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SEMIGROUPS IN LIE GROUPS, SEMIALGEBRAS IN LIE ALGEBRAS

BY

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ABSTRACT. Consider a subsemigroup of a Lie group containing the identity and being ruled by one-parameter semigroups near the identity. We associate with it the set W of its tangent vectors at the identity and obtain a subset of the Lie algebra L of the group. The set W has the following properties: (i) $W + W = W$, (ii) $\mathbf{R}^+ \cdot W \subset W$, (iii) $W^- = W$, and, the crucial property, (iv) for all sufficiently small elements x and y in W one has $x * y = x + y + \frac{1}{2}[x, y] + \cdots$ (Campbell-Hausdorff!) $\in W$. We call a subset W of a finite-dimensional real Lie algebra L a Lie semialgebra if it satisfies these conditions, and develop a theory of Lie semialgebras. In particular, we show that a subset W satisfying (i)–(iii) is a Lie semialgebra if and only if, for each point x of W and the (appropriately defined) tangent space T_x to W in x, one has $[x, T_x] \subset T_x$. (The Lie semialgebra W of a subgroup is always a vector space, and for vector spaces W we have $T_x = W$ for all x in W, and thus the condition reduces to the old property that W is a Lie algebra.) In the introduction we fully discuss all Lie semialgebras of dimension not exceeding three. Our methods include a full duality theory for closed convex wedges, basic Lie group theory, and certain aspects of ordinary differential equations.

We are interested in subsemigroups of Lie groups. In particular, we propose Lie's program to characterize infinitesimally generated (local) subsemigroups in terms of their tangent vectors at the origin.

Semigroups in Lie groups have been observed in a variety of contexts. Loewner **[Lo]** studied them in the context of partial differential equations, and investigated their role in the geometry of pseudo-Riemannian manifolds. More recently, Rothkrantz treated semigroups and hermitian symmetric spaces in a dissertation written under the direction of van Est **[Ro]**. Lie semigroups occur in geometric control theory as was explained in a survey article by Brockett **[Br]**. This topic was the object of several papers in recent years, so, e.g., of papers by Hirschhorn **[Hi]**, Jurdjevic and Sussmann **[JS]**.

Vinberg **[Vi]** and Ol'shanskii **[Ol1, Ol2]** characterized convex closed cones which are invariant under the action of a semisimple Lie group. In particular, they determined closed cones in a semisimple Lie algebra which are invariant under the adjoint action. Every invariant cone in a Lie algebra is the precise tangent object of a local semigroup in the corresponding Lie group. Their interest in this matter arose from the representation theory of semisimple groups and their results overlap those

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of Rothkrantz's. Foundations for a Lie semigroup theory were attempted by Hille and Philips [**HP**], and Langlands [**La**] and, more recently and more systematically, Graham [**Gr1**, **Gr2**] and Hofmann and Lawson [**HL1**–**HL5**].

We are still a long way from having a full and satisfactory theory, but the characteristic problems and difficulties begin to emerge. We appear to have a semigroup equivalent of Sophus Lie's Fundamental Theorem: The set of tangent vectors at the origin at a local semigroup is a *wedge*, i.e. a closed convex set which is closed under positive scalar multiplication and addition. This was observed early, certainly by Loewner, but very probably by S. Lie himself. Much more recently it was observed, independently, by Ol'shanskii [**Ol2**] and Hofmann and Lawson [**HL3**] that the following additional condition is satisfied by any wedge W arising as the tangent object of a local semigroup:

(L) $\quad e^{\operatorname{ad} X}W = W \quad$ for all X from the edge $W \cap -W$ of the wedge W.

Conversely, Ol'shankii has announced [**Ol2**] without proof, that any wedge satisfying condition (L) is the precise tangent object of the local semigroup it generates. For a special class of wedges this was proved, independently, by Hofmann and Lawson [**HL3**]. In full generality, the claim was recently verified by the authors [**HH2**].

In general, the exponential function is not locally surjective from an (L)-wedge onto the local semigroup generated by it. This is the case if and only if the local semigroup generated by the wedge is locally divisible; see [**HL2**, **HL4**]. It was known that a wedge generates a semigroup with this property if and only if it is a so-called Lie semialgebra:

DEFINITION 0. A subset W of a Lie algebra L (of finite dimension over $\mathbf{R}$) is called a *Lie-semialgebra* if and only if the following conditions are satisfied:

(1) W is a wedge (i.e. (i) $W + W \subseteq W$, (ii) $\mathbf{R}^+ \cdot W \subseteq W$, (iii) $\overline{W} = W$).

(2) There is an open symmetric convex neighborhood B of 0 in L on which the Campbell-Hausdorff series $X * Y = X + Y + \frac{1}{2}[X, Y] + \cdots$ converges absolutely for $X, Y \in B$, and which satisfies

$$(W \cap B) * (W \cap B) \subseteq W.$$

We observe that if $\exp\colon L(G) \to G$ is the exponential function of a Lie group, then $\exp(W \cap B)$ is a local semigroup in G in which every element lies on a local one-parameter semigroup, and all of these are so obtained.

As an example, we will presently describe all semialgebras of dimension three or less. Any invariant cone, according to Vinberg and Ol'shanskii, is a Lie semialgebra. The Lie wedge of a closed divisible subsemigroup of a Lie group is a semialgebra provided it has no invertible elements other than the identity (see [**HL4**]; this last condition is possibly superfluous).

Doubtlessly, Lie semialgebras are an important concept in the study of Lie semigroups. However, on the basis of their definition, they are not easily handled. In particular, their definition not only depends on the Campbell-Hausdorff series, but indeed on the existence of a fixed neighborhood B on which the Campbell-Hausdorff multiplication is defined. It is not clear at all from the definition that, for another larger Campbell-Hausdorff neighborhood B', the condition $(W \cap B')*(W \cap B') \subseteq W$ is still satisfied.

It is therefore highly desirable to have a characterization of semialgebras which is, firstly, global in nature; secondly, which depends only on the Lie bracket and not on the Campbell-Hausdorff series; and, finally, which reflects the rich geometric structure of a wedge in a finite-dimensional vector space. The objective of this paper is to provide such a characterization. Before we describe the main result, we recall a few concepts on the geometry of a wedge. In the proof of the main theorem we will have to study the geometry of a wedge in much greater detail.

If W is a wedge in a finite-dimensional vector space L, then we say that a hyperplane T of L is a *tangent hyperplane* of W if the following three conditions are satisfied:

(a) $\dim(T \cap W) \geqslant 1$.

(b) W is contained in one of the closed half-spaces bounded by T.

(c) If S is a hyperplane of L satisfying conditions (a) and (b) with S in plane of T, then $S = T$.

One expresses these facts sometimes by saying that T is a unique support-hyperplane of W. By a classical theorem of Straszewicz **[St]** there are enough points $x \in W$ such that there is a unique tangent hyperplane T_x of W through x in the sense that W is the intersection of half-spaces bounded by tangent hyperplanes T_x. We can now formulate our main theorem as follows:

THEOREM A. *Let W be a wedge in a finite-dimensional Lie algebra L such that $L = W - W$. Then the following statements are equivalent*:

(1) *W is a Lie semialgebra in L.*

(2) *$[x, T_x] \subseteq T_x$ for all $x \in W$ for which T_x is a tangent hyperplane of W in L.*

(3) *For any neighborhood U of O in L such that $X * Y = X + Y + \frac{1}{2}[X, Y] + \cdots$ is defined for all $X, Y \in U$, we have $(W \cap U)*(W \cap U) \subseteq W$.*

It is known that, for any Lie semialgebra W in a finite-dimensional Lie algebra L, the vector space $W - W$ is a Lie algebra **[HL2]**. Therefore the hypothesis $L = W - W$ does not restrict the generality of the theorem.

Let us test this theorem by deriving a number of consequences. On the basis of Definition 0 it is not clear at all that the intersection of an arbitrary family of Lie semialgebras is a Lie semialgebra. Now, however, it follows that the set of Lie semialgebras in a Lie algebra is a complete lattice:

COROLLARY B. *The collection of Lie semialgebras in a finite-dimensional Lie algebra is closed under arbitrary intersection.*

PROOF. This is an immediate consequence of conditions (3) in Theorem A. Let U be as in Theorem A(3). If $\{W_j\colon j \in J\}$ is a family of Lie semialgebras and $X, Y \in \bigcap\{(W_j \cap U)\colon j \in J\}$, then $X * Y \in W_i$ for all $i \in J$ by Theorem A(3), and thus $X * Y \in \bigcap\{W_i\colon i \in J\}$. $\square$

We derive a result of Ol'shanskii and Vinberg [**Ol2**].

COROLLARY C. *Let W be an invariant wedge in a finite-dimensional Lie algebra, i.e. a closed wedge satisfying $e^{\operatorname{ad} X}W = W$ for all $X \in L$. Then W is a Lie semialgebra.*

PROOF. Let $x \in W$ be such that T_x is a tangent hyperplane at W in x. Then T_x is also a tangent hyperplane in all points tx with $t \geqslant 0$. If f is an arbitrary automorphism of W, then $f(T_x) = T_{f(x)}$. If we take $f = e^{t \operatorname{ad} x}$, then $f(x) = x$, and we conclude that

(i) $e^{t \operatorname{ad} x}T_x = T_x$ for all $t \geqslant 0$.

Then

(ii) $\frac{1}{t}(e^{t \operatorname{ad} x_y} - y)$ for all $y \in T_x$ and $t > 0$.

If we let t tend to 0 we obtain

(iii) $(\operatorname{ad} x)T_x \subseteq T_x$,

which is condition (3) of Theorem A. $\square$

A few comments illuminate the special attention we must pay to the fact that we are dealing with *semigroups* rather than with groups. Firstly, it is essential that the adjoint group acts as a group of *automorphisms* of the wedge W. Thus is would not suffice to know that the relation $e^{\operatorname{ad} x}W \subseteq W$ is satisfied for all $x \in W$. For an endomorphism f of W it is not true in general that $f(T_x)$ is again a tangent hyperplane.

Secondly, we passed from information (ii) to information (iii) by differentiating. This is possible because T_x is a vector space. The relation $e^{tF}W \subseteq W$ for a vector space endomorphism F of L does not yield the relation $FW \subseteq W$ since the differentiation process does not apply. It is instructive to realize very clearly where the proof of (ii) breaks down in this present case.

Thirdly, Corollary C has no general converse as the example of $\mathrm{sl}(2, \mathbf{R})$ will show. However, in any compact Lie algebra L, each semialgebra is invariant [**HH3**].

