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ON THE IDEALS OF A NOETHERIAN RING

BY

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ABSTRACT. We construct various examples of Noetherian rings with peculiar ideal structure. For example, there exists a Noetherian domain R with a minimal, nonzero ideal I, such that R/I is a commutative polynomial ring in n variables, and a Noetherian domain S with a (second layer) clique that is not locally finite. The key step in the construction of these rings is to idealize at a right ideal I in a Noetherian domain T such that T/I is not Artinian.

Introduction. The aim of this paper is to construct several examples concerned with the ideal structure of a Noetherian ring. We will delay discussion of their relevance until we are ready to construct them (in §§2, 3, and 4) and simply describe their properties at this stage. However, for the most part, these examples complement the positive results in the author's survey article [**12**].

EXAMPLE 1 (see THEOREM 2.3). *Let m be a positive integer. There exists a Noetherian domain R with a unique, minimal, nonzero ideal I such that R/I is a polynomial ring in m variables. In particular, R has infinitely many prime ideals.*

EXAMPLE 2 (THEOREM 3.2). *There exists a prime Noetherian ring R with Goldie rank two, written* $\operatorname{Grk} R = 2$, *yet for each $n \in \mathbf{N}$ there exists a prime ideal Q_n of R with* $\operatorname{Grk} R/Q_n = 2n + 1$. *Further,* $\bigcap Q_n = 0$.

EXAMPLE 3 (THEOREM 4.1). *There exists a Noetherian ring R and two prime ideals P_1 and P_2 of R such that there exists an ideal link between P_1 and P_2 but such that P_1 and P_2 do not belong to the same (second layer) clique.*

EXAMPLE 4 (THEOREM 4.4). *There exists a Noetherian domain R and a (second layer) clique of prime ideals of R that is not locally finite. This clique also satisfies the second layer condition.*

The connecting theme between these examples, and the crucial step in their construction, is the concept of the idealizer $\mathbf{I}(I) = \{ f \in S: fI \subseteq I \}$ of a right ideal I in a Noetherian ring S. Of course, since Robson's seminal paper [**8**] this has been a well-known and productive method of producing interesting examples. However, in previous examples S/I has been Artinian. Unfortunately this only produces finitely many "new" ideals, whereas the present examples are concerned with the properties of an infinite set of prime ideals. Thus we are forced to idealize at a right ideal I such that S/I is not Artinian. In general, such an idealizer will not be left or right

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Noetherian. Even worse, one seems to need very detailed information about the lattice of subfactors of S/I to even be able to test for Noetherianness, as can be seen from the results in §1. Fortunately, such problems can be bypassed by a judicious choice of the ring S and right ideal I, and we believe that this method will prove useful whenever one wishes to test problems concerned with an infinite set of ideals.

Examples 1 and 2, together with some of the results from [**12**], appeared in an earlier, unpublished version of this paper that was circulated privately in 1981.

1. Idealizers. In this section we collect various elementary results about the Noetherianness of idealizers that will prove useful for our examples. If I is a right ideal in a ring R, then the *idealizer* of I, written $\mathbf{I}(I)$, is the subring $\{f \in R: fI \subseteq I\}$. If A and B are subsets of R, write $(B : A) = (B : A)_R = \{f \in R: fA \subseteq B\}$. Observe that, if A and R are right ideals of R, then $(B:A)/B$ and $\operatorname{Hom}_R(R/A, R/B)$ are naturally isomorphic as right $\mathbf{I}(A)$-modules [**8**, Proposition 1.1]. In particular, $\mathbf{I}(A)/A \cong \operatorname{End}_R(R/A)$.

LEMMA 1.1. *Let I be a right ideal of a right Noetherian ring R and suppose that R/I is a Noetherian right $\mathbf{I}(I)$-module. Then $\mathbf{I}(I)$ is a right Noetherian ring.*

PROOF. This is a slightly stronger version of [**8**, Proposition 2.3(i)], but the proof is the same.

Lemma 1.1 is not, in practice, very easy to apply, and the following consequence of it will be more appropriate to our needs.

LEMMA 1.2. *Let I be a right ideal of a right Noetherian ring R. Assume, for every right ideal $J \supseteq I$ of R, that* $\operatorname{Hom}(R/I, R/J)$ *is right Noetherian as a right $\mathbf{I}(I)$-module. Then $\mathbf{I}(I)$ is right Noetherian.*

PROOF. By Lemma 1.1 we need only prove that R/I is a Noetherian right $\mathbf{I}(I)$-module. By hypothesis, $\mathbf{I}(I)/I \cong \operatorname{Hom}_R(R/I, R/I)$ is Noetherian, so let B be a right $\mathbf{I}(I)$-submodule of R with $\mathbf{I}(I) \subseteq B$. By the Noetherianness of R, let J be the largest right ideal of R of the form $J = \sum_1^n b_i I$ for some $b_i \in B$. Of course, as $B \supseteq \mathbf{I}(I)$ we have $I = 1I \subseteq J$. Let $C = (J:I)$. There are two possibilities. First, suppose that $B \not\subseteq C$ and pick $b \in B \setminus C$. Then $bI \not\subseteq J$ so $J \subsetneqq bI + J \subseteq B$, contradicting the choice of J. Thus $B \subseteq C = C + \mathbf{I}(I)$. By hypothesis, C/J is Noetherian, and hence so is the subfactor $B/\sum_1^n b_i \mathbf{I}(I)$. Thus B is finitely generated.

