

INDEX THEORY ON CURVES

BY

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ABSTRACT. This paper constructs from the $\bar{\partial}$ -operator on the smooth part of a complex projective algebraic curve a cycle in the analytically defined K homology of the curve. The paper identifies the corresponding cycle in the topologically defined K homology.

1. Introduction. In [4] Baum and Douglas generalize the Atiyah-Singer index theorem. They give analytic and topological definitions of the K homology group $KU_0(X)$ for any finite simplicial complex X . They call these groups $K_0^a(X)$ and $K_0'(X)$, and using the Dirac operator they construct the natural isomorphism $K_0'(X) \rightarrow K_0^a(X)$.

When X is a smooth closed manifold, examples of Baum and Douglas's work arise from elliptic differential operators on X . When X is not a smooth manifold, intrinsically defined elements of $K_0^a(X)$ are harder to find. Thus for X not a smooth manifold two natural problems are:

- (a) define elements of $K_0^a(X)$ as intrinsically as possible;
- (b) for each such element find the associated element in $K_0'(X)$.

This paper solves these problems when X is an arbitrary complex projective algebraic curve, which for simplicity is assumed to be without isolated points. §2 describes the parts of Baum and Douglas's work that are needed here. §3 introduces the necessary function spaces. §4 describes the Laplacian and the applications of the functional calculus used later. §5 constructs a cycle for $K_0^a(X)$ from the $\bar{\partial}$ -operator on the smooth part of X . §6 constructs a corresponding cycle in $K_0'(X)$ from the desingularization of X .

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2. K homology. The details of the material in this section can be found in [4].

A cycle for $K_0^a(X)$ is given by a 5-tuple $(H_0, \psi_0, H_1, \psi_1, T)$ satisfying:

- (a) H_0 and H_1 are separable Hilbert spaces;
- (b) $\psi_i: C(X) \rightarrow \mathcal{L}(H_i)$ is a unital algebra $*$ -homomorphism for $i = 0, 1$ ($C(X)$ = continuous \mathbb{C} -valued functions on X , and $\mathcal{L}(H_i)$ = bounded linear operators on H_i);

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(c) $T: H_0 \rightarrow H_1$ is a bounded Fredholm operator with $T \circ \psi_0(f) - \psi_1(f) \circ T$ compact for each $f \in C(X)$.

The equivalence relation imposed on these cycles to get $K_0^a(X)$ is in the same spirit. These cycles push forward by the pullback of continuous functions.

A cycle for $K_0'(X)$ is a triple (M, E, f) satisfying:

- (a) M is a compact Spin^c manifold without boundary;
- (b) E is a complex vector bundle on M ;
- (c) f is a continuous map $M \rightarrow X$.

The equivalence relation, which includes bordism, leading to $K_0'(X)$ is in the same spirit.

The cap product $K^0(X) \times K_0(X) \rightarrow K_0(X)$ is defined for both definitions of the K homology group. If (M, E, f) is a cycle for $K_0'(X)$ and F is a vector bundle on X , then $[F] \cap [(M, E, f)] = [(M, E \otimes f^*F, f)]$. Let $(H_0, \psi_0, H_1, \psi_1, T)$ be a cycle for $K_0^a(X)$, and let F be a vector bundle on X . There is an integer n and a continuous map $P: X \rightarrow \{\text{Projections on } \mathbb{C}^n\}$ such that F is isomorphic to the bundle of images of the projections. Denote by P^{ij} the entry functions for P with respect to the standard basis $\{e_1, \dots, e_n\}$ of \mathbb{C}^n . Define projections P_0 on $H_0 \otimes \mathbb{C}^n$ and P_1 on $H_1 \otimes \mathbb{C}^n$ by

$$P_0(y \otimes e_i) = \sum_{j=1}^n (\psi_0(P^{ij}))(y) \otimes e_j$$

and

$$P_1(w \otimes e_i) = \sum_{j=1}^n (\psi_1(P^{ij}))(w) \otimes e_j.$$

Then

$$[F] \cap [(H_0, \psi_0, H_1, \psi_1, T)] = [(P_0(H_0 \otimes \mathbb{C}^n), P_0(\psi_0 \otimes 1)P_0, P_1(H_1 \otimes \mathbb{C}^n), P_1(\psi_1 \otimes 1)P_1, P_1(T \otimes 1)P_0)]$$

(see [1] and [9]). Despite the numerous choices, these cap products are well defined on the K homology and K cohomology groups.

The push forward to $K_0(\text{point}) = \mathbb{Z}$ of $[(H_0, \psi_0, H_1, \psi_1, T)]$ is $\text{index}(T)$. The push forward to $K_0(\text{point})$ of $[(M, E, f)]$ is $(ch(E) \cup Td(TM))[M]$.

The isomorphism $K_0'(X) \rightarrow K_0^a(X)$ uses the Dirac operators of manifolds appearing in the cycles for $K_0'(X)$. In particular if M is a nonsingular complex projective curve, 1 the trivial line bundle on M and ψ_0 and ψ_1 pointwise multiplication by functions, then the classes of $(M, 1, \text{identity})$ and $(L^2(M), \psi_0, L^2(M, \bar{T}^*), \psi_1, \bar{\partial} \circ (1 - \Delta_M)^{-1/2})$ are associated to each other by the isomorphism $K_0'(M) \rightarrow K_0^a(M)$.

3. Function spaces on curves. The following notation will be used henceforth:

X is a complex projective algebraic curve;

$\pi: \tilde{X} \rightarrow X$ is the desingularization of X ;

S is a finite subset of X containing the singular set of X ;

$U = X - S$.

The local structures of X and π are well known (see e.g. [12, p. 28]). The metric on U is the restriction of the Fubini-Study metric from the ambient projective space. For simplicity we frequently give \tilde{X} a metric that agrees with $\pi^*(\text{metric on } U)$ except possibly on a neighborhood of $\pi^{-1}(S)$, where $(\text{metric on } \tilde{X}) \geq \pi^*(\text{metric on } U)$.

Frequently in what follows U will be identified with $\pi^{-1}(U)$. It is in this sense that functions, all of which are \mathbb{C} -valued, or forms on \tilde{X} can be restricted to U . U and $\pi^{-1}(U)$ have the same structure as complex manifolds. However, the metric, volume form and $*$ -operator on U may differ from the metric, volume form and $*$ -operator on $\pi^{-1}(U)$, which come from those on \tilde{X} . Which metric, volume form and $*$ -operator are desired will be indicated by the space, U or $\pi^{-1}(U)$ or \tilde{X} , that appears in accompanying notation.

For V any open subset with smooth boundary of U or \tilde{X} , with the closure of V taken in U or \tilde{X} respectively, we make the following definitions.

DEFINITION 3.1. For $f, g \in C^\infty(\bar{V})$, $(f, g)_{L^2(V)} = \int_V f \wedge * \bar{g}$.

