FILTERING COHOMOLOGY AND LIFTING VECTOR BUNDLES

BY

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ABSTRACT. For a module M over a local Cohen-Macaulay ring R we develop a (finite) sequence of presentations of M which facilitates the study of invariants arising from the cohomology modules of M . As an application we use this data, in case R is regular and M represents a vector bundle on the punctured spectrum of R with a vanishing cohomology module, to obtain bounds on how far M can be lifted as a vector bundle.

In an earlier article [5] we used the dual of a filtration of Auslander and Bridger [1] to obtain some results for nontrivial k th syzygies of rank k . In this article we establish a different method for filtering a module over a local Cohen-Macaulay ring, or more precisely a method for filtering a free presentation of a module (see discussion following Theorem 1.8). This filtration is accomplished through a finite sequence of presentations.

To be specific, let R be a local Cohen-Macaulay ring and let M be a finitely generated R -module such that $\text{depth}(\text{Ext}^i(M, R)) \geq i$ for all i less than or equal to the dimension of R . In Theorem 1.2 we establish an exact sequence, which we term a q th presentation of M ,

$$0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0,$$

such that the projective dimension of $\mathcal{X}_q(M)$ is less than q and such that $\text{Ext}^i(\mathcal{E}_q(M), R)$ is zero for i from one through q . In addition we establish uniqueness results (see Corollary 1.6 and Theorem 1.7) in the sense of stable isomorphism. As a consequence we recapture a variation of our main result in [6] without the assumptions on rank or the existence of maximal Cohen-Macaulay modules. We also show, if R is Gorenstein, that the intermediate cohomology of a module is always carried by a module of finite projective dimension (see Corollary 1.9).

In §2 we turn our attention to the lifting of vector bundles on the punctured spectra of regular local rings and achieve a substantial improvement over our results in [5]. In particular, if $\rho_i(M)$ denotes the rank of the i th syzygy of $\text{Ext}^i(M, R)$ for $i \geq 0$, we show that a nontrivial vector bundle M with $\text{Ext}^q(M, R)$ zero for some q in the range $1 \leq q \leq \dim R - 2$, cannot be lifted more than $\sum_{i=0}^{q-1} \rho_i(M)$ times. For a complete discussion of the projective case the reader should consult Barth and Van de Ven [2], Sato [12] and Tyurin [14]. For additional information on this problem

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one should consult Hartshorne [7], our papers [4, 5] and the fundamental paper [9] by Horrocks.

1. The filtration. Serre in [13] and later Murthy in [11] use the following construction. Let M be a finitely generated R -module of projective dimension one. Then there is an exact sequence

$$0 \rightarrow R^n \rightarrow E \rightarrow M \rightarrow 0$$

in which the map $\text{Hom}(R^n, R) \rightarrow \text{Ext}^1(M, R)$ is surjective. One concludes that $\text{Ext}^1(E, R)$ is zero and hence, since the projective dimension of E is at most one, that E is projective. If one drops the requirement that M has projective dimension one, then one can nevertheless construct such a sequence where $\text{Ext}^1(E, R)$ is zero. This is the case $q = 1$ of our construction. We utilized this construction implicitly (for $q = 1$) in our article with Bruns [3, Lemma 2.1]. We present the full details here in order to give the flavor of our subsequent construction.

LEMMA 1.1. *Let R be a local ring and let M be a finitely generated R -module with $\text{Ext}^1(M, R)$ generated by n elements. Then there is an exact sequence*

$$0 \rightarrow \mathcal{X}_1(M) \rightarrow \mathcal{E}_1(M) \rightarrow M \rightarrow 0$$

such that $\mathcal{X}_1(M)$ is isomorphic to R^n and such that $\text{Ext}^1(\mathcal{E}_1(M), R)$ is zero.

PROOF. If $n = 0$ we take $\mathcal{E}_1(M) = M$ and $\mathcal{X}_1(M) = 0$. If $n > 0$ let $\xi_1, \dots, \xi_n$ be a generating set for $\text{Ext}^1(M, R)$ in which the class of ξ_i is represented by the extension $0 \rightarrow Re_i \rightarrow N_i \rightarrow M \rightarrow 0$. We form the direct sum of these extensions and the pullback diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \oplus Re_i & \rightarrow & N & \rightarrow & M \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \Delta \\ 0 & \rightarrow & \oplus Re_i & \rightarrow & \oplus N_i & \rightarrow & M^n \rightarrow 0 \end{array}$$

where Δ is the diagonal map. We note that the pushout of the top row along the i th projection gives back ξ_i . Hence, the natural map $\text{Hom}(\oplus Re_i, R) \rightarrow \text{Ext}^1(M, R)$ is surjective and therefore $\text{Ext}^1(N, R)$ is zero. Thus with $\mathcal{E}_1(M) = N$ and $\mathcal{X}_1(M) = \oplus Re_i$ we have the desired sequence.