It is worth making a few remarks about Lemma 1.2. First, we have not assumed that $RI = R$, as is typically the case, for example, in [**8**]. Indeed, if R/I is semisimple there is no loss of generality in this assumption [**8**, Proposition 1.7]. In the situation that concerns us, however, it would be a considerable restriction, since, in constructing Examples 2–4 we need to idealize at a right ideal I such that $RI \neq R$. Secondly, in order to apply Lemma 1.2, one needs to know a lot of information about the right ideals that contain the right ideal at which one wishes to idealize. Although this is easy enough for particular R, it does mean that the result is almost useless in the abstract. It would be interesting to know whether there is any result in this direction that is significantly more informative than Lemma 1.2.

Unfortunately the last two lemmas give no information about when the idealizer at a right ideal is *left* Noetherian (and the answer seems to be "rarely"). We will completely avoid this problem by using the following trick.

LEMMA 1.3. *Let x be a regular element in a prime, Noetherian ring R such that there exists an antiautomorphism σ of R satisfying $\sigma(xR) = Rx$. Then $\mathbf{I}(xR)$ is right Noetherian if and only if it is left Noetherian.*

PROOF. Let Q be the simple Artinian quotient ring of R and θ the automorphism $q \to xqx^{-1}$ of Q. The point behind the lemma is that, if $S = \theta(R) = xRx^{-1}$, then $\mathbf{I}(xR)$ is the left idealizer $\mathbf{I}_S(Sx)$. Thus if $\mathbf{I}(xR)$ is right Noetherian, then applying σ shows that $\mathbf{I}_R(Rx) = \sigma(\mathbf{I}(xR))$ is left Noetherian. Applying θ therefore implies that $\mathbf{I}_S(Sx) = \theta(\mathbf{I}_R(Rx))$ is also left Noetherian.

Another instance when a subring of a Noetherian ring is Noetherian is provided by the following lemma, which is presumably well known but does not seem to have appeared in the literature.

LEMMA 1.4. *Let $R \subset S$ be algebras over a central field k such that S is Noetherian and S/R is a finite-dimensional k-vector space. Then R is Noetherian.*

PROOF. Let $K = \{f \in S\colon SfS \subseteq R\}$. Thus K is an ideal of S such that $K \subseteq R$. We claim, further, that R/K is finite dimensional over k. For, let $S = \sum_1^m s_i k + R$, put $s_0 = 1$, and $K_{ij} = \{f \in S\colon s_i f s_j \in R\}$ for $0 \leqslant i, j \leqslant m$. If $f_1, \ldots, f_{m+1} \in R \setminus K_{ij}$ then $\{s_i f_k s_j\colon 1 \leqslant k \leqslant m+1\}$ must be linearly dependent in S/R. Thus $\sum \lambda_k s_i f_k s_j \in R$ for some $\lambda_k \in k$ and $\sum \lambda_k f_k \in K_{ij}$. This ensures that each R/K_{ij} is at most m-dimensional. As $K = \bigcap K_{ij}$, this proves the assertion.

Now let I be a right ideal of R. Then IK is a finitely generated right S-module, and hence is a finitely generated right R-module. Now IS/IK is a finitely generated right S/K-module and so is a finite-dimensional k-module. Thus I/IK is certainly finitely generated as an R-module.

We end this section by noting a triviality that will prove useful.

LEMMA 1.5. *Let R be a k-algebra, for some field k, and suppose that* $\operatorname{End}_R(M)$ *is a finite-dimensional k-vector space for every simple R-module M. Then* $\operatorname{Hom}_R(A, B)$ *is a finite-dimensional k-vector space for every pair of Artinian R-modules A and B.*

2. A subdirectly irreducible Noetherian domain. One of the basic building blocks for Noetherian rings is the class of subdirectly irreducible (SDI) rings. Moreover, the results in [**11**] suggest that one ought to be able to reduce many questions about Noetherian rings to the case of prime, Noetherian, SDI rings. Unfortunately, there seem to be no deep results about prime SDI rings and few examples. Indeed, the only concrete examples in the literature are the primitive factor rings of enveloping algebras of semisimple Lie algebras and idealizers inside simple rings. However, all of the former and all known examples of the latter have only finitely many prime ideals and even have DCC on ideals. In this section we provide an example of a Noetherian, SDI domain that has neither of these properties, since it has a commutative polynomial ring as a factor. This answers the question raised in [**5**, p. 167].

Let us begin by giving the idea behind the construction. So, we want a ring R with a unique, minimal, nonzero ideal E such that R/E has many ideals. The first of these two conditions is most easily satisfied by taking $R = \mathbf{I}(I)$, where I is a right ideal in a simple ring S, as then $E = I$. The obvious way to ensure that R/I has many ideals is to demand that R/I be a large commutative ring. This is likely to be the case if S is an Ore extension of a commutative subring C and $I = xS$ for some $x \in C$, since then the only obvious elements in $\mathbf{I}(I)/I$ will be those from C/xC.

Let $C = k[x_1,\ldots,x_n]$ be a polynomial extension of an algebraically closed field k of characteristic zero. Define a derivation δ on C by $\delta(x_1) = 1$ and $\delta(x_i) = x_i x_{i-1} - 1$ for $i > 1$. The *Ore extension* $S = C[y;\delta]$ is the ring that, additively, is isomorphic to the polynomial extension $C[y]$, but multiplication is defined by $yf = fy + \delta(f)$ for $f \in C$.