DEFINITION 3.2. For $f, g \in C^\infty(\bar{V})$, $(f, g)_{H^1(V)} = \int_V f \wedge * \bar{g} + \int_V df \wedge * \bar{dg}$.

Denote by T_C^* the complexified cotangent bundle and by \bar{T}^* the antiholomorphic cotangent bundle.

DEFINITION 3.3. For $\alpha, \beta \in C^\infty(\bar{V}, T_C^*)$, $(\alpha, \beta)_{L^2(V, T_C^*)} = \int_V \alpha \wedge * \bar{\beta}$.

DEFINITION 3.4. For $\alpha, \beta \in C^\infty(\bar{V}, \bar{T}^*)$, $(\alpha, \beta)_{L^2(V, \bar{T}^*)} = \int_V \alpha \wedge * \bar{\beta}$.

DEFINITION 3.5. $L^2(V) =$ completion of $\{f \in C^\infty(\bar{V}): (f, f)_{L^2(V)} < \infty\}$ in the norm given by $(\cdot, \cdot)_{L^2(V)}$.

DEFINITION 3.6. $H^1(V) =$ completion of $\{f \in C^\infty(\bar{V}): (f, f)_{H^1(V)} < \infty\}$ in the norm given by $(\cdot, \cdot)_{H^1(V)}$.

DEFINITION 3.7. $L^2(V, T_C^*) =$ completion of $\{\alpha \in C^\infty(\bar{V}, T_C^*): (\alpha, \alpha)_{L^2(V, T_C^*)} < \infty\}$ in the norm given by $(\cdot, \cdot)_{L^2(V, T_C^*)}$.

DEFINITION 3.8. $L^2(V, \bar{T}^*) =$ completion of $\{\alpha \in C^\infty(\bar{V}, \bar{T}^*): (\alpha, \alpha)_{L^2(V, \bar{T}^*)} < \infty\}$ in the norm given by $(\cdot, \cdot)_{L^2(V, \bar{T}^*)}$.

Clearly

LEMMA 3.9. *The spaces in Definitions 3.5–3.8 are separable Hilbert spaces with inner products extended from those of Definitions 3.1–3.4, respectively.*

DEFINITION 3.10 $\text{Hol}^1(V) = \{\text{functions holomorphic on } V \text{ and contained in } H^1(V)\}$.

LEMMA 3.11. *One can define an equivalent norm on $H^1(V)$ by starting with the inner product given by $\int_V f \wedge * \bar{g} + \int_V \bar{\partial} f \wedge * \bar{\partial} \bar{g} + \int_V \partial f \wedge * \bar{\partial} \bar{g}$.*

Let \tilde{Y} be an arbitrary open subset of \tilde{X} with smooth boundary and with $Y = \pi(\tilde{Y}) \cap U$.

LEMMA 3.12. *For $\alpha \in C^\infty(\tilde{Y}, T_C^*)$ and $\beta =$ restriction of α to Y , $\int_{\tilde{Y}} \alpha \wedge * \bar{\alpha} = \int_Y \beta \wedge * \bar{\beta}$.*

The proofs of Lemmas 3.11 and 3.12 are well known calculations.

LEMMA 3.13. *Restriction is an isometry $L^2(\tilde{Y}, \bar{T}^*) \rightarrow L^2(Y, \bar{T}^*)$.*

PROOF. Observe that $C_c^\infty(\tilde{Y} - \pi^{-1}(S) \cap \tilde{Y}, \bar{T}^*)$ and $C_c^\infty(Y, \bar{T}^*)$ are dense in $L^2(\tilde{Y}, \bar{T}^*)$ and $L^2(Y, \bar{T}^*)$, respectively, and apply Lemma 3.12.

LEMMA 3.14. *Restriction gives bounded linear maps $L^2(\tilde{Y}) \rightarrow L^2(Y)$ and $H^1(\tilde{Y}) \rightarrow H^1(Y)$.*

PROOF. This lemma follows immediately from our choice of metrics, which in fact makes these maps norm-decreasing, and Lemma 3.12.

LEMMA 3.15. *The metric on U is conical in the sense that each point in X has a neighborhood in which each branch is quasi-isometric to a cone.*

PROOF. See [7, p. 323] or calculate directly.

EXAMPLE 3.16. Most interesting aspects of this paper can be seen in the cuspidal cubic $x^2z = y^3$ in \mathbf{CP}^2 . Consider a coordinate chart centered at $\pi^{-1}(\text{singular point})$. Change to polar coordinates. For f supported near $\pi^{-1}(\text{singular point})$,

$$(f, f)_{L^2(U)} \sim \int |f|^2 r^3 dr d\theta \quad \text{and} \quad (f, f)_{L^2(\tilde{X})} \sim \int |f|^2 r dr d\theta.$$

If $|f|$ behaves like $1/r$ near $\pi^{-1}(\text{singular point})$, $f \in L^2(U)$ but $f \notin L^2(\tilde{X})$ and $f \notin H^1(U)$.

4. The Laplace operator. In this section a selfadjoint Laplace operator is identified, and the functional calculus is applied to it to prove Proposition 4.4, which is needed in the succeeding sections. Many arguments in this section are true on U or \tilde{X} . In statements that hold in either case, the space “ U ” or “ \tilde{X} ” is suppressed in the notation. For example the Laplace operator on U is Δ_U and that on \tilde{X} is $\Delta_{\tilde{X}}$; and when a statement is true of either operator, Δ will be used.

Cheeger [5, p. 93] shows that d with domain $\{f \in C^\infty \cap L^2: df \in L^2(T_C^*)\}$ is closable and that $\delta = -*d*$ with domain $\{\omega \in C^\infty(T_C^*) \cap L^2(T_C^*): \delta\omega \in L^2\}$ is closable. Note that the domain of \bar{d} is H^1 .

DEFINITION 4.1. $\Delta = -\bar{\delta}d$ as an operator $L^2 \rightarrow L^2$.

Since the metrics on U and \tilde{X} are conical, $\bar{d}^* = \bar{\delta}$ [7, p. 321], and by [13, p. 312].

LEMMA 4.2. Δ is selfadjoint.

Thus we can apply to Δ the functional calculus arising from the spectral theorem. By [13, p. 307] $(1 - \Delta)^{-1}$ is selfadjoint, has norm ≤ 1 and has positive spectrum. We will see shortly that $\text{range}((1 - \Delta)^{-1}) \subset H^1$. By Corollary 5.10, which is independent of arguments here, the inclusion $H^1 \hookrightarrow L^2$ is compact (see also [6]). Therefore, L^2 has an orthonormal basis of eigenvectors of $(1 - \Delta)^{-1}$ and thus of Δ . Also the spectrum of Δ is a discrete infinite set of nonpositive real numbers, each an eigenvalue of finite multiplicity. This information allows one to justify each application of the functional calculus that follows.

