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VARIETIES OF AUTOMORPHISM GROUPS OF ORDERS

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ABSTRACT. The group $A(\Omega)$ of automorphisms of a totally ordered set Ω must generate either the variety of all groups or the solvable variety of class n. In the former case, $A(\Omega)$ contains a free group of rank $2^{\aleph_0}$; in the latter case, $A(\Omega)$ contains a free solvable group of class $n-1$ and rank $2^{\aleph_0}$.

1. Introduction. This work was prompted by the following question of A. Ehrenfeucht (see the footnote on p. 47 of [**8**]); I thank J. Mycielski for calling it to my attention:

> Must the group of automorphisms of an ordered set have a free subgroup of rank $2^{\aleph_0}$ whenever it has a free subgroup of rank 2?

The answer is *yes*. In fact, if the group $A(\Omega)$ of automorphisms of the (totally) ordered set Ω fails to satisfy every nontrivial equational group law, then $A(\Omega)$ must contain a free subgroup of rank $2^{\aleph_0}$ (Theorem 5.6).

Phrased in the last way, the question suggests an analogous question for groups $A(\Omega)$ which do satisfy some equational laws. In the first place, one should ask what varieties (equationally defined classes) can be generated by the groups $A(\Omega)$. We will show that these are just the varieties $\mathfrak{S}_n$ of n-solvable groups, together with the variety $\mathfrak{G}$ of all groups (Theorem 2.10). Then we show that if $A(\Omega)$ generates $\mathfrak{S}_n$, $A(\Omega)$ contains a free-$\mathfrak{S}_{n-1}$ subgroup of rank $2^{\aleph_0}$ (Theorem 5.3). As a by-product of this investigation, we can describe precisely the structure of a group $A(\Omega)$ if $A(\Omega)$ generates $\mathfrak{S}_n$ (Corollaries 4.3 and 2.6), thus generalizing a theorem of Chang and Ehrenfeucht [**1**] from the case $\mathfrak{S}_1 =$ abelian. In the special cases that $A(\Omega)$ is transitive on Ω, we characterize not only the group $A(\Omega)$ but the set Ω if $A(\Omega)$ generates $\mathfrak{S}_n$ (Theorem 3.2 and Corollary 2.6), thus generalizing a theorem of Ohkuma [**10**], again from the case $\mathfrak{S}_1$.

In many ways the most natural context for the study of $A(\Omega)$ is as a lattice ordered group. The lattice operations are defined pointwise: for $f, g \in A(\Omega)$ and $\alpha \in \Omega$, $\alpha(f \vee g) = (\alpha f) \vee (\alpha g)$, and dually. The questions posed above have analogues in the language of lattice ordered groups (l-groups), where the operations $\wedge, \vee$ together with the group operations can be used in equational laws for $A(\Omega)$. In Corollary 2.6 we show that the group $A(\Omega)$ generates $\mathfrak{S}_n$ iff the l-group $A(\Omega)$ generates $\mathcal{A}^n$ (the l-group analogue of $\mathfrak{S}_n$), and in this case $A(\Omega)$ contains a free-$\mathcal{A}^{n-1}$ l-subgroup of rank $2^{\aleph_0}$ (Theorem 5.2). Finally, the group $A(\Omega)$ generates $\mathfrak{G}$ iff the l-group $A(\Omega)$ generates either $\mathcal{N}$ or $\mathcal{L}$, where $\mathcal{N}$ is the join of all $\mathcal{A}^n$

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and $\mathcal{L}$ is the variety of all l-groups (Corollary 2.8). In each case $A(\Omega)$ contains an appropriate free l-subgroup of rank $2^{\aleph_0}$ (Theorems 5.4 and 5.5).

2. The varieties of $A(\Omega)$. For general background in the theory of ordered permutation groups and l-groups, the reader is referred to Glass [**2**]. We deal first with varieties of l-groups. Let $\mathcal{A}$ denote the variety of abelian l-groups, $\mathcal{A}^0$ the variety of one-element l-groups, and, for each positive integer n, $\mathcal{A}^n$ the variety of those l-groups G which have an l-ideal H (convex normal sublattice subgroup) such that $H \in \mathcal{A}^{n-1}$ and $G/H \in \mathcal{A}$ (see Martinez [**6**]). Let $\mathcal{N}$ be the join of all $\mathcal{A}^n$. Then $\mathcal{N}$ is covered by the variety $\mathcal{L}$ of all l-groups [**4, 5**]. We show that the variety of l-groups generated by $A(\Omega)$, l-var $A(\Omega)$, is either $\mathcal{L}$, $\mathcal{N}$, or $\mathcal{A}^n$ for some n.

