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ON DERIVATIONS ANNIHILATING A MAXIMAL ABELIAN SUBALGEBRA

BY

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ABSTRACT. Let $\mathcal{A}$ be an AF C^*-algebra, and let δ be a closed $*$-derivation which annihilates the maximal abelian subalgebra $\mathcal{C}$ of diagonal elements of $\mathcal{A}$. Then we show that δ generates an approximately inner C^*-dynamics on $\mathcal{A}$, and that δ is a commutative $*$-derivation. Any two closed $*$-derivations vanishing on $\mathcal{C}$ are shown to be strongly commuting. More generally, if δ is a semiderivation on $\mathcal{A}$ which vanishes on $\mathcal{C}$, we prove that δ is a generator of a semigroup of strongly positive contractions of $\mathcal{A}$.

1. Introduction. Let δ be a densely-defined $*$-derivation on a C^*-algebra $\mathcal{A}$. One of the central problems in the theory of unbounded derivations is to determine whether a derivation is a generator, i.e., whether there exists a one-parameter group of $*$-automorphisms $\{\beta_t : t \in \mathbf{R}\}$ of $\mathcal{A}$ such that for all $x \in D(\delta)$, the domain of δ, $\delta(x) = \lim_{t\to 0}(1/t)(\beta_t(x) - x)$, and such that $D(\delta)$ coincides with the set of all x for which the limit above exists. Recently a number of articles have appeared considering this situation for the case where δ commutes with a group $\{\alpha_g : g \in G\}$ of automorphisms of $\mathcal{A}$, i.e., $\alpha_g : D(\delta) \to D(\delta)$, all $g \in G$, and $\alpha_g \circ \delta = \delta \circ \alpha_g$. For example, Bratteli and Jørgensen have shown in [**1**] that a closed $*$-derivation must be a generator if it commutes with a compact abelian group of automorphisms of $\mathcal{A}$ and annihilates the fixed point algebra $\mathcal{A}^\alpha$ of $\mathcal{A}$. Roughly speaking, their strategy is to decompose the algebra into spectral subspaces which are invariant under G (and also δ), to show that δ acts as a generator on each of these subspaces and to piece these results together to show that δ is a generator.

In this paper we consider an AF-algebra $\mathcal{A}$, $\mathcal{A} = \overline{\bigcup_n \mathcal{A}_n}$, a certain maximal abelian C^*-subalgebra $\mathcal{C}$ and a closed $*$-derivation δ which annihilates $\mathcal{C}$. For $n \in \mathbf{N}$, we construct conditional expectations Φ_n, mapping $\mathcal{A}$ onto the C^*-algebra $\langle \mathcal{A}_n, \mathcal{C}\rangle$ generated by $\mathcal{C}$ and the finite-dimensional subalgebra $\mathcal{A}_n$ of $\mathcal{A}$. These maps are shown to respect the action of δ in the sense that $\Phi_n : D(\delta) \to D(\delta)$ and $\delta(\Phi_n(x)) = \Phi_n(\delta x)$, all $x \in D(\delta)$. Using these techniques, we show that δ is a generator, and that any two derivations satisfying the above conditions are strongly commuting. Moreover, we exhibit a dense $*$-subalgebra of analytic elements of $\mathcal{A}$. We also show that δ is an approximately inner normal $*$-derivation which is commutative in the sense of Sakai [**9**]. Finally, we show that our techniques may be applied to prove analogous results in the more general case where δ is a $*$-semiderivation of $\mathcal{A}$.

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In a recent paper, [**11**], A. Kumjian uses somewhat different techniques to study similar questions on a class of C^*-algebras which includes the continuous trace AF-algebras. He shows that if δ is a closed $*$-derivation on a continuous trace AF-algebra which annihilates the diagonal subalgebra, then δ must be a generator [**11**, Theorem 5.2]. Moreover, a dense subalgebra of analytic elements for δ is also exhibited.

