GENERAL DEFECT RELATIONS OF HOLOMORPHIC CURVES

BY

KIYOSHI NIINO

Dedicated to Professor Yūsaku Komatu on his 70th birthday

ABSTRACT. Let $x: \mathbf{C} \rightarrow P_n\mathbf{C}$ be a holomorphic curve of finite lower order μ , and let $A = \{\alpha\}$ be an arbitrary finite family of holomorphic curves $\alpha: \mathbf{C} \rightarrow (P_n\mathbf{C})^*$ satisfying $T(r, \alpha) = o(T(r, x))$ ($r \rightarrow \infty$). Suppose x is nondegenerate with respect to A , and A is in general position. We show the following general defect relations: (1) x has at most n deficient curves in A if $\mu = 0$. (2) $\sum_{\alpha \in A} \delta(\alpha) \leq n$ if $0 < \mu \leq 1/2$. (3) $\sum_{\alpha \in A} \delta(\alpha) \leq [2n\mu] + 1$ if $1/2 < \mu < +\infty$.

1. Introduction. Let f be a transcendental meromorphic function on the complex plane $\mathbf{C}$ and let a_j ($j = 1, 2, \dots, q$) be distinct meromorphic functions on $\mathbf{C}$ satisfying

$$T(r, a_j) = o(T(r, f)), \quad r \rightarrow \infty.$$

Then the deficiency of a_j with respect to f is defined by

$$\delta(a_j, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, 0, f - a_j)}{T(r, f)},$$

and a_j is called a deficient function of f when $\delta(a_j, f) > 0$.

In 1929 Nevanlinna [10] proved that the defect relation

$$(1.1) \quad \sum_{j=1}^q \delta(a_j, f) \leq 2$$

is valid for $q = 3$ and asked if (1.1) is true for all positive integers q . In 1939 Dufresnoy [4] showed that $\sum_{j=1}^q \delta(p_j, f) \leq 2 + d$ for distinct polynomials p_j of degree $\leq d$. In 1964 Chuang [3] proved (1.1) when $\delta(\infty, f) = 1$. Recently, Yang [17] gave the following:

THEOREM A. Let μ be the lower order of f .

(1) If $\mu = 0$, f has at most one deficient function (cf. Mori [8, Theorem 1]).

(2) If $\mu > 0$ then

$$\sum_{j=1}^q \delta(a_j, f) \begin{cases} \leq 1 & (0 < \mu \leq 1/2, q = 1), \\ < 1 - \cos \pi \mu & (0 < \mu \leq 1/2, q \geq 2), \\ \leq 2 - \sin \pi \mu & (1/2 < \mu \leq 1), \\ \leq \min\{[2\mu] + 1, (\sqrt{2}/2)\pi\mu\} & (\mu > 1). \end{cases}$$

On the other hand, Mori [9] regarded $\delta(a_j, f)$ as a deficiency of the moving divisor and extended the above result of Nevanlinna to the case of holomorphic

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mappings of $\mathbf{C}^n$ into $P_m\mathbf{C}$. Further, under some conditions, Shiffman [12], Mori [9], and Stoll [13] obtained general defect relations for moving divisors with respect to meromorphic functions on $\mathbf{C}^n$, holomorphic mappings of $\mathbf{C}^n$ into $P_m\mathbf{C}$, and holomorphic mappings of M into $P_m\mathbf{C}$, respectively, where M is a parabolic complex manifold.

The purpose of this paper is to extend Theorem A to the case of holomorphic curves. We assume the reader is familiar with Nevanlinna theory of meromorphic functions and holomorphic curves (cf. [10, 15, 16]).

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2. Notations and main results. We denote complex projective n -space by $P_n\mathbf{C}$ and dual complex projective n -space by $(P_n\mathbf{C})^*$. Let $x: \mathbf{C} \rightarrow P_n\mathbf{C}$ be a holomorphic curve and $\tilde{x} = (x_0, x_1, \dots, x_n): \mathbf{C} \rightarrow \mathbf{C}^{n+1} - \{0\}$ its reduced representation. We define the characteristic (order) function $T(r, x)$ of x by

$$(2.1) \quad T(r, x) = \frac{1}{2\pi} \int_0^{2\pi} \log |\tilde{x}(re^{i\theta})| d\theta - \log |\tilde{x}(0)|,$$

where $|\tilde{x}(z)| = (\sum_{j=0}^n |x_j(z)|^2)^{1/2}$.

Let $\alpha: \mathbf{C} \rightarrow (P_n\mathbf{C})^*$ be a holomorphic curve and $\tilde{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_n): \mathbf{C} \rightarrow (\mathbf{C}^{n+1})^* - \{0\}$ its reduced representation. We denote by $N(r, \alpha) \equiv N(r, \alpha; x)$ the counting function of zeros of the entire function

$$F(z) \equiv \langle \tilde{x}(z), \tilde{\alpha}(z) \rangle = \sum_{j=0}^n \alpha_j(z) x_j(z) \neq 0,$$

that is, $N(r, \alpha) = N(r, 0, F)$. We define the deficiency $\delta(\alpha) \equiv \delta(\alpha, x)$ of α with respect to x by

$$(2.2) \quad \delta(\alpha) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, \alpha)}{T(r, \alpha) + T(r, x)}.$$

Since $|F(z)| \leq |\tilde{\alpha}| |\tilde{x}|$, (2.1) and Jensen's formula imply $0 \leq \delta(\alpha) \leq 1$. If $\delta(\alpha) > 0$ then we say that α is a deficient curve. We remark that if α satisfies

$$(2.3) \quad T(r, \alpha) = o(T(r, x)), \quad r \rightarrow \infty,$$

then (2.2) is reduced to

$$(2.4) \quad \delta(\alpha) = 1 - \limsup_{r \rightarrow \infty} \frac{T(r, \alpha)}{T(r, x)}.$$