LEMMA 2.1. *The Ore extension $S = C[y;\delta]$ is a simple Noetherian domain.*

PROOF. This appears in [**1**], but since it has not been published, we will outline the proof. The only nontrivial part of the proof is to show that S is simple, for which it suffices to show that no proper ideal of C is left invariant by δ.

Write $D_t = k[x_1,\ldots,x_t]$ for $1 \leqslant t \leqslant n$ and suppose, for some r, that $\Sigma\delta^i(g)D_{r-1} = D_{r-1}$ for all $g \neq 0 \in D_{r-1}$ (this is obviously true for $r = 2$). Let $f = \Sigma_0^m x_r^i f_i \in D_r$, where each $f_i \in D_{r-1}$ and $f_m \neq 0$. Then $\delta(f) - mx_{r-1}f$ has leading term $x_r^m\delta(f_m)$. By the inductive hypothesis we may therefore replace f by some element in $\Sigma\delta^i(f)$ and assume that $f_m = 1$. But now

$$\delta(f) - mx_{r-1}f = \sum_0^{m-1} x_r^i\{\delta(f_i) + (i-m)f_i x_{r-1} + (i+1)f_{i+1}\}.$$

An easy degree argument shows that the coefficient of x_r^{m-1} in this expression is nonzero, so induction completes the proof.

LEMMA 2.2. *Let J be a right ideal of S such that $x_1 \in J$. Then* (i) $J = (J \cap C)S$, *and* (ii) $(J : x_1S) = J + C$.

PROOF. (i) If $J \neq (J \cap C)S$, pick $f = \Sigma_0^m f_i y^i \in J \setminus (J \cap C)S$, where each $f_i \in C$ and m is as small as possible. Then

$$\sum_0^m i f_i y^{i-1} = fx_1 - x_1 f \in J.$$

Thus, by the inductive hypothesis, each f_i, for $i > 1$, lies in $(J \cap C)S$. In particular, $f_m \in (J \cap C)S$ and so $f - f_m y^m = \Sigma_0^{m-1} f_i y^i \in J \setminus (J \cap C)S$, a contradiction.

(ii) Let $K = (J : x_1S)$. Then certainly $J + C \subseteq K$. Thus, suppose that $f = \Sigma_0^m f_i y^i \in K \setminus J + C$ (of course this implies that $m > 0$). Then, by the definition of K,

$$\sum_0^m i f_i y^{i-1} = fx_1 - x_1 f \in J.$$

By part (i), therefore, $f_m \in J$ and once again induction completes the proof.

We can now construct our SDI ring.

THEOREM 2.3. *With S as above, let $R = \mathbf{I}(x_1S)$. Then $R = C + x_1S$. Further, R is a Noetherian domain with a unique, minimal, nonzero ideal $x_1S = I$. Thus, $R/I \cong k[x_1,\ldots,x_{n-1}]$ and certainly has infinitely many prime ideals.*

REMARKS. (i) Any multiplicatively closed subset of elements of C forms an Ore set in R. Thus, by inverting such a set we may, in the statement of the theorem, replace C by any localisation of itself. In particular, one can require that R/I is a local commutative ring of Krull dimension n.

(ii) One can generalise Theorem 2.3 in another direction and produce a Noetherian, SDI domain R with infinite (classical) Krull dimension. For, repeat the construction of R as in the theorem, but with C a polynomial ring in infinitely many variables. One may then make R Noetherian by means of a gang localisation in C.

PROOF. Clearly, R is a domain. Further, $R = C + x_1S$ by Lemma 2.2(ii) applied to $J = x_1S$. By Lemma 2.2(ii) and Lemma 1.2, R is right Noetherian. Since the map σ that fixes C and sends y to $-y$ is an antiautomorphism of S, Lemma 1.3 implies that R is also left Noetherian. Finally, if J is an ideal of R then $J \supseteq xSJxS = xS$, since S is simple. Thus xS is indeed the smallest nonzero ideal of R.

3. Goldie rank of prime factor rings. The *Goldie rank*, or *uniform dimension*, of a prime Goldie ring R will be denoted by $\mathrm{Grk}(R)$. Given any positive integers m and n it is easy to construct a prime Noetherian ring R and a prime ideal P in R such that $\mathrm{Grk}(R) = n$ while $\mathrm{Grk}(R/P) = m$; and so one might suppose that there exists no relationship between the Goldie rank of a prime ring R and those of its prime factors. Yet surprisingly, there is a relationship, in the sense that prime ideals with bad Goldie ranks do not occur too frequently.

PROPOSITION 3.1 [**12**, COROLLARY 3.9]. *Let R be a prime Noetherian ring and n a positive integer such that* $\mathrm{Grk}(R)$ *does not divide n. Then*

$$\bigcap\{P \text{ a prime ideal}: \mathrm{Grk}(R/P) = n\} \neq 0.$$

In particular,

$$\bigcap\{P \text{ a prime ideal}: \mathrm{Grk}(R/P) < \mathrm{Grk}(R)\} \neq 0.$$

If R is a fully bounded ring, then Proposition 3.1 can be improved to say that

$$\bigcap\{P \text{ a prime ideal of } R: \mathrm{Grk}(R) \text{ does not divide } \mathrm{Grk}(R/P)\} \neq 0$$

(see [**13**, Theorem 4]). This raises the question of whether this stronger result holds in general. In this section we show that the answer is "no" by proving

THEOREM 3.2. *There exists a prime Noetherian ring R, with* $\mathrm{Grk}\, R = 2$, *such that for each integer $n \geqslant 0$ there exists a prime ideal Q_n with* $\mathrm{Grk}\, R/Q_n = 2n + 1$. *Further,* $\bigcap Q_n = 0$.