THEOREM 1.2. *Let R be a Cohen-Macaulay local ring and let M be a finitely generated R -module. Assume that the grade of $\text{Ext}^i(M, R)$ is at least i for each i up to the dimension of R . Then for each q between 1 and the dimension of R there is a short exact sequence*

$$0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0$$

such that the projective dimension of $\mathcal{X}_q(M)$ is less than q and $\text{Ext}^i(\mathcal{E}_q(M), R)$ is zero for $1 \leq i \leq q$.

PROOF. We proceed by induction on q . The case $q = 1$ is equivalent to Lemma 1.1. Let

$$0 \rightarrow K \rightarrow F_{q-2} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow \mathcal{E}_{q-1}(M) \rightarrow 0$$

be an exact sequence with the F_i free R -modules and with $\mathcal{E}_{q-1}(M)$ already constructed with the desired properties. Then $\text{Ext}^q(M, R)$ is isomorphic to $\text{Ext}^q(\mathcal{E}_{q-1}(M), R)$. By a dimension shift we see that $\text{Ext}^q(M, R)$ is isomorphic to $\text{Ext}^1(K, R)$. We apply the case $q = 1$ to K and obtain a diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & K & \rightarrow & F_{q-2} & \rightarrow \cdots \rightarrow & F_0 & \rightarrow & \mathcal{E}_{q-1}(M) & \rightarrow & 0 \\
 & & \uparrow & & & & & & & & \\
 & & \mathcal{E}_1(K) & & & & & & & & \\
 & & \uparrow & & & & & & & & \\
 & & \mathcal{X}_1(K) & & & & & & & & \\
 & & \uparrow & & & & & & & & \\
 & & 0 & & & & & & & &
 \end{array}$$

where $\mathcal{X}_1(K)$ is R^n , the integer n being the number of generators of $\text{Ext}^1(K, R)$. Dualizing with respect to R and using that $\text{Ext}^i(\mathcal{E}_{q-1}(M), R) = 0$ for $1 \leq i \leq q-1$, we obtain the diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 0 & \leftarrow & K^* & \leftarrow & F_{q-2}^* & \leftarrow \cdots \leftarrow & F_0^* & \leftarrow & \mathcal{E}_{q-1}(M)^* & \leftarrow & 0 \\
 & & \downarrow & & & & & & & & \\
 & & \mathcal{E}_1(K)^* & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & (R^n)^* & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & \text{Ext}^1(K, R) & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
 \end{array}$$

with exact row and column. Let V be the image of $\mathcal{E}_1(K)^*$ in $(R^n)^*$. Since the grade of $\text{Ext}^1(K, R)$ is at least q we have that $\text{Ext}^j(\text{Ext}^1(K, R), R)$ is zero for $j = 1$ through $q-1$. Thus $\text{Ext}^i(V, R)$ is zero for $1 \leq i \leq q-2$. Using the usual Cartan-Eilenberg construction for forming a projective resolution of the middle term of an exact sequence given resolutions of the ends, we obtain a diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \leftarrow & K^* & \leftarrow & F_{q-2}^* & \leftarrow \cdots \leftarrow & F_0^* & \leftarrow & \mathcal{E}_{q-1}(M)^* & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \leftarrow & \mathcal{E}_1(K)^* & \leftarrow & F_{q-2}^* \oplus G_{q-2}^* & \leftarrow \cdots \leftarrow & F_0^* \oplus G_0^* & \leftarrow & W & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \leftarrow & V & \leftarrow & G_{q-2}^* & \leftarrow \cdots \leftarrow & G_0^* & \leftarrow & Z & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
 \end{array}$$

with exact rows and columns and with the F_i^* and G_i^* free R -modules. We now dualize this diagram with respect to R and use that

$$0 \rightarrow K \rightarrow F_{q-2} \rightarrow \cdots \rightarrow F_0 \rightarrow \mathcal{E}_{q-1}(M) \rightarrow 0$$

is exact, that $V^* \cong R^n$ and that $\text{Ext}^i(V, R) = 0$ for $1 \leq i \leq q-2$. This yields the diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \rightarrow & K & \rightarrow & F_{q-2} & \rightarrow \cdots \rightarrow & F_0 & \rightarrow & \mathcal{E}_{q-1}(M) & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \rightarrow & \mathcal{E}_1(K) & \rightarrow & F_{q-2} \oplus G_{q-2} & \rightarrow \cdots \rightarrow & F_0 \oplus G_0 & \rightarrow & W^* & \rightarrow & C \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \parallel \\
 0 & \rightarrow & R^n & \rightarrow & G_{q-2} & \rightarrow \cdots \rightarrow & G_0 & \rightarrow & Z^* & \rightarrow & C \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
 \end{array}$$

which has exact rows and columns. The cokernel C is isomorphic to $\text{Ext}^q(\text{Ext}^1(K, R), R)$ which is in turn isomorphic to $\text{Ext}^q(\text{Ext}^q(M, R), R)$. We let $\mathcal{E}_q(M)$ be the image of $F_0 \oplus G_0$ in W^* and let L be the image of G_0 in Z^* . Then the projective dimension of L is at most $q-1$. We have an exact sequence

$$0 \rightarrow L \rightarrow \mathcal{E}_q(M) \rightarrow \mathcal{E}_{q-1}(M) \rightarrow 0.$$