Now let $\alpha^{(k)}: \mathbf{C} \rightarrow (P_n\mathbf{C})^*$ ($k = 1, 2, \dots, q$; $q > n + 1$) be holomorphic curves and $\tilde{\alpha}^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \dots, \alpha_n^{(k)}): \mathbf{C} \rightarrow (\mathbf{C}^{n+1})^* - \{0\}$ their reduced representation. If $\det\{(\alpha^{(k_i)}(z))_{1 \leq i \leq n+1, 0 \leq j \leq n}\} \neq 0$ for any $\{k_1, k_2, \dots, k_{n+1}\} \subset \{1, 2, \dots, q\}$, then we say that $\{\alpha^{(k)}\}_{k=1}^q$ is in general position. A curve $x: \mathbf{C} \rightarrow P_n\mathbf{C}$ is called nondegenerate with respect to $\{\alpha^{(k)}\}$ if

$$F_k(z) = \sum_{j=0}^n \alpha_j^{(k)}(z) x_j(z) \neq 0 \quad \text{for all } k.$$

Using Yang's idea [17], we prove the following defect relations:

THEOREM 1. *Let $x: \mathbf{C} \rightarrow P_n \mathbf{C}$ be a holomorphic curve of lower order μ and $A = \{\alpha\}$ an arbitrary finite family of holomorphic curves $\alpha: \mathbf{C} \rightarrow (P_n \mathbf{C})^*$ satisfying*

$$(2.3) \quad T(r, \alpha) = o(T(r, x)), \quad r \rightarrow \infty.$$

Suppose x is nondegenerate with respect to A , and A is in general position. Then:

(I) *If $\mu = 0$ there are at most n deficient curves in A .*

(II) *Assume $0 < \mu \leq 1/2$ and put $B = \{\alpha \in A; \delta(\alpha) > 1 - \cos \pi \mu\}$ and $C = \{\alpha \in A; \delta(\alpha) = 1 - \cos \pi \mu\}$. If there are n curves belonging to $B \cup C$, then all the remaining deficiencies are equal to zero. If $\#B = p < n$, then*

$$\sum_{\alpha \in A-B} \delta(\alpha) \leq (n-p)(1 - \cos \pi \mu),$$

where equality holds if and only if $\#C = n-p$ and $\delta(\alpha) = 0$ for all $\alpha \in A - (B \cup C)$.

(III) *If $1/2 < \mu < +\infty$ we have*

$$\sum_{\alpha \in A} \delta(\alpha) \leq [2n\mu] + 1 - \cos(2n\mu - [2n\mu])(\pi/2).$$

As an immediate consequence of (I) and (II) we have

COROLLARY. *If $0 \leq \mu \leq 1/2$, then $\sum_{\alpha \in A} \delta(\alpha) \leq n$.*

3. Spread relation. Let $x: \mathbf{C} \rightarrow P_n \mathbf{C}$ be a holomorphic curve of lower order μ ($0 < \mu < +\infty$) and $\tilde{x} = (x_0, x_1, \dots, x_n): \mathbf{C} \rightarrow \mathbf{C}^{n+1} - \{0\}$ its reduced representation. Suppose that a holomorphic curve $\alpha: \mathbf{C} \rightarrow (P_n \mathbf{C})^*$ and its reduced representation $\tilde{\alpha}: \mathbf{C} \rightarrow (\mathbf{C}^{n+1})^* - \{0\}$ satisfy

$$(3.1) \quad T(r, \alpha) = o(T(r, x)), \quad r \rightarrow \infty,$$

and $\langle \tilde{x}(z), \tilde{\alpha}(z) \rangle \neq 0$. We put

$$\|x(z), \alpha(z)\| = |\langle \tilde{x}(z), \tilde{\alpha}(z) \rangle| / (|\tilde{x}(z)| |\tilde{\alpha}(z)|).$$

A positive, increasing, unbounded sequence $\{r_m\}$ is called a sequence of Pólya peaks of order μ of $T(r, x)$ if it is possible to find positive sequences $\{r'_m\}$, $\{r''_m\}$, and $\{\varepsilon_m\}$ such that, as $m \rightarrow \infty$, then $r'_m \rightarrow \infty$, $r_m/r'_m \rightarrow \infty$, $r''_m/r_m \rightarrow \infty$, $\varepsilon_m \rightarrow 0$, and

$$(3.2) \quad T(t, x)/T(r_m, x) \leq (t/r_m)^\mu (1 + \varepsilon_m) \quad (r'_m < t < r''_m).$$

We now define the spread $\sigma(\alpha) \equiv \sigma(\alpha, x)$ of α with respect to x . Let $\{r_m\}$ be a sequence of Pólya peaks of order μ of $T(r, x)$ and $\Lambda(r)$ a positive function satisfying

$$(3.3) \quad \Lambda(r) = o(T(r, x)), \quad r \rightarrow \infty.$$

Put

$$E_\Lambda(r, \alpha) \equiv E_\Lambda(r, \alpha; x) = \{\theta; \log \|x(re^{i\theta}), \alpha(re^{i\theta})\| < -\Lambda(r)\} \subset (-\pi, \pi]$$

and let

$$\sigma_\Lambda(\alpha) = \liminf_{m \rightarrow \infty} \text{meas } E_\Lambda(r_m, \alpha).$$

Then we define

$$\sigma(\alpha) = \inf_{\Lambda} \sigma_\Lambda(\alpha),$$

where the infimum is taken over all functions $\Lambda(r)$ satisfying (3.3).

We obtain the following spread relation, which is a generalization of Baernstein [1]—Yang [17] and the author [11]—Krytov [7].