We draw further conclusions from the main theorem:

COROLLARY D. *Let L be a finite-dimensional Lie algebra and W a closed half-space with the hyperplane T as boundary. Then the following two conditions are equivalent*:

(1) *W is a Lie semialgebra.*

(2) *T is a Lie algebra.*

PROOF. Since T is the boundary of a half-space, we have $T = T_x$ for all $x \in T \setminus \{0\}$. Hence Theorem A(2) is equivalent to saying that T is a subalgebra. $\square$

We say that a wedge W in a vector space L is *polyhedral* if it is the intersection of finitely many half-spaces. Its finitely many tangent hyperplanes are called its *bounding hyperplanes*. We then have the following generalization of Corollary D.

COROLLARY E. *A polyhedral wedge W in a Lie algebra L with $L = W - W$ is a Lie semialgebra if and only if its finitely many bounding hyperplanes are subalgebras.*

In order to prove this corollary, we use additional information which we will provide in this paper. If T is a tangent hyperplane at a wedge W then $W \cap T$ is a subwedge of W, and $E(T) = (W \cap T) - (W \cap T)$ is the subvector space generated by it. Now we complement the equivalent conditions (1), (2), (3) of Theorem A by a fourth condition:

THEOREM F. *If W is a wedge in a finite-dimensional Lie algebra L with $L = W - W$, then W is a Lie semialgebra if and only if*

(4) $[E(T), T] \subseteq T$ *for any tangent hyperplane T of W.*

If these conditions are satisfied, then $E(T)$ is a Lie algebra.

The last assertion is a relatively elementary consequence of (4), as we will see. (4) may be expressed by saying that T is an $E(T)$-module relative to the adjoint action of L on itself. For a polyhedral wedge we have, of course, $E(T) = T$ for all bounding hyperplanes, which gives the proof of Corollary E. At the end of the paper we give a construction of a class of examples which will show that, in general, the tangent hyperplanes T of a Lie semialgebra W are not subalgebras.

For our intuition, nevertheless, an inspection of low-dimensional Lie algebras L is helpful, and here we observe

COROLLARY G. *If W is a wedge in a Lie algebra L with* $\dim L \leqslant 3$, *then W is a Lie semialgebra if and only if all tangent hyperplanes of W are subalgebras.*

The proof of Corollary G follows directly from the following observation.

LEMMA H. *Let E be a subalgebra of a Lie algebra L and let T be a vector subspace of L containing E such that $[E, T] \subseteq T$. If* $\dim T \leqslant \dim E + 1$, *then T is a subalgebra.*

PROOF. Suppose $x \in T \setminus E$; then $T = E + \mathbf{R} \cdot x$ and, thus, $[T, T] = [E + \mathbf{R} \cdot x, E + \mathbf{R} \cdot x] \subseteq [E, E] + [E, x] \subseteq [E, T] \subseteq T$. □

This allows us to completely describe all Lie semialgebras of dimension $\leqslant 3$:

(a) *L abelian*. Every wedge trivially is a Lie semialgebra.

(b) *L nilpotent*. In the 3-dimensional Heisenberg algebra L any 2-dimensional subalgebra contains the central commutator algebra $[L, L]$. Hence a wedge W is a Lie semialgebra if $[L, L] \subseteq W$. (This remains true for nilpotent algebras of all dimensions as Hofmann and Lawson **[HL1]** have shown, but the 3-dimensional case was the hard portion of their proof.)

(c) *L solvable*. Here we have two separate cases to consider:

(i) Let A_n be a Lie algebra which is isomorphic to the Lie algebra of all matrices of degree $n + 1$ of the form

$$\begin{bmatrix} tE_n & X \\ 0 & 0 \end{bmatrix},$$

X an n-component column, $t \in \mathbf{R}$. Then any wedge contained in A_n is a semialgebra (see **[HL1]**). Now (a) $L = A_3$ or (b) $L = A_2 \oplus \mathbf{R}$. In case (b), let T be a two-dimensional subalgebra of L. Then either $T = \mathbf{R} \cdot a \oplus \mathbf{R}$ for some $0 \neq a \in A_2$ or $T = \{(a, f(a))\colon a \in A_2\}$ for some linear form $f\colon A_2 \to \mathbf{R}$. Then $([a, b], f[a, b]) = [(a, f(a)), (b, f(b))] = ([a, b], 0)$ shows that $f([A_2, A_2]) = \{0\}$ and that every f vanishing on $[A_2, A_2]$ gives a subalgebra T. Now $[L, L] = [A_2, A_2] \oplus \{0\}$. We conclude that the semialgebras W in L are of the following type: (1) W is trivial (i.e.

contains $[L, L]$), (2) $W = W_2 \oplus \mathbf{R}$ with an arbitrary wedge W_2 in A_2, (3) W is intersection of a wedge of type (1) with a wedge of type (2).

(ii) L is a Lie algebra which is isomorphic to the Lie algebra of complex 2 by 2 matrices of the form

$$\begin{bmatrix} ta & u \\ 0 & 0 \end{bmatrix}, \qquad t \in \mathbf{R}, u \in \mathbf{C},$$

with a fixed complex number $a \notin \mathbf{R}$. Then $[L, L]$, the set of matrices with $t = 0$, is the only two-dimensional subalgebra of L. Hence the only 3-dimensional Lie semialgebras in L are the two half-space semialgebras bounded by the plane $[L, L]$, and L.

(d) L *(semi-)simple.*

(i) $L = \mathrm{so}(3)$. This algebra has no 2-dimensional subalgebras, hence no Lie semialgebras of dimension > 1.

(ii) $L = \mathrm{sl}(2, \mathbf{R})$. The two-dimensional subalgebras are precisely the tangent planes of the double cone $\{x : \kappa(x, x) = 0\}$ with the Cartan-Killing form κ. Any family of half-spaces bounded by such planes will intersect in a Lie semialgebra, and every Lie semialgebra is so obtained. This holds, in particular, for the two ice cream cones Σ^+, Σ^-, with $\Sigma^+ \cup \Sigma^- = \{x : \kappa(x, x) \leqslant 0\}$, but also for infinitely many polyhedral cones with their bonding planes tangent to Σ^+ (say) and containing Σ^+. None of these semialgebras are tangent objects of global semigroups in $\mathrm{Sl}(2, \mathbf{R})$; but they do belong to global semigroups in the universal covering group of $\mathrm{Sl}(2, \mathbf{R})$. There are other polyhedral Lie semialgebras which generate global semigroups in $\mathrm{Sl}(2, \mathbf{R})$; e.g., the Lie semialgebra of all

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \quad \text{with } a, b, c \in \mathbf{R}, b, c \geqslant 0.$$

It is the intersection of the two half-space semialgebras

$$\left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbf{R} \text{ and } b \geqslant 0 \text{ [respectively, } c \geqslant 0] \right\},$$

and it generates the semigroup of all $\mathrm{Sl}(2, \mathbf{R})$ matrices with nonnegative entries. The semialgebras Σ^+ and Σ^- are the only invariant wedges in $\mathrm{sl}(2, \mathbf{R})$. The semialgebras of $\mathrm{sl}(2, \mathbf{R})$ may be considered as completely known. (Details were discussed in [**HH1**].)

The paper is organized as follows: In §1 we recall the analytic geometry of wedges and their duality theory. It appears difficult to find a comprehensive source for all the informations we need; we will therefore give a self-contained treatment and make reference to sources only where they are easily accessible. §2 will then deal with semialgebras of Lie algebras and will contain the proof of Theorems A and F. This proof will rest substantially on the information contained in §1, on some arguments involving Lie algebra arguments, and on some basic facts involving ordinary differential equations. §3 describes Lie semialgebras occuring in compact Lie algebras.

We will also apply the main result to answer in the affirmative a question posed in [**HL1**, p. 357]. We call a Lie algebra *exponential*, if the Campbell-Hausdorff multiplication $(x, y) \mapsto x + y + \frac{1}{2}[x, y] + \cdots$, which is always defined for sufficiently small x and y, allows an analytic extension to a multiplication $*: L \times L \to L$.

COROLLARY I. *If W is a Lie semialgebra in a finite-dimensional exponential Lie algebra L, then $W * W = W$.*

1. The analytic geometry of wedges.

The duality of wedges.

1.1. DEFINITION. A subset W of topological vector space L over $\mathbf{R}$ is called a *wedge* if it satisfies the following conditions, where we set $\mathbf{R}^+ = \{r \in \mathbf{R}: 0 \leqslant r\}$:

$$\text{(i)}\, W + W \subseteq W. \quad \text{(ii)}\, \mathbf{R}^+ W = W. \quad \text{(iii)}\, \overline{W} = W.$$

The set $H(W) = W \cap -W$ is called the *edge of the wedge*, and a wedge is called a *cone* if its edge is singleton.

If $\hat{L} = \mathrm{Hom}(L, \mathbf{R})$ denotes the topological dual of L, i.e. the vector space of all continuous functionals (with the weak *-topology), then we set

$$W^* = \{\omega \in \hat{L}: \langle \omega, x \rangle \geqslant 0 \text{ for all } x \in W\},$$
$$W^\perp = \{\omega \in \hat{L}: \langle \omega, x \rangle = 0 \text{ for all } x \in W\}$$

for any subset $W \subseteq L$. If W is a wedge, then W^* is called the *dual of* W, and $W^\perp$ is called the *anniliator of* W.

It is immediate that the dual W^* of a wedge W is again a wedge; it is therefore also called *the dual wedge*. The annihilator of any subset is a vector space. If we give L the weak *-topology, then $(W^*)^*$ makes sense in the bidual of L. However, we will always set

$$W^{**} = \{x \in L: \langle \omega, x \rangle \geqslant 0 \text{ for all } \omega \in W^*\}.$$

If L is locally convex, then we may identify L with a subspace of its bidual $L^{\hat{}\hat{}}$, in which case $W^{**} = L \cap (W^*)^*$. The Hahn-Banach Theorem immediately yields

1.2. PROPOSITION. *For a wedge W in locally convex space L we have $W^{**} = W$.* □

We will henceforth always consider the vector spaces L and $\hat{L}$ in their dual pairing and we will do the same for W and W^*. For finite-dimensional vector spaces L (which attract our principal interest), this duality is perfect, since L may be identified with the dual of L in this case and W^{**} is equal to $(W^*)^*$.

1.3. PROPOSITION. *Let $\{W_j: j \in J\}$ be a family of wedges in a locally convex space. Then*

(i) $(\bigcap\{W_j: j \in J\})^* = (\Sigma\{W_j^*: j \in J\})^-$,
(ii) $(\Sigma\{W_j: j \in J\})^* = \bigcap\{W_j^*: j \in J\}$.