Once again we begin by motivating the construction. Let S be a prime Noetherian ring, containing a central field k, such that $\mathrm{Grk}\, S = 2$ and such that for every $n > 0$ there exists a prime ideal Q_n such that $S/Q_n \simeq M_{2n}(k)$. (This is easy to arrange; we will use $S = M_2(U)$, where U is the enveloping algebra of the Lie algebra $\mathrm{Sl}_2(\mathbf{C})$.) If

one idealizes S at a maximal right ideal $V_n \supset Q_n$, then $R = \mathbf{I}(V_n)$ has exactly two maximal ideals that contain Q_n: one being V_n and the other (say) W_n. Since $\mathrm{Grk}(R/V_n) = 1$ this forces $\mathrm{Grk}\, R/W_n = 2n - 1$. The idea behind the proof is therefore to idealize at a right ideal I of S such that $\mathbf{I}(I) + Q_n = \mathbf{I}(V_n)$ for each n. This is likely to happen if, for each n, $I + Q_n$ is a maximal right ideal. Such right ideals I are easy to find inside $S = M_2(U)$.

Thus, write $U = U(\mathrm{Sl}_2(\mathbf{C}))$ for the enveloping algebra of $\mathrm{Sl}_2(\mathbf{C})$. We take $\{e, f, h\}$ for the standard basis of Sl_2; so $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$. The basic properties of U that we use can, for example, be found in [**4** or **10**]. We begin by considering idealizers in U.

LEMMA 3.3. *Let $T = \mathbf{I}_U(eU)$ be the idealizer. Then $T = eU + \mathbf{C}[h]$.*

PROOF. As $he = e(h + 2)$, certainly $A = eU + \mathbf{C}[h] \subseteq \mathbf{I}(eU) = T$. Conversely, suppose that $r = \sum_0^m f^i \lambda_i(h) \in T \setminus A$ for some $\lambda_i \in \mathbf{C}[h]$. Then $m \geqslant 1$ and

$$
\begin{aligned}
(1) \qquad re &= \sum_0^m f^i e \lambda_i(h + 2) \\
&= e \sum f^i \lambda_i(h + 2) - \sum_{i=0}^{m} \sum_{j=0}^{i-1} f^{i-j-1} h f^j \lambda_i(h + 2) \\
&\equiv \sum_0^m f^{i-1} \left\{ - \sum_{j=0}^{i-1} (h - 2j) \right\} \lambda_i(h + 2) \mod eU.
\end{aligned}
$$

As $m > 0$, this final term is nonzero and so cannot be contained in eU, a contradiction. Thus $A = T$ as required.

LEMMA 3.4. *Let $\Omega = h^2 - 2h + 4ef$ be the Casimir element. If I is a right ideal of U, with $eU \subsetneqq I$, then $I \cap \mathbf{C}[\Omega] \neq 0$.*

PROOF. The reader is reminded that Ω generates the center of U. It follows from (1) that $I \cap \mathbf{C}[h] \neq 0$, say $\prod(h - c_i) \in I$ for some $c_i \in \mathbf{C}$. Now

$$(h - c_i)U + eU \ni (h - c_i)(h + c_i - 2) + 4ef = \Omega - c_i(c_i - 2).$$

It follows easily that $I \cap \mathbf{C}[\Omega] \neq 0$.

It is now easy to prove

PROPOSITION 3.5. *The idealizer $T = \mathbf{I}_U(eU)$ is a Noetherian domain.*

PROOF. As U is a domain, so is T. In order to prove that T is right Noetherian we apply Lemma 1.2. By Lemma 3.3, $\mathrm{Hom}_U(U/eU, U/eU) \simeq \mathbf{C}[h]$ is certainly Noetherian. Let J be a right ideal of U with $eU \subsetneqq J$. By Lemma 3.4 there exists $f = \prod(\Omega - c_i) \in J$ for some $c_i \in \mathbf{C}$. Now $U/eU + (\Omega - c_i)U$ is certainly Artinian for each i [**10**, Theorem 3.2]. Since Ω is central, it follows easily that U/J is also Artinian. Further, as f is central,

$$X = \mathrm{Hom}_U(U/eU, U/J) \cong \mathrm{Hom}(U/eU + fU, U/J).$$

Thus, by Lemma 1.5 combined with Quillen's Lemma, X is a finite-dimensional $\mathbf{C}$-vector space. Thus, by Lemma 1.2, T is right Noetherian. Finally, multiplication

by -1 on Sl_2 induces an antiautomorphism on U. Thus Lemma 1.3 implies that T is also left Noetherian.

COROLLARY 3.6. *If* $R = \left(\begin{smallmatrix} T & eU \\ U & U \end{smallmatrix}\right) \subset M_2(U)$, *then* R *is a prime Noetherian ring, with* $\mathrm{Grk}(R) = 2$.