THEOREM 2 (SPREAD RELATION). *Let $x: \mathbf{C} \rightarrow P_n \mathbf{C}$ be a holomorphic curve of lower order μ ($0 < \mu < +\infty$). Then*

$$\sigma(\alpha) \geq \min\{2\pi, (4/\mu) \sin^{-1}(\delta(\alpha)/2)^{1/2}\}$$

for every holomorphic curve $\alpha: \mathbf{C} \rightarrow (P_n \mathbf{C})^$ satisfying (3.1) and $\|x(z), \alpha(z)\| \not\equiv 0$.*

PROOF. We may assume that the reduced representations $\tilde{x}$ and $\tilde{\alpha}$ of x and α , respectively, satisfy

$$(3.4) \quad |\tilde{x}(0)| = 1 \quad \text{and} \quad |\tilde{\alpha}(0)| = 1.$$

We have

$$E_\Lambda(r, \alpha) = \{\theta; \log |\tilde{x}(re^{i\theta})| + \log |\tilde{\alpha}(re^{i\theta})| - \log |F(re^{i\theta})| > \Lambda(r)\},$$

where

$$F(z) = \langle \tilde{x}(z), \tilde{\alpha}(z) \rangle = \sum_{j=0}^n \alpha_j(z) x_j(z).$$

Define the entire function $h(z)$ by

$$h(z) = cz^{-k} F(z),$$

where k is a nonnegative integer, c is a nonzero constant, and

$$(3.5) \quad h(0) = 1.$$

Then

$$(3.6) \quad N(r, \alpha) = N(r, 0, F) = N(r, 0, h) + k \log r.$$

We put

$$(3.7) \quad \begin{aligned} G(z) &= \log |\tilde{x}(z)| + \log |\tilde{\alpha}(z)| - \log |h(z)|, \\ \Lambda_1(r) &= \Lambda(r) + k \log r - \log |c|, \end{aligned}$$

and

$$E_{\Lambda_1}(r, G) = \{\theta; G(re^{i\theta}) > \Lambda_1(r)\}.$$

Then we deduce that

$$(3.8) \quad E_{\Lambda_1}(r, G) = E_\Lambda(r, \alpha),$$

and

$$(3.9) \quad \Lambda_1(r) = o(T(r, x)) \quad (r \rightarrow \infty).$$

We next write $T(r) = T(r, x) + T(r, \alpha)$. Then it follows from (3.1) that

$$(3.10) \quad T(r) \sim T(r, x), \quad r \rightarrow \infty,$$

that is, $T(r)/T(r, x) \rightarrow 1$ as $r \rightarrow \infty$. Hence, $T(r)$ and $T(r, x)$ have the same sequence of Pólya peaks, and, combining (3.6) and (3.10) with the definition of deficiency $\delta(\alpha)$, we have

$$(3.11) \quad \delta(\alpha) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, 0, h)}{T(r)}.$$

Therefore, in order to prove Theorem 2, from (3.8)–(3.11) it is sufficient to prove that

$$(3.12) \quad \liminf_{r \rightarrow \infty} \text{meas } E_{\Lambda_1}(r_m, G) \geq \min \left\{ 2\pi, \frac{4}{\mu} \sin^{-1} \left(\frac{\delta(\alpha)}{2} \right)^{1/2} \right\}.$$

Following Baernstein [1] we now define

$$T^*(z) = \sup_E \frac{1}{2\pi} \int_E G(re^{i\omega}) d\omega + N(r, 0, h) \quad (z = re^{i\theta}, \quad 0 < r < +\infty, \quad 0 \leq \theta \leq \pi),$$

where the supremum is taken over all measurable sets $E \subset (-\pi, \pi]$ whose measure equals 2θ . Further, we define

$$(3.13) \quad T^*(0) = 0.$$

Then since $\log |\tilde{x}(z)| + \log |\tilde{\alpha}(z)|$ and $\log |h(z)|$ are subharmonic, from (3.4), (3.5), (3.7), (3.13), and Theorem A' in Baernstein [2], it follows that $T^*(z)$ is subharmonic in $\{z; \text{Im } z > 0\}$ and continuous on $\{z; \text{Im } z \geq 0\}$. By definition of $T^*(z)$,

$$(3.14) \quad T^*(re^{i\pi}) = T^*(-r) = T(r), \quad T^*(r) = N(r, 0, h) \equiv N(r).$$

Since $|h(z)| \leq |c| |z|^{-k} |\tilde{\alpha}(z)| |\tilde{x}(z)|$, then $G(z) + \log |c| - k \log |z| \geq 0$ for all z . Hence, we deduce that

$$(3.15) \quad \begin{aligned} T^*(re^{i\theta}) &\leq T^*(re^{i\pi}) + ((\pi - \theta)/\pi) \{\log |c| - k \log r\} \\ &< T(r) + K \quad (r > 1), \end{aligned}$$

where K is a positive constant such that $\log^+ |c| < K$.

Now we can apply the arguments of Baernstein [1, pp. 430–433] and the author [11, pp. 364–365]. For the sake of clarification we sketch them here.

If $\delta(\alpha) = 0$ there is nothing to prove, so we may assume $\delta(\alpha) > 0$. Put

$$\gamma = (1/2\pi) \min\{2\pi, (4/\mu) \sin^{-1}(\delta(\alpha)/2)^{1/2}\}.$$

Then

$$(3.16) \quad 0 < \gamma \leq 1, \quad 0 < \gamma\mu \leq 1/2 \quad \text{and} \quad 1 - \delta(\alpha) \leq \cos \pi\gamma\mu.$$

Define

$$v(z) = T^*(z^\gamma) \quad (z = re^{i\theta}, \quad 0 < r < \infty, \quad 0 \leq \theta \leq \pi).$$

Then $v(z)$ is subharmonic in $\{z; \text{Im } z > 0\}$ and continuous in $\{z; \text{Im } z \geq 0\}$. Hence, when $re^{i\theta} \in D_R = \{z = re^{i\theta}; \quad 0 < r < R, \quad 0 < \theta < \pi\}$,

$$(3.17) \quad v(re^{i\theta}) \leq \int_{-R}^R v(t) A(t, r, \theta, R) dt + \int_0^\pi v(Re^{i\psi}) B(\psi, r, \theta, R) d\psi,$$

where

$$\begin{aligned} A(t, r, \theta, R) &= \frac{1}{\pi} \frac{r \sin \theta}{t^2 - 2tr \cos \theta + r^2} - \frac{1}{\pi} \frac{R^2 r \sin \theta}{R^4 - 2rR^2 \cos \theta + r^2 t^2}, \\ B(\psi, r, \theta, R) &= \frac{2Rr \sin \theta}{\pi} \frac{(R^2 - r^2) \sin \psi}{|R^2 e^{2i\psi} - 2rR e^{i\psi} \cos \theta + r^2|^2}. \end{aligned}$$