We make use of the (orbital) wreath product, defined as follows. Let (K, Λ) be an ordered permutation group, that is, K is a subgroup of $A(\Lambda)$. For each $\lambda \in \Lambda$ let H_λ be a group with $H_\lambda = H_\mu$ when λ and μ lie in the same K-orbit. Let $H = \prod_{\lambda\in\Lambda} H_\lambda$. The wreath product $\{H_\lambda\}\,\mathrm{Wr}\,(K,\Lambda)$ is the splitting extension of H by K where conjugation by an element of K permutes the indices of H in the obvious way. If K is a sublattice of $A(\Lambda)$ and each H_λ is an l-group, then G becomes an l-group with the order $e \leq hk$ (for e the identity element, $h \in H$, $k \in K$) iff for each $\lambda \in \Lambda$, $\lambda < \lambda k$ or both $\lambda = \lambda k$ and $h_\lambda \geq e$. If each H_λ is given as an l-subgroup of $A(\theta_\lambda)$ for some totally ordered set θ_λ, the wreath product can be described another way. Let $\Omega = \{(\sigma, \lambda) \in (\bigcup_{\lambda\in\Lambda} \theta_\lambda) \times \Lambda \colon \sigma \in \theta_\lambda\}$, and order Ω lexicographically from the right. Then $\{H_\lambda\}\,\mathrm{Wr}\,(K, \Lambda)$ consists of all $g \in A(\Omega)$ having the form $(\sigma, \lambda)g = (\sigma g_\lambda, \lambda\overline{g})$, where $\overline{g} \in K$ and $g_\lambda \in H_\lambda$ for each λ.

Of special interest are the iterated wreath products of the l-group Z of integers (permuting itself), $\mathrm{Wr}^0 Z = \{e\}$, $\mathrm{Wr}^1 Z = Z$, $\mathrm{Wr}^2 Z = \{Z\}\,\mathrm{Wr}\,(Z, Z)$, and, generally, $\mathrm{Wr}^n Z = \{\mathrm{Wr}^{n-1} Z\}\,\mathrm{Wr}\,(Z, Z)$.

An *orbital* of an ordered permutation group $A(\Omega)$ is the convexification of an orbit $\alpha A(\Omega)$, $\alpha \in \Omega$. Note that if Ω_i is an orbital of $A(\Omega)$, then the restriction of $A(\Omega)$ to Ω_i is $A(\Omega_i)$, and $A(\Omega)$ is the full direct product of its restrictions to orbitals. A *natural congruence* on an orbital Ω_i is an equivalence relation on Ω_i whose classes are convex, which is respected by the action of $A(\Omega_i)$, and which satisfies certain technical conditions (see McClearly [**7**] or Glass [**2**] for details). For our purpose here, the important properties are that the natural congruences on any orbital form a complete tower under containment, and if $\mathcal{C}_i \subsetneq \mathcal{C}^i$ are natural congruences such that $\mathcal{C}^i$ covers $\mathcal{C}_i$ (a *covering pair*) and if C is a $\mathcal{C}^i$ class, then $A(\Omega)$ induces a *primitive component* corresponding to C and $\mathcal{C}_i$ by restriction to C followed by the natural homomorphism arising from the congruence $\mathcal{C}_i$. The primitive component $K = K(C, \mathcal{C}_i)$ is again an ordered permutation group, and either (i) K is *regular*, that is, K is a subgroup of the real numbers R, and the set permuted by K is a union of cosets of K in R, or (ii) K is 0-2-*transitive* on one of its orbits, which means that if $\alpha < \beta$ and $\gamma < \delta$ are points of that orbit, then there exists $k \in K$ such that $\alpha k = \gamma$ and $\beta k = \delta$ (see McCleary [**7**] or Glass [**2**, especially Corollary 4.4.1]). It is also useful to note that if $\mathcal{C}$ is a congruence, then $A(\Omega_i) \approx \{A(C_\lambda)\}\,\mathrm{Wr}\,(G, \Omega_i/\mathcal{C})$, where $\{C_\lambda \colon \lambda \in \Omega_i/\mathcal{C}\}$ are the $\mathcal{C}$ classes and the induced group $G \approx A(\Omega_i/\mathcal{C})$ in case $A(\Omega_i)$ is transitive (though not in general).

We call an element $e < f \in A(\Omega)$ *join irreducible* if $f = a \vee b$ and $a \wedge b = e$ imply $a = e$ or $b = e$.

LEMMA 2.1. *If on one orbital of $A(\Omega)$ there are covering pairs $(\mathcal{C}_i, \mathcal{C}^i)$, $i = 1, 2, \ldots, n$, of natural congruences with $\mathcal{C}^i \subseteq \mathcal{C}_{i+1}$, and there are $\mathcal{C}^i$ classes C^i with $C^i \subseteq C^{i+1}$, and induced primitive components $K_i = K_i(C^i, \mathcal{C}_i)$ all nontrivial, then $A(\Omega)$ contains an l-subgroup isomorphic to $\mathrm{Wr}^n Z$; moreover, we can assume the generator of the upper copy of Z is join irreducible.*

PROOF. We prove by induction that $A(C^i)$ contains a copy of $\mathrm{Wr}^i Z$. Since K_1 is nontrivial, there exists a join irreducible $e < g \in A(C^1)$. Then g generates an infinite cyclic totally ordered subgroup, so $A(C^1)$ contains an l-subgroup isomorphic to $Z = \mathrm{Wr}^1 Z$. Assume $A(C^i)$ contains an l-subgroup H_0 isomorphic to $\mathrm{Wr}^i Z$. We may identify $A(C^i)$ with a subgroup of $A(\Omega)$ fixing each point not in C^i. Let C be the $\mathcal{C}_{i+1}$ class containing C^i. Since K_{i+1} is not trivial, there exists a join irreducible $e < g \in A(C^{i+1})$ such that $Cg \neq C$ (this follows from the technical definition of natural congruence; see [**7** or **2**]). Then the set $\{Cg^j\}$ is pairwise disjoint. Let $H_j = g^{-j} H_0 g^j$. Clearly $A(C^{i+1})$ contains the full direct product $\prod H_j$, and conjugation by g permutes the indices of this product. Thus $A(C^{i+1})$ contains an l-subgroup isomorphic to $H_0 \mathrm{Wr}\, Z \approx \mathrm{Wr}^{i+1} Z$.

