PRIMITIVE GROUP RINGS AND NOETHERIAN RINGS OF QUOTIENTS

BY

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ABSTRACT. Let k be a field, and let G be a countable nilpotent group with centre Z . We show that the group algebra kG is primitive if and only if k is countable, G is torsion free, and there exists an abelian subgroup A of G , of infinite rank, with $A \cap Z = 1$. Suppose now that G is torsion free. Then kG has a partial quotient ring $Q = kG(kZ)^{-1}$. The above characterisation of the primitivity of kG is intimately connected with the question: When is Q a Noetherian ring? We determine this for those groups G , as above, all of whose finite rank subgroups are finitely generated. In this case, Q is Noetherian if and only if G has no abelian subgroup A of infinite rank with $A \cap Z = 1$.

1. Introduction. This paper has two main themes. Both concern the group algebra kG of a torsion free nilpotent group G . The first—the one we had in mind when we began our work—is the characterisation of those algebras kG which are primitive, for G countable. Let Z denote the centre of G . The second theme is the determination of when the partial quotient ring $kG(kZ)^{-1}$ is Noetherian. (It is always simple.) It was a surprise to us that $kG(kZ)^{-1}$ frequently is Noetherian, and that this phenomenon is closely related to the primitivity of kG .

Let $\mathcal{P}$ denote the class of torsion free locally nilpotent groups whose finite rank subgroups are finitely generated. A subgroup H of a group G is *dense* if some positive power of every element of G lies in H . Countable locally nilpotent groups contain dense $\mathcal{P}$ -subgroups (Lemma 5.1(ii)). Our central result is

THEOREM D. *Let G be a countable hypercentral group with $G \neq 1$. Let H be a dense $\mathcal{P}$ -subgroup of G , let $Z = Z(G)$ and let $Z_0 = Z \cap H$. Let k be a field. Then the following are equivalent:*

- (a) kG is primitive;
- (b) k is countable, G is torsion-free and there exists an abelian subgroup A of G , of infinite rank, with $A \cap Z = 1$;
- (c) $kH(kZ_0)^{-1}$ is not Noetherian, G is torsion-free and k is countable.

Most of the arguments used in proving Theorem D actually apply to a somewhat larger class of groups, namely the class of countable torsion-free hyperabelian locally nilpotent groups. So Theorem D is a refinement of Theorem C, a less precise result concerning the group algebras of groups in this class.

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The sufficiency of the conditions in Theorem D(b) for primitivity is an easy consequence of one of Zalesskii's intersection theorems, and was already observed in [3, Lemma 3.1]. Nor is it at all difficult to see that (a) implies (c): The argument is in two parts. First one observes that if kG is primitive, then so is kH (Lemma 7.4). Secondly, if $kH(kZ_0)^{-1}$ is Noetherian, then any cyclic kZ_0 -torsion free kH -module V is easily seen to be induced from a cyclic module W for a subalgebra kC , where C/Z_0 is polycyclic, and one deduces from a stronger form [2] of P. Hall's generic flatness result [8, Lemma 4.1] that W (and so V) cannot be irreducible.

The deepest part of Theorem D is thus the fact that (c) implies (b). This implication is part of our second main result:

THEOREM B. *Let R be a commutative domain and let G be a torsion-free locally nilpotent hyperabelian $\mathcal{P}$ -group. Let B be a normal subgroup of G with $G/B \in \mathcal{P}$, and let QG denote $RG(RB)^{-1}$. Then the following are equivalent:*

- (a) QG is Noetherian;
- (b) the set of subgroups of G having trivial intersection with B satisfies the ascending chain condition;
- (c) G has no abelian subgroup A of infinite rank with $B \cap A = 1$.

Note that if G is a hypercentral $\mathcal{P}$ -group, then $G/Z(G) \in \mathcal{P}$ (Lemma 5.1(i)). To see that $kG(kB)^{-1}$ exists in the circumstances of Theorem B, note that kB is an Ore domain since it has these properties locally, B being locally-(poly-(infinite cyclic)). Then [14, Lemma 2.6] shows that $kB - \{0\}$ is an Ore set in kG .

The free nilpotent groups G_c of class c ($c \geq 2$) on a countably infinite generating set provide instructive examples for Theorems B and D. Each G_c is a $\mathcal{P}$ -group. Every abelian subgroup having trivial intersection with the centre Z of G_2 is cyclic; so, for any field k , $kG_2(kZ)^{-1}$ is Noetherian and kG is not primitive. However, if $c \geq 3$, G_c contains an abelian subgroup of infinite rank which meets the centre of G_c in $\{1\}$, so $kG_c(kZ)^{-1}$ is not Noetherian, and kG_c is primitive if and only if k is countable. (In his Ph.D. thesis (Leeds University, 1984), J. Pettit also shows that $kG_2(kZ)^{-1}$ is Noetherian.)

The difficult part of Theorem B is the statement that (b) implies (a). The crux of the proof rests in showing that, given (b), if M is a nonzero QG -module, then M contains a nonzero kG -module induced from a subalgebra kD , where $B \triangleleft D$ and D/B is polycyclic. The construction depends on properties of modules over group algebras of finitely generated nilpotent groups, and in particular on a result on the "lifting of induction" (Theorem A) which may have other applications.

The results are arranged as follows. §2 contains frequently used notation and terminology. §3 is a collection of module theoretic lemmas. Theorem A is proved in §4. A few group theoretic results are gathered together in §5. Theorem B is proved in §6, and Theorems C and D in §7.

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2. Notations and conventions. “Module” will always mean “unital right module”. If S is a ring and M is an S -module, $k\text{-dim}_S(M)$, or $k\text{-dim}(M)$, denotes the Krull dimension of M (if it exists) in the sense of Gabriel and Rentschler (see [6]); and $u\text{-dim}_S(M)$ or $u\text{-dim}(M)$ denotes the uniform or Goldie dimension of M [9, Theorem 10.4.4]. The annihilator of M , $\{s \in S: Ms = 0\}$, is denoted $\text{Ann}_S(M)$ or $\text{Ann}(M)$. The S -modules M and N are *similar* if their injective hulls are isomorphic. In other words, M is similar to N if there exist essential submodules M' of M and N' of N with $M' \cong N'$. A nonzero S -module is *critical* if it is finitely generated, has Krull dimension, and all its proper images have Krull dimension strictly less than that of M .

If I is an ideal of the group ring RG , then I^+ will denote the normal subgroup $\{g \in G: g - 1 \in I\}$ of G . Let G be polycyclic-by-finite and R commutative Noetherian. A nonzero RG -module M is called *impervious* if, for all subgroups H of infinite index in G , there is no nonzero submodule of M of the form $U \otimes_{RH} RG$ for an RH -module U .

The Hirsch number of the finitely generated nilpotent group G will be denoted by $h(G)$. If H is a subgroup of a locally nilpotent group G , the *isolator* of H in G , $\{x \in G: x^n \in H \text{ for some } n \geq 1\}$, will be denoted $\text{is}_G(H)$ or $\text{is}(H)$ —it is a subgroup of G . We say that H is *isolated* (in G) if $\text{is}_G(H) = H$, and *dense* in G if $\text{is}_G(H) = G$.

A commutative Noetherian domain R is *absolutely Hilbert* if each prime ideal of R is an intersection of maximal ideals, and all its field images are algebraic extensions of finite fields.

3. Module theoretic lemmas.

LEMMA 3.1. *Let S be a ring and M an S -module which is a sum of uniform similar S -modules W_i ($i \in I$). Suppose M contains no S -module isomorphic to a submodule of a proper quotient of one of the W_i . Let B be a uniform submodule of M . Then B is similar to each of the W_i .*

PROOF. There is no harm in assuming that B is finitely generated, so $M = W_1 + W_2 + \cdots + W_n$, say. We construct an essential submodule $N = N_1 \oplus \cdots \oplus N_t$ of M with each N_i similar to W_i . Put $N_1 = W_1$ and suppose $N_1 \oplus \cdots \oplus N_i$ has been defined and is essential in $V_j = W_1 + \cdots + W_j$, where $1 \leq i \leq j < n$. If $N_1 \oplus \cdots \oplus N_i$ is essential in $V_{j+1} = W_1 + \cdots + W_{j+1}$, no more need be said. Otherwise, there exists a nonzero submodule T of V_{j+1} with $T \cap V_j = 0$, so

$$T = T/T \cap V_j \cong T + V_j/V_j \subseteq V_{j+1}/V_j \cong W_{j+1}/W_{j+1} \cap V_j.$$

If $W_{j+1} \cap V_j \neq 0$, then T is an S -submodule of M isomorphic to a submodule of a proper quotient of W_{j+1} , contradicting the hypothesis. Thus $W_{j+1} \cap V_j = 0$ and we let N_{i+1} be W_{j+1} . In this way N can be constructed.