It follows from (3.14) and (3.15) that

$$(3.18) \quad v(t) = T^*(t^\gamma) = N(t^\gamma), \quad v(-t) \leq T(r) + K, \quad v(Re^{i\psi}) \leq T(r) + K.$$

We deduce that

$$(3.19) \quad 0 < A(t, r, \theta, R) \leq \frac{1}{\pi} \frac{r \sin \theta}{t^2 - 2tr \cos \theta + r^2} \equiv P(t, r, \pi - \theta) \quad (0 < r < R, 0 < \theta < \pi),$$

$$(3.20) \quad 0 < B(\psi, r, \theta, R) \leq 32r/\pi R \quad (0 < \theta < \pi, 0 < \psi < \pi, 0 < r < R/2),$$

and

$$(3.21) \quad \int_0^\infty P(t, r, \theta) d\theta = \frac{\theta}{\pi} < 1 \quad (0 < \theta < \pi, r > 0).$$

Using (3.18)–(3.21) in (3.17) we obtain

$$(3.22) \quad \begin{aligned} v(re^{i\theta}) &\leq \int_0^R N(t^\gamma)P(t, r, \pi - \theta) dt + \int_0^R T(t^\gamma)P(t, r, \theta) dt \\ &\quad + 32(r/R)(T(R^\gamma) + K) + K \quad (0 < \theta < \pi, 0 < r < R/2). \end{aligned}$$

Let $\{r_m\}$ be any sequence of Pólya peaks of order μ of $T(r, x)$ and put $s_m = r_m^{1/\gamma}$. $\{r_m\}$ is also a sequence of Pólya peaks of order μ of $T(r)$. Hence, it follows from the discussion of Baernstein [1, pp. 332–333] that

$$(3.23) \quad v(s_m e^{i\theta}) \leq T(r_m) \{\cos(\pi - \theta)\gamma\mu + \alpha_m\} \quad (m = 1, 2, \dots; 0 < \theta < \pi),$$

where $\{\alpha_m\}$ is a sequence tending to zero. Let

$$\sigma_m = \text{meas } E_{\Lambda_1}(r_m, G) \equiv \text{meas } E(r_m).$$

Then (3.12) is equivalent to the inequality

$$(3.24) \quad \liminf_{m \rightarrow \infty} \sigma_m \geq 2\pi\gamma.$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} T(r_m) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log |\tilde{x}(r_m e^{i\omega})| + \log |\tilde{\alpha}(r_m e^{i\omega})|) d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} G(r_m e^{i\omega}) d\omega + N(r_m) \\ &\leq \frac{1}{2\pi} \int_{E(r_m)} G(r_m e^{i\omega}) d\omega + N(r_m) + \Lambda_1(r_m) \\ &\leq T^*(r_m e^{i\sigma_m/2}) + \Lambda_1(r_m). \end{aligned}$$

Dividing by $T(r_m)$ and remembering (3.15), we find that

$$(3.25) \quad \lim_{m \rightarrow \infty} \frac{T^*(r_m e^{i\sigma_m/2})}{T(r_m)} = 1.$$

Let $M = \{m; \sigma_m < 2\pi\gamma\}$. If M is a finite set, then (3.24) holds and we have finished, so we assume that M is infinite. The point $(r_m e^{i\sigma_m/2})^{1/\gamma} = s_m e^{i\sigma_m/2\gamma}$ belongs to the domain of $v(z)$, i.e. the upper half-plane, if and only if $m \in M$, in which case we have

$$T^*(r_m e^{i\sigma_m/2}) = v(s_m e^{i\sigma_m/2\gamma}) \quad (m \in M).$$

Using this in (3.25), comparing with (3.23), and remembering (3.16), we deduce that $\lim_{m \rightarrow \infty, m \in M} \sigma_m/2\gamma = \pi$, which shows that (3.24) holds in this case also.

Thus the proof of Theorem 2 is complete.

4. Lemmas. In order to prove Theorem 1 we need several lemmas.

LEMMA 1. Let $x: \mathbf{C} \rightarrow P_n \mathbf{C}$ be a holomorphic curve, $\alpha^{(k)}: \mathbf{C} \rightarrow (P_n \mathbf{C})^*$ ($k = 1, 2, \dots, n+1$) holomorphic curves in general position, and $\Lambda(r)$ a positive function satisfying (3.2). Suppose x is nondegenerate with respect to $\{\alpha^{(k)}\}_{k=1}^{n+1}$. Then

$$\text{meas} \left\{ \bigcap_{k=1}^{n+1} E_\Lambda(r, \alpha^{(k)}, x) \right\} \leq \frac{2\pi}{\Lambda(r)} \left\{ \sum_{k=1}^{n+1} T(r, \alpha^{(k)}) + K \right\},$$

where K is a constant independent of Λ .