If $f \in A(\Omega)$, $\operatorname{supp} f = \{\alpha \in \Omega \colon \alpha f \neq \alpha\}$ and, for any subgroup $G \subseteq A(\Omega)$, $\operatorname{supp} G = \{\alpha \in \Omega \colon \exists f \in G,\ \alpha f \neq \alpha\}$.

LEMMA 2.2. *If some primitive component of $A(\Omega)$ is not regular then $A(\Omega)$ has l-subgroups $G_n \approx \mathrm{Wr}^n Z$, $n = 1, 2, \ldots$, such that if $n \neq m$, $\operatorname{supp} G_n \cap \operatorname{supp} G_m = \emptyset$.*

PROOF. If K is a nonregular primitive component, K is 0-2-transitive on one of its orbits. Let $\alpha < \beta$ be any two points of this orbit. We construct $G_n \approx \mathrm{Wr}^n Z$ with $\operatorname{supp} G_n \subseteq (\alpha, \beta)$. There exists $k \in K$ such that $e < k$ and $\operatorname{supp} k \subseteq (\alpha, \beta)$. Hence K contains $\langle k \rangle \approx Z$. If $\gamma k \neq \gamma$, the intervals $(\gamma k^i, \gamma k^{i+1})$ are pairwise disjoint, and $(\gamma, \gamma k)$ contains $\operatorname{supp} k'$ for some $e < k' \in K$. Then $\operatorname{supp} k^{-i} k' k^i \subseteq (\gamma k^i, \gamma k^{i+1})$ and K contains the full direct product of the groups $k^{-i} \langle k' \rangle k^i$. Hence K contains an l-subgroup isomorphic to $Z \,\mathrm{Wr}\, Z = \mathrm{Wr}^2 Z$. The construction is completed by induction. We can then choose disjoint intervals $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots$ and find $G_n \approx \mathrm{Wr}^n Z$ with $\operatorname{supp} G_n \subseteq (\alpha_n, \beta_n)$.

We let l-var G denote the variety of l-groups generated by G, and g-var G the variety of groups.

LEMMA 2.3. *l-var $\mathrm{Wr}^n Z = \mathcal{A}^n$ [**4**, Theorem 4.6] and g-var $\mathrm{Wr}^n Z = \mathfrak{S}_n$ (see Neumann [**9**, 22.24 plus 17.6]).*

LEMMA 2.4. *If $A(\Omega)$ has just one orbital and there are only a finite number of natural congruences, then $A(\Omega)$ can be embedded (as an l-group) in the iterated wreath product of its primitive components (see Glass [**2**]).*

We also note the following connection between wreath products and varieties. If $H \in \mathcal{A}$ and each $G_\lambda \in \mathcal{A}^i$ then $\{G_\lambda\}\, \mathrm{Wr}\, (H, \Lambda) \in \mathcal{A}^{i+1}$.

THEOREM 2.5. *If $A(\Omega)$ has just one orbital, the following are equivalent:*

(i) *There exist at least n nontrivial primitive components or some primitive component is not regular.*

(ii) $\mathrm{Wr}^n Z \subseteq A(\Omega)$.

(iii) $\mathcal{A}^n \subseteq l\text{-var}\, A(\Omega)$.

(iv) $\mathfrak{S}_n \subseteq g\text{-var}\, A(\Omega)$.

PROOF. That (i) implies (ii) follows from Lemmas 2.1 and 2.2, while (ii) implies both (iii) and (iv) by Lemma 2.3. If there are no more than $m < n$ covering pairs of natural congruences and all primitive components are regular, then Lemma 2.4 implies $A(\Omega)$ can be embedded in the iterated wreath product of m regular, hence abelian, l-groups, and so $A(\Omega) \in \mathcal{A}^m \subseteq \mathfrak{S}_m$, contradicting both (iii) and (iv). Hence (iii) implies (i), and (iv) implies (i).

COROLLARY 2.6. *If $A(\Omega)$ has just one orbital then the following are equivalent:*
(i) *There exist exactly n nontrivial primitive components and all are regular.*
(ii) $\mathrm{Wr}^n Z \subseteq A(\Omega)$ *but* $\mathrm{Wr}^{n+1} Z \not\subseteq A(\Omega)$.
(iii) l-var $A(\Omega) = \mathcal{A}^n$.
(iv) g-var $A(\Omega) = \mathfrak{S}_n$.