PROOF. In view of $K^* = \overline{K}^*$ for $K \subseteq L$, conclusion (ii) follows from (i) by duality, we therefore prove (i): We set $D = \bigcap\{W_j: j \in J\}$. Then $D \subseteq W_i$ for all $i \in J$, whence $W_i^* \subseteq D^*$ for all $i \in J$ and thus $(\Sigma\{W_j^*: j \in J\})^- \subseteq D^*$. Conversely, for each $i \in J$ we have $W_i^* \subseteq \Sigma\{W_j^*: j \in J\}$ and thus $(\Sigma\{W_j^*: j \in J\})^{-*} \subseteq W_i^{**} = W_i$ for all $i \in J$, whence $(\Sigma\{W_j^*: j \in J\})^{-*} \subseteq D$ and thus $D^* \subseteq (\Sigma\{W_j^*: j \in J\})^-$ in view of 1.2. □

We recall that $W - W$ is the vector space generated by W.

1.4. PROPOSITION. *Let W be a wedge in a locally convex space L. Then we have the following conclusions*:

$$\text{(i) } W^\perp = (W - W)^\perp = H(W^*). \quad \text{(ii) } H(W)^\perp = (W^* - W^*)^-.$$
$$\text{(iii) } (W^*)^\perp = H(W).$$

PROOF. (i) The relation $W^\perp = (W - W)^\perp = ((W - W)^-)^\perp$ is clear. But $\omega \in W^\perp$ iff $\langle \omega, x\rangle = 0$ for all $x \in W$ iff $\langle \omega, x\rangle \geqslant 0$, and $\langle -\omega, x\rangle \geqslant 0$ for all $x \in W$ iff $\omega \in W^* \cap -W^* = H(W^*)$.

(ii) By duality we have $H(W) = (W^* - W^*)^\perp$ from (i), whence

$$H(W)^\perp = (W^* - W^*)^{\perp\perp} = (W^* - W^*)^-.$$

(iii) follows from (ii), in view of $(W^* - W^*)^\perp = W^{*\perp}$. $\square$

1.5. PROPOSITION. *Let W be a wedge in a locally convex space L.*

(i) *If $M \subseteq W$, then $M^\perp \cap W^* = -M^* \cap W^*$.*

(ii) *If K is a wedge with $K \subseteq W$, then*

$$(K^\perp \cap W^*)^* = (W - K)^- \quad \textit{and} \quad (K^\perp \cap W^*)^\perp = H(W - K)^-).$$

PROOF. (i) From the definitions we have $M^\perp = M^* \cap -M^*$, and thus $M^\perp \cap W^* = -M^* \cap M^* \cap W^* = -M^* \cap W^*$ since $W^* \subseteq M^*$.

(ii) From (i) we know $(K^\perp \cap W^*)^* = (-K^* \cap W^*)^*$, and by Proposition 1.3(i) this equals $(-K^{**} + W^{**})^- = (W - K)^-$. By Proposition 1.4(i) the remainder follows. $\square$

Exposed faces. While in a general theory of convex bodies the concept of a *face* (see e.g. [**Ba**]) plays a role, in the theory of convex cones we need it is a special type of faces which are relevant, the so-called *exposed faces*. They are ideally adapted to be treated in terms of duality. The definition of an exposed face is as follows:

1.6. DEFINITION. A subset F of a wedge W in a locally convex space L is called an *exposed face* (cf. [**Ba**]) if and only if

$$\text{(EXP)} \quad F = (F^\perp \cap W^*)^\perp \cap W.$$

The set of all exposed faces of W is denoted $\mathrm{EXP}(W)$.

Every exposed face is a face, but not conversely. Exposed faces can be characterized in a variety of ways. The following characterization theorem will be important for our purposes.

1.7. THEOREM. *For a subset F of a wedge W in a locally convex space L, the following statements* (1)–(4) *are equivalent, and if* $\dim L$ *is finite, then* (1)–(7) *are equivalent*:

(1) $F \in \mathrm{EXP}(W)$.

(2) $F = (F - W)^- \cap W$.

(3) *There is a subset $\phi \subseteq W^*$ with $F = \phi^\perp \cap W$.*

(4) *There is a $\phi \in \mathrm{EXP}(W^*)$ with $F = \phi^\perp \cap W$.*

(5) *There is an element $\omega \in W^*$ with $F = \omega^\perp \cap W$.*

(6) *There is an element* $x \in F$ *with* $F = (x^\perp \cap W^*)^\perp \cap W$.

(7) *For each x in the interior of F in* $F - F$ *we have* $F = (x^\perp \cap W^*)^\perp \cap W$.

PROOF. (1) $\Leftrightarrow$ (2). $(F^\perp \cap W^*)^\perp \cap W = (F - W)^- \cap (W - F)^- \cap W = (F - W)^- \cap W$ by Proposition 1.5(ii), since $H((W - F)^-) = (F - W)^- \cap (W - F)^-$.

(1) $\Rightarrow$ (3). Choose $\phi = F^\perp \cap W^*$. (3) $\Rightarrow$ (1). $\phi^\perp \cap W$ is an exposed face iff $\phi^\perp \cap W = (\phi^\perp \cap W - W)^- \cap W$ by (1) $\Leftrightarrow$ (2). Trivially, we have the inclusion $\subseteq$ always. Thus we must show the converse containment. So let $z = \lim(x_n - y_n) \in W$ with $x_n \in \phi^\perp \cap W$ and $y_n \in W$. If $\omega \in \phi \subseteq W^*$, then

$$0 \leqslant \langle \omega, z \rangle = \lim \langle \omega, x_n - y_n \rangle = \lim \langle \omega, -y_n \rangle,$$

since $\langle \omega, x_n \rangle = 0$ on account of $x_n \in \phi^\perp$. But $\langle \omega, -y_n \rangle \leqslant 0$ since $y_n \in W$ and $\omega \in W^*$. It follows that $\langle \omega, z \rangle = 0$, i.e. $z \in \phi^\perp \cap W$.

(1) $\Rightarrow$ (4). We have $F = (F^\perp \cap W^*)^\perp \cap W$ by Definition 1.6, but $F^\perp \cap W^*$ is an exposed face of W^*, since $((F^\perp \cap W^*)^\perp \cap W)^\perp \cap W = F^\perp \cap W^*$ (by Definition 1.6).

(4) $\Rightarrow$ (3) is trivial.

Thus (1)–(4) are equivalent. Clearly, (5) $\Rightarrow$ (3) and (6) $\Rightarrow$ (3).

Now suppose that dim L is finite.

(1) $\Rightarrow$ (5). If ω is an inner point of $F^\perp \cap W^*$ in $(F^\perp \cap W^*) - (F^\perp \cap W^*)$, then $F^\perp \cap W^*$ is the exposed face generated by ω in W^*, Then $F^\perp \cap W^* = F_\omega \subseteq (w^\perp \cap W)^\perp \cap W^*$ since $(\omega^\perp \cap W)^\perp \cap W^*$ is an exposed face of W^* by the equivalence of (1) and (3), since it clearly contains ω. Taking annihilators we find $\omega^\perp \cap W \subseteq (\omega^\perp \cap W)^{\perp\perp} \subseteq ((\omega^\perp \cap W)^\perp \cap W^*)^\perp \subseteq (F^\perp \cap W^*)^\perp$, and thus $\omega^\perp \cap W \subseteq (F^\perp \cap W^*)^\perp \cap W = F$ in view of Definition 1.6. Since ω annihilates F, we have trivially $F \subseteq \omega^\perp \cap W$.

(1) $\Rightarrow$ (7). If we let $\phi = F^\perp \cap W^*$ be the exposed face of W^* corresponding to F, then by (1) $\Rightarrow$ (5) and duality, for all inner points x of F in $F - F$, we have $\phi = x^\perp \cap W^*$. By Definition 1.6 we then have $F = (x^\perp \cap W^*)^\perp \cap W$.

(7) $\Rightarrow$ (6) is trivial. □

1.8. COROLLARY. *For a wedge W in a locally convex space L the two functions* $F \to F^\perp \cap W^*$: $\mathrm{EXP}(W) \to \mathrm{EXP}(W^*)$ *and* $\phi \to \phi^\perp \cap W$: $\mathrm{EXP}(W^*) \to \mathrm{EXP}(W)$ *are mutually inverse containment reversing functions. In particular, the exposed faces of W and those of* W^* *are in bijective correspondence (which reverse containment).*

PROOF. It is clear that the functions are containment reversing. That they are mutually inverse and well defined follows from Theorem 1.7 and its proof. □

1.9. PROPOSITION. *Let W be a wedge in a locally convex space L and suppose* $M \subseteq W$. *Then* $(M^\perp \cap W^*)^\perp \cap W$ *is the smallest exposed face of W containing M.*

PROOF. By Theorem 1.7, $(M^\perp \cap W^*)^\perp \cap W \in \mathrm{EXP}(W)$ and it clearly contains M. If $F \in \mathrm{EXP}(W)$ with $M \subseteq F$, then $(M^\perp \cap W^*)^\perp \cap W \subseteq (F^\perp \cap W^*)^\perp \cap W = F$ (by Definition 1.6). □

1.10. DEFINITION. We say that $(M^\perp \cap W^*)^\perp \cap W$ is the *exposed face generated by* M and denote it by E_M. If $M = \{x\}$ we write E_x instead of $E_{\{x\}}$.

1.11. THEOREM. *Let W be a wedge in a locally convex space and let M be an arbitrary subset of W. Then the exposed face E_M generated by M satisfies the following condition*:

(i) $(M^\perp \cap W^*)^* = (E_M^\perp \cap W^*)^* = (W - E_M)^-$ $(= (W - M)^-$, *if M is a wedge*).

In particular, if x is an arbitrary element of W, we have

(ii) $(x^\perp \cap W^*)^* = (E_x^\perp \cap W^*)^* = (W - E_x)^- = (W - \mathbf{R}^+ \cdot x)^-$,

(iii) $x^\perp \cap W^* = E_x^\perp \cap W^* = (W - E_x)^* = (W - \mathbf{R}^+ \cdot x)^*$.

PROOF. By Proposition 1.5(ii) we have

(a) $(E_M^\perp \cap W^*)^* = (W - E_M)^-$, and

(b) $(M^\perp \cap W^*) = (W - M)^-$ if M is a wedge.

Moreover, since $E_M^\perp \cap W^* \subseteq M^\perp \cap W^*$ we have

(c) $(M^\perp \cap W^*)^* \subseteq (E_M^\perp \cap W^*)^*$.