Now $N \cap B$ is nonzero and uniform, so one of the projection maps from $N \cap B$ to N_i is a monomorphism. Thus $N \cap B$ can be embedded in N_i ; since N_i is similar to W_1 , the proof is complete.

The next lemma is an attempt to retrieve some aspects of Clifford's theorem (see [5, Theorem 11.1] for instance) for infinite groups and for modules which are not necessarily Artinian.

LEMMA 3.2. *Let R be a ring and H a normal subgroup of the group G . Let W be an RH -submodule of an RG -module and define $L = \{g \in G: Wg \text{ is } RH\text{-similar to } W\}$.*

(i) *L is a subgroup of G .*

(ii) *Suppose that W is a uniform RH -submodule of an RG -module M , and that every nonzero submodule of WRL contains a uniform RH -module. Suppose there are no RH -submodules of M isomorphic to a submodule of a proper quotient of W . Then $WRG = WRL \otimes_{RL} RG$.*

PROOF. (i) If $g \in L$ and $A \subseteq W$, $B \subseteq Wg$ with $\psi: A \rightarrow B$ an RH -isomorphism, then $Ag^{-1} \subseteq Wg^{-1}$ and $Bg^{-1} \subseteq W$ are isomorphic via the RH -map $\psi': ag^{-1} \mapsto \psi(a)g^{-1}$. Suppose that $g, u \in L$, so there are essential RH -submodules $A \subseteq Wg$, B and C in W , and $D \subseteq Wu$, with RH -isomorphisms $\psi: A \rightarrow B$ and $\varphi: C \rightarrow D$. As above, there is an RH -isomorphism φ' between $Cg \subseteq Wg$ and $Dg \subseteq W(ug)$. Now $A \cap Cg$ is essential in Wg . Thus $\varphi'\psi$ defines an RH -isomorphism between the essential submodules $\psi^{-1}(A \cap Cg)$ of W and $\varphi'(A \cap Cg)$ of $W(ug)$.

(ii) Let $\{g_i: i \in I\}$ be a right transversal to L in G . Suppose there exists an RH -module B with

$$0 \neq B \subseteq WRL \cap \sum_{\substack{i \in I \\ g_i \notin L}} WRLg_i.$$

By hypothesis we can choose B finitely generated and uniform, in such a way that (rewriting part of the index set I and with $g_1 \in L$),

$$0 \neq B \subseteq WRL \cap \sum_{i=2}^n WRLg_i,$$

with n as small as possible. Consequently, as RL -modules,

$$\sum_{i=2}^n WRLg_i = \sum_{i=2}^n {}^\oplus WRLg_i.$$

The RH -projections $\pi_i: B \rightarrow WRLg_i$ ($2 \leq i \leq n$) are thus well defined, and $\bigcap_{i=2}^n \ker \pi_i = 0$. The uniformity of B forces $\ker \pi_i = 0$ for some i ; hence, for this i , B is similar to a uniform RH -submodule of $WRLg_i$. Since $B \subseteq WRL$, B is similar to W by Lemma 3.1 and the supposition about sections of W . In view of Lemma 3.1 and the definition of L , no such B can exist. Thus (ii) is proved.

Recall that a nonzero RG -module is *impervious* if it contains no nonzero submodule which is induced from a subgroup of infinite index in G .

LEMMA 3.3. *Let H be a normal subgroup of a polycyclic-by-finite group G with G/H infinite cyclic. Let R be a commutative Noetherian ring and let W be an impervious RG -module. Then W contains a nonzero RG -submodule W_1 such that*

$$u\text{-dim}_{RH}(W_1) < \infty.$$

PROOF. Let $G = \langle H, x \rangle$ and let $U = uRH$ be a critical RH -submodule of W of smallest Krull dimension. Since W is impervious there exist $n \geq 1$ and $u' \in U$ such that $U' := U + Ux + \cdots + Ux^{n-1}$ is a direct sum and

$$(1) \quad 0 \neq u'\alpha_n x^n = \sum_{i=0}^{n-1} u'\alpha_i x^i \in Ux^n \cap U',$$

where $\alpha_0, \dots, \alpha_n \in RH$. It is clearly enough to prove that U' is RH -essential in URG .

To prove this, we argue by induction to show that, for all $m \geq n$ and $t \geq 0$,

$$(2) \quad U' \text{ is essential in } \sum_{i=-t}^m Ux^i.$$

Fix $t \geq 0$, $r > n$ and suppose that (2) has been proved for the chosen t and for $m < r$. Then by (1)

$$(3) \quad 0 \neq u'\alpha_n x^r = \sum_{i=0}^{n-1} u'\alpha_i x^{i+r-n} \in Ux^r \cap \left(\sum_{i=0}^{r-1} Ux^i \right).$$

Put $V_r = \sum_{i=0}^r Ux^i$, and let $0 \neq T \subseteq V_r$ satisfy $T \cap U' = 0$. Then

$$(4) \quad \begin{aligned} T &= T/T \cap V_{r-1} \cong T + V_{r-1}/V_{r-1} \\ &\subseteq V_r/V_{r-1} \cong Ux^r/Ux^r \cap V_{r-1}. \end{aligned}$$

Since Ux^r is RH -critical, $k\text{-dim}_{RH}(T) < k\text{-dim}_{RH}(U)$ by (4) and (3). This contradicts the choice of U , and so proves the induction step for m . Similar arguments deal with induction in t , and so (2) is proved. Thus the result follows.

LEMMA 3.4. *Let σ be an automorphism of a ring R and S a subring of R with $\sigma(S) = S$. Let $T = R[X, X^{-1}; \sigma]$ and $T_0 = S[X, X^{-1}; \sigma]$. Let V be an R -module with $u\text{-dim}_S(V) = n < \infty$. Then $u\text{-dim}_{T_0}(V \otimes_R T) = n$.*

PROOF. The argument used in [13, Proposition 2.4] can easily be adapted to show that if vS is a cyclic uniform S -submodule of V , then $vS[X, X^{-1}; \sigma]$ is a uniform $S[X, X^{-1}; \sigma]$ -module. It follows at once that $u\text{-dim}_S(V) = u\text{-dim}_{T_0}(VT_0)$. But V is an R -module, so $VT_0 = VT$.

LEMMA 3.5. *Let R be a commutative Noetherian ring and let H be an isolated normal subgroup of the finitely generated nilpotent group G . Let W be an RH -module. Then every nonzero RG -submodule V of $W \otimes_{RH} RG$ contains a nonzero submodule induced from RH .*

PROOF. Clearly we may assume that W (and so V) are finitely generated. Thus there exist a finitely generated subring S of R , an SH -submodule W_1 of W , and an SG -submodule V_1 of V such that $V = V_1 \otimes_{SG} RG$ and $W = W_1 \otimes_{SH} RH$, with $V_1 \subseteq W_1 \otimes_{SH} SG$. Note that S is absolutely Hilbert. Also, any conclusions about the existence of an induced module pass immediately from SG to RG . So we may assume that R is absolutely Hilbert.

Let $Y = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_t$ be an essential submodule of W , with each Y_i critical. Let V be a nonzero RG -submodule of $W \otimes RG$. Then $V \cap YRG \neq 0$ by

Lemma 3.4, so there is no loss in assuming V is a uniform submodule of $Y \otimes RG$. Then one of the projection maps $\pi_i: V \rightarrow Y_i \otimes RG$ is a monomorphism, so we may replace W by Y_i and assume that W is critical.