Then $\text{Ext}^q(\mathcal{E}_q(M), R) \cong \text{Ext}^1(\mathcal{E}_1(K), R) = 0$ and $\text{Ext}^i(\mathcal{E}_q(M), R) = 0$ for $1 \leq i \leq q$, by virtue of the dual exactness of the middle row of the above diagram. We let $\mathcal{X}_q(M)$ be the kernel of the map of $\mathcal{E}_q(M)$ onto M via the composition of $\mathcal{E}_q(M) \rightarrow \mathcal{E}_{q-1}(M) \rightarrow M$. Therefore there is an exact sequence $0 \rightarrow L \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{X}_{q-1}(M) \rightarrow 0$ and thus $\mathcal{X}_q(M)$ has projective dimension at most $q-1$ as desired.

REMARK 1.3. (a) The above short exact sequence $0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0$ will hereafter be referred to as a q th presentation of M .

(b) If M has finite projective dimension or if M is locally free on the punctured spectrum of R or if R is Gorenstein, then the cohomology modules $\text{Ext}^i(M, R)$ satisfy the grade assumptions of the theorem.

(c) If M has projective dimension $q < \infty$, then the sequence $0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0$ has $\mathcal{E}_q(M)$ free and thus is merely the beginning of a finite projective resolution of M . Thus the presentations $0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0$ begin with the Serre-Murthy exact sequence for $q=1$ and end with a free presentation of M for q equal to the projective dimension of M .

(d) The properties of a q th presentation yield isomorphisms

$$\text{Ext}^i(\mathcal{X}_q(M), R) \cong \text{Ext}^{i+1}(M, R) \quad \text{for } 1 \leq i \leq q-1,$$

and

$$\text{Ext}^j(M, R) \cong \text{Ext}^j(\mathcal{E}_q(M), R) \quad \text{for } j \geq q+1.$$

(e) If R is Gorenstein and if M is a reflexive R -module, then both $\mathcal{E}_q(M)$ and $\mathcal{X}_q(M)$ are reflexive. This follows because the cokernel C is necessarily zero, since $\text{Ext}^q(M, R)$ must have grade at least $q + 1$ in case M is reflexive.

The next results describe the uniqueness of the modules $\mathcal{E}_q(M)$ and $\mathcal{X}_q(M)$. The argument is in the same vein as many proofs of Schanuel's lemma.

THEOREM 1.4. *If $0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0$ and $0 \rightarrow \mathcal{X}'_q(M) \rightarrow \mathcal{E}'_q(M) \rightarrow M \rightarrow 0$ are both q th presentations of M , then $\mathcal{E}_q(M) \oplus \mathcal{X}'_q(M)$ is isomorphic to $\mathcal{E}'_q(M) \oplus \mathcal{X}_q(M)$.*

PROOF. We consider the following pullback diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{X}'_q(M) & = & \mathcal{X}'_q(M) & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & \mathcal{X}_q(M) & \rightarrow & P & \rightarrow & \mathcal{E}'_q(M) \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & \mathcal{X}_q(M) & \rightarrow & \mathcal{E}_q(M) & \rightarrow & M \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Clearly it suffices to show the middle row and column of the above diagram are split exact sequences. This is an immediate consequence of the following lemma.

LEMMA 1.5. *Let R be a local ring and suppose A and B are finitely generated R -modules such that the projective dimension of A is less than k and $\text{Ext}^i(B, R) = 0$ for $1 \leq i \leq k$. Then $\text{Ext}^i(B, A) = 0$ for $1 \leq i \leq k - \text{pd } A$. (pd A denotes the projective dimension of A .)*

PROOF. We proceed by induction on the projective dimension of A . The case $\text{pd } A = 0$ is immediate.

Let $\text{pd } A = n > 0$ and let $0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$ be an exact sequence with F a finitely generated free R -module. Applying the functor $\text{Hom}(B, -)$ and taking the long exact sequence in cohomology yields the exact sequence

$$\cdots \rightarrow \text{Ext}^i(B, F) \rightarrow \text{Ext}^i(B, A) \rightarrow \text{Ext}^{i+1}(B, K) \rightarrow \cdots$$

for each $i \geq 1$. If $1 \leq i \leq k - \text{pd } A$, then $2 \leq i + 1 \leq k - \text{pd } K$. So by induction $\text{Ext}^{i+1}(B, K) = 0$. Now $\text{Ext}^i(B, F) = 0$ since F is free and $\text{Ext}^i(B, R) = 0$. Thus $\text{Ext}^i(B, A) = 0$.

We now prove a stronger uniqueness which you would expect for the analogue of a free resolution.