We will now show

(d) $(W - E_M)^- \subseteq (M^\perp \cap W^*)^*$

and thereby conclude the proof. For this purpose, let $z \in (W - E_M)^-$. Then there are elements $x_n \in W$ and $y_n \in E_M$ with $z = \lim(x_n - y_n)$. But $E_M = (M^\perp \cap W^*)^\perp \cap W$ by Proposition 1.9. Hence, if ω is an arbitrary element of $M^\perp \cap W^*$ we have $\langle \omega, y_n \rangle = 0$. Thus $\langle \omega, z \rangle = \lim \langle \omega, x_n - y_n \rangle = \lim \langle \omega, x_n \rangle$. But $\langle \omega, x_n \rangle \geqslant 0$ since $\omega \in W^*$ and thus $\langle \omega, z \rangle \geqslant 0$. This shows $z \in (M^\perp \cap W^*)^*$ as asserted. Conclusion (ii) is a special case of (i), and (iii) follows from (ii) by duality. □

Tangent spaces. We now link the concept of *tangents* with the concepts considered so far.

1.12. DEFINITION. For a convex set S in a topological vector space L we define

$$L(S) = \{x \in L \colon \text{there are elements } x_n \in S \text{ with } X = \lim nx_n\}.$$

Since clearly $\lim x_n = 0$, the set $L(S)$ is empty if $0 \notin S^-$. The elements $x \in L(S)$ are the *tangent vectors at S in* 0 (defined in a "one-sided fashion").

Tangent vectors may be defined in various equivalent ways. We need the following alternative:

1.13. LEMMA. *If S is a convex set in a topological vector space L with $0 \in \overline{S}$, then the following conditions are equivalent for a vector $x \in L$*:

(1) $x \in L(S)$.

(2) $x = \lim m_n x_n$ *for an unbounded sequence of integers m_n and a sequence $x_n \in S$.*

PROOF. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Suppose that $x = \lim m_n x_n$ with m_n and x_n as in (2). By omitting terms from the sequence and reindexing, if necessary, we may assume that the m_n are increasing. If S is convex, then $\overline{S}$ is convex and, since $0 \in \overline{S}$, then for all $0 \leqslant r \leqslant 1$ and $s \in \overline{S}$, we have $rs = (1 - r)0 + rs \in \overline{S}$. If now $m_n \leqslant k \leqslant m_{n+1}$, we conclude that $y_k = (m_n/k)x_n \in \overline{S}$. We observe that $ky_k = m_n x_n$ for all $k \in \mathbf{N}$, and thus $x = \lim ky_k$. Hence $x \in L(\overline{S})$. It is a simple exercise to show that $L(S) = L(\overline{S})$ (see e.g. [**HL2**, p. 149] for a similar proof). Hence $x \in L(S)$ as we had to show. □

It is not hard to see that, under the hypotheses of Lemma 1.13, the set $L(S)$ is in fact a wedge:

1.14. REMARK. *If S is a convex set in a topological vector space L with $0 \in \bar{S}$, then $L(S)$ is a wedge.*

PROOF. If $x, y \in L(S)$, then $x = \lim nx_n$, $y = \lim ny_n$ with $x_n, y_n \in S$. Then $\frac{1}{2}(x_n + y_n) \in S$ since S is convex. Then $x + y = \lim 2n(\frac{1}{2}(x_n + y_n))$; and if we set $z_{2m} = z_{2m+1} = \frac{1}{2}(x_m + y_m)$ we find that $x + y = \lim nz_n$. Thus $L(S) + L(S) \subseteq L(S)$. Moreover, if $0 \leqslant r \leqslant 1$, then $rx_n = (1 - r)0 + rx_n \in \bar{S}$ since S, hence $\bar{S}$, is convex. But $rx = \lim n(rx_n)$, whence $rx \in L(S^-) = L(S)$. If $r \in \mathbf{R}^+$ we write $r = (r - [r]) + [r]$ and have $(r - [r]) \cdot x \in L(S)$ after what we just saw. But $[r] \cdot x \in L(S)$ since $L(S)$ is additively closed. It follows that $r \cdot x = (r - [r]) \cdot x + [r] \cdot x \in L(S) + L(S) \subseteq L(S)$. □

1.15. DEFINITION. Let W be a wedge in a topological vector space and $x \in W$. The *tangent space* at W in x is

$$T_x = H(L(W - x)) \quad (= L(W - x) \cap -L(W - x)).$$

1.16. THEOREM. *Let W be a wedge in a locally convex space L, and $x \in W$. Then*:

(i) $L(W - x) = (W - \mathbf{R}^+ \cdot x)^- = (W - E_x)^-$.

(ii) $T_x = H(W - E_x)^-$.

(iii) $x^\perp \cap W^* = L(W - x)^*$.

(iv) $H(L(W - x)^*) = (W - W)^\perp = H(W^*)$.

REMARK. $x^\perp \cap W^*$ is the exposed face in W^* corresponding to the exposed face E_x generated by x in W according to the correspondence in Corollary 1.8. From (i) and (ii) it follows that this face is a cone if W is generating, i.e. satisfies $W - W = L$. Theorem 1.16 links the concept of tangents with that of faces and duality.

PROOF. (i) We have $y \in L(W - x)$ iff $y = \lim n(w_n - x)$ with $w_n \in W$. But $n(w_n - x) \in W - \mathbf{R}^+ \cdot x$, and so $y \in (W - \mathbf{R}^+ \cdot x)^-$. Conversely, let

$$y \in (W - \mathbf{R}^+ \cdot x)^-.$$

Then $y = \lim(v_n - r_n \cdot x)$ with $v_n \in W$ and $0 \leqslant r_n$. If the r_n are bounded, w.l.o.g., $r = \lim r_n$ exists. Then $y + r \cdot x = \lim v_n \in W$ and thus $y \in W - \mathbf{R}^+ \cdot x \subseteq L(W - x)$. If the r_n are unbounded, w.l.o.g. assume that the r_n increase and set $w_n = (1/r_n) \cdot v_n \in W$. Then $x = \lim r_n(w_n - x) \in L(W - x)$ by Lemma 1.13. Thus (i) is proved in view of Theorem 1.11.

(ii)

$$\begin{aligned} T_x &= H(L(W - x)) \quad \text{(by Definition 1.15)} \\ &= H(W - E_x)^- \quad \text{(by (i))}. \end{aligned}$$

(iii) First we take an $\omega \in L(W - x)^*$ and show that $\omega \in x^\perp \cap W^*$. Since $\pm x \in L(W - x)$, we have $\langle \omega, \pm x \rangle \geqslant 0$, and $\langle \omega, x \rangle = 0$, i.e. $\omega \in x^\perp$. If $y \in W$, then $y - x \in W - x \subseteq L(W - y)$, whence $0 \leqslant \langle \omega, y - x \rangle = \langle \omega, y \rangle$. Thus $\omega \in W^*$ and therefore $\omega \in x^\perp \cap W^*$.

Secondly, we consider an $\omega \in x^\perp \cap W^*$ and show that $\omega \in L(W - x)^*$. If $z \in L(W - x)$, then $z = \lim nz_n$ with $z_n = x_n - x$, where $x_n \in W$. Now $\langle \omega, z \rangle = \lim n\langle \omega, x_n - x \rangle = \lim n\langle \omega, x_n \rangle \geqslant 0$, whence $\omega \in L(W - x)^*$.

(iv) The equality $(W - W)^\perp = H(W^*)$ follows from Proposition 1.4(i). Now

$$\begin{aligned} H\big(L(W-x)^*\big) &= H(x^\perp \cap W^*) \quad \text{(by (i))} \\ &= x^\perp \cap W^* \cap -W^* \cap -x^\perp = x^\perp \cap H(W^*) \\ &= x^\perp \cap (W-W)^\perp = (W-W)^\perp, \end{aligned}$$

since $x \in W - W$. $\square$

1.17. THEOREM. *Let W be a wedge in a locally convex space and $x \in W$. Then*:

(i) $T_x^\perp \cap W^* = x^\perp \cap W^* = E_x^\perp \cap W^* = L(W-x)^*$.

(ii) $T_x = (T_x^\perp \cap W^*)^\perp = (x^\perp \cap W^*)^\perp$.

(iii) $T_x \cap W = E_x$ (*the exposed face generated by* x).

(iv) $T_x^\perp = (x^\perp \cap W^*) - (x^\perp \cap W^*)$.

PROOF. (i) The equality $x^\perp \cap W^* = L(W-x)^*$ was established in Theorem 1.16(iii) and the equality $x^\perp \cap W^* = E_x^\perp \cap W^*$ in Theorem 1.11(iii). Since $x \in T_x$ we have $T_x^\perp \subseteq x^\perp$ and thus $T_x^\perp \cap W^* \subseteq x^\perp \cap W^*$. On the other hand,

$$\begin{aligned} x^\perp \cap W^* &\subseteq x^\perp \cap W^* - x^\perp \cap W^* \\ &= L(W-x)^* - L(W-x)^* \quad \text{(by Theorem 1.16(iii))} \\ &\subseteq H(L(W-x))^\perp \quad \text{(by Proposition 1.4(i))} \\ &= T_x^\perp \quad \text{(by Definition 1.15)}. \end{aligned}$$

This proves (i), and in the process we have shown (iv), too.

(ii) From $T_x = T_x^{\perp\perp} = (x^\perp \cap W^*)^\perp$ (by (iv)), we have $T_x = (x^\perp \cap W^*)^\perp$, and (by (i)) $T_x = (T_x^\perp \cap W^*)^\perp$.

(iii)

$$\begin{aligned} T_x \cap W &= (x^\perp \cap W^*)^\perp \cap W \quad \text{(by (ii))} \\ &= E_x \quad \text{(by Proposition 1.9)}. \quad \square \end{aligned}$$

By Theorem 1.17(iii), the tangent space T_x at W in x allows us to construct the exposed face E_x generated by x as $T_x \cap W$. In the finite-dimensional case, every exposed face F is principal, i.e. generated by one point. In general, every principal exposed face is obtained via the tangent space in its generator in this fashion:

1.18. THEOREM. *Let W be a wedge in a locally convex space. Let F be an exposed face of W and suppose that F is principal, i.e. $F = E_x$ for an $x \in F$. (If* $\dim W$ *is finite, then all exposed faces are principal by Theorem* 1.7 *and Proposition* 1.9.)

Then we have the following conclusions:

(i) $F = T_x \cap W$.

(ii) $T_x = (F^\perp \cap W^*)^\perp$.

(iii) *If T_y is any tangent space at W in y with $T_y \cap W = F$, then $T_y = T_x$.*

PROOF. (i) follows from Theorem 1.17(iii).

(ii)

$$T_x = (x^\perp \cap W^*)^\perp \quad \text{(by Theorem 1.17(ii))}$$
$$= (E_x^\perp \cap W^*)^\perp \quad \text{(by Theorem 1.11)}$$
$$= (F^\perp \cap W^*)^\perp .$$

(iii) The relation $T_y \cap W = F$ implies $(y^\perp \cap W^*)^\perp \cap W = F$ (by 1.17(ii)), and thus

$$y^\perp \cap W^* = \left((y^\perp \cap W^*)^\perp \cap W\right) \cap W^* \quad \text{(by Corollary 1.8 and Proposition 1.9)}$$
$$= F^\perp \cap W^*.$$

The same calculation yields $x^\perp \cap W^* = F^\perp \cap W^*$. Thus

$$T_x = (x^\perp \cap W^*)^\perp \quad \text{(by Theorem 1.17(ii))}$$
$$= (y^\perp \cap W^*)^\perp \quad \text{(by what we just saw)}$$
$$= T_y. \quad \square$$

The most important tangent spaces are those which are hyperplanes.