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2. Diagonalization of an AF-algebra. In this section we introduce some notation which shall be needed in the proofs of our main results, and we also recall some facts and notation from [**10**] on the diagonalization of an AF-algebra. To begin, suppose that $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ is an ascending union of finite-dimensional C^*-algebras, with common identity $\mathbf{1}$. Let $\mathcal{A}$ be the unital AF-algebra formed as the uniform closure of the union $\bigcup_n \mathcal{A}_n$. For any subsets $\mathcal{S}_1, \mathcal{S}_2, \ldots$ of $\mathcal{A}$, let $\langle \bigcup_n \mathcal{S}_n \rangle$ denote the smallest C^*-subalgebra of $\mathcal{A}$ containing each of the $\mathcal{S}_n$. In particular, $\mathcal{A} = \langle \bigcup_n \mathcal{A}_n \rangle$. Furthermore, if $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$, denote by $\mathcal{B}'$ the C^*-subalgebra of $\mathcal{A}$ given by $\{x \in \mathcal{A} : xy = yx, \text{ all } y \in \mathcal{B}\}$.

Following [**10**], we construct a maximal abelian C^*-subalgebra (m.a.s.a.) $\mathcal{C}$, the *diagonal subalgebra* of $\mathcal{A}$. $\mathcal{C}$ is defined inductively as follows: let $\mathcal{A}_0 = \mathbf{C} \cdot \mathbf{1}$, and for each $n \in \mathbf{N} \cup \{0\}$, choose a m.a.s.a. $\mathcal{D}_{n+1}$ of $\mathcal{A}'_n \cap \mathcal{A}_{n+1}$. Set $\mathcal{C}_0 = \mathbf{C} \cdot \mathbf{1}$ and define $\mathcal{C}_{n+1} = \langle \mathcal{C}_n, \mathcal{D}_{n+1} \rangle$. Then $\mathcal{C}_n$ is a m.a.s.a. of $\mathcal{A}_n$. Moreover, if $\mathcal{I}$ and $\mathcal{J}$ are index sets such that $\{p_i : i \in \mathcal{I}\}$, $\{q_j : j \in \mathcal{J}\}$ are the minimal projections of $\mathcal{C}_n$, $\mathcal{D}_{n+1}$, respectively, then $\{p_i q_j : i \in \mathcal{I}, j \in \mathcal{J}\}$ is the set of minimal projections of $\mathcal{C}_{n+1}$. A straightforward argument now shows that $\mathcal{C} = \langle \bigcup_n \mathcal{C}_n \rangle$ is a m.a.s.a. of $\mathcal{A}$. More generally, one also obtains that, for fixed n, and for any $k \in \mathbf{N}$, the abelian C^*-algebra $\mathcal{A}'_n \cap \mathcal{C}_{n+k}$ is a m.a.s.a. of $\mathcal{A}'_n \cap \mathcal{A}_{n+k}$, and the uniform closure $\langle \bigcup_k (\mathcal{A}'_n \cap \mathcal{C}_{n+k}) \rangle$ of the union is a m.a.s.a. of the AF-subalgebra $\mathcal{A}'_n$ of $\mathcal{A}$. We refer the reader to the exposition in [**10**] for details.

In [**10**] a conditional expectation Φ from $\mathcal{A}$ to its m.a.s.a. $\mathcal{C}$ is constructed. By conditional expectation it is meant that $\Phi : \mathcal{A} \to \mathcal{C}$ is a positive linear mapping which satisfies (i) $\Phi \circ \Phi = \Phi$, and (ii) $\Phi(xy) = \Phi(x)y$ and $\Phi(yx) = y\Phi(x)$ for $x \in \mathcal{A}$ and $y \in \mathcal{C}$. In particular it follows that $\Phi|_{\mathcal{C}}$ is the identity mapping. These properties determine Φ uniquely. We now derive a slight generalization of this notion.

PROPOSITION 2.1. *Let* $\mathcal{A} = \overline{\bigcup_n \mathcal{A}_n}$ *be a unital AF-algebra. Then for any* $n \in \mathbf{N} \cup \{0\}$ *there exists a conditional expectation* Φ_n *from* $\mathcal{A}$ *onto the* C^**-subalgebra of* $\mathcal{A}$ *generated by* $\mathcal{A}_n$ *and the (diagonal) m.a.s.a.* $\mathcal{C}$ *of* $\mathcal{A}$, *i.e.,* $\Phi_n : \mathcal{A} \to \langle \mathcal{A}_n, \mathcal{C} \rangle$.