COROLLARY 1.6. *Let $0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0$ and $0 \rightarrow \mathcal{X}'_q(M) \rightarrow \mathcal{E}'_q(M) \rightarrow M \rightarrow 0$ be two q th presentations for M . Then there are free R -modules F, F', G and G' such that $\mathcal{X}_q(M) \oplus F \cong \mathcal{X}'_q(M) \oplus F'$ and $\mathcal{E}_q(M) \oplus G \cong \mathcal{E}'_q(M) \oplus G'$.*

PROOF. We already know by Theorem 1.4 that $\mathcal{X}_q(M) \oplus \mathcal{E}'_q(M) \cong \mathcal{X}'_q(M) \oplus \mathcal{E}_q(M)$. Further, it is enough to consider the case of R complete, since if two finitely generated R -modules are isomorphic over the completion they are already isomorphic. Let $K_1 \oplus \cdots \oplus K_n$, $E_1 \oplus \cdots \oplus E_m$, $K'_1 \oplus \cdots \oplus K'_u$ and $E'_1 \oplus \cdots \oplus E'_v$ be the decompositions of $\mathcal{X}_q(M)$, $\mathcal{E}_q(M)$, $\mathcal{X}'_q(M)$ and $\mathcal{E}'_q(M)$ respectively. By the Krull-Schmidt Theorem for complete local rings and the above isomorphism, we have that $n + v = m + u$ and that the indecomposables from each side of the isomorphism are pairwise isomorphic. Furthermore, we can assume that $n \leq u$ and that the pairing is arranged so as to pair as many K_i with K'_j as possible. If any K_i is not paired with a K'_j , then it is paired with an E_j . Hence it must have projective dimension less than q while $\text{Ext}^s(K_i, R) = 0$ for $1 \leq s \leq q$. That is, in this case K_i must be free and therefore just R itself. Since the maximum number of K_i 's were paired with the K'_j 's we must have that $\mathcal{X}_q(M) \oplus F \cong \mathcal{X}'_q(M)$, where F is free. Similarly $\mathcal{E}_q(M) \oplus F \cong \mathcal{E}'_q(M)$, where F is the same free module since the rank of the largest free summand on either side of the isomorphism $\mathcal{X}_q(M) \oplus \mathcal{E}'_q(M) \cong \mathcal{X}'_q(M) \oplus \mathcal{E}_q(M)$ must be preserved.

If the q th presentation for a module M , $0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0$, has the property that there is a free R -module F with $\mathcal{X}_q(M) = K_q \oplus F$ and $\mathcal{E}_q(M) = E_q \oplus F$ in which the map $\mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M)$ induces an isomorphism of F into F , then one may obtain a q th presentation of the form $0 \rightarrow K_q \rightarrow E_q \rightarrow M \rightarrow 0$, that is, one may remove the free R -module F . In such a situation we refer to $\mathcal{X}_q(M)$ and $\mathcal{E}_q(M)$ as having a common free summand. Clearly at least one q th presentation exists in which $\mathcal{X}_q(M)$ and $\mathcal{E}_q(M)$ have no common (nonzero) free summand. Such a q th presentation is called a minimal q th presentation. Our next theorem asserts that all q th presentations are obtained from minimal ones by adding a common free summand to $\mathcal{X}_q(M)$ and $\mathcal{E}_q(M)$.

THEOREM 1.7. *Let $0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0$ be a minimal q th presentation. If $0 \rightarrow K \rightarrow E \rightarrow M \rightarrow 0$ represents another q th presentation then K and E have a common free summand F such that $K \cong \mathcal{X}_q(M) \oplus F$ and $E \cong \mathcal{E}_q(M) \oplus F$.*

PROOF. It suffices to show, if K and E have no common free summand, then $K \cong \mathcal{X}_q(M)$ and $E \cong \mathcal{E}_q(M)$. By Corollary 1.6, we can write $\mathcal{X}_q(M) = K_1 \oplus G_1$ and $K = K_1 \oplus G_2$ so that K_1 has no nontrivial free summand and where G_1 and G_2 are free. Similarly, we have that $\mathcal{E}_q(M) = E_1 \oplus F_1$ and $E = E_1 \oplus F_2$, where E_1 has no nontrivial free summand and where F_1 and F_2 are free. Now consider the pullback diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & K_1 \oplus G_2 & = & K_1 \oplus G_2 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \rightarrow & K_1 \oplus G_1 & \rightarrow & P & \rightarrow & E_1 \oplus F_2 \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & K_1 \oplus G_1 & \rightarrow & E_1 \oplus F_1 & \rightarrow & M \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

used in the proof of Theorem 1.4. The maps $K_1 \oplus G_1 \rightarrow F_1$ and $K_1 \oplus G_2 \rightarrow F_2$ necessarily have images in $\mathfrak{m}F_1$ and $\mathfrak{m}F_2$, respectively ($\mathfrak{m}$ denotes the maximal ideal of R), since there are no common free summands. Thus the induced map $F_1 \rightarrow F_2$, via the splitting of the middle column, must be surjective. By symmetry the induced map $F_2 \rightarrow F_1$ must also be surjective. Thus F_1 and F_2 are isomorphic. From this fact together with Theorem 1.4 we conclude that G_1 and G_2 are isomorphic. Thus $\mathcal{E}_q(M) \cong E$ and $\mathcal{X}_q(M) \cong K$.