Let $k\text{-dim}_{RH}(W) = s$. By $h(G/H)$ applications of [12, Lemma 10], WRG is critical of dimension $r = s + h(G/H)$. Let V be a nonzero submodule of WRG and let V_1 be a critical RH -submodule of V . Since V_1 lies in a finite direct sum of conjugates of W and projects nontrivially into one of the summands, $k\text{-dim}_{RH}(V_1) = s$. Therefore, applying [12, Lemma 10] $h(G/H)$ times, we deduce that

$$V_1 RG = V_1 \otimes_{RH} RG.$$

LEMMA 3.6. *Let R be a commutative Noetherian ring and G a finitely generated nilpotent group with isolated normal subgroup H . Let W_1 and W_2 be finitely generated RH -modules such that no nonzero submodule of any conjugate $W_1 g$ (where $g \in G$) is isomorphic to a submodule of a proper quotient of W_2 . Then no nonzero submodule of $W_1 \otimes_{RH} RG$ is isomorphic to a submodule of a proper quotient of $W_2 \otimes_{RH} RG$.*

PROOF. Since H is isolated in G there is a chain $H = H_1 \subseteq \cdots \subseteq H_r = G$ of normal subgroups of G with each H_{i+1}/H_i infinite cyclic. The proof is by an inductive argument along the chain. As our inductive hypothesis we suppose that no nonzero submodule of $W_1 g \otimes RH_i$, for any $g \in G$, is isomorphic to a submodule of a proper quotient of $W_2 \otimes RH_i$.

Let V be a nonzero RH_{i+1} -submodule of $W_1 g \otimes RH_{i+1}$. By Lemma 3.5, V contains a nonzero submodule induced from RH . But any nonzero RH -submodule of $W_1 g \otimes RH_{i+1}$ has a nonzero submodule isomorphic to $W' g h_{i+1}$, for some $h_{i+1} \in H_{i+1}$ and some RH -submodule W' of W_1 . So V has a nonzero submodule isomorphic to $W' g h_{i+1} \otimes RH_{i+1}$; that is, to $W' g \otimes RH_{i+1}$. If V were isomorphic to a submodule of a proper quotient of $W_2 \otimes RH_{i+1}$, then $W' g \otimes RH_{i+1}$ would be too.

Now let $H_{i+1} = \langle H_i, x \rangle$ and $U = W_2 \otimes RH_i$. We consider a proper quotient of $W_2 \otimes RH_{i+1} = U \otimes RH_{i+1}$. Denote images in this quotient by an overbar. For some $n > 0$ and $u_0, \dots, u_n \in U$,

$$(5) \quad \bar{u}_0 + \cdots + \bar{u}_n x^n = \bar{0}, \quad u_0, u_n \neq 0.$$

Suppose an RH_i -submodule $\bar{U}'$ of $\overline{URH_{i+1}}$ is isomorphic to $W' g h_{i+1} \otimes RH_i$. We claim that

$$(6) \quad \bar{U}' \cap \sum_{j=0}^{n-1} \bar{U} x^j \neq 0.$$

To prove (6), suppose

$$\bar{U}' \cap \sum_{j=-t}^s \bar{U} x^j \neq 0,$$

with $s \geq n - 1$ and $t \geq 0$. Then (5) implies that, if $s > n - 1$,

$$\sum_{j=-t}^s \bar{U} x^j / \sum_{j=-t}^{s-1} \bar{U} x^j$$

is isomorphic to a proper quotient of Ux^s . If $\bar{U}' \cap \sum_{j=-t}^{s-1} \bar{U}x^j = 0$, then $\bar{U}'$ is isomorphic to a submodule of a proper quotient of $\bar{U}x^s$. But $\bar{U}' \cong W'gh_{i+1} \otimes RH_i$, and so $W'gh_{i+1}x^{-s} \otimes RH_i$ is isomorphic to a submodule of a proper quotient of $\bar{U}$, and hence of U . This contradicts the inductive hypothesis. Similarly we can deduce that if $t > 0$, then $\bar{U}' \cap \sum_{j=-t+1}^s \bar{U}x^j \neq 0$. This proves (6).

Now $\sum_{j=0}^{n-1} \bar{U}x^j$ is a finitely generated RH_i -module, and is thus of finite uniform dimension. Also, $W'g \otimes RH_{i+1}$ is the direct sum of the RH_i -submodules $W'gx^j \otimes RH_i$ for $-\infty < j < \infty$. Thus, in view of (6), $W'g \otimes RH_{i+1}$ cannot be embedded in $\overline{URH}_{i+1}$, an arbitrary proper quotient of $W_2 \otimes RH_{i+1}$. Hence there is no nonzero submodule of $W_1g \otimes RH_{i+1}$ which can be embedded in a proper quotient of $W_2 \otimes RH_{i+1}$, and the inductive step of the proof is established.

4. A control theorem for certain modules. This section is devoted to the proof of the following result, which is needed for the proof of Theorem B.

THEOREM A. *Let R be a commutative Noetherian domain, and G a group. Let H be a finitely generated nilpotent normal subgroup of G and K an isolated subgroup of H . Let W be an RK -submodule of an RG -module M . Suppose that*

- (i) *W is uniform and impervious;*
- (ii) *there exist no nonzero RH -submodules of M isomorphic to a submodule of a proper quotient of WRH ;*
- (iii) *$WRH = W \otimes_{RK} RH$.*

Put $N = N_G(K)$. Then $WRG = WRN \otimes_{RN} RG$.

PROOF. *Step 1. The induction set-up.* For $i \geq 1$ we define subgroups K_i of H as follows. Put $K_1 = H$ and for $i \geq 2$ set $K_i = \text{is}_{K_{i-1}}(K(K_{i-1})')$. An easy induction shows that, for $i \geq 1$,

$$K_i \subseteq \text{is}_H \left(\underbrace{K[H, \dots, H]}_i \right).$$

Since H is nilpotent,

$$\underbrace{[H, \dots, H]}_s = 1$$

for some s , so that $K_s = \text{is}_H(K) = K$.

We shall also define subgroups L_i of G for $i \geq 0$, satisfying

$$(1; i) \quad K_{i+1} \triangleleft L_i.$$

Put $L_0 = G$, so that $(1; 0)$ is certainly true. The definition of the L_i 's is an inductive one. After each L_i is defined we must establish that $(1; i)$ is true; this constitutes Step 2 of the proof. Assuming $(1; i)$, we can define

$$L_{i+1} = \{ g \in L_i : WRK_{i+1}g \text{ is } RK_{i+1}\text{-similar to } WRK_{i+1} \}.$$

By $(1; i)$, this definition makes sense, and L_{i+1} is a subgroup of L_i by Lemma 3.2(i). We aim to apply part (ii) of that lemma with K_{i+1} in the role of H and L_i in that of G , so we must show that WRK_{i+1} is uniform. Since K is isolated in H , there is a

subnormal series between K and K_{i+1} with infinite cyclic factors. Thus $h(K_{i+1}) - h(K)$ applications of Lemma 3.4 establish that WRK_{i+1} is uniform. Then Lemma 3.2(ii) yields

$$(2; i) \quad WRL_i = WRL_{i+1} \otimes_{RL_{i+1}} RL_i.$$

Because $K_s = K$, $(1; s-1)$ shows that L_{s-1} normalizes K . So $(2; s-2)$ and its predecessors yield the theorem.

It remains to prove $(1; i)$ after L_i has been defined (so that L_{i+1} can be defined).

Step 2. Proof of $(1; i)$. It is clear from the definition of L_i that $K_i \subseteq L_i$. But $K_{i+1} \subseteq K_i$ and so $K_{i+1} \subseteq L_i$. Consider an isolated subgroup D of K_i with $K'_i \subseteq D$ and $h(K_{i+1}/D \cap K_{i+1}) = 1$. Since $K_{i+1} = \text{is}_{K_i}(K(K'_i))$ we have $h(K/D \cap K) = 1$; and since $K'_i \subseteq D$, $D \triangleleft K_i$ and so $D \cap K \triangleleft K$. Since K_{i+1} is isolated in K_i , the argument used in Step 1 to prove WRK_{i+1} uniform also shows that WRK_i is uniform. So for any nonzero submodule W_1 of W , W_1RK_i is essential in WRK_i , and

$$L_i = \{g \in L_{i-1} : W_1RK_i g \text{ is } RK_i\text{-similar to } W_1RK_i\}.$$

Hence, in proving $(1; i)$, we can replace W by any of its nonzero RK -submodules. In particular, by Lemma 3.3, we may assume that

$$(3) \quad u\text{-dim}_{R(D \cap K)}(W) < \infty.$$

There exist $t \geq 0$ and elements $x_0, x_1, \dots, x_t$ of D such that, writing $D_i = \langle K, x_0, \dots, x_i \rangle$ for $0 \leq i \leq t$, $D_0 = K$, $D_i = KD$, $D_i \triangleleft D_{i+1}$ and D_{i+1}/D_i is infinite cyclic for $0 \leq i < t$. Clearly, $D \cap D_i \triangleleft D_{i+1}$. Hence, by hypothesis (iii) and t applications of Lemma 3.4, starting from (3), $u\text{-dim}_{RD}(WR(KD)) < \infty$. Since $E := \text{is}_{K_i}(K_{i+1}D) = \text{is}_{K_i}(KD)$,

$$(4) \quad u\text{-dim}_{RD}(WRE) < \infty.$$

By (iii), WRK_i is a direct sum of conjugates of WRE as an RE -module. Each of these summands has finite uniform dimension as an RD -module since $D \triangleleft K_i$. Thus every finitely generated RE -submodule of WRK_i has finite uniform dimension as an RD -module.