PROOF. Let $\tilde{x} = (x_0, x_1, \dots, x_n): \mathbf{C} \rightarrow \mathbf{C}^{n+1} - \{0\}$ and

$$\tilde{\alpha}^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \dots, \alpha_n^{(k)}): \mathbf{C} \rightarrow (\mathbf{C}^{n+1})^* - \{0\}$$

be reduced representations of x and $\alpha^{(k)}$, respectively. By the linear equations

$$(4.1) \quad \sum_{j=0}^n \alpha_j^{(k)}(z) x_j(z) = F_k(z), \quad k = 1, 2, \dots, n+1,$$

we have $x_j = D_j/D$, where D is the coefficient determinant of (4.1) and

$$D_j = \sum_{k=1}^{n+1} (-1)^{j+k+1} D_{kj} F_k,$$

where D_{kj} is the minor of D obtained by omitting the k th row and $(j+1)$ th column from D . The Hadamard inequality yields

$$|D_{kj}| \leq \prod_{l=1, \neq k}^{n+1} |\tilde{\alpha}^{(l)}|,$$

so

$$|x_j| \leq \frac{1}{|D|} \sum_{k=1}^{n+1} \left(\prod_{l=1, \neq k}^{n+1} |\tilde{\alpha}^{(l)}| \right) |F_k|.$$

Hence,

$$(4.2) \quad \log |\tilde{x}| \leq \log \left\{ \sum_{k=1}^{n+1} \left(\prod_{l=1, \neq k}^{n+1} |\tilde{\alpha}^{(l)}| \right) |F_k| \right\} - \log |D| + \frac{1}{2} \log(n+1).$$

For $z = re^{i\theta}$, with $\theta \in E(r) \equiv \bigcap_{k=1}^{n+1} E_\Lambda(r, \alpha^{(k)})$, we have

$$|F_k(z)| / (|\tilde{x}(z)| |\tilde{\alpha}^{(k)}(z)|) < \exp(-\Lambda(r)), \quad k = 1, 2, \dots, n+1,$$

so

$$\left\{ \sum_{k=1}^{n+1} \left(\prod_{l=1, \neq k}^{n+1} |\tilde{\alpha}^{(l)}| \right) |F_k| \right\} / \left(|\tilde{x}| \prod_{k=1}^{n+1} |\alpha^{(k)}| \right) < (n+1) \exp(-\Lambda(r)).$$

Hence, combining this with (4.2) we obtain

$$(4.3) \quad \sum_{k=1}^{n+1} \log |\tilde{\alpha}^{(k)}(z)| - \log |D(z)| > \Lambda(r) - \frac{3}{2} \log(n+1) \text{ for } z = re^{i\theta}, \theta \in E(r).$$

The Hadamard inequality implies $|D(z)| \leq \prod_{k=1}^{n+1} |\tilde{\alpha}^{(k)}(z)|$, so

$$(4.4) \quad \sum_{k=1}^{n+1} \log |\tilde{\alpha}^{(k)}(z)| - \log |D(z)| \geq 0$$

for all z . Hence, it follows from (4.3) and (4.4) that

$$\begin{aligned} & \frac{1}{2\pi} (\Lambda(r) - \frac{3}{2} \log(n+1)) \text{meas } E(r) \\ & \leq \frac{1}{2\pi} \int_{E(r)} \left(\sum_{k=1}^{n+1} \log |\tilde{\alpha}^{(k)}(re^{i\theta})| - \log |D(re^{i\theta})| \right) d\theta \\ & \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=1}^{n+1} \log |\tilde{\alpha}^{(k)}(re^{i\theta})| - \log |D(re^{i\theta})| \right) d\theta \\ & = \sum_{k=1}^{n+1} (T(r, \alpha^{(k)}) + \log |\tilde{\alpha}^{(k)}(0)|) - (N(r, 0, D) + \log |c_\lambda|), \end{aligned}$$

where

$$D(z) = c_\lambda z^\lambda + c_{\lambda+1} z^{\lambda+1} + \cdots \quad (c_\lambda \neq 0).$$

Hence, choosing a constant K such that

$$\sum_{k=1}^{n+1} \log |\tilde{\alpha}^{(k)}(0)| - \log |c_\lambda| + \frac{3}{2} \log(n+1) < K,$$

we obtain

$$\text{meas } E(r) \leq \frac{2\pi}{\Lambda(r)} \left\{ \sum_{k=1}^{n+1} T(r, \alpha^{(k)}) + K \right\},$$

which proves Lemma 1.

Now let $A = \{\alpha^{(k)}\}_{k=1}^q$ ($q \geq n+1$) be a finite family of holomorphic curves in general position which satisfy

$$(4.5) \quad T(r, \alpha^{(k)}) = o(T(r, x)), \quad r \rightarrow \infty.$$

Further suppose that x is nondegenerate with respect to A . We define $I_\Lambda(r, \alpha^{(k)}) \equiv I_\Lambda(r, \alpha^{(k)}, A)$ by

$$(4.6) \quad I_\Lambda(r, \alpha^{(k)}) = E_\Lambda(r, \alpha^{(k)}) - \bigcup_{(l_1, \dots, l_n)} \left\{ E_\Lambda(r, \alpha^{(k)}) \cap \left(\bigcap_{j=1}^n E_\Lambda(r, \alpha^{(l_j)}) \right) \right\},$$

where the union is taken over all possible n -tuples $(l_1, \dots, l_n) \subset (1, \dots, k-1, k+1, \dots, q)$, and

$$(4.7) \quad \sigma'_\Lambda(\alpha^{(k)}) = \liminf_{m \rightarrow \infty} \text{meas } I_\Lambda(r_m, \alpha^{(k)}).$$

Then we can deduce

LEMMA 2. *For any $(k_1, \dots, k_{n+1}) \subset (1, 2, \dots, q)$ we have*

$$\bigcap_{j=1}^{n+1} I_\Lambda(r, \alpha^{(k_j)}) = \emptyset.$$

We now prove

LEMMA 3. *There is a sequence of positive functions $\Lambda_\nu(r)$ satisfying (3.3) such that*

$$\liminf_{\nu \rightarrow \infty} \sigma'_{\Lambda_\nu}(\alpha^{(k)}) \geq \min \left\{ 2\pi, \frac{4}{\mu} \sin^{-1} \left(\frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \right\}.$$

PROOF. We can choose a constant K such that Lemma 1 is valid for all $(n+1)$ -tuples $(k_1, \dots, k_{n+1}) \subset (1, \dots, q)$. Hence, it follows that

$$\begin{aligned} (4.8) \quad & \sum_{(l_1, \dots, l_n)} \text{meas} \left\{ E_\Lambda(r, \alpha^{(k)}) \cap \left(\bigcap_{j=1}^n E_\Lambda(r, \alpha^{(l_j)}) \right) \right\} \\ & \leq \sum_{(l_1, \dots, l_n)} \frac{2\pi}{\Lambda(r)} \left\{ T(r, \alpha^{(k)}) + \sum_{j=1}^n T(r, \alpha^{(l_j)}) + K \right\} \\ & \leq \frac{2\pi}{\Lambda(r)} \binom{q-1}{n} \left(\sum_{l=1}^q T(r, \alpha^{(l)}) + K \right). \end{aligned}$$