Let J be the inclusion $H^1 \hookrightarrow L^2$. Since J is bounded as a map $H^1 \rightarrow L^2$, its adjoint J^* is defined everywhere. Thus for all $f \in L^2$ and $g \in H^1$,

$$(f, Jg)_{L^2} = (J^*f, g)_{H^1} = (J^*f, g)_{L^2} + (\bar{d}J^*f, \bar{d}g)_{L^2(T_C^*)} = ((1 - \Delta)J^*f, g)_{L^2}.$$

Since $Jg = g$, $J^* = (1 - \Delta)^{-1}$. It follows that J 's partial isometry part, which is an isometry, is $(1 - \Delta)^{1/2}$.

LEMMA 4.3. $(1 - \Delta)^{1/2}: H^1 \rightarrow L^2$ is an isometry. I.e. $H^1 = \text{domain}((1 - \Delta)^{1/2})$.

Let W be the subset of \tilde{X} for which the metric on \tilde{X} agrees with π^* (metric on U). In what follows the support of an L^2 or H^1 function refers to the intersection of supports of functions in the equivalence class.

PROPOSITION 4.4. For arbitrary $\varepsilon > 0$, there exist maps $L_\varepsilon: L^2(U) \rightarrow H^1(U)$ and $\tilde{L}_\varepsilon: L^2(\tilde{X}) \rightarrow H^1(\tilde{X})$ satisfying:

- (a) for $u \in \text{support}(L_\varepsilon(f))$, $\text{distance}(u, \text{support}(f)) < \varepsilon$; for $x \in \text{support}(\tilde{L}_\varepsilon(g))$, $\text{distance}(x, \text{support}(g)) < \varepsilon$;
- (b) $(1 - \Delta_U)^{-1/2} - L_\varepsilon: L^2(U) \rightarrow H^1(U)$ is compact; $(1 - \Delta_{\tilde{X}})^{-1/2} - \tilde{L}_\varepsilon: L^2(\tilde{X}) \rightarrow H^1(\tilde{X})$ is compact (in fact these operators are continuous between domains of any two powers of $1 - \Delta_U$ or $1 - \Delta_{\tilde{X}}$ respectively);
- (c) if $f \in L^2(\tilde{X})$ has support contained in $\{x \in \tilde{X}: \text{distance}(x, \tilde{X} - W) > \varepsilon\}$, then $L_\varepsilon(f) = \tilde{L}_\varepsilon(f)$ (here we identify functions by the restriction convention of §3);
- (d) $L_\varepsilon: L^2(U) \rightarrow H^1(U)$ and $\tilde{L}_\varepsilon: L^2(\tilde{X}) \rightarrow H^1(\tilde{X})$ are isomorphisms of topological vector spaces.

The rest of this section is a proof of this proposition. Parts (a), (b) and (c) are taken from [8].

Let λ be the spectral representation of $(1 - \Delta)^{1/2}$, regarded as an unbounded operator from L^2 to L^2 . By change of variable and [10, 858.801],

$$(4.5) \quad \lambda^{-1/2} = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \frac{\cos(\lambda y)}{y^{1/2}} dy.$$

Let $\phi_\varepsilon(y)$ be a monotonically decreasing C^∞ bump function on $[0, \infty]$ that is identically 1 in a neighborhood of 0 and that has support contained in $\{y: 0 \leq y \leq \varepsilon/2\}$. Then

$$(4.6) \quad \begin{aligned} \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \frac{\cos(\lambda y)}{y^{1/2}} dy &= \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \frac{\cos(\lambda y)}{y^{1/2}} \phi_\varepsilon(y) dy \\ &\quad + \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \frac{\cos(\lambda y)}{y^{1/2}} (1 - \phi_\varepsilon(y)) dy. \end{aligned}$$

LEMMA 4.7. For any $k \in \mathbf{R}$ there is an operator T_k , bounded on the domain of any power of $(1 - \Delta)$, such that $(\sqrt{2}/\sqrt{\pi}) \int_0^\infty (\cos(\lambda y)/y^{1/2}) (1 - \phi_\varepsilon(y)) dy$ is the spectral representation of $(1 - \Delta)^k \circ T_k$.

PROOF. Integrate by parts.

To understand the properties of the operator represented by

$$\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \frac{\cos(\lambda y)}{y^{1/2}} \phi_\varepsilon(y) dy,$$

consider the hyperbolic equation $(\partial^2/\partial t^2 + 1 - \Delta)f(z, t) = 0$. For initial conditions $f_0(z) = f(z, 0) \in L^2(\text{z-space})$ and $(\partial/\partial t)f(z, t)|_{t=0} \equiv 0$, the solution of this

equation is $\cos((1 - \Delta)^{1/2}t)(f_0(z))$ [14, p. 70ff.]. When the domains of powers of the Laplacian are locally defined, this solution exhibits finite propagation speed [14, p. 79ff.]:

$$(4.8) \quad \text{for } z \in \text{support}(\cos((1 - \Delta)^{1/2}t)f_0(z)), \quad \text{distance}(z, \text{support}(f_0)) \leq |t|.$$

By (4.5) and (4.6)

$$(4.9) \quad \lambda^{-1} = \left[\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \frac{\cos(\lambda y)}{y^{1/2}} \phi_\epsilon(y) dy + \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \frac{\cos(\lambda y)}{y^{1/2}} (1 - \phi_\epsilon(y)) dy \right]^2.$$

Let L_ϵ be the operator represented by $[(\sqrt{2}/\sqrt{\pi}) \int_0^\infty (\cos(\lambda y)/y^{1/2}) \phi_\epsilon(y) dy]^2$ where λ is the spectral representation of $(1 - \Delta_U)^{1/2}$. Let \tilde{L}_ϵ be the operator represented by $[(\sqrt{2}/\sqrt{\pi}) \int_0^\infty (\cos(\lambda y)/y^{1/2}) \phi_\epsilon(y) dy]^2$, where λ is the spectral representation of $(1 - \Delta_{\tilde{X}})^{1/2}$. Since the domains of powers of these Laplacians are locally defined, the condition on $\text{support}(\phi_\epsilon)$ and (4.8) imply Proposition 4.4(a). Lemma 4.7 implies Proposition 4.4(b).

Finite propagation speed implies uniqueness of solutions [14]. In particular for f as in Proposition 4.4(c), $\cos((1 - \Delta_U)^{1/2}t)(f) = \cos((1 - \Delta_{\tilde{X}})^{1/2}t)(f)$ for $0 \leq t < \epsilon$ because each can be regarded as the unique solution to the hyperbolic equation on \tilde{X} for $0 \leq t < \epsilon$. Proposition 4.4(c) follows.