Now let $g \in L_i$. Thus WRK_i and $WRK_i g$ are similar uniform RK_i -modules. Let V be a nonzero RK_i -module isomorphic to a submodule of both of them. Suppose that $K_{i+1} \neq K_{i+1}^g$ and put $E = \text{is}_{K_i}(K_{i+1}K_{i+1}^g)$. There exists an isolated subgroup D of K_i with $K_{i+1}^g \subseteq D \triangleleft E$ and $h(E/D) = 1$. Thus D is of the type discussed in the previous paragraph. Note that $\text{is}_{K_i}(K_{i+1}D) = E$. As shown above, every finitely generated RE -submodule of WRK_i , and hence of V , has finite uniform dimension as an RD -module.

But as an RK_i -module,

$$(5) \quad WRK_i g = (W \otimes_{RK} RK_i)g \cong Wg \otimes_{RK^g} RK_i = (Wg \otimes_{RK^g} RD) \otimes RK_i,$$

where the isomorphism is given by $(w \otimes \alpha)g \mapsto wg \otimes \alpha^g$. Now Wg is a uniform RK^g -module, so the same is true of $Wg \otimes_{RK^g} RD$ by Lemma 3.4. In view of (5), V , being isomorphic to a submodule of $WRK_i g$, contains an RK_i -submodule induced from RD , by Lemma 3.5; hence V contains an RE -module T induced from RD . In particular, $u\text{-dim}_{RD}(T) = \infty$, contradicting the analysis of the previous paragraph. Thus $K_{i+1} = K_{i+1}^g$, $(1; i)$ is proved, and with it the theorem.

5. Group theoretic lemmas. Recall that a locally nilpotent group is a $\mathcal{P}$ -group if it is torsion free and all its finite rank subgroups are finitely generated. A subgroup H of a group G is *dense* in G if $\text{is}_G(H) = G$. The example of a free abelian group mapping onto $\mathbb{Q}^+$ shows that $\mathcal{P}$ is not a quotient-closed class. Nevertheless, we do have

LEMMA 5.1. *Let G be a locally nilpotent group with torsion subgroup T . Suppose that G/T is countable. Let $Z = Z(G)$.*

(i) *If $Z \in \mathcal{P}$, then the terms $Z_r(G)/Z_{r-1}(G)$ ($r \in \mathbb{N}^+$) of the upper central series of G are in $\mathcal{P}$. In particular, if G is hypercentral of length ω and $Z \in \mathcal{P}$, then G and G/B are in $\mathcal{P}$, where $B = Z_r(G)$ for any $r \in \mathbb{N}^+$.*

(ii) *If G is torsion free and in $\mathcal{P}$, then $G/Z \in \mathcal{P}$.*

(iii) *Let B be a normal subgroup of G . Then there exists a dense subgroup H of G with H and HB/B in $\mathcal{P}$.*

PROOF. (i) If $Z \in \mathcal{P}$, the hypercentre of G is torsion free, and so countable. The proof is by induction on r , the case $r = 1$ being our hypothesis. Suppose $r > 1$ and Z_{r-1}/Z_{r-2} is in $\mathcal{P}$ (where $Z_i = Z_i(G)$ for all $i \in \mathbb{N}^+$). Let $Z_{r-1} \subseteq H \subseteq Z_r$, with H/Z_{r-1} of finite rank. Let $x \in H \setminus Z_{r-1}$. There is an element y of G with $[x, y] \in Z_{r-1} \setminus Z_{r-2}$. The map $g \mapsto [g, y]$ induces a homomorphism $\theta: H/Z_{r-1} \rightarrow Z_{r-1}/Z_{r-2}$ having nontrivial image of finite rank. By the induction hypothesis, this image is finitely generated and torsion free. The kernel of θ does not contain $Z_{r-1}x$. But x was arbitrary, so H/Z_{r-1} is residually-(finitely generated and torsion free). Hence there is a group K with $Z_{r-1} \subseteq K \subseteq H$ and with H/K finitely generated and torsion free of the same rank as H/Z_{r-1} . Since H/Z_{r-1} is torsion free, $K = Z_{r-1}$. Hence H is finitely generated.

(ii) Since Z is torsion free, so is G/Z . This follows from the same result for nilpotent groups by an easy local argument. Let H/Z be a finite rank subgroup of G/Z . We claim that there exists a subgroup D of G , with D/Z of finite rank, $H \subseteq D$ and $H \cap Z(D) = Z$. Suppose that such a group D has been found. Then D is nilpotent by [10, Theorem 6.36] applied to the torsion free, locally nilpotent, finite rank group D/Z . By Lemma 5.1(i), $D/Z(D) \in \mathcal{P}$. Since $H/Z \subseteq D/Z(D)$, H/Z is finitely generated. Thus an arbitrary finite rank subgroup of G/Z is finitely generated; that is, $G/Z \in \mathcal{P}$.

To prove the claim, we define D as follows. We shall choose $g_1, \dots, g_n \in G$ such that $D = \langle H, g_1, \dots, g_n \rangle$. Suppose $i \geq 1$ and $g_1, \dots, g_{i-1}$ have been chosen, but $Z \subsetneq H \cap Z(\langle H, g_1, \dots, g_{i-1} \rangle)$. Then choose g_i to be any element of G which does not centralise $H \cap Z(\langle H, g_1, \dots, g_{i-1} \rangle)$. Thus

$$H \cap Z(\langle H, g_1, \dots, g_i \rangle) \subsetneq H \cap Z(\langle H, g_1, \dots, g_{i-1} \rangle),$$

and the factor here is torsion free since factors of subgroups of G by their centres always are, as observed at the start of the proof. Thus the process must stop since H/Z has finite rank. Define D to be the last group in the chain.

(iii) If $G = T$ take $H = \{1\}$. Otherwise, let $\{x_i; i \geq 1\}$ be a transversal to T in G . The countable subgroup $U = \langle x_i; i \geq 1 \rangle$ is dense in G , so we need only find a dense

subgroup H of U with H and $H/H \cap B$ both in $\mathcal{P}$. Since

$$H/H \cap B = H/H \cap (U \cap B) \cong H(U \cap B)/U \cap B,$$

we may assume that $G = U = \langle x_i : i \geq 1 \rangle$. Put $K_0 = 1$ and for $i \geq 1$ put $K_i = \langle x_1, \dots, x_i \rangle$. Thus K_i is nilpotent, so that K_{i-1} is subnormal in K_i . By refining the series $K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots$ we can therefore obtain a series $1 = G_0 \subseteq G_1 \subseteq \dots \subseteq G = \bigcup_i G_i$ with G_i/G_{i-1} cyclic for all $i \geq 1$.

Define dense subgroups H_i of G_i as follows. Set $H_0 = 1$ and suppose that $i \geq 1$ and H_{i-1} has been defined. There are three cases to consider.

Case (i). G_i/G_{i-1} finite. Put $H_i = H_{i-1}$.

Case (ii). G_i/G_{i-1} and $G_i B/G_{i-1} B$ both infinite. Let $G_i/G_{i-1} = \langle G_{i-1} g \rangle$, so $G_i B/G_{i-1} B = \langle G_{i-1} B g \rangle$. A dense subgroup of a finitely generated nilpotent group is of finite index, so H_{i-1} contains a characteristic subgroup L_{i-1} of G_{i-1} with G_{i-1}/L_{i-1} finite. Now g acts on G_{i-1} , and so on G_{i-1}/L_{i-1} . Thus some power, g^r say, of g fixes G_{i-1}/L_{i-1} and in particular leaves H_{i-1} invariant. Define $H_i = \langle H_{i-1}, g^r \rangle$, the split extension of H_{i-1} by $\langle g^r \rangle$, so H_i is dense in G_i , and H_i/H_{i-1} and $H_i B/H_{i-1} B$ are infinite cyclic.