COROLLARY 2.7. *For any $A(\Omega)$, l-var $A(\Omega) = \mathcal{A}^n$ iff g-var $A(\Omega) = \mathfrak{S}_n$.*

COROLLARY 2.8. *For any $A(\Omega)$, l-var $A(\Omega) = \mathcal{N}$ or $\mathcal{L}$ iff g-var $A(\Omega) = \mathfrak{G}$.*

PROOF. $\mathcal{N}$ is the join of all $\mathcal{A}^n$ and $\mathcal{L}$ covers $\mathcal{N}$. Hence l-var $A(\Omega) = \mathcal{N}$ or $\mathcal{L}$ is equivalent to the statement: for all n there is an orbital Ω_i such that l-var $A(\Omega_i) \supseteq \mathcal{A}^n$. In turn, this is equivalent to g-var $A(\Omega_i) \supseteq \mathfrak{S}_n$; and the corollary follows because $\mathfrak{G}$ is the join of all $\mathfrak{S}_n$ (any free group is a subdirect product of solvable groups).

THEOREM 2.9. *l-var $A(\Omega)$ is either $\mathcal{L}$, $\mathcal{N}$, or $\mathcal{A}^n$ for some n.*

PROOF. Since the named varieties form a complete tower, and since $A(\Omega)$ belongs to a variety $\mathcal{V}$ iff the restriction of $A(\Omega)$ to each of its orbitals belongs to $\mathcal{V}$, it suffices to prove the theorem assuming $A(\Omega)$ has just one orbital. If some primitive component is not regular, then l-var $A(\Omega) = \mathcal{L}$ [**5**]. If all primitive components are regular then $A(\Omega) \in \mathcal{N}$ (Read [**11**]). The theorem now follows from Corollary 2.6 and the fact that $\mathcal{N}$ is the join of all $\mathcal{A}^n$.

THEOREM 2.10. *g-var $A(\Omega)$ is either $\mathfrak{G}$ or $\mathfrak{S}_n$ for some n.*

PROOF. Immediate from Theorem 2.9 and Corollaries 2.7 and 2.8.

COROLLARY 2.11. *$A(\Omega) \in \mathcal{A}^n$ iff $A(\Omega) \in \mathfrak{S}_n$.*

It follows from Theorem 2.5 that if l-var $A(\Omega) = \mathcal{N}$ then, for each n, $A(\Omega)$ contains an l-subgroup isomorphic to $\mathrm{Wr}^n Z$. In §5 we need the following strong version of this fact.

LEMMA 2.12. *If l-var $A(\Omega) = \mathcal{N}$ then $A(\Omega)$ contains l-subgroups G_n, $n = 1, 2, \ldots$, such that $G_n \approx \mathrm{Wr}^n Z$ and $\operatorname{supp} G_i \cap \operatorname{supp} G_j = \emptyset$ if $i \neq j$.*

PROOF. *Case* 1. Suppose, for every orbital Ω_λ, l-var $A(\Omega_\lambda) \neq \mathcal{N}$. Inductively, suppose $\mathrm{Wr}^i Z \approx G_i \subseteq A(\Omega_{\lambda_i})$, $i = 1, 2, \ldots, n$, with $\Omega_{\lambda_i} \neq \Omega_{\lambda_j}$ when $i \neq j$. Then l-var $A(\Omega_{\lambda_i}) = \mathcal{A}^{m_i}$, so there exists $\Omega_{\lambda_{n+1}}$ such that l-var $A(\Omega_{\lambda_{n+1}}) \not\supseteq \mathcal{A}^{m_i}$, $i = 1, 2, \ldots, n$. Hence $\Omega_{\lambda_{n+1}}$ is different from all Ω_{λ_i}, $i \leq n$, and $A(\Omega_{n+1})$ contains an l-subgroup G_{n+1} isomorphic to $\mathrm{Wr}^{n+1} Z$ by Theorem 2.5.

Case 2. Suppose l-var $A(\Omega_\lambda) = \mathcal{N}$ for some orbital Ω_λ.

Subcase 2a. Suppose Ω_λ has no largest proper natural congruence. Inductively, suppose $\mathrm{Wr}^i Z \approx G_i \subseteq A(\Omega_\lambda)$, $i = 1, 2 \dots, n$, with each G_i having a join-irreducible generator g_i for its uppermost copy of Z. Then $\operatorname{supp} g_i$ is a congruence class B_i (because $A(\Omega_i) \in \mathcal{N}$; see [**11**]). There exists a join irreducible g'_{n+1} which moves B_n. Let $\operatorname{supp} g'_{n+1} = B'_{n+1}$. Then the conjugates of G_n by powers of g'_{n+1} have disjoint support, so $A(\Omega_\lambda)$ contains their full direct product which, together with g'_{n+1}, generates an l-subgroup G'_{n+1} isomorphic to $\mathrm{Wr}^{n+1} Z$. There is a proper congruence class $B \supseteq B'_{n+1}$. Let g be any element of $A(\Omega_\lambda)$ which moves B. Let $G_{n+1} = g^{-1} G'_{n+1} g$.