By Lemma 4.3 and change of variable, $s = \lambda y$, to prove Proposition 4.4(d) it suffices to show that there exist finite positive k_1 and k_2 such that for each value of λ , also denoted λ ,

$$(4.10) \quad k_1 \leq \int_0^\infty \cos(s)/s^{1/2} ds / \int_0^\infty (\cos(s)/s^{1/2}) \phi_\epsilon(s/\lambda) ds \leq k_2.$$

By [10, 858.801] $(\sqrt{2}/\sqrt{\pi}) \int_0^\infty \cos(s)/s^{1/2} ds = 1$. Let $A = \sup\{y: \phi_\epsilon(y) = 1\}$ and let $B = \sup\{y: \phi_\epsilon(y) \neq 0\}$. Let $F(r) = \int_0^r \cos(s)/s^{1/2} ds$. Let λ_1 = smallest value of λ such that $(\pi/2)/\lambda < B$. Then for $\lambda < \lambda_1$

$$(4.11) \quad F(A) \leq \int_0^\infty (\cos(s)/s^{1/2}) \phi_\epsilon(s/\lambda) ds \leq F(\pi/2)$$

and for $\lambda \geq \lambda_1$

$$(4.12) \quad \phi_\epsilon(\pi/2\lambda_1) \cdot F(3\pi/2) \leq \int_0^\infty (\cos(s)/s^{1/2}) \phi_\epsilon(s/\lambda) ds \leq F(\pi/2).$$

One checks (4.11) and (4.12) by using the alternating sum associated with each integral by virtue of the behavior of cosine. By [15, p. 55, 745] (4.11) and (4.12) imply (4.10).

COROLLARY 4.13. *Statements analogous to those in Proposition 4.4 are true for $(1 - \Delta)^{1/2}$. (For (d) the maps are $H^1 \rightarrow L^2$.)*

PROOF. $(1 - \Delta)^{1/2} = (1 - \Delta) \circ (1 - \Delta)^{-1/2} = (1 - \Delta) \circ (L_\epsilon^{(\cdot)} + ((1 - \Delta)^{-1/2} - L_\epsilon^{(\cdot)}))$ and $1 - \Delta$ is local.

5. Defining a cycle.

THEOREM 5.1. *Let X be a complex projective algebraic curve. Let $U = X - S$, where S is a finite subset of X which contains the singularities of X . Let $\psi_0: C(X) \rightarrow \mathcal{L}(L^2(U))$ be multiplication of L^2 functions by restrictions of continuous functions. Let $\psi_1: C(X) \rightarrow \mathcal{L}(L^2(U, \bar{T}^*))$ be multiplication of L^2 forms by restrictions of continuous functions. Let $\bar{\partial}$ be the natural extension of the $\bar{\partial}$ -operator from $C^\infty H^1$ functions to H^1 functions. Then $(L^2(U), \psi_0, L^2(U, \bar{T}^*), \psi_1, \bar{\partial} \circ (1 - \Delta_U)^{-1/2})$ is a cycle representing an element of $K_0^a(X)$.*

PROOF. By Lemma 3.9 the proof reduces to proofs of three propositions.

PROPOSITION 5.2. *Give $C(X)$ a C^* -algebra structure with conjugation as $*$ and the sup norm as norm. Then ψ_0 and ψ_1 are unital algebra $*$ -homomorphisms.*

PROOF. Since X is compact, each element of $C(X)$ has finite sup, and the proposition follows immediately from definitions.

PROPOSITION 5.3. $\bar{\partial} \circ (1 - \Delta_U)^{-1/2}: L^2(U) \rightarrow L^2(U, \bar{T}^*)$ is a bounded Fredholm operator.

PROOF. By Lemma 4.3 it suffices to show that $\bar{\partial}: H^1(U) \rightarrow L^2(U, \bar{T}^*)$ is bounded and Fredholm. Lemma 3.11 shows that $\bar{\partial}$ is bounded. Recall that $\pi: \tilde{X} \rightarrow X$ is the desingularization of X . By compactness of X , the $\bar{\partial}$ -Poincaré lemma for currents [11, p. 385] and elliptic regularity, X can be covered with a finite collection of open sets $\{W_i\}$ such that for each i :

(a) $\partial W_i \cap S = \emptyset$;

(b) $0 \rightarrow \text{Hol}^1(\pi^{-1}(W_i)) \rightarrow H^1(\pi^{-1}(W_i)) \xrightarrow{\bar{\partial}} L^2(\pi^{-1}(W_i), \bar{T}^*) \rightarrow 0$ is exact.

Denote by W an arbitrary one of the W_i . Consider the commutative diagram

(5.4)

$$\begin{array}{ccccccc}
 0 & \rightarrow & \text{Hol}^1(\pi^{-1}(W)) & \rightarrow & H^1(\pi^{-1}(W)) & \xrightarrow{\bar{\partial}} & L^2(\pi^{-1}(W), \bar{T}^*) \rightarrow 0 \\
 & & \downarrow R_1 & & \downarrow R_2 & & \downarrow R_3 \\
 0 & \rightarrow & \text{Hol}^1(W \cap U) & \rightarrow & H^1(W \cap U) & \xrightarrow{\bar{\partial}} & L^2(W \cap U, \bar{T}^*) \rightarrow 0
 \end{array}$$

where the vertical maps are restrictions (under convention of §3) of functions or forms. By Lemmas 3.13 and 3.14 the vertical maps are continuous and R_3 is an isometry.

LEMMA 5.5. *The bottom row of (5.4) is exact.*

PROOF. Elliptic regularity determines the kernel of $\bar{\partial}$. A diagram chase shows that $\bar{\partial}$ is surjective.

LEMMA 5.6. R_1 is a bijection.

PROOF. If K is a compact subset of X that is disjoint from S , the norms on $H^1(\tilde{X})$ and $H^1(U)$ are equivalent on functions with support in K . Thus this lemma reduces

to the statement: if f has a pole or essential singularity at a point in $\pi^{-1}(S)$, $f \notin H^1(U)$. This statement can be checked by using the norm described in Lemma 3.11 with special attention paid to the term involving ∂f .