Case (iii). $G_i/G_{i-1} = \langle G_{i-1} g \rangle$ infinite, and $G_i B/G_{i-1} B$ finite. Then $|G_i B : H_{i-1} B| < \infty$, and we can choose $r \geq 1$ in a similar manner to Case (ii) so that $H_i := \langle H_{i-1}, g^r \rangle$ is the split extension of H_{i-1} by $\langle g^r \rangle$, and so that $g^r \in H_{i-1} B$. Thus H_i/H_{i-1} is infinite cyclic, and $BH_i = BH_{i-1}$.

Each factor H_i/H_{i-1} is either trivial or infinite cyclic. Put $H = \bigcup_i H_i$, so H is torsion free and dense in G . A subgroup of H having finite rank must lie in some H_i , and so be finitely generated. Similarly, $BH/B \in \mathcal{P}$. Thus H is the required subgroup of G .

LEMMA 5.2. *Let G be a locally nilpotent group with $\langle g \rangle^G$ polycyclic for all $g \in G$. Let $B \triangleleft G$ be such that G has the maximal condition on subgroups intersecting trivially with B . Then G/B is of finite torsion free rank.*

PROOF. Let A be an abelian subgroup of G maximal such that $A \cap B = 1$. Since A has the maximal condition on subgroups, it is finitely generated. Thus A^G is finitely generated. Let $C = C_G(A^G)$, so G/C has the maximal condition on abelian subgroups, being a group of automorphisms of a polycyclic group [10, proof of Theorem 3.27]. Since G/C is locally nilpotent, it is polycyclic by [10, Theorem 3.31].

Suppose that G/B has infinite torsion free rank. Then BC/B has infinite rank, so there exists $g \in C$ such that BD/B has strictly greater rank than BA/B , where D is the abelian group $\langle A, g \rangle$. Let $B_0 = B \cap D$, so $|D : B_0 \cap A| = \infty$. Hence there is a subgroup E of D with $A \subsetneq E$ and $B_0 \cap E = 1$. This contradicts our choice of A . Therefore G/B has finite rank.

6. Noetherian quotient rings. This section is devoted to the proof of Theorem B. The strategy employed here for proving that certain rings of fractions are Noetherian is embodied in the following lemma.

LEMMA 6.1. *Let R be a ring which is the union of an ascending chain $\{S_\lambda: 0 \neq \lambda < \rho\}$ of right Noetherian subrings S_λ (where ρ is an ordinal). Assume that R is a flat left S_λ -module for all λ . Then R is right Noetherian if and only if, given a nonzero R -module M , there exists $\lambda < \rho$ and a nonzero S_λ -submodule N of M with $NR = N \otimes_{S_\lambda} R$.*

PROOF. If R is right Noetherian and M is a nonzero R -module, then any nonzero cyclic submodule M' of M has the form R/I for a finitely generated right ideal I of R . There exists λ such that $I = (I \cap S_\lambda)R = (I \cap S_\lambda) \otimes_{S_\lambda} R$; so $M' = N \otimes_{S_\lambda} R$, where $N = S_\lambda/I \cap S_\lambda$.

Suppose, on the other hand, that R is not right Noetherian. Choose a right ideal I of R which is maximal with respect to not being finitely generated. (Such an ideal exists by Zorn's Lemma.) By hypothesis, there exist λ and a nonzero cyclic S_λ -submodule N of R/I such that $NR = N \otimes_{S_\lambda} R$. Let $NR = K/I$, so, by choice of I , K is finitely generated. By taking a greater λ if necessary, we may assume that $K = K'R = K' \otimes_{S_\lambda} R$ for some right ideal K' of S_λ . There exists a right ideal I' of S_λ with $I' \subseteq K'$ and $K'/I' = N$. Notice that in fact $I' = I \cap K'$. Since R is a flat left S_λ -module, $K/I = (K'/I')R = K' \otimes R/I' \otimes R$, and so $I = I' \otimes R$ is finitely generated. This is a contradiction, so R is right Noetherian.

Let M be a finitely generated module over the Noetherian ring S . A *critical composition series* (c.c.s.) of M is a series $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ of submodules of M defined as follows. Set $M_0 = 0$, and if M_i has been defined, let M_{i+1} be a maximal member of the set of submodules X of M with $M_i \subset X$ and X/M_i critical of smallest possible Krull dimension. Associated with each c.c.s. of M is a sequence of nonnegative integers $(a_0, a_1, \dots, a_i, \dots)$, where a_i indicates that there are a_i factors in the c.c.s. of M of Krull dimension i . (Note that these i factors all occur together—the sequence of Krull dimensions of the factors of a c.c.s. is nondecreasing.) The sequence $(a_1, a_2, \dots)$ is called the *critical sequence* of M ; it follows from the analogue of the Jordan-Hölder theorem for c.c.s. [7, Corollary 2.8] that the critical sequence is an invariant of M . Critical sequences can be ordered lexicographically; this done, it is easy to see that the critical sequence of any proper factor of M is strictly less than that of M . We shall need these ideas to prove

THEOREM B. *Let R be a commutative domain and let G be a torsion free locally nilpotent hyperabelian $\mathcal{P}$ -group. Let $B \triangleleft G$ with $G/B \in \mathcal{P}$, and let QG denote $RG(RB)^{-1}$. Then the following are equivalent:*

- (a) QG is Noetherian;
- (b) the set of subgroups of G having trivial intersection with B satisfies the ascending chain condition;
- (c) G has no abelian subgroup A of infinite rank with $B \cap A = 1$.

PROOF. Replacing R by its quotient field, we may assume that R is a field.

(a) $\Rightarrow$ (c) If a group A exists as in (c), then RA is not Noetherian. Since QG is a faithfully flat extension of RA , QG is not Noetherian.

(c) $\Rightarrow$ (b) Suppose (c) holds but (b) fails to hold, and choose a subgroup H of G which is not finitely generated and satisfies $H \cap B = 1$. Since $H \in \mathcal{P}$, H is finitely generated nilpotent by (c) and [10, Theorem 6.36]. This is a contradiction.

(b) $\Rightarrow$ (a) We shall use Lemma 6.1.

Step 1. The induction set-up. There is a chain $B = B_0 \subset \cdots \subset B_\alpha = G$ of normal subgroups of G with $B_{\beta+1}/B_\beta$ abelian for all ordinals β with $0 \leq \beta < \alpha$. This chain can be refined to a chain $B = G_0 \subset \cdots \subset G_\rho = G$ such that $G_\lambda \triangleleft G_{\lambda+1}$ and $G_{\lambda+1}/G_\lambda$ is cyclic for $0 \leq \lambda < \rho$. The proof is by induction on the ordinal ρ , so by hypothesis

$$(1) \quad RG_\lambda(RB)^{-1} \text{ is Noetherian for all } \lambda < \rho.$$

If $\rho = \lambda + 1$, then $RG_\rho(RB)^{-1}$ is Noetherian by (1) and, if $|G_{\lambda+1} : G_\lambda| = \infty$, a twisted version of the Hilbert basis theorem [9, Theorem 10.2.6]. So we may assume that ρ is a limit ordinal. Thus, by construction of the chain of G_λ 's, G is the ascending union of those G_λ , $\lambda < \rho$, which are *normal* in G .