Subcase 2b. Suppose Ω_λ has a largest proper natural congruence $\mathcal{C}$. Then $A(\Omega_\lambda)$ induces $A(C)$ on each $\mathcal{C}$ class C. It cannot be that $l\text{-var}\, A(C) \subseteq \mathcal{A}^n$ for a fixed n and all C, for then $A(\Omega_\lambda) \in \mathcal{A}^{n+1}$. Hence for each n there exists a $\mathcal{C}$ class C_n such that $\mathrm{Wr}^n Z \subseteq A(C_n)$, and we may assume $C_n \neq C_m$ when $n \neq m$, since each $\mathcal{C}$ class has infinitely many isomorphic translates. The proof can now be completed just as in Case 1.

3. The structure of transitive $A(\Omega) \in \mathcal{A}^n$. Ohkuma [**10**] showed that if $A(\Omega)$ is uniquely transitive (for each $\alpha, \beta \in \Omega$ there exists a unique $g \in A(\Omega)$ such that $\alpha g = \beta$), then $A(\Omega)$ is isomorphic to a subgroup of the additive ordered group of real numbers and $\Omega \approx A(\Omega)$ as ordered sets. Nonzero subgroups of the reals with this property have since been called *Ohkuma groups* and extensively studied ([**3**]; see also Glass [**2**]). Since it is easily seen that a transitive abelian $A(\Omega)$ must be uniquely transitive, Ohkuma's result may be stated as follows: if $A(\Omega)$ is transitive and nontrivial, then $l\text{-var}\, A(\Omega) = \mathcal{A}$ iff Ω is an Ohkuma group, and in this case $A(\Omega) \approx \Omega$. In this section we study the analogous problem when $l\text{-var}\, A(\Omega) = \mathcal{A}^n$.

The heart of the main theorem of this section is contained in the following technical lemma.

LEMMA 3.1. *Let $O_1, O_2, \dots, O_n$ be Ohkuma groups, and $\Omega = O_1 \times O_2 \times \cdots \times O_n$ ordered lexicographically from the right. Then* (i) *Ω is not isomorphic to any proper segment of itself, and* (ii) *no proper initial segment of Ω is isomorphic to any proper final segment of Ω.*

PROOF. We deal first with the case $n = 1$. The result is clearly true if $\Omega = O_1 \approx Z$. Since all other (noncyclic) subgroups of the reals are dense, we assume $\Omega = O_1$ is a dense subgroup of the reals. If $\phi\colon \Omega \to \Omega'$ is an isomorphism (of ordered sets) where Ω' is a proper segment of Ω, let f be any nontrivial member of $A(\Omega)$. Then $\phi^{-1} f \phi$ is a nontrivial member of $A(\Omega')$, which can be extended to $f' \in A(\Omega)$, fixing all points of $\Omega \backslash \Omega'$. This contradicts the unique transitivity of $A(\Omega)$, hence (i) holds.

Since Ω is a dense subset of the real numbers, each initial segment of Ω has the form $(-\infty, \alpha) = \{x \in \Omega\colon x < \alpha\}$, where α is a real number (not necessarily in Ω). Likewise, each final segment has the form (β, ∞). Suppose $\psi\colon (-\infty, \alpha) \approx (\beta, \infty)$. Let g be any negative member of $A(\Omega)$. Then $(-\infty, \alpha) \approx (-\infty, \alpha g)$, and if we let $\beta' = \alpha g \psi$, there is an isomorphism $\phi\colon (\beta, \beta') \approx (\beta, \infty)$, which can be extended to an isomorphism of the *real* intervals $(\beta, \beta') \approx (\beta, \infty)$. Because Ω is dense, there exists a positive $t \in A(\Omega)$ such that $\beta < \beta t < \beta'$. We may also consider t extended to a map from the reals to the reals. Because both t and ϕ preserve order, their extensions are continuous. Since $\beta t < \beta' = \beta \phi$ and $\beta' t < \infty = \beta' \phi$, there exists a real number γ such that $\gamma t = \gamma\phi$ and $\beta < \gamma < \beta'$. Now define $f\colon \Omega \to \Omega$ by

$$xf = \begin{cases} x & \text{if } x < \beta, \\ x\phi & \text{if } \beta \le x \le \beta', \\ xt & \text{if } \beta' < x. \end{cases}$$

Then $f \in A(\Omega)$. But f fixes some points without fixing all, denying unique transitivity of $A(\Omega)$. Hence (ii) is satisifed. This takes care of the case $n = 1$.

Now let $\Omega = O_1 \times O_2 \times \cdots \times O_n$, $n > 1$, and suppose the result is true for $\Lambda = O_1 \times O_2 \times \cdots \times O_{n-1}$. Let $\mathcal{C}_n$ be the equivalence relation on Ω given by $(a_1, a_2, \ldots, a_n)\, \mathcal{C}_1\, (b_1, b_2, \ldots, b_n)$ iff $a_n = b_n$. If $\phi\colon \Omega \to \Omega'$ is an isomorphism of Ω onto a proper segment of itself and ϕ preserves $\mathcal{C}_n$ classes, then ϕ induces an isomorphism of O_n onto a proper segment of itself, contradicting the result for $n = 1$. Hence for some $\mathcal{C}_n$ class $x\,\mathcal{C}_n$, $x\phi\,\mathcal{C}_1 \neq x\,\mathcal{C}_n\,\phi$. Each of these two classes is a convex segment of Ω and is isomorphic to Λ. Further, $x\phi$ belongs to both. Hence either one is contained in the other, or they overlap so that an initial segment of one is a final segment of the other. But each of these possibilities contradicts the inductive assumption about Λ. Hence (i) is satisfied by Ω. A similar argument shows that (ii) is satisfied, completing the proof of the lemma.