1.19. DEFINITION. *A tangent hyperplane* T of a wedge W is a hyperplane for which there is a point $x \in W$ with $T = T_x$. Any such point will be called a C^1-point, and the set of all such points in W will be written $C^1(W)$.

The following result characterizes tangent hyperplanes through duality.

1.20. THEOREM. *Let W be a wedge in a locally convex space L and let T be a support hyperplane of W, i.e., $T = \omega^{-1}(0) = \omega^\perp$ with $\omega \in W^*$. Let $\phi = \mathbf{R}^+\cdot \omega$. Then the following statements are equivalent*:

(1) $\phi^* = (W - (T \cap W))^-$.

(2) $\phi \in \mathrm{EXP}(W)$.

(3) *If T' is a support hyperplane with $T \cap W \subseteq T' \cap W$, then $T = T'$. These statements are implied by*

(4) *T is a tangent hyperplane.*

If dim L *is finite, then they are all equivalent.*

PROOF. (1) $\Leftrightarrow$ (2). We have (2) iff

$$\phi = (\phi^\perp \cap W)^\perp \cap W' \quad \text{(by Definition 1.6)}$$
$$= (T \cap W)^\perp \cap W^* \quad (\text{since } \phi^\perp = \omega^\perp = T).$$

This is equivalent to $\phi^* = ((T \cap W)^\perp \cap W^*)^* = ((T \cap W)^{\perp *} + W^{**})^-$ (by Proposition 1.3). But

$$\left((T \cap W)^{\perp *} + W^{**}\right)^- = \left(W + (T \cap W)^{\perp\perp}\right)^- = \left(W + ((T \cap W) - (T \cap W))^-\right)^-$$
$$= (W + (T \cap W) - (T \cap W))^- = (W - (T \cap W))^-.$$

Hence (2) is equivalent to (1).

(1) $\Rightarrow$ (3). Suppose that T' satisfies $T' = \omega'^\perp$ and $T \cap W \subseteq T' \cap W$. Then $\phi^* = (W - (T \cap W))^- \subseteq (W - (T' \cap W))^-$ which is equal to ω'^* by the same calculation which showed $\phi' = (W - (T \cap W))^-$. Thus $\omega' \in \phi^{**} = \phi = \mathbf{R}^+\cdot \omega$. Hence $T' = \omega'^\perp = \omega^\perp = T$.

(3) $\Rightarrow$ (2). We have (3) iff, for all $\omega' \in W'$ with $T \cap W \subseteq \omega'^{\perp} \cap W$, we have $\omega' \in \phi$. This means $(T \cap W)^{\perp} \cap W^* = \phi$. Since $T = \phi^{\perp}$, this is (2) by Definition 1.6. Now suppose (4). Then $T = T_x$ for a suitable $x \in W$. Then $\phi = T^{\perp} \cap W^* = T_x^{\perp} \cap W^* = x^{\perp} \cap W^*$ (by Theorem 1.17(i)). By Theorem 1.7 this implies $\phi \in \mathrm{EXP}(W)$, i.e., (2). Now suppose that L is finite dimensional. Then, by Theorem 1.7, condition (2) implies the existence of an $x \in W$ with $\phi = x^{\perp} \cap W^*$. Then $T = \phi^{\perp} = (w^{\perp} \cap W^*)^{\perp} = T_x$ by Theorem 1.17(ii). Hence (2) implies (4) in this case. $\square$

We point out that occasionally one calls a hyperplane a tangent hyperplane of W iff it satisfies conditions (1)–(3) of Theorem 1.20. We have just seen that in the finite-dimensional case this amounts to the same definition as ours in Definition 1.19.

Wedges in finite-dimensional vector spaces.

1.21. DEFINITION. We call a wedge W in a vector space L *generating* iff $L = W - W$. By Proposition 1.4(i), a wedge is generating iff W^* is a cone.

1.22. THEOREM. *Let W be a generating wedge in a finite-dimensional vector space L. Then*:

(i) $W^* = (\sum\{\mathbf{R}^+ \cdot \omega\colon \omega \in W^*$ *and* $\mathbf{R}^+ \cdot \omega \in \mathrm{EXP}(W^*)\})^-$ (*Theorem of Straszewicz*).

(ii) $W = \bigcap\{\omega^*\colon \omega \in W^*$ *and* $\mathbf{R}^+ \cdot \omega \in \mathrm{EXP}(W^*)\} = \bigcap\{S\colon S$ *is a closed half-space containing W whose boundary is a tangent hyperplane of W*$\}$.

PROOF. (i) is proved in [**St**] and (ii) follows from (i) by duality (see Proposition 1.3). $\square$

1.23. LEMMA. *If $V \subseteq W$ are wedges in a locally convex space L and X is in the interior of V relative to W, then $x^{\perp} \cap V^* = x^{\perp} \cap W^*$.*

PROOF. Trivially $x^{\perp} \cap W^* \subset x^{\perp} \cap V^*$. We have to show the reverse inclusion. Since $(x^{\perp} \cap W^*)^* = (W - \mathbf{R}^+ \cdot x)^-$ and $(x^{\perp} \cap V^*)^* = (V - \mathbf{R}^+ \cdot x)^-$ by Proposition 1.5(i), it suffices to show that $W \subseteq V - \mathbf{R}^+ \cdot x$. Let $z \in W$. Since V is a neighborhood of x in W, there is an $r > 0$ such that $x + \frac{1}{r} \cdot z \in V$. Then $z \in V - r \cdot x \subseteq V - \mathbf{R}^+ \cdot x$. $\square$

The following is a technical lemma which will play a crucial role in the proof of the main theorem.

1.24. LEMMA. *Let $V \subseteq W$ be generating wedges in a finite-dimensional vector space and let x be in the interior of V relative to W. Suppose that $\omega \neq 0$ is an extremal point of $x^{\perp} \cap W$. Then there exists a sequence $(\omega_n, x_n) \in W^* \times V$ satisfying the following conditions*:

(i) $\langle \omega_n, x_n \rangle = 0$ *for all n.*

(ii) $\mathbf{R}^+ \cdot \omega_n \in \mathrm{EXP}(W)$ *for all n.*

(iii) $\omega = \lim \omega_n$.

(iv) $T_{x_n} = \omega_n^{\perp}$ *is a tangent hyperplane at W in x_n.*

(v) $x_n \in C^1(W)$ (*cf. Definition* 1.19).

PROOF. Since V is generating, then V^* is a cone by Proposition 1.4(i). By Lemma 1.23, we have $x^\perp \cap V^* = x^\perp \cap W^*$; by Theorem 1.7 this is an exposed face of V^*. We now show that ω is an extremal point of V^*:

Indeed let $\omega = \alpha + \beta$ with $\alpha, \beta \in V^*$; then $\alpha = \omega - \beta \in ((x^\perp \cap V^*) - V^*) \cap V^* \subseteq ((x^\perp \cap V^*) \cap V^*)^- - V^* = x^\perp \cap V^*$ by Theorem 1.7(2). Likewise, $\beta \in x^\perp \cap V^*$. Since ω is an extremal point of $x^\perp \cap W^* = x^\perp \cap V^*$, we have $\alpha, \beta \in \mathbf{R}^+ \cdot \omega$ as was to be shown. It now follows from Theorem 1.22(i) that there is a sequence of points ω_n in V^* with $\mathbf{R}^* \cdot \omega_n \in \mathrm{EXP}(V^*)$ and with $\omega = \lim \omega_n$. By Theorem 1.20 for each n there is an $x_n \in V$ such that $\omega_n^\perp$ is the tangent hyperplane at V in x_n. In particular, $\langle \omega_n, x_n \rangle = 0$ for all n.

The only thing which remains to show is that we may choose the ω_n, in fact, in such a fashion that $\mathbf{R}^+ \cdot \omega_n \in \mathrm{EXP}(W^*)$. Since L is finite dimensional, we may assume that V is obtained from W by intersection with finitely many closed half-spaces , i.e. $V = W \in \sigma_1^* \cap \cdots \cap \sigma_m^*$ for suitable elements $\sigma_k \in V^*$. (Otherwise we could pass to a smaller wedge neighborhood of x with this property.) We now claim that the x_n eventually miss all $\sigma_k^\perp$; otherwise there would be a k such that $\langle \sigma_k, x_n \rangle = 0$ for infinitely many n. Thus $\sigma_k \in x_n^\perp \cap V^* = \mathbf{R}^+ \cdot \omega_n$ infinitely often. Then $\omega = \lim \omega_n \in \mathbf{R}^+ \cdot \sigma_k$, i.e. $\omega = s \cdot \sigma_k$ for some $s > 0$. But $\langle \sigma_k, x \rangle = s^{-1} \langle \omega, x \rangle = 0$, and this would mean that x is the boundary of V which is not the case. By omitting finitely many of the x_n, if necessary, we may now assume that all x_n are in the interior of V relative to W. Then Lemma 1.23 shows $x_n^\perp \cap W^* = x_n^\perp \cap V^* = \mathbf{R}^+ \cdot \omega_n$. But then $T_{x_n} = (x_n^\perp \cap W^*)^\perp = \omega_n^\perp$. Thus $\mathbf{R}^+ \cdot \omega_n \in \mathrm{EXP}(W_n)$ by Theorem 1.20. Note that (v) is immediate from (iv). □

2. Semialgebras in Lie algebras.

Background results. For the record, we first repeat the basic definitions.

2.1. DEFINITION. Let L be a Dynkin algebra, i.e. a Lie algebra over $\mathbf{R}$ which is completely normable and has a continuous Lie multiplication.

A Campbell-Hausdorff neighborhood (*C-H neighborhood*) is an open convex symmetric neighborhood of 0 in L such that for $x, y \in B$ the Baker-Campbell-Dynkin-Hausdorff series $X * Y = X + Y + \frac{1}{2}[X, Y] + \cdots$ converges absolutely and defines a partial multiplication $*: B \times B \to L$. A *Lie semialgebra W in L with respect to B* is a wedge in L (see Definition 1.1) satisfying $(W \cap B) * (W \cap B) \subseteq W$.

Quite generally, *a Lie semialgebra* W is a wedge in some Dynkin algebra L in which there is some C-H neighborhood B with respect to which W is a Lie semialgebra in L.