The five lemma implies that R_2 is bijective. The open mapping theorem implies that R_2 is an isomorphism of topological vector spaces. One can then use a partition of unity subordinate to the $\pi^{-1}(W_i)$ to prove

LEMMA 5.7. *Restriction gives an isomorphism of topological vector spaces $H^1(\tilde{X}) \rightarrow H^1(U)$.*

Consider the commutative diagram

$$(5.8) \quad \begin{array}{ccc} H^1(\tilde{X}) & \xrightarrow{\bar{\partial}} & L^2(\tilde{X}, \bar{T}^*) \\ \downarrow & & \downarrow \\ H^1(U) & \xrightarrow{\bar{\partial}} & L^2(U, \bar{T}^*) \end{array}$$

where each vertical map is restriction. By the $\bar{\partial}$ -Poincaré lemma for currents and elliptic regularity the top map is Fredholm and has index equal to the Euler characteristic of $H_{\bar{\partial}}^{0,*}(\tilde{X})$. It follows by Lemmas 3.13 and 5.7 that $\bar{\partial}: H^1(U) \rightarrow L^2(U, \bar{T}^*)$ is Fredholm, which finishes the proof of Proposition 5.3, and that

COROLLARY 5.9. *The index of $\bar{\partial}: H^1(U) \rightarrow L^2(U, \bar{T}^*)$ equals the Euler characteristic of $H_{\bar{\partial}}^{0,*}(\tilde{X})$.*

The arguments above help establish the following (see also [6]):

COROLLARY 5.10. *The inclusion $H^1(U) \hookrightarrow L^2(U)$ is compact.*

PROOF. Consider the commutative diagram

$$(5.11) \quad \begin{array}{ccc} H^1(\tilde{X}) & \xrightarrow{i_1} & L^2(\tilde{X}) \\ \uparrow h_1 & & \downarrow h_2 \\ H^1(U) & \xrightarrow{i_2} & L^2(U) \end{array}$$

where i_1 and i_2 are inclusions, h_1 is (restriction) $^{-1}$, and h_2 is restriction. Apply Rellich's lemma on \tilde{X} .

PROPOSITION 5.12. *For each $f \in C(X)$, $\bar{\partial} \circ (1 - \Delta_U)^{-1/2} \circ \psi_0(f) - \psi_1(f) \circ \bar{\partial} \circ (1 - \Delta_U)^{-1/2}: L^2(U) \rightarrow L^2(U, \bar{T}^*)$ is compact.*

PROOF. For simplicity let D denote $\bar{\partial} \circ (1 - \Delta_U)^{-1/2}: L^2(U) \rightarrow L^2(U, \bar{T}^*)$. The finite set $\pi^{-1}(S)$ is denoted $\{s_j\}_{j \in J}$. All distances come from the metric on U .

The map $C(X) \hookrightarrow C(\tilde{X})$ given by $f \mapsto f \circ \pi$ leads to extensions of ψ_0 and ψ_1 to $C(\tilde{X})$, both of which are unital algebra $*$ -homomorphisms. By abuse of notation, both extensions will be denoted by m . That is, $f \in C(\tilde{X})$ goes to $m_f \in \mathcal{L}(L^2(U))$ and $m_f \in \mathcal{L}(L^2(U, \bar{T}^*))$.

Choose an arbitrary $f \in C(X)$. Choose $\{g_n\}_{n \in \mathbb{Z}, n > 0}$ such that for each n :

(a) $g_n \in C^\infty(\tilde{X})$;

(b) $\sup |g_n - f \circ \pi| < 1/n$.

(c) for each $j \in J$ there exists $\delta_{nj} > 0$ such that g_n is constant in a neighborhood of s_j with radius at least δ_{nj} .

A standard construction allows such choices. Condition (b) implies that $m_{g_n} \rightarrow m_{f \circ \pi}$ in operator norm. Since D is bounded, compactness of $D \circ m_{g_n} \rightarrow m_{g_n} \circ D$ for each n implies compactness of $D \circ \psi_0(f) - \psi_1(f) \circ D$. Choose an arbitrary element of $\{g_n\}$ and call it g . Call its δ_{nj} 's δ_j 's.

LEMMA 5.13. $D \circ m_g - m_g \circ D = \bar{\partial} \circ (1 - \Delta_U)^{-1/2} \circ m_g - \bar{\partial} \circ m_g \circ (1 - \Delta_U)^{-1/2} + \bar{\partial} \circ m_g \circ (1 - \Delta_U)^{-1/2} - m_g \circ \bar{\partial} \circ (1 - \Delta_U)^{-1/2}$.

PROOF. A middle term has been subtracted and added. Because $g \in C^\infty(\tilde{X})$ and $dg = 0$ in a neighborhood of $\pi^{-1}(S)$, $m_g: H^1(U) \rightarrow H^1(U)$ is a bounded linear map and the middle term is well defined.

LEMMA 5.14. $\bar{\partial} \circ m_g \circ (1 - \Delta_U)^{-1/2} - m_g \circ \bar{\partial} \circ (1 - \Delta_U)^{-1/2}: L^2(U) \rightarrow L^2(U, \bar{T}^*)$ is compact.

PROOF. By Lemma 4.3 this lemma reduces to: $\bar{\partial} \circ m_g - m_g \circ \bar{\partial}: H^1(U) \rightarrow L^2(U, \bar{T}^*)$ is compact. That $\bar{\partial} \circ m_g - m_g \circ \bar{\partial} = m_{\bar{\partial}g}$ is seen by applying the product rule to $(\bar{\partial} \circ m_g - m_g \circ \bar{\partial})$ restricted to $C^\infty(U) \cap H^1(U)$ and extending by continuity. By choice of g , $\bar{\partial}g \in C_c^\infty(U, \bar{T}^*)$. Therefore, denoting by $(\cdot, \cdot)_u$ pointwise evaluation of the metric on T_C^* , we have $\sup_{u \in U} (\bar{\partial}g(u), \bar{\partial}g(u))_u^{1/2} < \infty$, and for all $h \in L^2(U)$

$$(5.15) \quad (\bar{\partial}g \cdot h, \bar{\partial}g \cdot h)_{L^2(U, \bar{T}^*)}^{1/2} \leq \left(\sup_{u \in U} (\bar{\partial}g(u), \bar{\partial}g(u))_u^{1/2} \right) \cdot (h, h)_{L^2(U)}^{1/2}.$$

Therefore, the domain $m_{\bar{\partial}g}$ can be extended to $L^2(U)$ and so $m_{\bar{\partial}g}: H^1(U) \rightarrow L^2(U, \bar{T}^*)$ is compact.

LEMMA 5.16. $\bar{\partial} \circ (1 - \Delta_U)^{-1/2} \circ m_g - \bar{\partial} \circ m_g \circ (1 - \Delta_U)^{-1/2}: L^2(U) \rightarrow L^2(U, \bar{T}^*)$ is compact.

PROOF. Since $\bar{\partial}: H^1(U) \rightarrow L^2(U, \bar{T}^*)$ is bounded, it suffices to show that $(1 - \Delta_U)^{-1/2} \circ m_g - m_g \circ (1 - \Delta_U)^{-1/2}: L^2(U) \rightarrow H^1(U)$ is compact. We prove the equivalent statement:

$$(5.17) \quad m_g - (1 - \Delta_U)^{1/2} \circ m_g \circ (1 - \Delta_U)^{-1/2}: L^2(U) \rightarrow L^2(U) \text{ is compact.}$$

For convenience let L denote $1 - \Delta_U$.