In the following theorem we collect some properties of these presentations.

THEOREM 1.8. *Let R be a local Cohen-Macaulay ring and let M be a finitely generated R -module.*

(a) *If M has finite projective dimension less than or equal to q and if $0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0$ is a q th presentation of M , then $\mathcal{E}_q(M)$ is free and $\mathcal{X}_q(M)$ is the module of relations on M .*

(b) *If $\text{Ext}^{q+1}(M, R) = 0$, then a q th presentation of M is also a $(q+1)$ st presentation.*

(c) *If $f: M \rightarrow N$ is a homomorphism of R -modules, then for each q , there is an induced commutative diagram:*

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{X}_q(M) & \rightarrow & \mathcal{E}_q(M) & \rightarrow & M \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow f \\ 0 & \rightarrow & \mathcal{X}_q(N) & \rightarrow & \mathcal{E}_q(N) & \rightarrow & N \rightarrow 0 \end{array}$$

(d) *If M represents a vector bundle on the punctured spectrum of R , then each q th presentation $0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0$ is an exact sequence of vector bundles.*

(e) *If for some $q \geq 2$ we have that $\mathcal{X}_q(M)$ is free, then $\text{Ext}^i(M, R) = 0$ for $2 \leq i \leq q$.*

PROOF. Straightforward (see Remarks 1.3 and Lemma 1.5).

Let M be a module of finite projective dimension n over R . As noted above the presentation $0 \rightarrow \mathcal{X}_n(M) \rightarrow \mathcal{E}_n(M) \rightarrow M \rightarrow 0$ is a free presentation of M . Moreover, for $0 \leq i \leq j \leq n$ and taking $\mathcal{E}_0(M) = M$, $\mathcal{X}_0(M) = 0$, one may obtain commutative triangles (see beginning of the proof of Theorem 1.2)

$$\begin{array}{ccc} \mathcal{E}_n(M) & \rightarrow & \mathcal{E}_i(M) \\ \searrow & & \nearrow \\ & \mathcal{E}_j(M) & \end{array}$$

where the maps are surjective (however, the presentations are not in general minimal in this setup). One also obtains corresponding commutative triangles

$$\begin{array}{ccc} \mathcal{X}_n(M) & \rightarrow & \mathcal{X}_i(M) \\ \searrow & & \nearrow \\ & \mathcal{X}_j(M) & \end{array}$$

with surjective maps. In this fashion we obtain a filtration based on cohomology of a (nonminimal, in general) free presentation of M . We end this section with two corollaries resulting from this filtration.

COROLLARY 1.9. *Let R be a Gorenstein ring of dimension n and let M be a finitely generated R -module of depth d . Then $\text{Ext}^{i+1}(M, R) \cong \text{Ext}^i(\mathcal{X}_{n-d}(M), R)$ for $1 \leq i \leq n - d - 1$. That is the cohomology modules $\text{Ext}^i(M, R)$, for $2 \leq i \leq n - d$, are isomorphic to those of a module of finite projective dimension.*

PROOF. For R Gorenstein of dimension n and M of depth d , the only possible nonzero $\text{Ext}^i(M, R)$ lie in the range $1 \leq i \leq n - d$. Then $\text{Ext}^i(\mathcal{E}_{n-d}(M), R) = 0$ for $i > 0$ while $\text{Ext}^i(\mathcal{X}_{n-d}(M), R)$ is isomorphic to $\text{Ext}^{i+1}(M, R)$ for $i > 0$ (see 1.3(d)).

Our remaining corollary contains a weaker version of our result [6, Theorem 2.1]. This is moderately surprising since our proof there uses our syzygy theorem [4] which in turn uses Hochster's big Cohen-Macaulay modules [8], while the result below only uses Theorem 1.2. However, it should be stressed that we do not recover the full generality of [6, Theorem 2.1] but only the case of a vector bundle and without the minimality implicit in the earlier result.

In Corollary 1.10 as well as the next section we need to make use of Matlis duality. For an R -module L , the Matlis dual of L is $\text{Hom}(L, I)$, where I is the injective envelope of the residue field of R . We use the notation L^v to indicate the dual module $\text{Hom}(L, I)$. If L is an Artinian module, then L is naturally isomorphic to L^{vv} .

COROLLARY 1.10. *Let R be a regular local ring of dimension n with $n \geq 4$. Let M be a k th syzygy which is not a $(k + 1)$ th syzygy, for $2 \leq k \leq n - 2$, which represents a vector bundle on the punctured spectrum of R . Then in the $(n - k - 1)$ th presentation $\mathcal{X}_{n-k-1}(M)$ is a $(k + 2)$ th syzygy and $\mathcal{E}_{n-k-1}(M)$ is a k th syzygy of $\text{Ext}^{k-1}(M^*, R)$.*

REMARK. Unlike [6, Theorem 2.1] $\mathcal{E}_{n-k-1}(M)$ may not be the minimal k th syzygy of $\text{Ext}^{k-1}(M^*, R)$.