2.2. REMARK. *For any Lie semialgebra W in L the vector space $(W - W)^-$ is a Dynkin algebra. If W is finite dimensional, then $W - W$ is a Lie algebra.*

PROOF. See [**HL2**, 3.19, p. 155]. □

In the next we shall use a power series which we formally introduce in the following definition.

2.3. DEFINITION. In the ring of power series in one variable X we set

$$g(X) = 1 + \frac{1}{2}X + \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n)!} X^{2n} \quad \text{with the Bernoulli numbers } b_{2n}.$$

2.4. LEMMA. *In the ring of two noncommuting variables we have* $x * y \equiv x + g(\mathrm{ad}\, x) y \bmod (y^2)$, *where* (y^2) *is the closed ideal generated by* y^2.

PROOF. See [**Bo**]. □

The following propositions are steps towards a proof of the main results.

2.5. PROPOSITION. *Let* W *be a Lie semialgebra in a Dynkin algebra* L. *Let* $x \in W$ *and let* $E_x = (E_x^\perp \cap W^*)^\perp \cap W$ *be the exposed face of* W *generated by* x. *Take* $y \in E_x \cap B$ *so that* $E_y = E_x$. *Then*

(i) $g(\mathrm{ad}\, y)(W - E_x)^- \subseteq (W - E_x)^-$.

If T_x *is the tangent space at* W *in* x, *then*

(ii) $g(\mathrm{ad}\, y) T_x \subseteq T_x$.

PROOF. (i) We fix a C-H neighborhood B such that W is a Lie semialgebra in L with respect to B. First we show

$$g(\mathrm{ad}\, y)(W) \subseteq (W - E_x)^-.$$

Therefore, let $w \in W$. Consider $t \in]0, \varepsilon]$ with an $\varepsilon > 0$ so that $\langle 0, \varepsilon\rangle \cdot w \subseteq B$. We know from Lemma 2.4 that $y * t \cdot w = y + tg(\mathrm{ad}\, y)w + R(t)$ with $\lim_{t \to 0+} \frac{1}{t} R(t) = 0$. Since W is a Lie semialgebra with respect to B, we have $y * t \cdot w \in W$.

Now take $\omega \in E_x^\perp \cap W^*$. Then

$$0 \leqslant \tfrac{1}{t}\langle \omega, y * t \cdot w\rangle = \tfrac{1}{t}\langle \omega, y\rangle + \langle \omega, g(\mathrm{ad}\, y)w\rangle + O(t)$$

with $\lim_{t \to 0+} O(t) = 0$. Since $\omega \in E_x^\perp$, we have $\langle \omega, y\rangle = 0$. It follows that $0 \leqslant \langle \omega, g(\mathrm{ad}\, y)w\rangle$, i.e., $g(\mathrm{ad}\, y)W \subseteq (E_x^\perp \cap W^*)^* = (W - E_x)^-$ by Theorem 1.11, as was asserted. Now let y be such that $E_y = E_x$. Then

$$\begin{aligned} g(\mathrm{ad}\, y)(W - E_x) - g(\mathrm{ad}\, y)(W - E_y) &= g(\mathrm{ad}\, y)(W - \mathbf{R} \cdot y) \\ &\subseteq g(\mathrm{ad}\, y)W - g(\mathrm{ad}\, y)(\mathbf{R} \cdot y) \subseteq (W - E_x)^- - \mathbf{R} \cdot y = (W - E_x). \end{aligned}$$

Since $g(\mathrm{ad}\, y)$ is continuous, we conclude that $g(\mathrm{ad}\, y)(W - E_x)^- \subseteq (W - E_x)^-$, as was to be shown.

(ii) By Theorem 1.16(ii) we have $T_x = H((W - E_x)^-)$. Since any vector space automorphism preserving the wedge $(W - E_x)^-$ must preserve its edge, (i) implies (ii). □

For a better understanding of Proposition 2.5, the following reminder is in order:

2.6. REMARK. *If* W *is a finite-dimensional wedge in a topological vector space, then the equation* $E_x = E_y$ *holds for any elements* $x, y \in W$ *for which* y *is in the interior of* E_x *in* $E_x - E_x$.

PROOF. See Theorem 1.7(7). □

2.7. LEMMA. *Let* W *be a finite-dimensional wedge and* B *a neighborhood of* 0 *in a Dynkin algebra. If*

(i) $g(\mathrm{ad}\, x) T_x \subseteq T_x$ *for all* $x \in W \cap B$, *then*

(ii) $[E_x, T_x] \subseteq T_x$ *for all* $x \in W$.

PROOF. Fix $x \in W$ and let $y \in B$ be in the interior of E_x in $E_x - E_x$. Then

$$\begin{aligned} g(\mathrm{ad}\, y) T_x &= g(\mathrm{ad}\, y) T_y \quad \text{(by Remark 2.6 and Theorem 1.16(ii))} \\ &\subseteq T_y \quad \text{(by (i))} \\ &= T_x. \end{aligned}$$

Since the interior of E_x in $E_x - E_x$ is dense in E_x, we conclude that

(iii) $g(\operatorname{ad} y)T_x \subseteq T_x$ for all $x \in W$ and all $y \in E_x \cap B$.

Now we take any $\omega \in E_x^\perp \cap W^*$. By (iii) we have $g(\operatorname{ad} ty)z \in T_x$ for all $t \in]0, \varepsilon[$, all $y \in E_x$ and all $z \in T_x$ (with a sufficiently small $\varepsilon > 0$ so that $]0, \varepsilon[y \subseteq B$). Since $E_x^\perp \cap W^* = T_x^\perp \cap W^*$ by Theorem 1.17(i), for all $t \in]0, \varepsilon[$ we have

$$0 = \left\langle \omega, \tfrac{1}{t} g(\operatorname{ad} ty)z \right\rangle = \tfrac{1}{t}\langle \omega, z\rangle + \tfrac{1}{2}\langle \omega, [y, z]\rangle + O(t)$$

with $\lim_{t\to 0+} O(t) = 0$. As $\omega \in T_x^\perp$, we have $\langle \omega, z\rangle = 0$ and, thus, upon passing to the limit $t \to 0+$, we conclude that $\langle \omega, [y, z]\rangle = 0$. Thus $[y, z] \in (x^\perp \cap W^*)^\perp = T_x$ according to Theorem 1.17(ii). This shows (ii). □

In Remark 2.2, we observed that for a Lie semialgebra W the vector space $(W - W)^-$ is a Dynkin algebra. The following lemma shows that weaker hypotheses on a wedge W suffice for $(W - W)^-$ to be a Dynkin algebra.

2.8. Lemma. *Let W be a wedge in a Dynkin algebra L which satisfies the following condition*:

$$g(\operatorname{ad} x)(W) \subseteq (W - E_x)^- \quad \textit{for all } x \in W \cap B \textit{ with some C-H neighborhood } B.$$

Then $(W - W)^-$ is a Dynkin algebra.

Proof. It suffices to show that $[W, W] \subseteq W^{\perp\perp} = (W - W)^-$. Thus we take $x, y \in W$ and $\omega \in W^\perp$. We assume $x \in B$ and consider all $t \in]0, 1]$. We note $g(\operatorname{ad} tx)y \in (W - E_x)^- \subseteq (W - W)^- = W^{\perp\perp}$ by hypothesis. Thus $0 = \frac{1}{t}\langle \omega, y\rangle + \langle \omega, [x, y]\rangle + O(t)$ with $\lim_{t\to 0+} O(t) = 0$. But $\langle \omega, y\rangle = 0$, since $\omega \in W^\perp$. Once again conclude $\langle \omega, [x, y]\rangle = 0$, i.e. $[x, y] \in W^{\perp\perp}$, which we had to show. □

In the finite-dimensional situation, we can now prove a converse of Proposition 2.5:

2.9. Proposition. *Let W be a finite-dimensional wedge in a Dynkin algebra L and let B be an arbitrary C-H neighborhood of L. Suppose that the following condition is satisfied*:

$$g(\operatorname{ad} x)(W) \subseteq (W - E_x)^- \quad \textit{for all } x \in W \cap B * B.$$

Then W is a Lie semialgebra in L with respect to B. In fact, if W^0 is the interior of W in $W - W$, then $(W^0 \cap B)*(W^0 \cap B) \subseteq W$.

Proof. By Lemma 2.8 we may assume $L = W - W$; then W has nonempty interior W^0. Let $x, y \in W^0 \cap B$. We define a function u: $[0, 1] \to L$ by $u(t) = x * ty$. Then u is differentiable and satisfies the following differential equation with initial condition:

$$u'(t) = g(\operatorname{ad} u(t))y, \qquad u(0) = x.$$

See **[HL5**, Proposition 4]. (The assertion can also be derived directly from Lemma 2.4.) Let $U = \{t \in [0, 1]\colon u(t) \in W^0\}$. Since $u(0) = x \in W^0$, the set U is an open neighborhood of 0 in $[0, 1]$. We claim $U = [0, 1]$; then in particular $x * y = u(1) \in W^0$, which will prove the proposition. If the claim were false, let $s = \min[0, 1] \setminus U$.

Then $s > 0$. We abbreviate $u(s)$ with z and find

$$u'(s) = \lim_{t \to s-} \frac{1}{s-t}(u(s) - u(t)) \in (\mathbf{R}^+ \cdot z - W)^-.$$

On the other hand, by our hypothesis, we have $u'(s) = g(\operatorname{ad} u(s))y \in (W - E_{u(s)})^- = (W - \mathbf{R}^+ \cdot z)^-$. in view of Theorem 1.11. Thus we derive $u'(s) \in H(W - \mathbf{R}^+ \cdot z)^- = T_z$ by Theorem 1.16. According to our hypothesis, the vector space automorphism $g(\operatorname{ad} z)$ preserves the wedge $(W - \mathbf{R}^+ \cdot z)^-$, hence must also preserve its edge T_z. Since T_z is finite dimensional, $g(\operatorname{ad} z)$ will induce an automorphism of the edge. The relations $u'(s) = g(\operatorname{ad} z)y$ and $u'(s) \in T_z$ then imply $y = g(\operatorname{ad} z)^{-1}u'(s) \in T_z$ and hence $y \in T_z \cap W^0$. However, since z is a boundary point of W by the definition of s and $z = u(s)$ (according to which there is a sequence of numbers $t_n \in [0, s[$ with $u(t_n) \in W^0$ and $s = \lim t_n$), we claim $T_z \cap W^0 = \varnothing$: Indeed, $w \in T_z \cap W^0$ would imply the existence of a neighborhood V of w with $V \subseteq T_z \cap W$ by the definition of T_z. This would entail $T_z = L$. But for a boundary point z of W, this is impossible. Having proved the claim, we have arrived at the desired contradiction. □

We have used the fact that W is finite dimensional in two places: Firstly in securing the existence of inner points of W in $W - W = (W - W)^-$; secondly in deriving from $g(\operatorname{ad} z)T_z \subseteq T_z$ the equality $g(\operatorname{ad} z)T_z = T_z$. The first conclusion could probably be circumvented by using an appropriate version of inner point adjusted to convexity theory; such concepts do exist. It is not clear how the second use of finite-dimensionality could be circumvented with the present line of proof. However, it seems likely that some variation of the proof might yield the result even in the absence of finite-dimensionality of the wedge. Later developments below, however, will use finite-dimensionality more seriously due the use of Theorem 1.22 and Lemmas 1.23 and 1.24.