Choose $\epsilon > 0$ such that $5\epsilon < \min_{j \in J} (\delta_j)$. Choose a C^∞ partition of unity μ_0 and μ_j for all $j \in J$ satisfying:

- (a) $\text{distance}(\text{support}(\mu_0), \pi^{-1}(S)) > 3\epsilon$;
- (b) $\text{distance}(\tilde{X} - \text{support}(\mu_j), s_j) < 4\epsilon$ for each $j \in J$.

(Remember distances come from metric on U .)

$$(5.18) \quad m_g - L^{1/2} \circ m_g \circ L^{-1/2} = (m_g - L^{1/2} \circ m_g \circ L^{-1/2}) \circ m_{\mu_0} + \sum_{j \in J} (m_g - L^{1/2} \circ m_g \circ L^{-1/2}) \circ m_{\mu_j}.$$

Let K_j be the constant function with value equal to g 's constant value in the δ_j neighborhood of s_j . Then for each $j \in J$

$$(5.19) \quad \left(m_g - L^{1/2} \circ m_g \circ L^{-1/2}\right) \circ m_{\mu_j} = \left(m_{g-K_j} - L^{1/2} \circ m_{g-K_j} \circ L^{-1/2}\right) \circ m_{\mu_j} \\ + \left(m_{K_j} - L^{1/2} \circ m_{K_j} \circ L^{-1/2}\right) \circ m_{\mu_j}.$$

Linearity of powers of L implies that the last term of (5.19) is zero. By condition (b) on the partition of unity, $m_{g-K_j} \circ m_{\mu_j} = 0$. Therefore

$$(5.20) \quad \left(m_g - L^{1/2} \circ m_g \circ L^{-1/2}\right) \circ m_{\mu_j} = -L^{1/2} \circ m_{g-K_j} \circ L^{-1/2} \circ m_{\mu_j}.$$

Replace $L^{-1/2}$ by $L_\epsilon + (L^{-1/2} - L_\epsilon)$ (see Proposition 4.4). By condition (b) on the partition of unity and by Proposition 4.4(a), $m_{g-K_j} \circ L_\epsilon \circ m_{\mu_j} = 0$. By Proposition 4.4(b) $L^{1/2} \circ m_{g-K_j} \circ (L^{-1/2} - L_\epsilon) \circ m_{\mu_j}$ maps $L^2(U)$ continuously to $H^1(U)$. By Corollary 5.10 it is compact as a map to $L^2(U)$.

To prove Lemma 5.16 only compactness of $(m_g - L^{1/2} \circ m_g \circ L^{-1/2}) \circ m_{\mu_0}$ remains to be shown. Choose another partition of unity ν_0 and ν_j for all $j \in J$ satisfying:

- (a) each ν_j is constant on components of a neighborhood of $\pi^{-1}(S)$;
- (b) $\text{distance}(\text{support}(\nu_0), \pi^{-1}(S)) > 2\epsilon$;
- (c) $\text{distance}(\text{support}(\mu_0), \text{support}(\nu_j)) > \epsilon$ for each $j \in J$.

$$(5.21) \quad \left(m_g - L^{1/2} \circ m_g \circ L^{-1/2}\right) \circ m_{\mu_0} = m_{\nu_0} \circ m_g \circ m_{\mu_0} + \sum_{j \in J} m_{\nu_j} \circ m_g \circ m_{\mu_0} \\ - L^{1/2} \circ m_{\nu_0} \circ m_g \circ L^{-1/2} \circ m_{\mu_0} \\ - \sum_{j \in J} L^{1/2} \circ m_{\nu_j} \circ m_g \circ L^{-1/2} \circ m_{\mu_0}.$$

Condition (c) on the ν -partition of unity implies that $m_{\nu_j} \circ m_g \circ m_{\mu_0} = 0$ and $L^{1/2} \circ m_{\nu_j} \circ m_g \circ L^{-1/2} \circ m_{\mu_0} = L^{1/2} \circ m_{\nu_j} \circ m_g \circ (L^{-1/2} - L_\epsilon) \circ m_{\mu_0}$ for each j . Condition (a) on the ν -partition of unity, Proposition 4.4(b) and Corollary 5.10 imply that this last expression is compact.

Let $h = \nu_0 \cdot g$. It remains to be shown that $m_h \circ m_{\mu_0} - L^{1/2} \circ m_h \circ L^{-1/2} \circ m_{\mu_0}$; $L^2(U) \rightarrow L^2(U)$ is compact. Let $N = \{u \in U: \text{distance}(u, S) > \epsilon\}$. Extend the metric on N to one on \tilde{X} . Call the nearly local operators of Corollary 4.13 \tilde{L}'_ϵ and \tilde{L}_ϵ . As maps from $L^2(\tilde{X})$ to $L^2(\tilde{X})$

$$(5.22) \quad m_h \circ m_{\mu_0} - (1 - \Delta_{\tilde{X}})^{1/2} \circ m_h \circ (1 - \Delta_{\tilde{X}})^{-1/2} \circ m_{\mu_0} \\ = m_h \circ m_{\mu_0} - \left((1 - \Delta_{\tilde{X}})^{1/2} - \tilde{L}'_\epsilon\right) \circ m_h \circ (1 - \Delta_{\tilde{X}})^{-1/2} \circ m_{\mu_0} \\ - \tilde{L}'_\epsilon \circ m_h \circ \left((1 - \Delta_{\tilde{X}})^{-1/2} - \tilde{L}_\epsilon\right) \circ m_{\mu_0} - \tilde{L}'_\epsilon \circ m_h \circ \tilde{L}_\epsilon \circ m_{\mu_0}.$$

The left side of (5.22) is compact by the theory of pseudodifferential operators on closed manifolds and Rellich's lemma. The second and third terms of the right side of (5.22) are compact by Proposition 4.4(b) and Corollary 4.13. Thus

$$(5.23) \quad m_h \circ m_{\mu_0} - \tilde{L}'_\epsilon \circ m_h \circ \tilde{L}_\epsilon \circ m_{\mu_0}; L^2(\tilde{X}) \rightarrow L^2(\tilde{X}) \text{ is compact.}$$

The operator of (5.23) actually maps $L^2(N)$ to $L^2(N)$. Rewrite (5.22) with \tilde{X} replaced by U , \tilde{L}_ϵ by L'_ϵ , and \tilde{L}_ϵ by L_ϵ . Once again the second and third terms on the right are compact by Proposition 4.4(b) and Corollary 4.13. By Proposition 4.4(c) and Corollary 4.13 the rewritten version of the first and fourth terms equals the original version as a map of $L^2(N)$ to $L^2(N)$. Thus by (5.23) it is compact, and the left side of rewritten (5.22) is compact.