PROOF. Since M represents a vector bundle on the punctured spectrum of R , each $\text{Ext}^i(M, R)$ has finite length. The projective dimension of M is $n - k$. Thus by Theorem 1.2 and Theorem 1.8, $\mathcal{X}_{n-k-1}(M)$ and $\mathcal{E}_{n-k-1}(M)$ also represent vector bundles. Furthermore $\mathcal{X}_{n-k-1}(M)$ has projective dimension $n - k - 2$ and $\mathcal{E}_{n-k-1}(M)$ has a unique nonvanishing Ext^i , namely $\text{Ext}^{n-k}(\mathcal{E}_{n-k-1}(M), R)$. Hence $\mathcal{X}_{n-k-1}(M)$ is a $n - (n - k - 2) = (k + 2)$ th syzygy and $\mathcal{E}_{n-k-1}(M)$ is a k th syzygy of $\text{Ext}^{n-k}(\mathcal{E}_{n-k-1}(M), R)^v$ (see discussion of Horrocks and Matlis duality in §2) which is isomorphic to $\text{Ext}^{n-k}(M, R)^v$. However, the duality of Horrocks [9] yields that $\text{Ext}^{n-k}(M, R)^v = \text{Ext}^{k-1}(M^*, R)$, which is nonzero because M is not a $(k + 1)$ th syzygy.

2. Lifting vector bundles. Throughout this section R will denote a regular local ring of dimension n with $n \geq 3$. We also assume that R contains a field in order that we may apply our results in [4, 5]. The R -module M will be finitely generated and

reflexive and will denote a vector bundle on the punctured spectrum of R , that is, M_P is a free R_P -module for each nonmaximal prime ideal P of R . We shall use S to denote a regular local ring with parameter t such that $S/tS = R$.

If there is a finitely generated reflexive S -module M , which is a vector bundle on the punctured spectrum of S , such that $(M'/tM')^{**}$ is isomorphic to M (the double dual here is taken with respect to R), then we say that M lifts as a bundle to S . Horrocks [10] has asked if a bundle which lifts to arbitrarily high dimension must be trivial, that is, must the module M be free? (For an affirmative answer in the case of projective space see [2, 12, 14].) In [5] we have established this fact in case $\text{Ext}^i(M, R)$ vanishes for $i = 1$ or $i = 2$. In this section we extend this result to include any i for $2 \leq i \leq n - 3$. Hence in view of the duality of Horrocks [9] and our result [5, Theorem 2.5] we establish an affirmative answer to Horrocks' question in case any cohomology $\text{Ext}^i(M, R)$ vanishes for some i in the range $1 \leq i \leq n - 2$. Often we shall express these statements in terms of the sheaf cohomology of M on the punctured spectrum (viewing M as a vector bundle), and so we remind the reader of Horrocks' article [9] and specifically of the isomorphisms

$$H^i(M) \cong \text{Ext}^i(M^*, R) \quad \text{and} \quad H^{n-j-1}(M^*) \cong H^j(M)^v,$$

where $(-)^v$ denotes the Matlis dual (see the end of §1 for a definition of Matlis duality).

Our technique is to show that a lifting of M together with the vanishing of $\text{Ext}^i(M, R)$ for some i with $2 \leq i \leq n - 3$, induces a lifting of $\mathcal{X}_i(M)$. However, in this situation $\mathcal{X}_i(M)$ is at least a third syzygy and thus our result [5, Theorem 2.5] may be applied to $\mathcal{X}_i(M)$. Furthermore we provide bounds derived from intrinsic data on M as to how far one may expect to lift M as a bundle.

The notation $\overline{K}$ indicates the reduction of an S -module K modulo the parameter t , i.e., $\overline{K} = K/tK$.

LEMMA 2.1. *Suppose that the reflexive S -module L lifts M as a bundle to S . Let $0 \rightarrow \mathcal{X}_{q+1}(L) \rightarrow \mathcal{E}_{q+1}(L) \rightarrow L \rightarrow 0$ be a $(q+1)$ th presentation of L over S , where $1 \leq q \leq n - 2$.*

(a) *If $\overline{\mathcal{X}_{q+1}(L)}$ and $\overline{\mathcal{E}_{q+1}(L)}^{**}$ (the double dual with respect to R) have a common free R -summand, then this free summand lifts to a common free S -summand of $\mathcal{X}_{q+1}(L)$ and $\mathcal{E}_{q+1}(L)$.*

(b) *If $\text{Ext}_R^q(M, R) = H^q(M^*) = 0$ for some q with $1 \leq q \leq n - 3$, then $\text{Ext}_S^i(L, S) = H^i(L^*) = 0$ for $i = q, q + 1$ and the sequence*

$$0 \rightarrow \overline{\mathcal{X}_{q+1}(L)} \rightarrow \overline{\mathcal{E}_{q+1}(L)}^{**} \rightarrow M \rightarrow 0$$

is a q th presentation of M over R .