Let M be a nonzero QG -module. We shall construct, in Step 3, sequences $1 = H_0 \subseteq H_1 \subseteq \cdots$ and $G = L_0 \supseteq L_1 \supseteq \cdots$ of subgroups of G with H_i finitely generated, $H_i \triangleleft L_i$ for all $i \geq 0$, and $BH_i \subsetneq BH_{i+1}$ unless $L_{i+1} = H_{i+1}$; and elements $m_i \in M$ such that for $i \geq 0$,

$$(2; i) \quad m_i RG = m_i RL_i \otimes RG;$$

$$(3; i) \quad m_i RH_i \text{ is critical and impervious};$$

$$(4; i) \quad m_i RH_i(L_i \cap B) = m_i RH_i \otimes RH_i(L_i \cap B), \text{ and there exists no } RH_i(L_i \cap B)\text{-submodule of } m_i RL_i \text{ isomorphic to a submodule of a proper quotient of } m_i RH_i(L_i \cap B);$$

and

$$(5; i) \quad \text{each uniform } RH_i\text{-submodule of } m_i RL_i \text{ is similar to } m_i RH_i.$$

Step 2. Deduction of the theorem from the construction. Let $H = \bigcup_i H_i$ and $L = \bigcap_i L_i$, so H is a normal subgroup of L . For each i , $H_i \triangleleft L_i$, so $H_i \triangleleft H$, and hence H is a locally nilpotent group with $\langle h \rangle^H$ polycyclic for all $h \in H$. Since $G/B \in \mathcal{P}$, BH/B is polycyclic by Lemma 5.2(ii). Hence there exists $t \geq 0$ such that $BH = BH_t = BH_{t+i}$ for all $i \geq 0$. By construction, $L_{t+1} = H_{t+1}$, so that BL_{t+1}/B is finitely generated. Therefore $R(BL_{t+1})(RB)^{-1}$ is Noetherian and in view of (2; $t+1$) we have achieved the conditions specified by Lemma 6.1. Thus QG is Noetherian.

Step 3. The construction of H_i , L_i and m_i . Put $H_0 = 1$, $L_0 = G$ and let m_0 be an arbitrary nonzero element of M . Let $i \geq 0$ and suppose H_i , L_i and m_i have been chosen satisfying the stated conditions. Now either (a) $m_i RL_i = m_i RH_i \otimes RL_i$, or (b) $m_i RL_i \neq m_i RH_i \otimes RL_i$. In case (a), put $m_{i+1} = m_i$ and $L_{i+1} = H_i = H_{i+1}$, so completing the $(i+1)$ st step in the construction (and in fact all subsequent steps). Suppose (b) holds. Then $m_i RG \neq m_i RH_i \otimes RG$, so there exists some $\lambda = \lambda(i) < \rho$ such that $H_i \subseteq G_\lambda$ and $m_i RG_\lambda \neq m_i RH_i \otimes RG_\lambda$.

Since QG_λ is Noetherian, we can choose $m_{i+1} \in m_i RL_i$ such that the critical sequence of $m_{i+1} QG_\lambda$ is minimal among all $m' QG_\lambda$ with $m' \in m_i RL_i$. Hence there is no m' in $m_i RL_i$ with $m' QG_\lambda$ isomorphic to a nonzero submodule of a proper

quotient of $m_{i+1}RG_\lambda$. From (2; i), $m'RG = m'RL_i \otimes RG$ for any $m' \in m_iRL_i$, and therefore $m'RG_\lambda = m'R(G_\lambda \cap L_i) \otimes RG_\lambda$. Hence,

$$(6) \quad m_{i+1}RG_\lambda = m_{i+1}R(G_\lambda \cap L_i) \otimes RG_\lambda,$$

and

$$(7) \quad \text{no nonzero } R(G_\lambda \cap L_i)\text{-submodule of } m_iRL_i \text{ is isomorphic to a submodule of a proper quotient of } m_{i+1}R(G_\lambda \cap L_i).$$

Since QG_λ is Noetherian, $m_{i+1}QG_\lambda = QG_\lambda/I$ for some finitely generated right ideal I of QG_λ . Let H_{i+1} be a finitely generated subgroup of G such that RH_{i+1} contains a set of generators for I . It is easy to check that $I \cap RG_\lambda = (I \cap RH_{i+1})RG_\lambda$, so that

$$(8) \quad m_{i+1}RG_\lambda = m_{i+1}RH_{i+1} \otimes_{RH_{i+1}} RG_\lambda.$$

In fact, in view of (6), H_{i+1} may be made smaller, if necessary, so that (8) still holds and H_{i+1} is a finitely generated subgroup of $L_i \cap G_\lambda$. Also, choosing a new m_{i+1} inside the original $m_{i+1}RH_{i+1}$, and a yet smaller H_{i+1} if necessary, we may suppose that $m_{i+1}RH_{i+1}$ is impervious and critical. This is (3; $i+1$). Because we are considering a submodule of the original $m_{i+1}R(L_i \cap G_\lambda)$, this new choice of m_{i+1} still satisfies (6) and (7).

We now define L_{i+1} and establish (2; $i+1$), (4; $i+1$) and (5; $i+1$).

Let C be an arbitrary finitely generated subgroup of L_i with $H_{i+1} \subseteq C$. We wish to apply Theorem A with $H_{i+1} = K$, $C \cap G_\lambda = H$ and $C = G$, and with $m_{i+1}RH_{i+1} = W$ and $m_{i+1}RC = M$. All the hypotheses other than (ii) are evidently satisfied. We must show that

$$(9) \quad \text{there exists no nonzero } R(C \cap G_\lambda)\text{-submodule of } m_{i+1}RC \\ \text{isomorphic to a submodule of a proper quotient of } m_{i+1}R(C \cap G_\lambda).$$

Suppose (9) is false; let $U \subseteq m_{i+1}RC$ be a nonzero $R(C \cap G_\lambda)$ -module with U isomorphic to a submodule U'/U_1 of $m_{i+1}R(C \cap G_\lambda)/U_1$ with $U_1 \neq 0$. Then the $R(L_i \cap G_\lambda)$ -submodule $UR(L_i \cap G_\lambda)$ of $m_{i+1}R(L_i \cap G_\lambda)$ would be isomorphic to some quotient of $(U'/U_1) \otimes R(L_i \cap G_\lambda)$. But C contains H_{i+1} and so

$$m_{i+1}R(C \cap G_\lambda) = m_{i+1}RH_{i+1} \otimes R(C \cap G_\lambda)$$

by (8). Thus

$$(m_{i+1}R(C \cap G_\lambda)/U_1) \otimes R(L_i \cap G_\lambda)$$

is isomorphic to a proper quotient of $m_{i+1}RH_{i+1} \otimes R(L_i \cap G_\lambda)$. But, by (8), the latter is isomorphic to $m_{i+1}R(L_i \cap G_\lambda)$. Hence $UR(L_i \cap G_\lambda)$ is isomorphic to a submodule of a proper quotient of $m_{i+1}R(L_i \cap G_\lambda)$. This contradicts (7). Hence (9) holds.

Applying Theorem A, we deduce that $m_{i+1}RC = m_{i+1}RE \otimes_{RE} RC$, where $E = N_C(H_{i+1})$. Since C was arbitrary in L_i it follows that

$$(10) \quad m_{i+1}RL_i = m_{i+1}RY_{i+1} \otimes RL_i,$$

where $Y_{i+1} = N_{L_i}(H_{i+1})$.

Now set $L_{i+1} = \{g \in Y_{i+1} : m_{i+1}RH_{i+1} \text{ is similar to } m_{i+1}RH_{i+1}g\}$. By Lemma 3.2(i), L_{i+1} is a group. Clearly, $H_{i+1} \triangleleft L_{i+1} \subseteq L_i$. By (7), no $R(L_i \cap G_\lambda)$ -submodule of $m_{i+1}RY_{i+1}$ is isomorphic to a submodule of a proper quotient of $m_{i+1}R(L_i \cap G_\lambda)$. Just as was argued in proving (9), one may deduce from this, using (8), that

$$(11) \quad \text{no nonzero } RH_{i+1}\text{-submodule of } m_{i+1}RY_{i+1} \text{ is isomorphic to a submodule of a proper quotient of } m_{i+1}RH_{i+1}.$$

From (11) and Lemma 3.1 follows (6; $i+1$).

Now apply Lemma 3.2(ii) with $L_{i+1} = L$, $H_{i+1} = H$ and $Y_{i+1} = G$, and with $m_{i+1}RH_{i+1} = W$. In view of (11), we obtain

$$(12) \quad m_{i+1}RY_{i+1} = m_{i+1}RL_{i+1} \otimes RY_{i+1}.$$

Recalling that $m_{i+1} \in m_iRL_i$, we now deduce (2; $i+1$) from (12), (10) and (2; i).

It remains to prove (4; $i+1$) and that $H_i \subseteq H_{i+1}$ and $H_{i+1} \not\subseteq BH_i$.

First we prove (4; $i+1$). Since $H_{i+1}(L_{i+1} \cap B) \subseteq G_\lambda$, (8) implies that

$$m_{i+1}RH_{i+1}(L_{i+1} \cap B) = m_{i+1}RH_{i+1} \otimes RH_{i+1}(L_{i+1} \cap B).$$

The by now familiar argument now leads from (7) and (8) to the remainder of (4; $i+1$). Hence (4; $i+1$) is established.