THEOREM 3.2. *If $A(\Omega)$ is transitive,*

$$l\text{-var}\, A(\Omega) = \mathcal{A}^n \quad \textit{iff}\ \Omega \approx O_1 \times O_2 \times \cdots \times O_n,$$

a lexicographically ordered product of Ohkuma groups; in this case $A(\Omega) \approx O_1 \operatorname{Wr} O_2 \operatorname{Wr} \cdots \operatorname{Wr} O_n$.

PROOF. If l-var $A(\Omega) = \mathcal{A}^n$, then there are exactly n primitive components O_i and are all regular by Corollary 2.6. But a transitive regular primitive component must be an Ohkuma group (see Glass [**2**]). Hence $A(\Omega)$ can be embedded in the wreath product $O_1 \operatorname{Wr} O_2 \operatorname{Wr} \cdots \operatorname{Wr} O_n$, which acts as an ordered permutation group on the set $O_1 \times O_2 \times \cdots \times O_n$ ordered lexicographically from the right. In the embedding $\Omega \approx O_1 \times O_2 \times \cdots \times O_n$, however, so $A(\Omega) \approx O_1 \operatorname{Wr} O_2 \operatorname{Wr} \cdots \operatorname{Wr} O_n$.

For the converse, suppose that $O_1, O_2, \ldots, O_n$ are given Ohkuma groups and $\Omega = O_1 \times O_2 \times \cdots \times O_n$. We claim that each of the relations $\mathcal{C}_i$, given by $(a_1, a_2, \ldots, a_n)\, \mathcal{C}_i\, (b_1, b_2, \ldots, b_n)$ iff $a_j = b_j$ for all $j \geq i$, is a congruence for $A(\Omega)$. This is the case because each of the $\mathcal{C}_i$ classes is isomorphic to the set $\Omega = O_1 \times O_2 \times \cdots \times O_{i-1}$, to which Lemma 3.1 applies. It follows that the primitive components of $A(\Omega)$ are $A(O_i) \approx O_i$, each of which is nontrivial and regular, so l-var $A(\Omega) = \mathcal{A}^n$ by Corollary 2.6.

The case of transitive $A(\Omega)$ with l-var $A(\Omega) = \mathcal{N}$ appears to be much more complicated. It is still true, just as before, that Ω is embedded in a lexicographically ordered product of Ohkuma groups, but the embedding need not be onto. Worse yet, even the full product of Ohkuma groups need not have the natural congruences. For example, $\cdots \times Z \times Z \times Z$ is isomorphic to the set of irrational real numbers, which has no proper natural congruence.

4. The group structure of arbitrary $A(\Omega) \in \mathcal{A}^n$. A theorem of Chang and Ehrenfeucht [**1**] states that $A(\Omega) \in \mathcal{A}$ iff $A(\Omega)$, as a group, is a full direct product of subgroups of the real numbers R. In order to extend this to higher powers, we define a group G to be of *wreal height* $\leq n$ inductively as follows. If $n = 0$, $G = \{e\}$;

in general, $G \approx \{G_\lambda\}\,\mathrm{Wr}\,(H,\Lambda)$, where H is a subgroup of R acting regularly on Λ, a union of orbits of H in R, $G_\lambda = \prod_k G_{\lambda,k}$, and each $G_{\lambda,k}$ is a group of wreal height $\leq n-1$. Note that G has wreal height ≤ 1 iff G is isomorphic to a subgroup of R. We will show that $A(\Omega) \in \mathcal{A}^n$ iff $A(\Omega)$ if a full direct product of groups of wreal height $\leq n$.

THEOREM 4.1. *If $A(\Omega) \in \mathcal{A}^n$, then $A(\Omega)$ is a full direct product of groups of wreal height $\leq n$.*

PROOF. It suffices to prove that if $A(\Omega)$ has just one orbital and belongs to $\mathcal{A}^n$, then $A(\Omega)$ is a group of wreal height $\leq n$. The theorem is trivially true if $n=0$. In the general case, by Corollary 2.6 and Theorem 2.9, $A(\Omega)$ has exactly $m \leq n$ nontrivial primitive components, and each is regular. If H is the component corresponding to the uppermost covering pair of natural congruences $(\mathcal{C}_i, \mathcal{C}^i)$, then H is a subgroup of R acting on Λ, a union of its orbits. On each of the $\mathcal{C}_i$ classes θ_λ, $A(\Omega)$ induces $A(\theta_\lambda) \in \mathcal{A}^{n-1}$, and so by induction $A(\theta_\lambda)$ is a product of groups of wreal height $\leq n-1$. Since $A(\Omega) \approx \{A(\theta_\lambda)\}\,\mathrm{Wr}\,(H,\Lambda)$, $A(\Omega)$ has wreal height $\leq n$.