This ring R is the one that interests us, as we have

PROPOSITION 3.7. *Let* P_n *be the annihilator of the n-dimensional simple right U-module and set* $I_n = R \cap M_2(P_n)$. *Then for all* $n > 1$ *there exist exactly two maximal ideals, say* B_n *and* C_n, *of* R *that contain* I_n. *Further (after relabelling),* $\mathrm{Grk}(R/C_n) = 2n - 1$, *while* $\mathrm{Grk}(R/B_n) = 1$, *for each* n. *For any infinite subset* Y *of* $\{2, 3, 4, \dots\}$ *we have* $\bigcap\{C_n : n \in Y\} = 0$.

PROOF. We first show that $T + P_n = \mathbf{I}_U(eU + P_n)$. Note that, by [**4**, p. 31], $U/P_n \cong M_n(\mathbf{C})$ and $e^n \in P_n$, while $e^{n-1} \notin P_n$. It follows easily that $eU + P_n$ is a maximal right ideal of U. Thus, $\mathrm{End}_U(U/eU + P_n) \simeq \mathbf{C}$ by Quillen's Lemma, and so $\mathbf{I}(eU + P_n) = \mathbf{C} + eU + P_n$. Since $eU + P_n \ni h - c$ for some $c \in \mathbf{C}$, this implies that $\mathbf{I}(eU + P_n) = \mathbf{C}[h] + eU + P_n = T + P_n$, as claimed.

Note that this implies that

$$\begin{aligned} R + M_2(P_n) &= \begin{pmatrix} \mathbf{C} + eU + P_n & eU + P_n \\ U & U \end{pmatrix} \\ &= \mathbf{I}_{M_2(U)} \begin{pmatrix} eU + P_n & eU + P_n \\ U & U \end{pmatrix}. \end{aligned}$$

Thus, identifying $eU + P_n/P_n$ with the bottom $n - 1$ rows of $M_n(\mathbf{C}) \cong U/P_n$, we obtain that

$$R + M_2(P_n)/M_2(P_n) \cong \left\{(f_{ij}) \in M_{2n}(\mathbf{C}) : f_{1j} = 0 \text{ for } j > 1\right\} \subset M_{2n}(\mathbf{C})$$

(see [**8**, Lemma 3.1 and Proposition 3.3]). This means that there are exactly two prime ideals of $R + M_2(P_n)$ containing $M_2(P_n)$, say B_n' and C_n', where $R + M_2(P_n)/B_n' \cong \mathbf{C}$ and $R + M_2(P_n)/C_n' \cong M_{2n-1}(\mathbf{C})$.

Let B_n and C_n be the inverse images of B_n' and C_n' in R. Thus, $\mathrm{Grk}(R/C_n) = 2n - 1$, while $\mathrm{Grk}(R/B_n) = 1$ for each n. Let Y be an infinite subset of $\{2, 3, \dots\}$. Then it only remains to show that, if $K = \bigcap\{C_n : n \in Y\}$, then $K = 0$. Since $\mathrm{Grk}\, R/B_n < 2 = \mathrm{Grk}\, R$, for each n, Proposition 3.1 implies that $J = \bigcap B_n \neq 0$ (alternatively, one may check that $\left(\begin{smallmatrix} e & 0 \\ 0 & 1 \end{smallmatrix}\right) \in J$). Now suppose that $K \neq 0$. Then

$$M_2(\bigcap\{P_n : n \in Y\}) \supseteq \bigcap\left\{(B_n \cap C_n)^2 : n \in Y\right\} \supseteq (J \cap K)^2 \neq 0.$$

Thus $\bigcap\{P_n : n \in Y\} \neq 0$, which is absurd.

4. Links between prime ideals. The most obvious obstacle to being able to localise at a prime ideal P in a Noetherian ring R is the existence of a link between the ideal P and a second prime ideal Q. Actually, there are two definitions of a link in the literature (both of which prevent localisations). First it could be a *second layer* link, written $P \rightsquigarrow Q$; that is, there exists an ideal A with $PQ \subseteq A \subset P \cap Q$ such that $P \cap Q/A$ is nonzero and torsion-free as both a left R/P- and right R/Q-module.

More generally, it could be an *ideal link*; that is, there exist ideals $B \subsetneqq A$ of R such that $PA + AQ \subseteq B$, but, again, A/B is torsion-free as both a left R/P- and right R/Q-module. In the two cases define the *clique* or link class to be the equivalence class defined by the relevant definition of link. Surprisingly, for most of the standard classes of Noetherian rings—for example, fully bounded (FBN) rings, group rings of polycyclic-by-finite groups, and enveloping algebras of finite-dimensional solvable, or semisimple, Lie algebras—the two concepts coincide. (For the first three cases see [**6**, Appendix]. The final case is an unpublished result of W. Borho and, as far as the author is aware, requires the Kazhdan-Lusztig conjecture in its proof.) This prompts the question of whether the two kinds of cliques are always equal—a question which we answer in the negative in this section.

Although cliques are usually infinite, they are not that large, for second layer cliques are always countable [**12**, Corollary 3.13]. Furthermore, for each of the four classes of rings mentioned above, the second layer cliques are *locally finite*; that is, for any prime ideal P there exist only finitely many prime ideals Q such that either $P \rightsquigarrow Q$ or $Q \rightsquigarrow P$ (use [**12**, Corollary 3.10; **2**, Theorem 6.4]). In this section we also provide an example of a ring in which second layer cliques are not locally finite. For further information on cliques and localisation the reader is referred to [**6**] or, for a brief survey, [**12**].