Choose a sequence of positive numbers ε_ν such that $\lim_{\nu \rightarrow \infty} \varepsilon_\nu = 0$. Put

$$(4.9) \quad \Lambda_\nu(r) = \frac{2\pi}{\varepsilon_\nu} \binom{q-1}{n} \left(\sum_{l=1}^q T(r, \alpha^{(l)}) + K \right).$$

Then it follows from (4.5) that for every fixed ν , $\Lambda_\nu(r)$ satisfies (3.3). From the definition of $\sigma_\Lambda(\alpha)$ and (4.6)–(4.9) we deduce that, for every fixed ν ,

$$\begin{aligned} \text{meas } I_{\Lambda_\nu}(r_m, \alpha^{(k)}) & \geq \text{meas } E_{\Lambda_\nu}(r_m, \alpha^{(k)}) \\ & \quad - \sum_{(l_1, \dots, l_n)} \text{meas} \left\{ E_{\Lambda_\nu}(r_m, \alpha^{(k)}) \cap \left(\bigcap_{j=1}^n E_{\Lambda_\nu}(r_m, \alpha^{(l_j)}) \right) \right\} \\ & \geq \text{meas } E_{\Lambda_\nu}(r_m, \alpha^{(k)}) - \varepsilon_\nu, \end{aligned}$$

so

$$\sigma'_{\Lambda_\nu}(\alpha^{(k)}) \geq \sigma_{\Lambda_\nu}(\alpha^{(k)}) - \varepsilon_\nu.$$

Therefore, our spread relation implies

$$\sigma'_{\Lambda_\nu}(\alpha^{(k)}) \geq \min \left\{ 2\pi, \frac{4}{\mu} \sin^{-1} \left(\frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \right\} - \varepsilon_\nu$$

for every fixed ν . Here let $\nu \rightarrow \infty$. Then we obtain Lemma 3.

We deduce the following obvious

LEMMA 4. Let $E_1, \dots, E_q$ be measurable subsets of $\{|z| = 1\}$. Define $E = E_1 \cup \dots \cup E_q$. Take an integer p with $1 \leq p \leq q$. Assume that for any selection of integers $1 \leq i_1 < i_2 < \dots < i_p \leq q$, we have $E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_p} = \emptyset$. Then

$$\sum_{j=1}^q \text{meas } E_j \leq (p-1) \text{meas } E.$$

We finally prove

LEMMA 5. Let $x: \mathbf{C} \rightarrow P_n \mathbf{C}$ be a holomorphic curve of lower order μ ($0 < \mu < +\infty$) and $\alpha^{(k)}: \mathbf{C} \rightarrow (P_n \mathbf{C})^*$, $k = 1, \dots, q$ ($q \geq n+1$) holomorphic curves satisfying (4.5). Suppose that x is nondegenerate with respect to $\{\alpha^{(k)}\}$, and $\{\alpha^{(k)}\}$ is in general position. Then

$$\sum_{k=1}^q \min \left\{ 2\pi, \frac{4}{\mu} \sin^{-1} \left(\frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \right\} \leq 2n\pi.$$

PROOF. Lemmas 2 and 4 imply

$$\sum_{k=1}^q \text{meas } I_\Lambda(r, \alpha^{(k)}) \leq 2n\pi,$$

and, consequently,

$$\sum_{k=1}^q \sigma'_\Lambda(\alpha^{(k)}) \leq 2n\pi.$$

Hence, we deduce from Lemma 3 that

$$\sum_{k=1}^q \min \left\{ 2\pi, \frac{4}{\mu} \sin^{-1} \left(\frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \right\} \leq 2n\pi,$$

which proves Lemma 5.

5. Proof of Theorem 1(I). Theorem 1(I) is an immediate consequence of the Corollary to Theorem 3, which is a generalization of results of Edrei-Fuchs [6], Toda [14], and Mori [8] (cf. Yang [17]).

THEOREM 3. Let $x: \mathbf{C} \rightarrow P_n \mathbf{C}$ be a holomorphic curve of lower order μ and $\alpha^{(k)}: \mathbf{C} \rightarrow (P_n \mathbf{C})^*$ ($k = 1, \dots, n+1$) holomorphic curves satisfying $T(r, \alpha^{(k)}) = o(T(r, x))$ ($r \rightarrow \infty$). Assume that x is nondegenerate with respect to $\{\alpha^{(k)}\}_{k=1}^{n+1}$, $\{\alpha^{(k)}\}_{k=1}^{n+1}$ is in general position, and $\alpha^{(k)}$ ($k = 1, \dots, n+1$) are deficient curves with respect to x . If $\gamma = \max_{1 \leq k \leq n+1} \{1 - \delta(\alpha^{(k)})\}$, then

$$\mu \geq \begin{cases} \frac{\log(1/\gamma(2-\gamma))}{\log(1+4/\gamma(1-\gamma))} & (\gamma \neq 0), \\ 1 & (\gamma = 0). \end{cases}$$

COROLLARY. Holomorphic curves with more than n deficient curves in general position have a positive lower order.

PROOF OF THEOREM 3. Let $\tilde{x} = (x_0, x_1, \dots, x_n): \mathbf{C} \rightarrow \mathbf{C}^{n+1} - \{0\}$ and $\tilde{\alpha}^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \dots, \alpha_n^{(k)}): \mathbf{C} \rightarrow (\mathbf{C}^{n+1})^* - \{0\}$ be reduced representations of x

and $\alpha^{(k)}$, respectively. Then we can choose a nonzero complex number β such that a reduced representation of a constant curve $\alpha^{(n+2)}: \mathbf{C} \rightarrow (P_n \mathbf{C})^*$ is $\tilde{\alpha}^{(n+2)} = (1, \beta, \beta^2, \dots, \beta^n)$, $\{\alpha^{(k)}\}_{k=1}^{n+2}$ is in general position, and x is nondegenerate with respect to $\{\alpha^{(k)}\}_{k=1}^{n+2}$.