THEOREM 4.2. *If G is a product of groups of wreal height $\leq n$, then, for some Ω, $G \approx A(\Omega)$ and $A(\Omega) \in \mathcal{A}^n$.*

PROOF. The theorem is trivial for $n=0$. We proceed by induction. We may assume G has wreal height $\leq n$, for if $G = \prod P_i$ and each P_i has wreal height $\leq n$, and if we know $P_i \approx A(\Omega_i) \in \mathcal{A}^n$, let $\{\pi_i\}$ be ordinals each greater than $\bigvee_i |P_i|$, and all different, and let $\Omega_i' = \pi_i \times \Omega_i$ ordered lexicographically from the right. Let Ω be the disjoint union of all Ω_i' ordered in any way so that each Ω_i' (as previously ordered) is convex. Then $A(\Omega) = \prod A(\Omega_i') \approx \prod A(\Omega_i) = G$. Hence, we now assume G has wreal height $\leq n$ and is a wreath product as in the definition. By induction, each $G_\lambda \approx A(\theta_\lambda) \in \mathcal{A}^{n-1}$. Let $\Omega^* = \{(\sigma,\lambda) \in (\bigcup \theta_\lambda) \times \Lambda \colon \sigma \in \theta_\lambda\}$. Then

$$G = \{g \in A(\Omega^*) \colon (\sigma,\lambda)g = (\sigma g_\lambda, \bar{g}),\ g_\lambda \in A(\theta_\lambda),\ \bar{g} \in H\}.$$

Suppose, in the first case, that H is a proper subgroup of R. We consider the regular action (H,R). Following Chang and Ehrenfeucht, for each orbit (coset) J_i of H in R, we let π_i be an ordinal greater than $\bigvee_\lambda |\theta_\lambda|$ and all π_i different. If we replace each real number r by the ordinal π_i when $r \in J_i$, we get a totally ordered set T and a natural action (H,T) such that $H = A(T)$ (see [**1**]). Moreover, we can assume $\Lambda \subseteq R \subseteq T$. For $\lambda \in \Lambda$ we already have θ_λ defined. For the other $t \in T\backslash\Lambda$, let θ_t be a single point, so $A(\theta_t) = \{e\}$. Then

$$G \approx \{A(\theta_t) \colon t \in T\}\,\mathrm{Wr}\,(A(T),T).$$

Let $\Lambda = \{(\sigma,t) \in (\bigcup \theta_t) \times T \colon \sigma \in \theta_t\}$ be the set acted on by this wreath product. Clearly each $\theta_t \times \{t\}$ is a congruence class for $A(\Omega)$ because if $x < y$ with $x,y \in \theta_t \times \{t\}$, then $|(x,y)| < \pi_i$ for all i, while if $x \in \theta_t \times \{t\}$ and $y \in \theta_{t'} \times \{t'\}$ with $t \neq t'$, then some ordinal π_i lies between x and y. It follows that

$$A(\Omega) \approx \{A(\theta_\lambda)\}\,\mathrm{Wr}\,(A(T),T) \approx G.$$

In the second case, $H = R$. Then H is isomorphic to a proper subgroup H' of R, and as permutation groups (thought not as *ordered* permutation groups), $(H,H) \approx (H',H')$. It follows that $G \approx \{G_h\}\,\mathrm{Wr}\,(H,H) \approx \{G_{h'}\}\,\mathrm{Wr}\,(H',H')$, and we proceed as in the first case.

Finally, observe that it follows from induction that any group of wreal height $\leq n$ belongs to the solvable variety $\mathfrak{S}_n$. Hence, by Corollary 2.11, if $A(\Omega)$ is a product of groups of wreal height $\leq n$, then $A(\Omega) \in \mathcal{A}^n$.

COROLLARY 4.3. *If G is a group, there exists Ω such that $G \approx A(\Omega) \in \mathcal{A}^n$ iff G is a full direct product of groups of wreal height $\leq n$.*

COROLLARY 4.4 (CHANG AND EHRENFEUCHT [**1**]). *If G is a group, there exists Ω such that $G \approx A(\Omega) \in \mathcal{A}$ iff G is a full direct product of subgroups of the real numbers.*

5. Free subgroups of $A(\Omega)$. The following lemma is only a slight modification of a theorem of J. Mycielski applied to the special case of l-groups. We invoke Mycielski's theorem in the proof.