Moreover, if $y \in \mathcal{A}'_n$, *then* $\Phi_n(y) \in \mathcal{A}'_n \cap \mathcal{C}$.

(REMARK. Observe that Φ_0 coincides with the conditional expectation Φ described above.)

PROOF. Fix positive integers $m > n$. We begin by defining a conditional expectation $\Phi_{m,n}$ from $\mathcal{A}$ onto $\langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle$. First recall (see, e.g., [**5**]) that for any $x \in \mathcal{A}$ there exist elements $u_i \in \mathcal{A}_n$, $v_i \in (\mathcal{A}'_n \cap \mathcal{A}_m)$ and $w_i \in \mathcal{A}'_m$, $i = 1, 2, \ldots, r$, such that $x = \sum_{i=1}^r u_i v_i w_i$. Now let $\{p_j : j \in \mathcal{J}\}$, some index set $\mathcal{J}$, be the minimal projections of $\mathcal{A}'_n \cap \mathcal{C}_m$. Consider the linear map $\Phi_{m,n}$ defined by

$\Phi_{m,n}(x) = \sum_{j\in J} p_j x p_j$. Then

$$\Phi_{m,n}(x) = \sum_{j\in J}\sum_{i=1}^{r} p_j u_i v_i w_i p_j = \sum_{j\in J}\sum_{i=1}^{r} u_i(p_j v_i p_j) w_i$$
$$= \sum_{i=1}^{r}\sum_{j\in J} u_i(p_j v_i p_j) w_i = \sum_{i=1}^{r} u_i(\Phi_{m,n}(v_i)) w_i.$$

Observe that for each i, $\Phi_{m,n}(v_i) \in \mathcal{A}'_n \cap \mathcal{A}_m$, and furthermore, $\Phi_{m,n}(v_i)$ commutes with $\mathcal{A}'_n \cap \mathcal{C}_m$. But $\mathcal{A}'_n \cap \mathcal{C}_m$ is maximal abelian in $\mathcal{A}'_n \cap \mathcal{C}_m$ [**10**], so that $\Phi_{m,n}(v_i) \in (\mathcal{A}'_n \cap \mathcal{C}_m)$. Hence $\Phi_{m,n}(x) \in \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m\rangle$. Moreover, one checks easily that the latter algebra is fixed by $\Phi_{m,n}$, so that $\Phi_{m,n}$ is a conditional expectation onto $\langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m\rangle$.

It is straightforward to verify that for $x \in \mathcal{A}_r$ and for $q > m \geq r$, $\Phi_{m,n}(x) = \Phi_{q,n}(x)$. To see this, note that since

$$\mathcal{A}'_n \cap \mathcal{C}_q = \mathcal{A}'_n \cap \langle \mathcal{C}_m, \mathcal{D}_{m+1}, \ldots, \mathcal{D}_q\rangle = \langle \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{D}_{m+1}, \ldots, \mathcal{D}_q\rangle,$$

there exist minimal projections $e_1, \ldots, e_s$ in $\langle \mathcal{D}_{m+1}, \ldots, \mathcal{D}_q\rangle$ such that $\{p_j e_k : j \in \mathcal{J},\ 1 \leq k \leq s\}$ is the set of minimal projections of $\mathcal{A}'_n \cap \mathcal{C}_q$. But since $e_k \in \mathcal{A}'_r$, all k, we have

$$\Phi_{q,n}(x) = \sum_{j\in J}\sum_{k=1}^{s} p_j e_k x p_j e_k = \sum_{j\in J}\sum_{k=1}^{s} p_j x p_j e_k$$
$$= \sum_{j\in J} p_j x p_j = \Phi_{m,n}(x).$$

Similarly, one may show that for $q \geq m \geq n$, $\Phi_{m,n} \circ \Phi_{q,n} = \Phi_{q,n}$.