We now consider the holomorphic curve $y: \mathbf{C} \rightarrow P_n \mathbf{C}$ which Mori [9] constructed from x and $\{\alpha^{(k)}\}_{k=1}^{n+2}$. We put

$$F_k(z) = \sum_{j=0}^n \alpha_j^{(k)}(z) x_j(z).$$

Let $G = \det\{\alpha_j^{(k)}\}_{0 \leq j \leq n, 1 \leq k \leq n+1}$, G_k be the determinant of the matrix replaced the k th column vector ${}^t(\alpha_0^{(k)}, \dots, \alpha_n^{(k)})$ of the matrix $\{\alpha_j^{(k)}\}_{0 \leq j \leq n, 1 \leq k \leq n+1}$ by the column vector ${}^t(\alpha_0^{(n+2)}, \dots, \alpha_n^{(n+2)})$, and $g_k = G_k/G$. We denote by $h(z)$ a common factor among $g_1(z)F_1(z), \dots, g_n(z)F_n(z)$ and $g_{n+1}(z)F_{n+1}(z)$ up to non-vanishing entire functions such that $y_{k-1}(z) = g_k(z)F_k(z)/h(z)$ ($k = 1, \dots, n+1$) are entire. Then, by the reasoning of Mori [9, proof¹ of Theorem 3.4], the holomorphic curve $y: \mathbf{C} \rightarrow P_n \mathbf{C}$ induced by $\tilde{y} = (y_0, y_1, \dots, y_n): \mathbf{C} \rightarrow \mathbf{C}^{n+1} - \{0\}$ has the properties

$$(5.2) \quad T(r, x) \sim T(r, y) \quad (r \rightarrow \infty),$$

and, for $k = 1, 2, \dots, n+1$,

$$(5.3) \quad N(r, 0, y_{k-1}) = N(r, 0, F_k) + o(T(r, x)) \quad (r \rightarrow \infty).$$

Now we prove (5.2) and (5.3). Hadamard's inequality implies

$$(5.4) \quad |G_k| \leq B \prod_{j=1, j \neq k}^{n+1} |\tilde{\alpha}^{(j)}|,$$

where $B = |\tilde{\alpha}^{(n+2)}|$ is a constant. Using $|F_k| \leq |\tilde{x}| |\tilde{\alpha}^{(k)}|$, together with (5.4), we have

$$|y_{k-1}| = \left| \frac{G_k F_k}{hG} \right| \leq B |\tilde{x}| \prod_{j=1}^{n+1} \frac{|\tilde{\alpha}^{(j)}|}{|hG|},$$

so

$$|\tilde{y}| \leq (n+1)^{1/2} B |\tilde{x}| \prod_{j=1}^{n+1} \frac{|\tilde{\alpha}^{(j)}|}{|hG|}.$$

Hence, we have, with a suitable constant K_1 ,

$$T(r, y) \leq T(r, x) + \sum_{j=1}^{n+1} T(r, \alpha^{(j)}) - N(r, 0, G) - N(r, 0, h) + N(r, \infty, h) + K_1.$$

Since $N(r, \infty, h) \leq N(r, 0, G)$, we obtain

$$(5.5) \quad T(r, y) \leq T(r, x) + \sum_{j=1}^{n+1} T(r, \alpha^{(j)}) + K_1.$$

¹There is a mistake in Mori's proof of his Theorem 3.4 [9], but he provides a correct proof in the "Correction to [9]". Unfortunately, his correction will not appear in the published paper, so we prove (5.2) and (5.3) along the lines of his correct proof.

On the other hand, it follows from

$$G_k \sum_{j=0}^n \alpha_j^{(k)} x_j = Gh y_{k-1} \quad (k = 1, \dots, n+1)$$

that $x_j = GhH_j/H$, where $H = \det\{G_k \alpha_j^{(k)}\}_{1 \leq k \leq n+1, 0 \leq j \leq n}$ and H_j is the determinant of the matrix obtained by replacing the j th column vector

$${}^t(G_1 \alpha_j^{(1)}, \dots, G_{n+1} \alpha_j^{(n+1)})$$

of the matrix $\{G_k \alpha_j^{(k)}\}_{1 \leq k \leq n+1, 0 \leq j \leq n}$ by the column vector ${}^t(y_0, \dots, y_n)$. We put $\tilde{\eta}_k = {}^t(G_1 \alpha_k^{(1)}, \dots, G_{n+1} \alpha_k^{(n+1)})$. Then using Hadamard's inequality together with (5.4), we have

$$|H_j| \leq |\tilde{y}| \prod_{k=0, \neq j}^n |\tilde{\eta}_k|,$$

and

$$|\tilde{\eta}_k|^2 = \sum_{l=1}^{n+1} |G_l \alpha_k^{(l)}|^2 \leq \sum_{l=1}^{n+1} \left(B \prod_{j=1, \neq l}^{n+1} |\tilde{\alpha}^{(j)}| \right)^2 |\tilde{\alpha}^{(l)}|^2 = (n+1) B^2 \prod_{j=1}^{n+1} |\tilde{\alpha}^{(j)}|^2,$$

so

$$|H_j| \leq |\tilde{y}| \left\{ (n+1)^{1/2} B \prod_{k=1}^{n+1} |\tilde{\alpha}^{(k)}| \right\}^n.$$