We point out further that even for finite-dimensional W we have not yet freed the definition of a Lie semialgebra from the dependence of the existence of a particular C-H neighborhood. In Proposition 2.9 the sets W and B are still linked.

The following is the crucial step to eliminate this link.

2.10. LEMMA. *Let W be a finite-dimensional wedge W in a Dynkin algebra L. Then the following conditions are equivalent*:

(1) $[E_x, T_x] \subseteq T_x$ *for all* $x \in W$ *for which* T_x *is a tangent hyperplane.*

(2) $g(\operatorname{ad} x)T_x \subseteq T_x$ *for all* $x \in W$ *for which* T_x *is a tangent hyperplane.*

(3) $g(\operatorname{ad} x)W \subseteq (W - E_x)^-$ *for all* $x \in W$ *for which* T_x *is a tangent hyperplane.*

(4) $g(\operatorname{ad} x)W \subseteq (W - E_x)^-$ *for all* $x \in W$.

(5) $[E_x, T_x] \subseteq T_x$ *for all* $x \in W$.

PROOF. (1) ⇒ (2). By (1), T_x is invariant under $\operatorname{ad} x$, and this implies (2).

(2) ⇒ (3). By Theorem 1.16(ii), T_x is the edge of the wedge $(W - E_x)^-$. If T_x is a tangent hyperplane, then the wedge $(W - E_x)^-$ is a half-space. If the vector space automorphism $g(\operatorname{ad} x)$ preserves the hyperplane T_x, then it preserves or interchanges the two closed half-spaces bounded by T_x. If T_x is a hyperplane, then so is $T_{tx} = T_x$ for $0 < t$. If $g(\operatorname{ad} tx)$ interchanges the two half-spaces for one $t > 0$, then this holds

for all t by continuity. But then $1_L = \lim_{t\to 0+} g(\operatorname{ad} tx)$ would also interchange these half-spaces which is not the case. Hence (3) follows.

(4) $\Rightarrow$ (5). Lemma 2.7.

(5) $\Rightarrow$ (1) is trivial.

It remains to show that

(3) $\Rightarrow$ (4). We assume (3) and the existence of an $x \in W$ with $g(\operatorname{ad} x)W \not\subseteq (W - E_x)^-$; if we derive a contradiction, we will have finished the proof. By assumption we have a $y \in W$ with $g(\operatorname{ad} x)y \not\subseteq (W - E_x)^-$. We may assume without loss of generality that $W - W = L$. Set $K = (W - E_x)^-$, then a fortiori $K - K = L$. Now we apply Theorem 1.22(ii) and find a $\omega \in K^*$ with $\mathbf{R}^+ \cdot \omega \in \operatorname{EXP}(K^*)$ such that $z = g(\operatorname{ad} x)y \notin \omega'$, i.e. $\langle \omega, z\rangle < 0$. In particular, ω is an extreme point of $K^* = (W - E_x)^* = x^\perp \cap W^*$ (cf. Theorem 1.11(iii)). This set is an exposed face of W^* by Theorem 1.7. Hence ω is also an extreme point of W^*. Now we choose a cone V in W containing x in its interior w.r.t. W such that

(i) for all $v \in V$ with $\|v\| = 1$ (relative to a fixed norm on L) we have $\langle \omega, g(\operatorname{ad} v)y\rangle$ $\frac{1}{2}\langle \omega, z\rangle < 0$.

Now we apply the technical Lemma 1.24 and find a sequence $(\omega_n, x_n) \in W^* \times V$ with $\langle \omega_n, x_n\rangle = 0$ and $\omega = \lim \omega_n$; moreover $\mathbf{R}^+ \cdot \omega_n \in \operatorname{EXP}(W^*)$. We may now assume that we normalize the x_n so that $\|x_n\| = 1$. It is no loss of generality to assume condition (4) for all x_n and to conclude that $g(\operatorname{ad} x_n)y \in (W - E_{x_n})^-$ for all n. Since $(\mathbf{R}^+ \cdot \omega_n)^* = (x_n^\perp \cap W^*)^* = (W - E_{x_n})^-$ by Theorem 1.11(ii) we may conclude that

(ii) $\langle \omega_n, g(\operatorname{ad} x_n)y\rangle \geqslant 0$ for all n.

The set $\{v \in V\colon \|v\| = 1\}$ is compact, because V is a cone and $\dim V \leqslant \dim W < \infty$. Hence there is a cluster point x' of the sequence x_n. Then $\|x'\| = 1$ and $x' \in V$, whence $\langle \omega, g(\operatorname{ad} x')y\rangle \leqslant \frac{1}{2}\langle \omega, z\rangle < 0$ by (i) above. On the other hand we then have

$$\langle \omega, g(\operatorname{ad} x')y\rangle = \lim \langle \omega_{n(k)}, g\big(\operatorname{ad} x_{n(k)}\big)y\rangle \geqslant 0$$

for a suitable sequence $n(k)$ of natural numbers by (ii). This is the desired contradiction. $\square$

We finally have all ingredients to prove the Main Theorem of this paper.

The principal result. In order to understand the notation of the Main Theorem we refer to Definition 2.1 for the concepts "Lie semialgebra" and "C-H neighborhood", to Definition 1.10 for E_x, to Definition 1.15, Theorem 1.16 and 1.17 for T_x and to Definition 1.19 for $C^1(W)$.

2.11. THE MAIN THEOREM. *Let W be a finite-dimensional wedge in a Dynkin algebra D and set $L = W - W$; then the following conditions are equivalent*:

(1) *W is a Lie semialgebra in D.*

(2) *For any C-H neighborhood B in D we have $(W \cap B) * (W \cap B) \subseteq W$.*

(3) $[E_x, T_x] \subseteq T_x$ *for all* $x \in W$.

(3′) $[x, T_x] \subseteq T_x$ *for all* $x \in W$.

(4) $[E_x, T_x] \subseteq T_x$ *for all* $x \in C^1(W)$.

(4′) $[x, T_x] \subseteq T_x$ *for all* $x \in C^1(W)$.

For the applications it is convenient to have the equivalent conditions of Lemma 2.10 together. For the proof, however, we pass through various other conditions, which we record in the following

2.12. COMPLEMENT TO THE MAIN THEOREM. *The conditions* (1)–(4) *are also equivalent to the following conditions*:

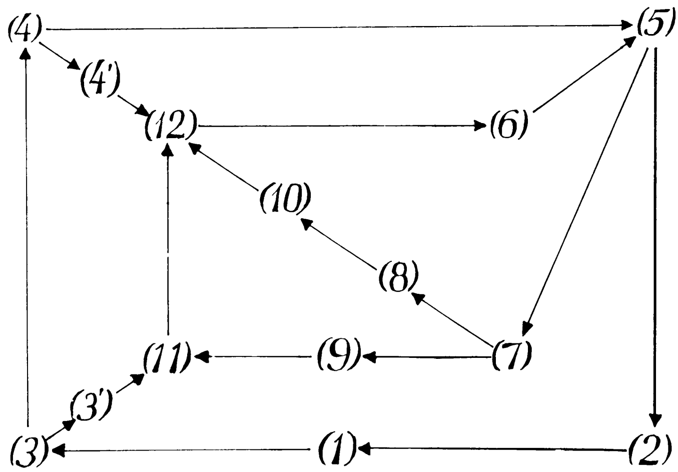
(5) (*resp.*, (6)) $g(\operatorname{ad} x)W \subseteq (W - \mathbf{R}^+ \cdot x)^-$ *for all* $x \in W$ (*resp., for all* $x \in C^1(W)$).

(7) (*resp.*, (8)) $g(\operatorname{ad} y)(W - E_x)^- \subseteq (W - E_x)^-$ *for all* $y \in E_x$ *and all* $x \in W$ (*resp., all* $x \in C^1(W)$).

(9) (*resp.*, (10)) $g(\operatorname{ad} y)T_x = T_x$ *for all* $y \in E_x$ *and all* $x \in W$ (*resp., all* $x \in C^1(W)$).

(11) (*resp.*, (12)) $g(\operatorname{ad} x)T_x \subseteq T_x$ *for all* $x \in W$ (*resp., all* $x \in C^1(W)$).

PROOF OF 2.11 AND 2.12. The proof is organized according to the following plan:



The following implications are trivial: (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (3′), (4) $\Rightarrow$ (4′), (3) $\Rightarrow$ (4), (3′) $\Rightarrow$ (11), (4′) $\Rightarrow$ (12), (7) $\Rightarrow$ (8), (9) $\Rightarrow$ (11), (11) $\Rightarrow$ (12), (10) $\Rightarrow$ (12).

(1) $\Rightarrow$ (3). See Proposition 2.5(ii) and Lemma 2.7.

(4) $\Rightarrow$ (5). See Lemma 2.10 (cf. Theorem 1.11(ii)).

(5) $\Rightarrow$ (2). See Proposition 2.9 (cf. also Theorem 1.11(ii)).

This concludes the outer circle.

(5) $\Rightarrow$ (7). Apply Lemma 2.6 to conclude $g(\operatorname{ad} y)(W - E_x)^- = g(\operatorname{ad} x)(W - E_y)^-$ for y in the interior of E_x in $E_x - E_x$. But $g(\operatorname{ad} y)(W - E_y)^- \subseteq (W - E_y)^-$ by (5) (and Theorem 1.11(ii)) and this last term is equal to $(W - E_x)^-$. A density and continuity argument then yields (7).

(7) $\Rightarrow$ (9). If the vector space automorphism $g(\operatorname{ad} y)$ leaves the wedge $(W - E_x)^-$ invariant, it must also preserve its edge T_x (see Theorem 1.16(ii)).

(12) $\Rightarrow$ (6). See Lemma 2.10 (cf. also Theorem 1.11(ii)).

(6) $\Rightarrow$ (5). See Lemma 2.10.