PROOF. Since $\mathcal{X}_{q+1}(L)$ necessarily has depth at least 4 (see Theorem 1.2) it follows that $\overline{\mathcal{X}_{q+1}(L)}$ is necessarily R -reflexive and that the sequence

$$0 \rightarrow \overline{\mathcal{X}_{q+1}(L)} \rightarrow \overline{\mathcal{E}_{q+1}(L)}^{**} \rightarrow M \rightarrow 0$$

is exact (note $\bar{L}^{**} \cong M$). In regard to part (b), the vanishing of the cohomology $H^i(L^*)$ for $i = q, q + 1$ is just a result of the long exact sequence for cohomology (see [5, 1.6]). For the same reason it follows that $\text{Ext}_R^i(\overline{\mathcal{E}_{q+1}(L)}^{**}, R) = 0$ for $1 \leq i \leq q$. Since $\text{Ext}_S^i(L, S) = 0$ for $i = q, q + 1$ it also follows that $\mathcal{X}_{q+1}(L)$ has projective dimension at most $q - 1$ over S . However, the projective dimension of $\mathcal{X}_{q+1}(L)$ over S is the same as the projective dimension of $\overline{\mathcal{X}_{q+1}(L)}$ over R . Part (b) now follows from the uniqueness result (Theorem 1.4). Returning to part (a), we observe that a common free summand F of $\overline{\mathcal{X}_{q+1}(L)}$ and $\overline{\mathcal{E}_{q+1}(L)}^{**}$ as described by the commutative triangle

$$\begin{array}{ccc} \overline{\mathcal{X}_{q+1}(L)} & \rightarrow & \overline{\mathcal{E}_{q+1}(L)}^{**} \\ & \searrow & \swarrow f \\ & F & \\ & \swarrow & \searrow \\ 0 & & 0 \end{array}$$

actually yields an identical one in which the R -double dual of $\overline{\mathcal{E}_{q+1}(L)}$ is removed. For the extension modules $\mathcal{E}_{q+1}(L)$ one always has an epimorphism $\text{Hom}(\mathcal{E}_{q+1}(L), S) \rightarrow \text{Hom}_R(\mathcal{E}_{q+1}(L), R)$, since one always has that the cohomology module $\text{Ext}_S^1(\mathcal{E}_{q+1}(L), S) = 0$. Therefore the map $f: \overline{\mathcal{E}_{q+1}(L)} \rightarrow F$ lifts to an epimorphism $g: \mathcal{E}_{q+1}(L) \rightarrow G$, where $G/\iota G = F$. Further we note that g maps $\mathcal{X}_{q+1}(L)$ onto G , by Nakayama's Lemma, since f takes $\mathcal{X}_{q+1}(L)$ onto F . Thus part (a) is also established.

COROLLARY 2.2 (NOTATION AS ABOVE). *If $0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0$ and $0 \rightarrow \mathcal{X}_{q+1}(L) \rightarrow \mathcal{E}_{q+1}(L) \rightarrow L \rightarrow 0$ are minimal presentations over R and S , respectively, and if $H^q(M^*) = 0$, then $H^q(L^*) = 0$ and $\mathcal{X}_{q+1}(L)$ lifts $\mathcal{X}_q(M)$ as a bundle.*

PROOF. The proof is an immediate consequence of the uniqueness result Theorem 1.7 and Lemma 2.1. For in this situation, Lemma 2.1 guarantees (when $H^q(L^*) = 0$) that $0 \rightarrow \mathcal{X}_{q+1}(L) \rightarrow \overline{\mathcal{E}_{q+1}(L)} \rightarrow L \rightarrow 0$ is a minimal $(q + 1)$ th presentation over S if and only if $0 \rightarrow \mathcal{X}_{q+1}(L) \rightarrow \overline{\mathcal{E}_{q+1}(L)}^{**} \rightarrow M \rightarrow 0$ is a minimal q th presentation over R .

In order to apply our result [5, Theorem 2.5] we would like to obtain an estimate of the ranks of the $\mathcal{X}_q(M)$ in terms of more familiar invariants.

LEMMA 2.3. *Let $\rho_i(M)$ be the rank of the i th syzygy of $\text{Ext}^i(M, R)$ for $i \geq 0$ and let $0 \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{E}_q(M) \rightarrow M \rightarrow 0$ be a minimal q th presentation for M . Then*

$$\text{rank}(\mathcal{X}_q(M)) \leq \sum_{i=1}^q \rho_i(M) \quad \text{for } q \geq 1.$$

PROOF. We induct on q . If $q = 1$, then $\mathcal{X}_1(M)$ is isomorphic to R^μ , where μ is the number of generators of $\text{Ext}^1(M, R)$. However, since $\text{Ext}^1(M, R)$ is torsion, then μ is also the rank of the first syzygy of $\text{Ext}^1(M, R)$.