6. A corresponding topological cycle.

THEOREM 6.1. *Let conditions and notation be as in Theorem 5.1. The element of $K'_0(X)$ associated to $(L^2(U), \psi_0, L^2(U, \bar{T}^*), \psi_1, \bar{\partial} \circ (1 - \Delta_U)^{-1/2})$ by the isomorphism $K'_0(X) \rightarrow K^a_0(X)$ is represented by $(\tilde{X}, 1, \pi)$, where $\pi: \tilde{X} \rightarrow X$ is the desingularization of X and 1 is the trivial line bundle $\tilde{X} \times \mathbb{C}$ on \tilde{X} .*

PROOF. Choose neighborhoods of S , A_1 and A_2 , with $\bar{A}_1 \subset \text{interior}(A_2)$ and with the intersection of $X - \bar{A}_2$ with each irreducible component of X nonempty. Choose $\epsilon > 0$ so that $\text{distance}(A_1, X - A_2) > 4\epsilon$. (Distance comes from metric on U .)

Choose an open set B such that:

- (a) $A_1 \subset X - B \subset A_2$;
- (b) $\text{distance}(A_1, B) > \epsilon$;
- (c) $\text{distance}(X - B, X - A_2) > 3\epsilon$.

Choose a set A_3 such that:

- (a) $X - B \subset A_3 \subset A_2$;
- (b) $\text{distance}(X - B, X - A_3) > \epsilon$;
- (c) $\text{distance}(A_3, X - A_2) > 2\epsilon$.

Choose a set A_4 such that:

- (a) $A_3 \subset A_4 \subset A_2$;
- (b) $\text{distance}(A_3, X - A_4) > \epsilon$;
- (c) $\text{distance}(A_4, X - A_2) > \epsilon$.

Choose a set A_5 such that:

- (a) $A_4 \subset A_5 \subset A_2$;
- (b) $\text{distance}(A_4, X - A_5) > \epsilon$;
- (c) $\bar{A}_5 \subset \text{interior}(A_2)$.

Give X a metric that agrees with $\pi^*(\text{metric on } U)$ except possibly on $\pi^{-1}(A_1)$. Let $\{X_i\}_{i \in I}$ be the set of irreducible components of X . Let $\pi_i: \tilde{X}_i \rightarrow X_i$ be the desingularization of each X_i . Let $\{X^k\}_{k \in K}$ be the set of connected components of X . For each k let α_k be a fixed map of a point into X^k . Again let 1 denote the trivial line bundle.

LEMMA 6.2. *As an abelian group $K'_0(X)$ is freely generated by $\{(\tilde{X}_i, 1, \pi_i)\}_{i \in I} \cup \{(\text{point}, 1, \alpha_k)\}_{k \in K}$.*

PROOF. Use Mayer-Vietoris sequences and compare to $H_*(X)$.

LEMMA 6.3. *An element β of $K'_0(X)$ is determined by $\{p_*([E] \cap \beta): [E] \text{ is an element of } K^0(X) \text{ represented by a bundle } E \text{ of fiber dimension one or zero on } X\}$. Here p_* is the map $K_0(X) \rightarrow K_0(\text{point}) = \mathbb{Z}$ induced by $p: X \rightarrow \text{point}$.*

PROOF. For $\beta = [\sum_{k \in K} a_k(\text{point}, 1, \alpha_k) + \sum_{i \in I} b_i(\tilde{X}_i, 1, \pi_i)]$ and E a vector bundle on X with $d_k(E) = \text{fiber dimension of } E \text{ over } X^k$,

$$p_*([E] \cap \beta) = \sum_{k \in K} a_k \cdot d_k(E) + \sum_{i \in I} b_i(ch(\pi_i^*E) \cup Td(T\tilde{X}_i))[\tilde{X}_i]$$

(see §2). For each $k \in K$ let $E^k = X^k \times \mathbb{C}$. For each pair $(i, \underline{k}) \in I \times K$ such that $X_i \subset X^{\underline{k}}$, define E_{ik} to be the bundle satisfying:

- (a) fiber dimension of E_{ik} is one over X^k and is zero over each X^k with $k \neq \underline{k}$;
- (b) $c_1(\pi_i^*E_{ik}) = 1$;
- (c) $c_1(\pi_i^*E_{ik}) = 0$ for $i \neq \underline{i}$.

Using only $\{E^k\}_{k \in K}$ and $\{E_{ik}\}$ as defined above, one can form enough linear equations to determine all of the a_k and b_i .

Note that for E a vector bundle on X that occurs as the image of a family of projection matrices acting on \mathbb{C}^n (see §2), π^*E occurs as the image of the pullback under π of the family of projection matrices. Note also that each element of $K^0(X)$ described in Lemma 6.3 can be associated with a family of projection matrices that at every point in A_2 have 1 or 0 as upper left entry and zeros elsewhere. Call such a family distinguished. Recall from §2 that $(\tilde{X}, 1, \pi)$ is associated to $(L^2(\tilde{X}), \tilde{\psi}_0, L^2(\tilde{X}, \bar{T}^*), \tilde{\psi}_1, \bar{\partial} \circ (1 - \Delta_{\tilde{X}})^{-1/2})$, where $\tilde{\psi}_i(f) = \text{multiplication by } f \circ \pi$. Switching to the cap product and push-forward in the analytic theory (see §2), we see by Lemma 6.3 that Theorem 6.1 is implied by the following: for each $[E]$ of Lemma 6.3, with $P_0, P_1, \tilde{P}_0, \tilde{P}_1$ formed from a distinguished family of matrices for E and $\psi_0, \psi_1, \tilde{\psi}_0, \tilde{\psi}_1$ respectively,

$$\begin{aligned} & \text{index} \left[P_1 \circ \left((\bar{\partial} \circ (1 - \Delta_U)^{-1/2}) \otimes I \right) \circ P_0 : \right. \\ & \quad \left. P_0(L^2(U) \otimes \mathbb{C}^n) \rightarrow P_1(L^2(U, \bar{T}^*) \otimes \mathbb{C}^n) \right] \\ (6.4) \quad & = \text{index} \left[\tilde{P}_1 \circ \left((\bar{\partial} \circ (1 - \Delta_{\tilde{X}})^{-1/2}) \otimes I \right) \circ \tilde{P}_0 : \right. \\ & \quad \left. \tilde{P}_0(L^2(\tilde{X}) \otimes \mathbb{C}^n) \rightarrow \tilde{P}_1(L^2(\tilde{X}, \bar{T}^*) \otimes \mathbb{C}^n) \right]. \end{aligned}$$

By Proposition 4.4(b) (6.4) is equivalent to

$$(6.5) \quad \text{index} \left[P_1 \circ \left((\bar{\partial} \circ L_e) \otimes I \right) \circ P_0 \right] = \text{index} \left[\tilde{P}_1 \circ \left((\bar{\partial} \circ \tilde{L}_e) \otimes I \right) \circ \tilde{P}_0 \right],$$

where the domains and ranges are as in (6.4).