We begin by showing that the two concepts of cliques are distinct, for which we slightly modify the example from the last section. Thus, let $U = U(\mathrm{Sl}_2(\mathbf{C}))$, $T = \mathbf{I}_U(eU) = eU + \mathbf{C}[h]$, and $\Omega = h^2 - 2h + 4ef$. Let $M = eU + fU + hU$ be the augmentation ideal of U and set $R = eM + \mathbf{C}[h] \subset T$. Observe that, as $f \in M$, certainly $\Omega \in R$. The ring R provides the counterexample, although for simplicity we state the result for a factor ring.

THEOREM 4.1. *With R as above, write $V = R/\Omega R$. Then V is a Noetherian ring with exactly two maximal ideals, say P_1 and P_2. Further, P_1 and P_2 lie in the same ideal clique but distinct second layer cliques. The lattice of ideals of V is*

$$\begin{array}{c}
V \\
\wedge \\
P_1 \; P_2 \\
\vee \\
P_1 \cap P_2 = (P_1 \cap P_2)^2 = P_1P_2 = P_2P_1 \\
| \\
N(V) = N \\
| \\
0 = P_1N = NP_2
\end{array}$$

PROOF. R is a Noetherian domain, by Lemma 1.4 and Proposition 3.5. Next, it is easy to write the four ideals of R that strictly contain ΩR. Observe that $e \notin R$, so

$\Omega e \notin \Omega R$. However, since Ω is a central element and $\Omega \in M$, we have $\Omega eU = e\Omega U \subset R$. Thus, $I = \Omega eU + \Omega R$ is an ideal of R that strictly contains ΩR. In fact, since $T = eU + \mathbf{C}[h]$, we have $I = \Omega T$. Strictly containing I is the ideal

$$J = eM + \Omega R = eM + h(h-2)\mathbf{C}[h].$$

Obviously, J is the intersection of the two maximal ideals $Q_1 = eM + (h-2)\mathbf{C}[h]$ and $Q_2 = eM + h\mathbf{C}[h]$.

We next show that these ideals satisfy the relations demanded by the theorem. It is well known (and easy to prove) that $UeU = M = M^2$. Thus, $(eM)^2 = eM(UeU)M = eM$. Therefore, certainly, $J = J^2 = Q_1Q_2 = Q_2Q_1$. Direct calculations show that $TQ_2 \subset R$ and $Q_1T \subset R$ (the latter follows easily from the fact that $(h-2)e = eh$). Thus, $IQ_2 \subset \Omega R$ and $Q_1I = \Omega Q_1T \subset \Omega R$.

To complete the proof we must show that the four ideals described above are the only ideals of R that contain ΩR. We begin by showing that $I = \Omega U \cap R$, for which it suffices to prove that $\Omega U \cap T = \Omega T$. Now $\Omega T = \Omega eU + \Omega\mathbf{C}[h]$. Thus, if $p \in \Omega U \cap T \setminus \Omega T$, then $p = \Omega q$ for some $q \in \mathbf{C}[f, h] \setminus \mathbf{C}[h]$. But then

$$h(h-2)q = \Omega q - 4efq \in T \cap \mathbf{C}[f, h] = \mathbf{C}[h],$$

a contradiction. Thus, $I = \Omega U \cap R$. Note that, as $U/\Omega U$ is a domain, this forces $I/\Omega R = N(V)$, the nilradical of V. Also, since $I/\Omega R = \Omega T/\Omega R \cong T/R$ is a one-dimensional $\mathbf{C}$-vector space, there are no ideals lying between I and ΩR. Now, let $x \in R \setminus I$. As ΩU is completely prime and $e \notin \Omega U$, certainly $xe \notin \Omega U$. Since M is the unique ideal of U that strictly contains ΩU (see, for example, [**10**, §2]), this implies that $UxeU + \Omega U \supseteq M$. Thus,

$$\begin{aligned} RxR + \Omega R &\supseteq eMxeM + \Omega R \supseteq eM\{UxeU + \Omega U\}M + \Omega R \\ &\supseteq eM^3 + \Omega R = J. \end{aligned}$$

This completes the proof.

We next construct an example of a Noetherian ring with a clique that is not locally finite, but as in the previous sections we begin with the idea behind the example. Consider, for a moment, the rings $T = \mathbf{I}(eU) \subset U$ of §3. By idealizing eU, each of the maximal ideals P_n of U was split into two linked prime ideals; call them V_n and W_n, where $eU \subset V_n$. Unfortunately there are no links between the ideals V_n, so this does not provide the appropriate example. However, each V_n does contain eU so the V_n are only distinct because $\mathbf{I}(eU)/eU \cong \mathbf{C}[h]$ is large. Thus, instead of U we need a Noetherian k-algebra S and a right ideal I such that (a) as before, S/I is a 1-critical module with infinitely many, nonisomorphic, finite-dimensional factor modules N_n; but now, (b) $\mathrm{End}_S(S/I) = k$. For, in passing to $\mathbf{I}(I)$ each $\mathrm{ann}(N_n)$ will again split into a pair of linked prime ideals, but now one of these pairs must be I. Thus, the clique of I will not be locally finite.