If $q > 1$, we appeal to the inductive description of the filtration (see discussion after Theorem 1.8 as well as the proof of Theorem 1.2) to obtain a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & L & \rightarrow & \mathcal{X}_q(M) & \rightarrow & \mathcal{X}_{q-1}(M) \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & L & \rightarrow & \mathcal{E}_q(M) & \rightarrow & \mathcal{E}_{q-1}(M) \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & M & = & M \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

in which L has at most one nonzero cohomology, namely $\text{Ext}^{q-1}(L, R)$. This property of L is inherent in the inductive step in the proof of Theorem 1.2 (particularly in view of the dual exactness properties of the complex G). While we may assume that the $(q-1)$ th presentation in the above diagram is minimal we cannot assume the same about the q th presentation (middle column), since it may be necessary to add on a free module in order to keep the rows exact. However, the desired inequality will follow from the subsequent argument since the rank of the “minimal $\mathcal{X}_q(M)$ ” cannot be larger than that of $\mathcal{X}_q(M)$ in the diagram.

Returning to the module L we have that

$$\text{Ext}^{q-1}(L, R) \cong \text{Ext}^{q-1}(\mathcal{X}_q(M), R) \cong \text{Ext}^q(M, R) \quad \text{for } q \geq 2.$$

Since $\text{Ext}^1(\mathcal{E}_{q-1}(M), R) = 0$, the sequence $0 \rightarrow \mathcal{E}_{q-1}(M)^* \rightarrow \mathcal{E}_q(M)^* \rightarrow L^* \rightarrow 0$ is exact. As a consequence we may assume that L^* and therefore also L is constructed without nontrivial free summands. Thus, if L is nonzero, then L is a bundle with one nonzero cohomology module, $\text{Ext}^{q-1}(L, R)$. Therefore L^* is a q th syzygy for $\text{Ext}^{q-1}(L, R) = \text{Ext}^q(M, R)$. But L and L^* have the same rank. It follows from the exact sequence $0 \rightarrow L \rightarrow \mathcal{X}_q(M) \rightarrow \mathcal{X}_{q-1}(M) \rightarrow 0$ that $\text{rank } \mathcal{X}_q(M) = \text{rank } L + \text{rank } \mathcal{X}_{q-1}(M)$. Finally by induction we obtain the desired inequality $\text{rank } \mathcal{X}_q(M) \leq \sum_{i=1}^q \rho_i(M)$, where $\mathcal{X}_q(M)$ is minimal.

We can now establish the bounds on the lifting of a vector bundle M having a vanishing cohomology module. We keep the notation of the previous discussion.

THEOREM 2.4. *Let M be a nonfree, reflexive R -module which represents a vector bundle on the punctured spectrum of R . If $H^q(M^*) \cong \text{Ext}^q(M, R) = 0$ for some q with $1 \leq q \leq \dim R - 2$, then M can be lifted at most $\sum_{i=0}^{q-1} \rho_i(M)$ times.*

PROOF. If $q = 0$ or 1 , then the result follows from our calculations in [5, Theorems 2.5 and 2.6]. So suppose that neither $\text{Ext}^1(M, R)$ nor $\text{Ext}^2(M, R)$ are zero, but that $\text{Ext}^q(M, R)$ is zero for some $q > 2$. By duality with M^* and the case $q = 0$ we may assume that $q \leq \dim R - 4$. By Theorem 1.8(c), $\mathcal{X}_q(M)$ cannot be free. However, by Corollary 2.2, we have that $\mathcal{X}_q(M)$ lifts every time that M does. From Lemma 2.3, $\mathcal{X}_q(M)$ has rank less than or equal to $\sum_{i=0}^{q-1} \rho_i(M)$ (note $\rho_q(M) = 0$) and has depth at

least $\dim R - q + 1 \geq 3$. Again we appeal to [5, Theorem 2.5] and find that $\mathcal{K}_q(M)$ cannot be lifted more than $[\sum_{i=0}^{q-1} \rho_i(M)] - (\dim R - q + 1)$ times. Thus our argument is complete.

We remark that one could obtain finer bounds depending on which (or how many) $\text{Ext}^q(M, R)$ vanish. However these can readily be obtained if needed by combining Theorems 2.5 and 2.6 of [5] together with Corollary 2.2 and Lemma 2.3 in individual cases. We also remark that M lifts if and only if M^* does, and moreover the cohomology modules are dual via Matlis duality. Thus one may restrict to the case of $\text{Ext}^q(M, R) = 0$ for q at most $\frac{1}{2}(\dim R + 1)$. This suggests that perhaps the number $\sum_{i=0}^h \rho_i(M)$ is an upper bound for the number of times M can be lifted for general M , where h is the greatest integer in $\frac{1}{2}(\dim R + 1)$.

ADDED IN PROOF. We have recently learned that the existence of q th presentations (Theorem 1.2) and also Corollary 1.7 were first noted by Auslander and Bridger [1].

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