LEMMA 5.1. *Let $\mathcal{V}$ be a variety of groups (or of l-groups). Suppose there are subgroups (or l-subgroups) $G_i \subseteq A(\Omega)$, $i = 1, 2, \ldots$, such that $\operatorname{supp} G_i \cap \operatorname{supp} G_j = \emptyset$ if $i \neq j$, and whenever $(w = e)$ is not a law of $\mathcal{V}$ (where w is a word in the free group (or free l-group) on $x_1, x_2, \ldots$), then there exists $i = i(w)$ such that if $j \geq i$, G_j does not satisfy $(w = e)$. Then $A(\Omega)$ contains $\prod G_i$ which contains a free $\mathcal{V}$ group (or l-group) on $2^{\aleph_0}$ generators.*

PROOF. Clearly $A(\Omega) \supseteq \prod G_i$. We first show $\prod G_i$ has a free subgroup (l-subgroup) on $\aleph_0$ generators. List all those words $w_1, w_2, \ldots$ such that $(w_i = e)$ is *not* a law of $\mathcal{V}$. Inductively, there exists $i(n) > i(n-1)$ such that $G_{i(n)}$ does not satisfy $(w_n = e)$; that is, there are $g(n,1), g(n,2), \ldots \in G_{i(n)}$ such that $w_n(\bar{g}(n,-)) \neq e$. Let $g_i = (g(1,i), g(2,i), \ldots) \in \prod G_n$. Then $\{g_1, g_2, \ldots\}$ is free.

Next, partition $Z^+ = P_1 \dot{\cup} P_2 \dot{\cup} \cdots$ into an infinite number of infinite sets. Then for each j, $\{G_i : i \in P_j\}$ satisfies the hypotheses of the lemma. Hence $G'_j = \prod_{i \in P_j} G_i$ contains a free $\mathcal{V}$ subgroup (or l-subgroup) on a countable set. Moreover, $\operatorname{supp} G'_j \cap \operatorname{supp} G'_k = \emptyset$ when $j \neq k$, so $A(\Omega) \supseteq \prod G'_j$. Mycielski's theorem ([**8**, Corollary 3] and the ensuing discussion) implies that $\prod G'_j$ contains a free $\mathcal{V}$ subgroup (or l-subgroup) on $2^{\aleph_0}$ generators.

THEOREM 5.2. *If $l\text{-var}\, A(\Omega) = \mathcal{A}^n$ then $A(\Omega)$ contains a free $\mathcal{A}^{n-1}$ l-subgroup on $2^{\aleph_0}$ generators.*

PROOF. We may suppose $A(\Omega)$ has just one orbital and a maximal proper natural congruence $\mathcal{C}$ (Corollary 2.6). For some $\mathcal{C}$ class C, $l\text{-var}\, A(C) = \mathcal{A}^{n-1}$ (otherwise $A(\Omega) \in \mathcal{A}^{n-1}$). There are infinitely many disjoint translates of C, so $A(\Omega)$ contains l-subgroups $G_i \approx \operatorname{Wr}^{n-1} Z$ satisfying the hypotheses of Lemma 5.1 for $\mathcal{V} = \mathcal{A}^{n-1}$ (by Lemma 2.3).

THEOREM 5.3. *If $g\text{-var}\, A(\Omega) = \mathfrak{S}_n$, then $A(\Omega)$ contains a free $\mathfrak{S}_{n-1}$ subgroup on $2^{\aleph_0}$ generators.*

PROOF. The hypothesis is equivalent to that of Theorem 5.2 (by Corollary 2.6). In the proof of Theorem 5.2, since $l\text{-var}\, A(C) = \mathcal{A}^{n-1}$, $g\text{-var}\, A(C) = \mathfrak{S}_{n-1}$ by Corollary 2.6, and the hypotheses of Lemma 5.1 are satisfied for $\mathcal{V} = \mathfrak{S}_{n-1}$.

THEOREM 5.4. *If $l\text{-var}\, A(\Omega) = \mathcal{N}$ then $A(\Omega)$ contains a free $\mathcal{N}$ l-subgroup on $2^{\aleph_0}$ generators.*

PROOF. $\mathcal{N}$ is generated by the set of all $\mathrm{Wr}^n Z$ since $\mathcal{N}$ is the join of all $\mathcal{A}^n$ (and using Lemma 2.3). From Lemma 2.12 the hypotheses of Lemma 5.1 are satisfied with $\mathcal{V} = \mathcal{N}$.

THEOREM 5.5. *If l-var $A(\Omega) = \mathcal{L}$ then $A(\Omega)$ contains a free l-subgroup on $2^{\aleph_0}$ generators.*

PROOF. Some primitive component of $A(\Omega)$ is not regular and so is 0-2-transitive on one of its orbits (see remarks preceding Lemma 2.1). Let $\alpha < \beta$ be points of that orbit. Then the restriction of $A(\Omega)$ to the interval (α, β) satisfies no nontrivial l-group law [**5**]. Hence, choosing disjoint intervals, we have l-subgroups $G_i \subseteq A(\Omega)$ satisfying the hypotheses of Lemma 5.1 for $\mathcal{V} = \mathcal{L}$.

THEOREM 5.6. *If g-var $A(\Omega) = \mathfrak{G}$, then $A(\Omega)$ contains a free subgroup on $2^{\aleph_0}$ generators.*

PROOF. We have either l-var $A(\Omega) = \mathcal{N}$ or l-var $A(\Omega) = \mathcal{L}$ by Corollary 2.8. In either case $A(\Omega)$ has subgroups $G_n \approx \mathrm{Wr}^n Z$ satisfying the hypotheses of Lemma 5.1 for $\mathcal{V} = \mathfrak{G}$ (by Lemmas 2.2, 2.3 and 2.12).

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