A ring S with the required properties can be obtained as a group ring. Let $G = H\langle z\rangle$ be an infinite cyclic extension of a free abelian group H, such that H has finite rank $n \geqslant 2$ and H is a plinth in G. This means that, for all subgroups G' of finite index in G/H, the ring $\mathbf{Q}G'$ acts irreducibly on $H \otimes_{\mathbf{Z}} \mathbf{Q}$. For example, take $H = \langle x, y\rangle$ and let $x^z = xy$ and $y^z = x^2y$. Write $S = kG$, where k is an algebraically closed, absolute field. The next lemma describes the relevant properties of S.

LEMMA 4.2. *Let $M = S/(1-z)S$. Then M is a 1-critical S-module. Furthermore, M has infinitely many nonisomorphic simple factor modules N_i, each of which is finite dimensional over k.*

PROOF. Any right ideal J which strictly contains $(1-z)S$ is of the form $J = (1-z)S + (J \cap kH)S$. Since $J \cap kH$ is then G/H-invariant, Bergman's Theorem [7, Corollary 3.9, p. 386] implies that J is of finite codimension in S. Thus, M is either 1-critical or Artinian. Conversely, by [7, Lemma 3.5, p. 548] there are infinitely many comaximal, G/H-invariant ideals T_i in kH. Thus, there exist infinitely many nonisomorphic simple factors $N_i = S/T_iS + (1-z)S$ of M.

LEMMA 4.3. *Let $R = \mathbf{I}_S(1-z)S$. Then $R = k + (1-z)S$ and is a Noetherian domain.*

PROOF. Since G is poly (infinite cyclic), S and, hence, R are domains. If $f \in kH \setminus k$ then $f(z-1) = (z-1)f^z + f^z - f$. Since rank $H \geqslant 2$, Bergman's Theorem implies that $f^z \neq f$. Thus, $\mathbf{I}_S(1-z)S = k + (1-z)S$. If $J \supsetneqq (1-z)S$ then, by Lemma 4.2, S/J is finite dimensional and, hence, so is $S/\operatorname{ann} J$. Thus,

$$\operatorname{Hom}_S(S/(1-z)S, S/J) \cong \operatorname{Hom}_S(S/(1-z)S + \operatorname{ann} J, S/J)$$

is also finite dimensional. By Lemma 1.2, therefore, R is right Noetherian. The antiautomorphism σ of S that fixes H and sends z to z^{-1} certainly sends $(1-z)S$ to $S(1-z)$. Thus, by Lemma 1.3, R is also left Noetherian, as required.

Finally we can show that R contains a clique that is not locally finite.

THEOREM 4.4. *Let $R = k + (1-z)S$ and $P = (1-z)S$. Then R is a Noetherian domain such that the second layer clique Ω of P is not locally finite.*

REMARKS. (i) We actually prove that there exist infinitely many prime ideals $\{Q_i: i \in K\}$ such that $Q_i \rightsquigarrow P$. By replacing R by R^{op} one obtains an example of a prime ideal P' and infinitely many prime ideals Q_i' such that $P' \rightsquigarrow Q_i'$.

(ii) Note that, by [12, Corollary 3.10], $\{\operatorname{Grk} R/Q_i: i \in K\}$ is necessarily unbounded.

PROOF. Let $\{T_i: i \in J\}$ be the annihilators in S of the nonisomorphic, simple factor modules of $S/(1-z)S$. By Lemma 4.2, J is an infinite set. Now $S(1-z)S$ contains $h^z - h$ for any $h \in H$, so $S(1-z)S \neq (1-z)S$. Thus, by Lemma 4.2, $S/S(1-z)S$ is of finite length. Thus, if $K = \{i \in J: (1-z) \notin T_i\}$, then K is still an infinite set.

For each $i \in K$, set $V_i = T_i \cap R$. By the choice of K, $P + T_i \neq S$, so $P + V_i \neq R$. As P is a maximal (right) ideal of R this forces $V_i \subseteq P$ for each i. The proof is now similar to that of Proposition 3.7. We want to show that each V_i is contained in exactly two maximal ideals, one being P and the other (say) Q_i, and that $Q_i \rightsquigarrow P$.

Let $i \in K$. Since $P \not\subseteq T_i$, we have that $P + T_i/T_i = \overline{P}_i$ is a proper right ideal of $S/T_i \cong M_{n_i}(k)$. Thus, if we identify $\overline{P}_i$ with the top r rows of $M_{n_i}(k)$ then

$R/V_i \cong k + P + T_i/T_i$ has exactly two prime ideals, one being $\overline{P}_i$ and the other being

$$\overline{Q}_i = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} \cdot k$$

(where the top left-hand corner is always an $r \times r$ matrix). Furthermore, $\overline{Q}_i \overline{P}_i = 0$, whereas

$$\overline{P}_i \cap \overline{Q}_i = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \neq 0.$$

Thus, $\overline{Q}_i \rightsquigarrow \overline{P}_i$. So, if Q_i is the inverse image of $\overline{Q}_i$ in R, we have shown that $Q_i \rightsquigarrow P$ for each of the infinitely many $i \in K$.

A set Ω of prime ideals in a Noetherian ring R satisfies the *second layer condition* if, for each prime ideal $P \in \Omega$, the only *prime* ideal of R that is the annihilator of a finitely generated submodule of the injective hull of R/P is P itself. The final result of this section shows that the clique Ω of Theorem 4.4 does satisfy the second layer condition. The interest in this property stems from the theory of (classical) localisation. For, any localisable prime ideal (or clique of prime ideals) must satisfy the second layer condition [**6**, §7]. Further, there is a sensible definition of a localisation at an infinite clique (see, for example, [**6**, §7, or **14**]). Given this definition, it becomes an interesting open problem as to whether or not every second layer clique that satisfies the second layer condition must be localisable. It is also clear from the partial results in, say, [**12** or **14**] that our clique Ω is a good example on which to test this question, but, unfortunately, we have been unable to prove it either way. In contrast, Brown has recently shown that cliques are always localisable in group rings of polycyclic groups over uncountable fields [**3**].