Suppose next that $H_i \not\subseteq H_{i+1}$. Then $H_i \cap H_{i+1}$ is an isolated proper subgroup of H_i . So there is a normal isolated subgroup H of H_i satisfying $H_i \cap H_{i+1} \subseteq H \triangleleft H_i$. Since G_λ contains H_i , (8) shows that $m_{i+1}RH_i = m_{i+1}RH \otimes RH_i$. Let W be a uniform RH -submodule of $m_{i+1}RH$. Now (5; i) says that there is a nonzero submodule W_1 of $W \otimes RH_i$ isomorphic to a submodule of m_iRH_i , and hence impervious. On the other hand, by Lemma 3.5 applied to $W \otimes RH_i$, W_1 contains a nonzero submodule induced from RH . This is a contradiction, and so $H_i \subseteq H_{i+1}$.

We prove next that

$$(13) \quad H_i \subset H_{i+1}.$$

We consider m_iRG , which as an RG_λ -module is a quotient of a sum of conjugates of m_iRG_λ . Suppose (13) is false, so that $H_i = H_{i+1}$. Then, by (8),

$$m_{i+1}RG_\lambda = m_{i+1}RH_i \otimes RG_\lambda.$$

Since $m_iRG \neq m_iRH_i \otimes RG_\lambda$, there is some finitely generated subgroup D of L_i , containing H_i , for which $m_{i+1}RD \subseteq m_iRD$ and

$$(14) \quad m_iR(D \cap G_\lambda) \not\cong m_iRH_i \otimes R(D \cap G_\lambda).$$

But $m_{i+1}R(D \cap G_\lambda) = m_{i+1}RH_i \otimes R(D \cap G_\lambda)$, and this module is a submodule of $\sum_{d \in D} m_iR(D \cap G_\lambda)d$. It therefore has a nonzero submodule isomorphic to a submodule of a quotient of $m_iR(D \cap G_\lambda)d$ for some $d \in D$. With (14) we can deduce that

$$(15) \quad m_{i+1}R(D \cap G_\lambda) \text{ has a nonzero submodule isomorphic to a proper quotient of } (m_iRH_id) \otimes R(D \cap G_\lambda).$$

Let $W_1 = m_{i+1}RH_i$ and $W_2 = m_iRH_i$. By (5; i) there is a nonzero RH_i -submodule W_1 isomorphic to a submodule of W_2 . So W_1 and W_2 are critical modules of the same Krull dimension. So no nonzero submodule of W_1d_1 ($d_1 \in D \cap G_\lambda$) can be

isomorphic to a submodule of a proper quotient of W_2d ($d \in D$). It follows from Lemma 3.6 that no nonzero submodule of $W_1 \otimes R(D \cap G_\lambda)$ can be embedded in a proper quotient of $W_2d \otimes R(D \cap G_\lambda)$ for any $d \in D$. This contradicts (15) and thereby (13) is proved. This completes the proof of the theorem.

7. Primitive group rings. For the proof of Theorem C we need a variant of one of Zalesskii's intersection theorems. Let G be a hyperabelian torsion free locally nilpotent group, so G contains a series $1 = G_0 \subset G_1 \subset \cdots \subset G_\lambda \subset \cdots \subseteq \bigcup_\lambda G_\lambda = G$, where each G_λ is normal in G , and $G_{\lambda+1}/G_\lambda$ is abelian for all ordinals λ . Define a characteristic subgroup $E(G)$ by induction on λ as follows: (i) Put $E(G) = G$ if G is abelian; (ii) if ρ is a limit ordinal, put $E(G_\rho) = \bigcup_{\lambda < \rho} E(G_\lambda)$; (iii) put $E(G_{\lambda+1}) = E(G_\lambda)C_{G_{\lambda+1}}(E(G_\lambda))$. It is clear that $E(G)$ is a characteristic hypercentral subgroup of G , and that

$$(1) \quad C_G(E(G)) \subseteq E(G).$$

Let a group H act on a group U ; we write

$$\Delta_H(U) = \{u \in U: |H: C_H(u)| < \infty\}.$$

If x and y are elements of a torsion free nilpotent group, and there exists $n \geq 1$ such that $[x, y^n] = 1$, then $[x, y] = 1$. Hence

$$(2) \quad C_G(E(G)) = \Delta_{E(G)}(G),$$

and so by (1) and (2)

$$\Delta_{E(G)}(G) \subseteq E(G).$$

Put $\mathfrak{Z}(G) = \Delta_{E(G)}(G)$, the Zalesskii subgroup of G . Note that $\mathfrak{Z}(G)$ is the centre of $E(G)$. Applying [9, Theorem 8.4.10] with $H = E(G)$ and $W = \mathfrak{Z}(G)$, we deduce part (i) of

PROPOSITION 7.1. (i) *Let G be a torsion free hyperabelian locally nilpotent group with Zalesskii subgroup $\mathfrak{Z}(G)$. If k is a field and I is a nonzero ideal of kG , then $I \cap k\mathfrak{Z}(G) \neq 0$.*

(ii) *If G is torsion free and hypercentral, then the same conclusion holds as in (i), with $\mathfrak{Z}(G)$ in this case taken to be the centre of G .*

For Proposition 7.1(ii), see [9, Theorem 8.4.9].

The connection between Theorems B and C is afforded by the following special case of the results of [2].

LEMMA 7.2. *Let Z be a torsion free central subgroup of the group H , with H/Z polycyclic. Let k be a field and let V be an irreducible kH -module. Then V is not a faithful kZ -module.*

PROOF. By [2, Theorem 2.1], there exists $0 \neq \alpha \in kZ$ such that V is α -free; that is, there is a free kZ -submodule M of V with V/M α -torsion. However, V is irreducible and so, if $\text{Ann}_{kZ}(V) = 0$, V is kZ -divisible. And, as first observed by P. Hall [8, Lemma 4.1], a nonzero kZ -module cannot be both α -free and divisible.

LEMMA 7.3. *Let G be a torsion free locally nilpotent group with centre Z . Let k be a field and let A be an abelian intersection subgroup for kG (so that $Z \subseteq A$). Suppose that either $\text{rank}(A/Z) \geq |kZ|$, or that $Z = 1$. If k is an algebraic extension of a finite field, assume that A/Z has infinite rank. Then kG is primitive.*

PROOF. By [1, Lemma 2], the hypotheses on k , A and Z allow one to find a maximal ideal I of kA with

$$(3) \quad \{a \in A : a - 1 \in I\} = 1$$

and

$$(4) \quad I \cap kZ = 0.$$

Put $J = \bigcap_{g \in G} I^g$. We shall show that

$$(5) \quad J = 0.$$

Let H be a finitely generated subgroup of A , let $x \in G$ and let $\bar{H}$ be the normal closure of H in $\langle H, x \rangle$. Since G is locally nilpotent, $\bar{H}$ is a finitely generated subgroup of A . Hence $J \cap k\bar{H} = P_1 \cap \cdots \cap P_n$, a finite intersection of prime ideals, each of which lies in $I^{g_i} \cap k\bar{H}$ for some $g_i \in G$. But $(I^{g_i})^+ = 1$ from (3). Hence $P_i^+ = 1$ for each i , so

$$P_i = (P_i \cap kC_{\bar{H}}(x))k\bar{H}$$

for each i , by [9, Corollary 11.4.6]. Therefore

$$(6) \quad J \cap k\bar{H} = (J \cap kC_{\bar{H}}(x))k\bar{H}.$$

Now allow H to range over all the finitely generated subgroups of A . From the resulting equalities (6), we deduce that $J = (J \cap kC_A(x))kA$. And these equalities, as x ranges over G , yield

$$(7) \quad J = (J \cap kZ)kA.$$

Now (5) follows from (4) and (7).

Let X be a maximal right ideal of kG containing IG , and let $V = kG/X$ have annihilator P . Then $P \cap kA \subseteq I$ and $P \cap kA$ is G -invariant, so $P \cap kA = 0$ by (5). Since A is an intersection subgroup, $P = 0$; that is, V is faithful.

