $F(t,x,y)$ the distribution $\widehat{dE_\lambda}/d\lambda$. Let $h_u = \langle Fu, u\rangle$.

We will calculate the singularities of ρh_u using the calculus of composition of Lagrangian distributions.

It is shown in [**DGu**] that $F \in I^{-1/4}(\mathbf{R}\times X, X; C)$, where C is the homogeneous canonical relation $\{(t,\tau),(x,\xi),(y,\eta) \in T^*(\mathbf{R}\times X\times X)\backslash\{0\} | \tau + q(x,\xi) = 0,\ (x,\xi) = \Phi^t(y,\eta)\}$, where Φ^t is the flow of the Hamiltonian vector field H_q. Therefore $\rho F \in I^{-1/4}(\mathbf{R}\times X, X; C')$, where $C' = \{(t,\tau),(x,\xi),(y,\eta) \in C \,|\, |t| < \varepsilon\}$.

Since the map $\pi\colon C' \to T^*X\backslash\{0\}$,

$$(t,\tau),(x,\xi),(y,\eta) \to (y,\eta)$$

is surjective, the following fiber product diagram is transversal:

$$\begin{array}{ccccc} (2n+1) & C' & \leftarrow & G & (n+1) \\ & \pi\downarrow & & \downarrow & \\ (2n) & T^*X & \overset{i}{\leftarrow} & \Lambda & (n) \end{array}$$

The numbers in parentheses are the dimensions of the spaces. G is the fiber product of C' and Λ,

$$G = \{(t,\tau),(x,\xi),(y,\eta) \in C' | (y,\eta) \in \Lambda\},$$

$$C' \circ \Lambda = \{(t,\tau),(x,\xi) \in T^*(\mathbf{R}\times X)\backslash\{0\} | \tau + q(x,\xi) = 0, \\ (x,\xi) = \Phi^t(y,\eta), (y,\eta) \in \Lambda, |t| < \varepsilon\}.$$

By Theorem 1.1,

$$\rho F u \in I^{-n/4-1/4}(\mathbf{R}\times X;\ C'\circ\Lambda).$$

Now we wish to evaluate $\langle \rho F u, u\rangle = \int_X \rho F u \overline{u} = \pi_*\Delta^*(\rho F u \boxtimes \overline{u})$, where $\Delta\colon \mathbf{R}\times X \to \mathbf{R}\times X\times X$ is the diagonal map, and $\pi\colon \mathbf{R}\times X\times \mathbf{R}$ is projection. Since $\pi_*\Delta^*$ is a Fourier integral operator, we would like to treat $\pi_*\Delta^*(\rho F u \boxtimes \overline{u})$ in the calculus of Lagrangian distributions. Unfortunately, $\rho Fu \boxtimes \overline{u}$ is not Lagrangian because of the presence of edge terms in its wavefront set. That is,

$$\mathrm{WF}(\rho Fu \times \overline{u}) \subset ((C' \circ \tilde{\Lambda}) \times \Lambda) \cup (0_{\mathbf{R}\times X} \times \tilde{\Lambda}) \cup ((C' \circ \Lambda) \times 0_X),$$

where $0_{\mathbf{R}\times X}$, 0_X are the zero sections of $T^*(\mathbf{R} \times X)$, T^*X, respectively, and $\tilde{\Lambda} = \{(x,\xi) \in T^*X | (x,-\xi) \in \Lambda\}$. From [**DGu**], we have $\pi_*\Delta^* \in I^0(\mathbf{R}, \mathbf{R} \times X \times X; D)$, where $D = \{(t,\tau),(t,\tau),(x,\xi),(x,-\xi) \in T^*(\mathbf{R} \times \mathbf{R} \times X \times X)\backslash\{0\}\}$. But $D \circ (0_{\mathbf{R}\times X} \times \Lambda)$ is empty, as is $D \circ ((C' \circ \Lambda) \times 0_X)$. Therefore, $\pi_*\Delta^*$ is smoothing on the edge singularities, so $\rho Fu \boxtimes \overline{u}$ can be treated as if it were Lagrangian distribution $I^{-n/2-1/4}(\mathbf{R} \times X \times X; (C' \circ \Lambda) \times \Lambda)$, with symbol $\sigma(\rho Fu) \otimes \sigma(\overline{u})$, when calculating $\pi_*\Delta^*(\rho Fu \boxtimes \overline{u})$. Form the fiber product diagram:

$$\begin{array}{ccccc} (2n+2) & D & \longleftarrow & G_2 & (n) \\ & \downarrow & & \downarrow & \\ (4n+2) & T^*(\mathbf{R} \times X \times X) & \longleftarrow & (C' \circ \Lambda) \times \tilde{\Lambda} & (2n+1) \end{array}$$

$$\begin{aligned} G_2 = \{(t,\tau),(t,\tau),(x,\xi),(x,-\xi) &\in T^*(\mathbf{R} \times \mathbf{R} \times X \times X)\backslash\{0\} | \\ &(x,\xi) \in \Lambda, (x,\xi) = \Phi^t(y,\eta), (y,\eta) \in \Lambda, |t| < \varepsilon, \tau + q(x,\xi) = 0\} \\ = \{(0,\tau),(0,\tau),&(x,\xi),(x,-\xi) | (x,\xi) \in \Lambda, \tau + q(x,\xi) = 0\}. \end{aligned}$$

Therefore, $D \circ (C' \circ \tilde{\Lambda}) = \{(0,\tau) | \tau < 0\} = N^*_-\{0\}$. The condition on τ comes from $\tau + q(x,\xi) = 0$ and the positivity of Q (and hence q). This diagram is clean with excess $n-1$. By Lemma 1.1, the projection map $G_2 \to N^*_-\{0\}$ is a fiber map with fiber $\Lambda/\mathbf{R}^+ = \Lambda^\#$. Therefore, by Theorem 1.1, $\rho h_u \in I^{-3/4}(\mathbf{R}, N^*_-\{0\})$ and

$$\sigma(\rho h_u) = (\sigma(\rho Fu) \times \sigma(\overline{u})) \circ g = \int_{\Lambda^\#} \sigma(\rho Fu) \otimes \overline{\sigma(u)} \otimes g \otimes \omega_1^{-1/2},$$

where $g = \sigma(\pi_*\Delta^*) = \pi_1^*|dt \wedge d\tau \wedge dx \wedge d\xi|^{1/2}$, $\pi_1 \colon D \to T^*(\mathbf{R} \times X)\backslash\{0\}$,

$$((t,\tau),(t,\tau)(x,\xi),(x,-\xi)) \xrightarrow{\pi_1} ((t,\tau),(x,\xi))$$

and ω_1 is the canonical density on $T^*(\mathbf{R} \times X \times X)\backslash\{0\}$ At $t = 0$, $\sigma(\rho F) = f = \pi_2^*(|dt|^{1/2} \otimes |dx \wedge d\xi|^{1/2})$. $\pi_2 \colon C' \to \mathbf{R} \times T^*X\backslash\{0\}$ is the diffeomorphism $((t,\tau),(x,\xi),(y,\eta)) \to (t,x,\xi)$. At $t = 0$, $\sigma(\rho Fu) = \sigma(\rho F) \circ \sigma(u) = f \otimes a(x,\xi) \otimes \omega_2^{-1/2}$. ω_2 is the canonical density on $T^*X\backslash\{0\}$.

Hence, $\sigma(\rho Fu) \otimes \sigma(\overline{u}) = f \otimes a(x,\xi) \otimes \omega_2^{-1/2} \otimes \overline{a}(y,-\eta)$, where $(x,\xi) \in \Lambda$, $(y,-\eta) \in \Lambda$.

Therefore,

$$\begin{aligned} \sigma(\rho h_u) &= \int_{\Lambda^\#} |a(x,\xi)|^2 \otimes f \otimes \omega_1^{-1/2} \otimes g \otimes \omega_2^{-1/2} \\ &= \begin{cases} \displaystyle\int_{\Lambda^\#} \frac{d\tau^{1/2}}{\tau} (|\sigma(u)|^2)^\#, & \tau < 0, \\ 0, & \tau > 0, \end{cases} \end{aligned}$$

since all of the other density factors cancel. Note also that Maslov factors do not enter because the Maslov bundles of both C' and D have canonical trivializations, and the Maslov factors on Λ are cancelled canonically in the product $|\sigma(u)|^2$.