To make the proof easier, we assume that every ideal of kG satisfies the Artin-Rees (AR) property. (This is not much of a restriction. By [**7**, Lemma 2.16, p. 499], we may replace G by a subgroup G' of finite index in G such that G' still satisfies our earlier conditions on G, but such that G' is now p-nilpotent for $p = \operatorname{char} k$. Now [**9**] implies that kG' is AR. For our group $G = H\langle z \rangle$, the proof of [**7**, Lemma 2.16] shows that $G' = H\langle z^n \rangle$ for some $n = n(p)$.)

COROLLARY 4.5. *Let $R = k + (1 - z)kG$ and $P = (1 - z)kG$ be as in Theorem* 4.4 *and assume that $S = kG$ is AR. Then the clique Ω of P satisfies the second layer condition.*

PROOF. By [**7**, Theorem 3.7, p. 549] every simple S-module is finite dimensional. We first check that the same is true for every simple right R-module. Let R/K be a simple right R-module, where, to avoid triviality, we assume that $K \neq P$. Since K/KP is a finitely generated, and hence Artinian, R/P-module, certainly P/KP is an Artinian R-module. Therefore it is Artinian and, hence, finite dimensional as an S-module. Thus, $R/K \cong P/K \cap P$ is also finite dimensional.

In order to prove the corollary, it suffices to show that every finitely generated, or even cyclic, essential extension of a simple right R-module is itself finite dimensional. Thus, let R/K be an essential extension of the simple submodule J/K. Write

$Q = \operatorname{ann}_R J/K$. Note that R/Q and, hence, J/JQP are finite dimensional. Let $\{V_i: i \in I\}$ be the annihilators, in S, of the simple factors of S/QP and set $W_i = \bigcap_k \{J \cap V_i^k + JQP\}$ for each i. Suppose first that $W = \bigcap W_i \neq JQP$, and pick $w \in W \setminus JQP$. For each $i \in I$, $w \in \bigcap(V_i^k + JQP)$ and V_i has the AR property. Thus, $wS + JQP = wV_i + JQP$ and $wt \in JQP$ for some $t \in \mathscr{C}(V_i)$. So, set $H = \{f \in S: wf \in JQP\}$. As $w \in J$, certainly $PQ \subseteq H$. Thus, if N is a maximal right ideal of S that contains H, then $N \supset V_j$ for some j. Since $N \cap \mathscr{C}(V_j) \neq \varnothing$ and S/V_j is finite dimensional, this is absurd.

Thus, $W = \bigcap W_i = JQP$. Since J/JQP is finite dimensional, there exist integers $u(i)$ and $v(i)$ such that

$$JQP = W_{u(1)} \cap \cdots \cap W_{u(r)} \quad \text{and} \quad W_{u(i)} = J \cap V_{u(i)}^{v(i)} + JQP.$$

Finally, set $V = R \cap V_{u(1)}^{v(1)} \cap V_{u(2)}^{v(2)} \cap \cdots \cap V_{u(r)}^{v(r)}$. Then $J \cap V \subseteq JQP \subseteq K$, so $J \cap (V + K) = K$. Since R/V is finite dimensional, this implies that R/K must be finite dimensional, as required.

References

1. J. Archer, *Derivations on commutative rings and projective modules over skew polynomial rings*, Ph.D. Thesis, Leeds Univ., 1980.

2. K. A. Brown, *The structure of modules over polycyclic groups*, Math. Proc. Cambridge Philos. Soc. **89** (1981), 257–283.

3. _____, *Localisation at cliques in group rings* (to appear).

4. J. Dixmier, *Enveloping algebras*, North-Holland, Amsterdam, 1977.

5. A. V. Jategaonkar, *Noetherian bimodules*, Noetherian Rings and Rings with Polynomial Identity (Proc. Durham Conf.), mimeographed notes, Leeds Univ., 1979.

6. _____, *Localisation in Noetherian rings*, London Math. Soc. Lecture Note Series (to appear).

7. D. S. Passman, *The algebraic structure of group rings*, Wiley, New York, 1977.

8. J. C. Robson, *Idealizers and hereditary Noetherian prime rings*, J. Algebra **22** (1972), 45–81.

9. J. E. Roseblade and P. F. Smith, *A note on the Artin-Rees property of certain polycyclic group algebras*, Bull. London Math. Soc. **11** (1979), 184–185.

10. S. P. Smith, *The primitive factor rings of the enveloping algebra of* Sl(2, **C**), J. London Math. Soc. **24** (1981), 97–108.

11. J. T. Stafford, *Generating modules efficiently: Algebraic K-theory for non-commutative Noetherian rings*, J. Algebra **69** (1981), 312–346; Corrigendum, ibid. **82** (1983), 294–296.

12. _____, *The Goldie rank of a module* (to appear).

13. R. B. Warfield, Jr., *The number of generators of a module over a fully bounded ring*, J. Algebra **66** (1980), 425–447.

14. _____, *Noncommutative, localized rings* (to appear).

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