THEOREM C. *Let G be a countable torsion free hyperabelian locally nilpotent group and let k be a field. Let $Z = Z(G)$.*

(i) *Suppose k is countable. Let H be a dense $\mathcal{P}$ -subgroup of G . Let $Z(H) = Z_0$ (so $Z_0 = Z \cap H$). Consider the statements*

- (a) *kG is primitive;*
- (b) *there exists an abelian subgroup A of G , of infinite rank, with $Z \cap A = 1$;*
- (c) *$kH(kZ_0)^{-1}$ is not Noetherian.*

Then (c) $\Leftrightarrow$ (b) $\Rightarrow$ (a). If G and G/Z are in $\mathcal{P}$, then (a) $\Rightarrow$ (c).

(ii) *Suppose k is uncountable. Then kG is primitive if and only if $Z = 1$.*

PROOF. Note that H exists, by Lemma 5.1.

(i) (c) $\Leftrightarrow$ (b) Theorem B and Lemma 5.1(ii).

(b) $\Rightarrow$ (a) Let $\mathfrak{Z}(G)$ be the Zalesskii subgroup constructed before Proposition 7.1, so $Z \subseteq \mathfrak{Z}(G)$. If $\mathfrak{Z}(G)/Z$ has infinite rank, then kG is primitive by Lemma 7.3 with $A = \mathfrak{Z}(G)$.

Suppose that $W := \mathfrak{Z}(G)/Z$ has finite rank. By [10, Lemma 6.37] applied to the (locally nilpotent) group $\mathfrak{Z}(G)A/Z$, $A/C_A(W)$ has finite rank. We can thus assume that A centralises W , so that

$$(8) \quad \mathfrak{Z}(G)A \text{ is nilpotent.}$$

Put

$$Z_1 = \{x \in \mathfrak{Z}(G) : [x, A] = 1\} = \Delta(\mathfrak{Z}(G)A) \cap \mathfrak{Z}(G).$$

Replacing A by a proper subgroup of infinite rank if necessary, we can assume that $Z_1 \cap A = [Z_1, A] = 1$. As in [1, Lemma 2], construct a maximal ideal X of $k(Z_1A)$ with $X \cap kZ_1 = 0$ and $X^+ = 1$. Let Y be a maximal right ideal of kG with $XkG \subseteq Y$, and set $V = kG/Y$. Let $P = \text{Ann}(V)$. Suppose that $P \neq 0$. Then $J := P \cap k\mathfrak{Z}(G) \neq 0$. If $J^+ \neq 1$, then $B := (Jk(\mathfrak{Z}(G)A))^+ \neq 1$ and so $B \subseteq \mathfrak{Z}(G)$ and $B \cap Z(\mathfrak{Z}(G)A) \neq 1$ by (8). Thus

$$(9) \quad J^+ \cap Z_1 \neq 1.$$

But

$$(10) \quad J \cap kZ_1 \subseteq X \cap kZ_1 = 0,$$

so (9) is impossible. Therefore $J^+ = 1$. The argument used in the proof of Lemma 7.3 can now be imitated to show that $J = (J \cap kZ_1)k\mathfrak{Z}(G)$. By (10), $J = 0$. This is a contradiction, so $P = 0$ and V is a faithful irreducible kG -module.

Suppose now that G (and so G/Z) are in $\mathcal{P}$. By the equivalence of (b) and (c), to prove that (a) implies (c) it is enough to show that

$$(11) \quad \text{if } kG \text{ is primitive, then } kG(kZ)^{-1} \text{ is not Noetherian.}$$

Suppose that kG is primitive with a faithful irreducible module $V = kG/Y$. Write QG for $kG(kZ)^{-1}$ and assume that QG is Noetherian. Then YQG is a finitely generated right ideal, so there exists a subgroup D of G with $Z \subseteq D$ and D/Z finitely generated such that

$$(12) \quad YQG = (Y \cap kD)QG.$$

We claim that

$$(13) \quad Y = (Y \cap kD)kG.$$

Let $y \in Y$. By (12) there exists $0 \neq \alpha \in kZ$ such that $Y = \sum_i y_i g_i \alpha^{-1}$, where $y_i \in Y \cap kD$ and $\{g_i\}$ is part of a right transversal to D in G . Write $y = \sum_i \beta_i g_i$, with $\beta_i \in kD$. Therefore $\sum_i \beta_i \alpha g_i = \sum_i y_i g_i$, and so $\beta_i \alpha = y_i \in Y \cap kD$ for all i . Since $Y \cap kZ = 0$, kZ acts torsion freely on V , and so $\beta_i \in Y$ for all i . Thus $y \in (Y \cap kD)kG$, and (13) is proved.

Put $M = kD/Y \cap kD$, so $V = M \otimes_{kD} kG$ by (13), and M is irreducible. By Lemma 7.2, $0 \neq I = \text{Ann}_{kZ}(M)$. But $VI = 0$, and so V is not faithful. This is a contradiction, so (11) is proved.

(ii) If k is uncountable and kG is primitive, $Z = 1$ by [9, Theorem 9.1.6]. Suppose now that k is uncountable and $Z = 1$. Then G contains an abelian subgroup of infinite rank by [10, Theorem 6.36]. Let F be the prime subfield of k . Now FG is primitive by (b) $\Rightarrow$ (a) of part (i). Since $\Delta(G) = Z = 1$, kG is primitive by [9, Theorem 9.1.5].

Notes. (i) In fact, (a) $\Rightarrow$ (b) of Theorem C(i) remains true even if $G \notin \mathcal{P}$. For, a result of Brookes and Heineken (to appear) states that if a group G satisfying the hypotheses of Theorem C has no subgroups A as in (b), then G is nilpotent; so one can prove (a) $\Rightarrow$ (b) by appealing to Theorem D.

(ii) The only reason for requiring G to be countable in Theorem C(i) is to be sure of the existence of a dense $\mathcal{P}$ -subgroup H . We do not know whether such an H always exists.

LEMMA 7.4. *Let H be a dense subgroup of the torsion free locally nilpotent group G . Let k be a field.*

(i) *If kH is primitive, then kG is primitive.*

(ii) *If G is hypercentral and if kG is primitive, then kH is primitive.*

PROOF. (i) Let X be a maximal right ideal of kH containing no nonzero ideal. Let $XkG \subseteq Y$, Y a maximal right ideal of kG , and let P be the annihilator of kG/Y . The ring $kG(kH)^{-1}$ is locally Artinian and is therefore the quotient division ring of kG . Thus if $P \neq 0$, $P \cap kH$ is a nonzero ideal of kH contained in $Y \cap kH = X$. Since this is impossible, $P = 0$.

(ii) Let Z be the centre of H , so $Z = Z(G) \cap H$ since G is torsion free. Let kG/Y be a faithful irreducible kG -module. Let $0 \neq f \in kZ$, so there exists $\gamma \in kG$ with

$$(14) \quad f\gamma \equiv \gamma f \equiv 1 \pmod{Y}.$$

The argument used in the proof of [2, Theorem C] shows that γ may be chosen in kH . Let X be a maximal right ideal of kH containing $Y \cap kH$. By (14) (with $\gamma \in kH$), $f \notin X$. Hence, $X \cap kZ = 0$. By Proposition 7.1(ii), kH/X is a faithful irreducible kH -module.

THEOREM D. *Let G be a countable hypercentral group with $G \neq 1$ and let k be a field. Let H be a dense $\mathcal{P}$ -subgroup of G , let $Z = Z(G)$ and let $Z_0 = Z \cap H$. Then the following are equivalent:*

(a) *kG is primitive;*

(b) *k is countable, G is torsion free and there is an abelian subgroup A of G , of infinite rank, with $A \cap Z = 1$;*

(c) *$kH(kZ_0)^{-1}$ is not Noetherian, G is torsion free and k is countable.*

PROOF. (b) $\Rightarrow$ (a) This follows from Proposition 7.1(ii) and [3, Lemma 3.1].

(a) $\Rightarrow$ (c) If kG is primitive, then it is prime, and so G has no nonidentity finite normal subgroups. Thus G is torsion free. By [9, Theorem 9.1.6], k is countable. By Lemma 7.4(ii), kH is primitive. Since $Z(H) = Z_0$, $H/Z_0 \in \mathcal{P}$ by Lemma 5.1(i); and (a) $\Rightarrow$ (c) of Theorem C(i) shows that $kH(kZ_0)^{-1}$ is not Noetherian.

(c) $\Rightarrow$ (b) This follows from Theorem C(i).

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