So

$$\check{\rho} * \frac{dg_u}{d\lambda}(\lambda) = (2\pi)^{-n}\left(\check{\rho} * \frac{dg_u}{d\lambda}(-\lambda)\right)^{\wedge\wedge} = (2\pi)^{-n}(\rho h_u)^{\wedge}$$
$$= (2\pi)^{-n}\sigma(\rho h_u)(-\lambda) + O(\lambda^{-2}) \quad \text{as } \lambda \to \infty,$$

because the total symbol of a distribution in $I(\mathbf{R}, N^*\{0\})$ is just its Fourier transform. Therefore

$$\check{\rho} * \frac{dg_u}{d\lambda}(\lambda) = \begin{cases} (2\pi)^{-n}(d\lambda^{1/2}/\lambda)\mathrm{res}_\Lambda|a|^2 + O(\lambda^{-2})\, d\lambda^{1/2}, & \lambda > 0, \\ O(\lambda^{-2})\, d\lambda^{1/2}, & \lambda < 0. \end{cases}$$

Cancelling the half-density $d\lambda^{1/2}$ then gives

$$\check{\rho} * \frac{dg_u}{d\lambda}(\lambda) = \|u\|_R^2\lambda^{-1} + O(\lambda^{-2}) \quad \text{as } \lambda \to \infty, \tag{3.2}$$

completing the proof of Theorem 3.1.

COROLLARY 3.1.

$$\check{\rho} * \frac{dg_{u,v}}{d\lambda}(\lambda) = \langle u, v\rangle_R\lambda^{-1} + O(\lambda^{-2}).$$

PROOF. Use a polarization argument on (3.2) like that in Corollary 1.1.

REMARK 3.1. $\check{\rho} * dg_u/d\lambda$ actually has a full asymptotic expansion $\sum_{i=1}^{\infty} a_i\lambda^{-i}$, with $a_1 = (2\pi)^{-n}\mathrm{res}_\Lambda|a|^2$, which comes from the asymptotic expansion of the total symbol of ρh_u.

Let Q, u, v, u_j, v_j, ρ be as above. Extend the definition zeta function $Z_{Q,u,v}(s) = \langle Q^{-s/2}u, Q^{-s/2}v\rangle$ to now include complex s. From §2, we have that, as $s \to 0^+$ along $\mathbf{R}^+$,

$$Z_{Q,u,v}(s) = \frac{\langle u, v\rangle_R}{s} + \langle u, v\rangle_{R,Q^{-1}} + O(s).$$

We will now rederive this by an alternate method using the above results. In fact, we will prove the somewhat stronger

THEOREM 3.2. *$Z_{Q,u,v}(s)$ is holomorphic for* $\mathrm{re}(s) > 0$ *and can be extended meromorphically to the whole complex plane with only simple poles at the nonnegative integers. Excluding neighborhoods of the poles, $Z_{Q,u,v}(s)$ has at worst polynomial growth in the half-planes* $\mathrm{re}(s) \geq s_0$ *for all $s_0 \in \mathbf{R}$. At $s = 0$, the residue is $\langle u, v\rangle_R$.*

PROOF.

$$g_{u,v}(\lambda) = \sum_{\substack{\lambda_j \in \mathrm{spec}(Q) \\ \lambda_j \leq \lambda}} u_j\overline{v}_j = \int_{-\infty}^{\lambda} \sum_{\lambda_j \in \mathrm{spec}(Q)} u_j\overline{v}_j\delta(\lambda - \lambda_j)\, d\lambda,$$

$$\check{\rho} * g_{u,v} = \check{\rho} * \int_{-\infty}^{\lambda} \sum_{\lambda_j \in \mathrm{spec}(Q)} u_j\overline{v}_j\delta(\lambda - \lambda_j)\, d\lambda$$
$$= \int_{-\infty}^{\lambda} \sum_{\lambda_j \in \mathrm{spec}(Q)} u_j\overline{v}_j\check{\rho}(\lambda - \lambda_j)\, d\lambda,$$

$$\check{\rho} * \frac{dg_{u,v}}{d\lambda} = \sum_{\lambda_j \in \mathrm{spec}(Q)} u_j\overline{v}_j\check{\rho}(\lambda - \lambda_j).$$

Denote by Σ the sum $\sum_{\lambda_j \in \mathrm{spec}(Q)} u_j \bar{v}_j \delta(\lambda - \lambda_j)$. Then $\check{\rho} * dg_{u,v}/d\lambda = \check{\rho} * \Sigma = \langle u, v\rangle_R/\lambda + O(\lambda^{-2})$ as $\lambda \to \infty$, by Corollary 3.1.

Let $\varepsilon_1 < \lambda_1$, λ_1 the first eigenvalue of Q. Let $\chi(\lambda) \in C^\infty(\mathbf{R})$, $\chi(\lambda) = 0$ if $\lambda < \varepsilon_1$, $\chi(\lambda) = 1$ if $\lambda \geq \lambda_1$ and $\chi_s(\lambda) = \chi(\lambda)\lambda^{-s}$. Then

$$Z_{Q,u,v}(s) = \sum_{\lambda_j \in \mathrm{spec}(Q)} \lambda_j^{-s} u_j \bar{v}_j = \langle \Sigma, \chi_s\rangle.$$

The theorem is now a consequence to the following two lemmas.