Henceforth restrict attention to an arbitrary $[E]$ and a distinguished family of matrices for E , and denote by R and \tilde{R} the operators of (6.5). $\text{Ker}(R)$ and $\text{Ker}(\tilde{R})$ denote the kernels of R and \tilde{R} . Each kernel is finite dimensional. P_B denotes restriction to B . Restriction is a projection on a direct summand of any Hilbert space to which P_B is applied. B is implicitly identified with $\pi^{-1}(B)$. Note that as long as we work with Hilbert space elements supported in A_4 or in $\pi^{-1}(A_4)$, R can be viewed as $\bar{\partial} \circ L_e: L^2(U) \rightarrow L^2(U, \bar{T}^*)$ and \tilde{R} as $\bar{\partial} \circ \tilde{L}_e: L^2(\tilde{X}) \rightarrow L^2(\tilde{X}, \bar{T}^*)$.

LEMMA 6.6. $\dim(\text{Ker}(R)) = \dim(P_B(\text{Ker}(R)))$.

PROOF. $P_B: \text{Ker}(R) \rightarrow P_B(\text{Ker}(R))$ is surjective by definition.

Assume $f, g \in \text{Ker}(R)$ and $P_B(f) = P_B(g)$. Then $\text{support}(f - g) \subset U - B$ (support is defined in the paragraph preceding Proposition 4.4).

$$0 = R(f) - R(g) = R(f - g) = (\bar{\partial} \circ L_\epsilon)(f - g) = \bar{\partial}(L_\epsilon(f - g)).$$

Since $\text{support}(L_\epsilon(f - g)) \subset A_3$, $L_\epsilon(f - g)$ is a holomorphic function in $H^1(U)$ with support in $\text{interior}(A_2 \cap U)$. Therefore, $L_\epsilon(f - g) = 0$, and so $f - g = 0$. Thus $P_B: \text{Ker}(R) \rightarrow P_B(\text{Ker}(R))$ is injective.

Analogous arguments on \tilde{X} give

LEMMA 6.7. $\dim(\text{Ker}(\tilde{R})) = \dim(P_B(\text{Ker}(\tilde{R})))$.

LEMMA 6.8. *If $f \in P_B(\text{Ker}(R))$, there exists a unique $g \in L^2(\pi^{-1}(A_4)) \subset L^2(\tilde{X})$ such that $f + g \in \text{Ker}(\tilde{R})$.*

PROOF. *Existence:* $f \in P_B(\text{Ker}(R)) \Rightarrow$ there exists $h \in L^2(U - B)$ such that $0 = R(f + h) = R(f) + R(h)$. $R(h) = -R(f)$. $\text{support}(L_\epsilon(h)) \subset A_3$. Therefore, there exists $g \in L^2(\pi^{-1}(A_4))$ with $\tilde{L}_\epsilon(g) = L_\epsilon(h)$. $\tilde{R}(g) = R(h) = -R(f)$. Proposition 4.4(c) implies that $R(f) = \tilde{R}(f)$. Thus $0 = \tilde{R}(g) + \tilde{R}(f) = \tilde{R}(g + f)$. $g + f \in \text{Ker}(\tilde{R})$.

Uniqueness: Suppose g_1 and g_2 satisfy the above conditions on g . Then $0 = \tilde{R}(g_1) - \tilde{R}(g_2) = \tilde{R}(g_1 - g_2)$. $\tilde{L}_\epsilon(g_1 - g_2)$ is holomorphic with compact support in $\text{interior}(\pi^{-1}(A_2))$. $\tilde{L}_\epsilon(g_1 - g_2) = 0$. $g_1 - g_2 = 0$.

LEMMA 6.9. $\dim(P_B(\text{Ker}(R))) \leq \dim(\text{Ker}(\tilde{R}))$.

PROOF. Lemma 6.8 defines a linear map from $P_B(\text{Ker}(R))$ to $\text{Ker}(\tilde{R})$. We proceed by showing that this map is injective.

Suppose we have f_1 and f_2 in $P_B(\text{Ker}(R))$ whose images under the above map, $f_1 + g_1$ and $f_2 + g_2$, are equal. Then $f_1 = f_2$ on $U - A_4$.

There exist $h_1, h_2 \in L^2(U - B)$ such that $f_1 + h_1$ and $f_2 + h_2$ are in $\text{Ker}(R)$. Since

$$\text{support}(f_1 + h_1 - f_2 - h_2) \subset A_4 \quad \text{and} \quad R(f_1 + h_1 - f_2 - h_2) = 0,$$

$L_\epsilon(f_1 + h_1 - f_2 - h_2)$ is holomorphic with support in $\text{interior}(A_2 \cap U)$. Thus it equals 0 and so does $f_1 + h_1 - f_2 - h_2$. It follows that $f_1 - f_2 = h_2 - h_1$, and by the nature of the supports of these functions, $f_1 = f_2$.

Reversing the roles of R and \tilde{R} in the preceding arguments, one proves

LEMMA 6.10. $\dim(P_B(\text{Ker}(\tilde{R}))) \leq \dim(\text{Ker}(R))$.

Lemmas 6.6, 6.7, 6.9 and 6.10 give

LEMMA 6.11. $\dim(\text{Ker}(R)) = \dim(\text{Ker}(\tilde{R}))$.

LEMMA 6.12. $\text{Image}(R) = \text{Image}(\tilde{R})$.

(Equality refers to identification by restriction from \tilde{X} to U . Recall that the range spaces of R and \tilde{R} are isomorphic as Hilbert spaces under this restriction.)

PROOF. Let P_{U-B} and $P_{\tilde{X}-B}$ denote restriction to $U - B$ and $\tilde{X} - B$ respectively. $\text{domain}(R) = P_B(\text{domain}(R)) \oplus P_{U-B}(\text{domain}(R))$. $\text{domain}(\tilde{R}) = P_B(\text{domain}(\tilde{R})) \oplus P_{U-B}(\text{domain}(\tilde{R}))$. $P_B(\text{domain}(R)) = P_B(\text{domain}(\tilde{R}))$ and R restricted to $P_B(\text{domain}(R))$ equals \tilde{R} restricted to $P_B(\text{domain}(\tilde{R}))$. $w \in R(P_{U-B}(\text{domain}(R))) \Rightarrow$ there exists $w_1 \in H^1(U)$ with support in $A_3 \cap U$ such that $\bar{\partial}w_1 = w \Rightarrow$ there exists $w_2 \in L^2(\pi^{-1}(A_4))$ such that $\tilde{L}_\epsilon(w_2) = w_1 \Rightarrow w \in \text{Image}(\tilde{R})$. An analogous chain of implications shows that $\tilde{R}(P_{U-B}(\text{domain}(\tilde{R}))) \subset \text{Image}(R)$.

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