(8) $\Rightarrow$ (10). Same as (7) $\Rightarrow$ (9). □

At last, the definition of a semialgebra (2.1) is finally made independent of a Campbell-Hausdorff neighborhood of reference. In fact, we note that Theorem A of the introduction is proved. In order to complete the proof of Theorem F of the introduction we first observe a simple technical fact:

2.13. LEMMA. *Let T be a vector subspace of a finite-dimensional Lie algebra L and let L_T be the set $\{x \in T\colon [x, T] \subseteq T\}$. Then L_T is a Lie algebra, and T is an L_T-submodule of L. Note that $L_T \subseteq T$.*

PROOF. Evidently L_T is a vector subspace of T. Suppose that $x, y \in L_T$. Then $[x, y] \in [x, L_T] \subseteq [x, T] \subseteq T$. Moreover, for any $t \in T$ we have

$$\begin{aligned}[[x, y], t] &= -[[y, t], x] - [[t, x], y] \quad \text{(by the Jacobi identity)}\\ &\in [x, T] + [y, T] \subseteq T + T \subseteq T.\end{aligned}$$

Thus $[x, y] \in L_T$. This proves the lemma. □

2.14. THEOREM. *Let W be a Lie semialgebra in a finite-dimensional Lie algebra. Then for all $x \in W$ we have*

(a) E_x is a Lie semialgebra, and

(b) $E_x - E_x$ is a Lie algebra. In particular, T_x is an $(E_x - E_x)$-module.

PROOF. We let $L_x = \{u \in T_x\colon [u, T_x] \subseteq T_x\}$. Then $E_x \subseteq L_x \subseteq T_x$ by Lemma 2.10 and L_x is a Lie algebra by Lemma 2.13. Now $E_x \subseteq W \cap L_x \subseteq W \cap T_x = E_x$ by Theorem 1.18(i). Thus $E_x = W \cap L_x$ is the intersection of a Lie semialgebra with a Lie algebra and is, therefore, a Lie semialgebra. This proves (a). But (b) is a consequence of (a) (see **[HL2]**). The remainder is a direct consquence of Theorem 2.11(3). □

2.15. COROLLARY. *Let W be a Lie-semialgebra in a finite-dimensional Lie algebra. Then every exposed face F of W is a Lie semialgebra and, as a consequence, $F - F$ is a Lie algebra.*

PROOF. This is an immediate consquence of Theorem 2.14, since every exposed face F is of the form E_x for any point x in the $(F - F)$-interior of F. □

We note that Theorem F of the introduction is now proved in view of Theorem 2.11 and Corollary 2.15. We recall, however, that the definition of a tangent hyperplane given in the introduction preceding Theorem A formally differs from Definition 1.15. But Theorem 1.20 showed that the two definitions are, in fact, equivalent.

We draw a conclusion for the Lie algebra on which the Campbell-Hausdorff multiplication has an analytic extension to a global multiplication $L \times L \to L$.

2.16. DEFINITION. A Dynkin algebra L is called *exponential* if and only if there is an analytic function $*\colon L \times L \to L$ such that for all sufficiently small $x, y \in L$ we have $x * y = x + y + \frac{1}{2}[x, y] + \cdots$. (If $\|[x, y]\| \leqslant \|x\|\|y\|$, then $\|x\| + \|y\| < \log 2$ suffices for the absolute convergence of the Campbell-Hausdorff series.)

A Dynkin algebra L is exponential if and only if there is a Lie group G such that L may be identified with the Lie algebra $L(G)$ and $\exp\colon L(G) \to G$ is a diffeomorphism. Indeed, if L is exponential we take $G = (L, *)$ and $\exp = 1_L$. Conversely, if $\exp\colon L(G) \to G$ is a diffeomorphism we set $x * y = \exp^{-1}(\exp x \exp y)$. Certainly all nilpotent Dynkin algebras are exponential, but there are also nonnilpotent solvable exponential algebras (such as e.g. the algebras A_n of example (c)(ii) in the introduction).

2.17. THEOREM. *Let L be a finite-dimensional exponential Lie algebra and let W be a Lie semialgebra in L. Then $W * W = W$.*

PROOF. We will show that the proof of Proposition 2.9 applies for arbitrary $x, y \in W^0$. For this purpose it suffices to verify the following assertion:

(A) If $x, y \in L$ and $t \in \mathbf{R}$, then the function u: $\mathbf{R} \to L$ defined by $u(t) = x * ty$ satisfies

$$u'(t) = g(\operatorname{ad} u(t))\, y \quad \text{for all } t \in \mathbf{R}.$$

We prove assertion (A): We set $u(x, y, t) = x * ty$ and define two analytical functions v, w: $L \times L * \mathbf{R} \to L$ by $v(x, y, t) = (\partial u/\partial t)(x, y, t)$ and $w(x, y, t) = g(\operatorname{ad} u(t))y = g(\operatorname{ad}(x * ty))y$. We define $B = \{x \in L\colon \|x\| < \frac{1}{2}\log 2\}$ with a norm satisfying $\|[x, y]\| \leqslant \|x\|\,\|y\|$. Then the Campbell-Hausdorff series $x + y + \frac{1}{2}[x, y] + \cdots$ converges absolutely on $B \times B$, and its sum agrees with $x * y$. This implies that the functions v and w agree on the open set $B \times B \times]-1, 1[$ of $L \times L \times \mathbf{R}$. Since they are analytic, $v = w$ follows. This proves (A).

Now if W is a semialgebra, by 2.12 we have $g(\operatorname{ad} x)(W) \subseteq (W - E_x)^-$ for all $x \in W$. The proof of Proposition 2.9 applies for all $x, y \in W^0$ after what we just saw and shows that $W^0 * W^0 \subseteq W^0$. By continuity, the assertion then follows from $W = (W^0)^-$. □

We have, in fact, shown a bit more:

2.18. COROLLARY (TO THE PROOF OF THEOREM 2.17). *If W a Lie semialgebra in a finite-dimensional exponential Lie algebra, then $W^0 * W^0 \subseteq W^0$ for the interior W^0 of W in $W - W$.* □

3. A construction. In the introduction we noted, in Corollary C, Ol′shanskii's result that an invariant wedge must necessarily be a semialgebra (cf. also Vinberg [**Vi**]). Ol′shanskii and Vinberg investigated invariant cones in semisimple Lie algebras. It is useful for the construction of examples to note another class of examples yielding invariant cones and, therefore Lie semialgebras. In particular, this class will illustrate the fact that the tangent hyperplanes T_x of a Lie semialgebra need not themselves be subalgebras. Since in the Lie algebras of low dimension, which determine much of our intuition, these tangent hyperplanes are indeed subalgebras as we observed in Corollary G of the introduction, this is perhaps a worthwhile warning. In another paper we show that the class of invariant wedges we construct here is essentially the only type of semialgebra which can occur in a compact Lie algebra [**HH3**].

3.1. DEFINITION. Let L be a Dynkin algebra and $\| \ \|$ a norm compatible with its structure. We say that this norm is *invariant* iff

(i) $\|e^{\operatorname{ad} x} y\| = \|y\|$ for all $x, y \in L$.

If L happens to be, in addition, a real Hibert space with inner product $(\ |\ )$, we say that this inner product is *invariant* iff

(ii) $([x, y]|z) = (x|[y, z])$ for all $x, y, z \in L$.

3.2. REMARK. *If L is a Dynkin algebra which is also a real Hilbert space, and if $\| \ \|$ denotes the norm associated with the inner product $(\ |\ )$, then $\| \ \|$ is invariant iff $(\ |\ )$ is invariant.*

PROOF. By Definition 3.1(i), the norm $\| \ \|$ is invariant iff

(iii) each $e^{\operatorname{ad} x}$, $x \in L$, is an orthogonal transformation of L. This is the case iff the function $t \to (e^{t \operatorname{ad} x} y | e^{t \operatorname{ad} y} z)$: $\mathbf{R} \to \mathbf{R}$ is constant for all x, y, z. Upon differentiating and observing the product rule we note that this condition is equivalent to (ii). □

The well-known fact that the Lie algebra of a compact Lie group is a direct sum of its center and its semisimple commutator algebra and the fact that the Cartan-Killing form on a Lie algebra is negative definite iff it is the Lie algebra of a compact semisimple Lie group immediately yield the well-known observation.

3.3. REMARK. *The following statements are equivalent for a finite-dimensional Lie algebra*:

(1) *L is the Lie algebra of a compact Lie group.*

(2) *L possesses an invariant inner product.* □

3.4. DEFINITION. A Lie algebra is called *compact* if it satisfies the two equivalent conditions of Remark 3.3.

3.5. EXAMPLE. *Let $L_1, \ldots, L_n$ be compact Lie algebras whose norms $\| \ \|_k$, $k = 1, \ldots, n$, are derived from invariant inner products. Then the direct product $L_1 \times \cdots \times L_n$ has an invariant norm given by*

$$\|(x_1, \ldots, x_n)\| = \max\{\|x_k\| : k = 1, 2, \ldots, n\}. \quad \square$$

Trivially, on an abelian Lie algebra every norm is invariant. The following construction now gives us a class of invariant cones:

3.6. CONSTRUCTION. Let L be a Dynkin algebra with an invariant norm $\| \ \|$. Define a cone W in $L \times \mathbf{R}$ by

$$W = \{(x, r) \in L \times \mathbf{R} : \|x\| \leqslant r\}.$$

Then W is invariant and hence is a generating Lie semialgebra.

PROOF. Since $e^{\operatorname{ad}(x,r)}(y, s) = (e^{\operatorname{ad} x} y, s)$, invariance follows immmediately, by Corollary C of the introduction, W is then a Lie semialgebra. Since it has inner points, it is generating. □

3.7. OBSERVATION. *Let $w = (x, 1)$ in W with $x = 1$. Let H_x be that vector subspace of L for which $x + H_x$ is the tangent space in x at the unit ball of L. Then $H_x \times \{0\} + \mathbf{R} \cdot w = \mathbf{R}((H_x + x) \times \{1\})$ is the tangent space T_w of W at w.*

PROOF. Exercise. □

3.8. OBSERVATION. *In the notation of Observation* 3.7, *the tangent space T_w is a Lie algebra if and only if H_x is a subalgebra of L.*

Proof is straightforward. □

3.9. EXAMPLE. *Let L be a compact semisimple Lie algebra equipped with an invariant inner product. Then no tangent space of the wedge $W = \{(x, r) \in L \times \mathbf{R} : (x|x) \leqslant r^2, r \geqslant 0\}$ is a subalgebra.*

PROOF. This is a consequence of the fact that all spaces H_x in this case are the hyperplanes $x^{\perp}$, and that a compact semisimple Lie algebra has no subalgebras of codimension 1. □

The lowest dimension we can produce in this fashion is 4, arising from $L = \mathrm{so}(3)$. The construction shows, in particular, that any compact Lie algebra with nontrivial center always contains generating Lie semialgebras. In **[HH3]** we remark that *a compact Lie algebra cannot contain a generating Lie semialgebra.*

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