LEMMA 3.1.

$$\langle \Sigma, \chi_s\rangle - \langle \check{\rho} * \Sigma, \chi_s\rangle = \langle (1-\rho)\hat{\Sigma}, \check{\chi}\rangle$$

is entire in s and has at most polynomial growth on any half-space $\mathrm{re}(s) \geq s_0$.

PROOF. See [**DGu**, Corollary 2.2].

LEMMA 3.2. *$\langle \check{\rho} * \Sigma, \chi_s\rangle$ is holomorphic for* $\mathrm{Re}(s) > 0$ *and extends meromorphically to $\mathbf{C}$ with only simple poles at $s = 0, -1, -2, \ldots$. It is bounded in the half-planes* $\mathrm{re}(s) \geq s_0$ *provided we avoid neighborhoods of the poles. The residue at $s = 0$ is $\langle u, v\rangle_R$.*

PROOF. The argument here is essentially the same as the one given in the proof of Corollary 2.2 in [**DGu**]. Suppose $f(\lambda) = O(\lambda^{-1-k})$. Then $\langle f, \chi_s\rangle$ is bounded and holomorphic for $\mathrm{re}(s) \geq s_0$, $s_0 > -k$. But

$$\begin{aligned}\langle \lambda^{-1-k}, \chi_s\rangle &= \int_{-\infty}^{\infty} \lambda^{-1-k}\chi(\lambda)\lambda^{-s}\,d\lambda \\ &= \frac{1}{s+k}\int_{-\infty}^{\infty} \lambda^{-k-s}\frac{d\chi}{d\lambda}\,d\lambda = \frac{\Psi(s)}{s+k},\end{aligned}$$

where $\Psi(s) = \int_{-\infty}^{\infty} \lambda^{-k-s}(d\chi/d\lambda)\,d\lambda$ is entire in s, and is bounded for $\mathrm{Re}(s) \geq s_0$ since $d\chi/d\lambda$ has compact support. Also, $\Psi(-k) = 1$. Hence each term in the asymptotic expansion of $\rho * \Sigma$ leads to a pole in $\langle \rho * \Sigma, \chi_s\rangle$, the residue of which is just the term's coefficient.

REMARK. Note that the constant term in the Laurent series of $Z_{Q,u,v}(s)$ about 0 is $\langle u, v\rangle_{R,Q^{-1}}$.

Lemma 3.1 tells us that $Z_{Q,u,v}(s) = \langle \Sigma, \chi_s\rangle$ has the same poles and residues as $\langle \rho * \Sigma, \chi_s\rangle$, and gives us the stated growth of $Z_{Q,u,v}$ on the half-spaces $\mathrm{Re}(s) \geq s_0$. We now present the third regularization scheme.

Let $Q \in \Psi^1(X)$ be elliptic and selfadjoint as above, and let $u, v \in I(X, \Lambda)$. Consider the heat operator e^{-tQ}, $t \geq 0$. For $t = 0$, e^{tQ} reduces to the identity. Define the function $\Theta_{Q,u,v}(t)$ for $t > 0$:

$$\Theta_{Q,u,v}(t) = \langle e^{-tQ}u, v\rangle = \sum_{\lambda_j \in \mathrm{spec}(Q)} e^{-t\lambda_j} u_j \bar{v}_j.$$

If $u, v \in L^2(X)$ then $\lim_{t\to 0^+} \Theta_{Q,u,v}(t) = \langle u, v\rangle$. If $u, v \in I_{\mathrm{cr}}(X, \Lambda)$, then $\Theta_{Q,u,v}$ will usually approach ∞ as $t \to 0^+$. However, the growth is controlled by $\langle u, v\rangle_R$:

THEOREM 3.3. *$\Theta_{Q,u,v}(t) = -\langle u,v\rangle_R \log t + \langle u,v\rangle_{R,Q^{-1}} + o(t^\varepsilon)$, $0 < t < 1$, as $t \to 0^+$.*

Theorem 3.3 shows that heat operator regularization of $\langle u,v\rangle$ is equivalent to the zeta function regularization. In fact, $Z_{Q,u,v}$ and $\Theta_{Q,u,v}$ contain essentially the same information, since $\Theta_{Q,u,v}$ is just the inverse Mellin transform of $Z_{Q,u,v}$:

LEMMA 3.3. *Let $c > 0$. Then*

$$\Theta_{Q,u,v}(t) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c} t^{-s} Z_{Q,u,v}(s)\Gamma(s)\,ds. \tag{3.3}$$

PROOF. From [**DGu**] we have

$$e^{-t\lambda_j} = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c} t^{-s}\lambda_j^{-s}\Gamma(s)\,ds.$$

So obviously,

$$e^{-t\lambda_j}u_j\bar{v}_j = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c} t^{-s}\lambda_j^{-s}u_j\bar{v}_j\Gamma(s)\,ds.$$

Summing over $\lambda_j \in \mathrm{spec}(Q)$ for $c > 0$ gives the desired formula.

PROOF OF THEOREM 3.3. We have that $\Gamma(s)$ exponentially decays in any vertical strip, as long as we avoid neighborhoods of the poles. This, combined with the polynomial growth of $Z_{Q,u,v}$, tells us $\Gamma(s)Z_Q(s)$ decays in any vertical strip of $\mathrm{Re}(s) > -1$ if we avoid a neighborhood of the origin. Therefore, we can shift the contour of integration in (3.3) to $\mathrm{re}(s) = -\varepsilon$, $0 < \varepsilon - 1$, provided we take the contribution from the pole at zero into account. So,

$$\Theta_{Q,u,v}(t) = I_1 + I_2,$$

$$I_1 = \frac{1}{2\pi i}\int_{\substack{\mathrm{Re}(s)=-\varepsilon\\ 0<\varepsilon<1}} t^{-s}Z_{Q,u,v}(s)\Gamma(s)\,ds,$$

$$I_2 = \int_{C_0} t^{-s}Z_{Q,u,v}(s)\Gamma(s)\,ds,$$

where C_0 is a small counterclockwise loop about $s = 0$. Clearly, $I_1 = O(t^\varepsilon)$ as $t \to 0^+$.

By the Cauchy-integral formula,

$$I_2 = -\langle u,v\rangle_R \log t + \langle u,v\rangle_{R,Q^{-1}} \quad \text{for } t > 0$$

since the residue of $Z_{Q,u,v}(s)$ at $s = 0$ is $\langle u,v\rangle_R$, and the residue of $\Gamma(s)$ at zero is 1. Therefore, $\Theta_{Q,u,v}(t) = -\langle u,v\rangle \log t + \langle u,v\rangle_{R,Q^{-1}} + O(t^\varepsilon)$. But the condition on ε is open, so we have

$$\Theta_{Q,u,v}(t) = \langle u,v\rangle \log t + \langle u,v\rangle_{R,Q^{-1}} + o(t^\varepsilon).$$

REFERENCES

[**Du**] J. J. Duistermaat, *Fourier integral operators*, Courant Institute Lecture Notes, New York, 1973.

[**DGu**] J. J. Duistermaat and V. W. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975), 29–39.

[**DHo**] J. J. Duistermaat and L. Hörmander, *Fourier integral operators*. II, Acta Math. **128** (1972), 184–269.

[**Gu1**] V. W. Guillemin, *Some classical theorems in spectral theory revisited*, Seminar on Singularities, Princeton Univ. Press, Princeton, N.J., 1978, pp. 219–259.

[**Gu2**] ——, *The Leray residue symbol and traces of pseudodifferential operators*, M.I.T. Notes, 1981.

[**GuSt**] V. W. Guillemin and S. Sternberg, *Geometric asymptotics*, Math. Surveys, no. 14, Amer. Math. Soc., Providence, R.I., 1977.

[**Hoc**] H. Hochstadt, *Integral equations*, Wiley, New York, 1973.

[**Hor1**] L. Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218.

[**Hor2**] ——, *Fourier integral operators*. I, Acta Math. **127** (1971), 79–183.

[**Iz**] S. Izen, Ph.D. Thesis, M.I.T., 1983.

[**Ni**] L. Nirenberg, *Lectures on linear partial differential equations*, CBMS Regional Conf. Ser. Math., no. 17, Amer. Math. Soc., Providence, R.I., 1972.

[**Se**] R. T. Seeley, *Complex powers of an elliptic operator*, Proc. Sympos. Pure Math., Vol. 10, Amer. Math. Soc., Providence, R.I., 1967, pp. 288–307.

[**Tr**] F. Treves, *Introduction to pseudodifferential and Fourier integral operators*, Plenum Press, New York, 1980.

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