By Hadamard's inequality we also have

$$(5.6) \quad |G| \leq \prod_{k=1}^{n+1} |\tilde{\alpha}^{(k)}|.$$

Hence,

$$\begin{aligned} \log |\tilde{x}| &= \log \left(\sum_{j=0}^n \left| \frac{hGH_j}{H} \right|^2 \right)^{1/2} \\ &= \log |G| + \log \left(\sum_{j=0}^n |H_j|^2 \right)^{1/2} + \log |h| - \log |H| \\ &\leq \log |\tilde{y}| + (n+1) \sum_{k=1}^{n+1} \log |\tilde{\alpha}^{(k)}| + \log |h| - \log |H| + \log B^n (n+1)^{(n+1)/2}, \end{aligned}$$

so, with a suitable constant K_2 ,

$$\begin{aligned} T(r, x) &\leq T(r, y) + (n+1) \sum_{k=1}^{n+1} T(r, \alpha^{(k)}) + N(r, 0, h) \\ &\quad - N(r, \infty, h) - N(r, 0, H) + K_2. \end{aligned}$$

Since $N(r, 0, h) \leq N(r, 0, H)$, we obtain

$$(5.7) \quad T(r, x) \leq T(r, y) + (n+1) \sum_{k=1}^{n+1} T(r, \alpha^{(k)}) + K_2.$$

Next (5.4) and (5.6) imply

$$(5.8) \quad N(r, 0, G_k) \leq \sum_{j=0, \neq k}^n T(r, \alpha^{(j)}) + K_3$$

and

$$(5.9) \quad N(r, 0, G) \leq \sum_{j=0}^n T(r, \alpha^{(j)}) + K_4,$$

where K_3 and K_4 are constants. Hence, using (5.8) we have

$$(5.10) \quad \begin{aligned} N(r, 0, y_{k-1}) &\leq N(r, 0, F_k) + N(r, 0, G_k) \\ &\leq N(r, 0, F_k) + \sum_{j=0, \neq k}^n T(r, \alpha^{(j)}) + K_4. \end{aligned}$$

Since $|H| \leq (\prod_{j=1}^{n+1} |G_j|)|G|$, we have

$$(5.11) \quad N(r, 0, H) \leq N(r, 0, G) + \sum_{j=1}^{n+1} N(r, 0, G_k) + K_5,$$

where K_5 is a constant. Hence, using $N(r, 0, h) \leq N(r, 0, H)$, together with (5.8), (5.9), and (5.11), we have

$$(5.12) \quad \begin{aligned} N(r, 0, F_k) &\leq N(r, 0, y_{k-1}) + N(r, 0, G) + N(r, 0, h) \\ &\leq N(r, 0, y_{k-1}) + 2N(r, 0, G) + \sum_{j=1}^{n+1} N(r, 0, G_j) + K_5 \\ &\leq N(r, 0, y_{k-1}) + (n+2) \sum_{j=1}^{n+1} T(r, \alpha^{(j)}) + K_6, \end{aligned}$$

where K_6 is a suitable constant. Therefore using (5.1) in (5.5), (5.7), (5.10), and (5.12), we obtain (5.2) and (5.3).

Now it follows from (5.2) and (5.3) that

$$\delta(\alpha^{(k)}, x) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, 0, y_{k-1})}{T(r, y)} = \delta(a^{(k)}, y) \quad (k = 1, 2, \dots, n+1),$$

where $a^{(k)} \in (P_n C)^*$ are induced by $(0, \dots, 0, 1, 0, \dots, 0)$ ($(k-1)$ th spot). Therefore, Theorem 3 follows Theorem 2 in Toda [14].

6. Proof of Theorem 1(II). We may assume that $\delta(\alpha^{(k)}) \geq 1 - \cos \pi \mu$, $k = 1, 2, \dots, n$. Then

$$(4/\mu) \sin^{-1}(\delta(\alpha^{(k)})/2)^{1/2} \geq 2\pi,$$

so

$$\min\{2\pi, (4/\mu) \sin^{-1}(\delta(\alpha^{(k)})/2)^{1/2}\} = 2\pi, \quad k = 1, \dots, n.$$

Hence, it follows from Lemma 5 that

$$\sum_{k=n+1}^q \min \left\{ 2\pi, \frac{4}{\mu} \sin^{-1} \left(\frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \right\} = 0$$

and, consequently,

$$\delta(\alpha^{(k)}) = 0, \quad k = n+1, \dots, q.$$

We next assume that $\delta(\alpha^{(k)}) > 1 - \cos \pi \mu$ ($k = 1, \dots, p$; $0 \leq p < n$) and $\delta(\alpha^{(k)}) \leq 1 - \cos \pi \mu$ ($k = p+1, \dots, q$). Then it follows from Lemma 5 that

$$\sum_{k=p+1}^q \frac{4}{\mu} \sin^{-1} \left(\frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \leq 2(n-p)\pi.$$

Hence, from the reasoning of §3 of Edrei [5] we can deduce that

$$(6.1) \quad \sum_{k=p+1}^q \delta(\alpha^{(k)}) \leq (n-p)(1 - \cos \pi \mu).$$

It follows from Lemma 1 of Edrei [5] that equality in (6.1) is possible if and only if exactly $n-p$ of $\{\delta(\alpha^{(k)})\}_{k=p+1}^q$ are equal to $1 - \cos \pi \mu$ and all other $\delta(\alpha^{(k)})$ are equal to zero.

Thus the proof of Theorem 1(II) is complete.

7. Proof of Theorem 1(III). Our assumption $\mu > 1/2$ implies

$$(4/\mu) \sin^{-1}(\delta(\alpha^{(k)})/2)^{1/2} < 2\pi,$$

so it follows from Lemma 5 that

$$\sum_{k=1}^q \frac{4}{\mu} \sin^{-1} \left(\frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \leq 2n\pi.$$

Hence, from the reasoning of §2 of Edrei [5] we can deduce that

$$\sum_{k=1}^q \delta(\alpha^{(k)}) \leq [2n\mu] + 1 - \cos(2n\mu - [2n\mu])(\pi/2),$$

which proves Theorem 1(III).

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FACULTY OF TECHNOLOGY, KANAZAWA UNIVERSITY, 2-40-20, KODATSUNO, KANAZAWA 920, JAPAN