We now use the results above to establish that for all $y \in \mathcal{A}$, and for all fixed n, the sequence $\{\Phi_{m,n}(y) : m > n\}$ converges uniformly. For, if $y \in \mathcal{A}$, and $\varepsilon > 0$, there exists an index r and $x \in \mathcal{A}_r$ such that $\|y - x\| < \varepsilon/2$. Using $\|\Phi_{m,n}\| = 1$, all $m > n$, we have for $q \geq m \geq r$,

$$\|\Phi_{q,n}(y) - \Phi_{m,n}(y)\| \leq \|\Phi_{q,n}(y - x)\| + \|\Phi_{q,n}(x) - \Phi_{m,n}(x)\| + \|\Phi_{m,n}(y - x)\|$$
$$\leq \|y - x\| + 0 + \|y - x\| < \varepsilon.$$

Hence the sequence $\{\Phi_{m,n}(y) : m > n\}$ has a uniform limit $\Phi_n(y)$. From the identity $\Phi_{m,n} \circ \Phi_{q,n} = \Phi_{q,n}$, $q \geq m \geq n$, it is straightforward to show that $\Phi_{m,n} \circ \Phi_n = \Phi_n$, and therefore, for all $y \in \mathcal{A}$,

$$\Phi_n(y) \in \bigcap_{m\geq n} \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m\rangle.$$

We show $\langle \mathcal{A}_n, \mathcal{C}\rangle = \bigcap_{m\geq n}\langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m\rangle$. First, we have, for all $m \geq n$ and $q \geq n$,

$$(\mathcal{A}'_n \cap \mathcal{C}_q) \subseteq (\mathcal{A}'_n \cap \mathcal{C}_m) \quad \text{if } m \geq q,$$

and

$$(\mathcal{A}'_n \cap \mathcal{C}_q) = \langle \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{D}_{m+1}, \ldots, \mathcal{D}_{q+1}\rangle$$
$$\subseteq \langle \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m\rangle \quad \text{if } m < q,$$

so that, in either case, we have $(\mathcal{A}'_n \cap \mathcal{C}_q) \subseteq \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle$, and therefore,

$$\langle \mathcal{A}_n, \mathcal{C} \rangle = \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C} \rangle = \left\langle \mathcal{A}_n, \bigcup_{q>n} (\mathcal{A}'_n \cap \mathcal{C}_q) \right\rangle \quad \text{(from [10])}$$
$$\subseteq \bigcap_{m \geq n} \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle.$$

For the reverse inclusion, suppose $x \in \langle \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle$, all $m \geq n$. Then clearly $x \in \mathcal{A}'_n$ and $x \in (\mathcal{A}'_n \cap \mathcal{C}_m)'$, all m, hence $\mathcal{A}'_n \cap [(\mathcal{A}'_n \cap \mathcal{C})']$. But $\mathcal{A}'_n \cap \mathcal{C}$ is maximal abelian in $\mathcal{A}'_n$ [**10**, Proposition I.1.3], so that $x \in \mathcal{A}'_n \cap \mathcal{C}$. The equality

$$\bigcap_{m \geq n} \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle = \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C} \rangle = \langle \mathcal{A}_n, \mathcal{C} \rangle$$

now follows immediately. Therefore Φ_n is a conditional expectation from $\mathcal{A}$ to $\langle \mathcal{A}_n, \mathcal{C} \rangle$, as asserted.

Finally, observe that if $y \in \mathcal{A}'_n$, then for $m > n$, $\Phi_{m,n}(y) = \sum_{j \in J} p_j y p_j$ also lies in $\mathcal{A}'_n$. Since $\Phi_n(y) = \lim_m \Phi_{m,n}(y)$, the last statement of the proposition must hold. □

3. Derivations annihilating $\mathcal{C}$. We now apply the preceding results to show that any closed $*$-derivation δ on $\mathcal{A}$ which vanishes on the diagonal m.a.s.a. $\mathcal{C}$ must necessarily be a generator. Our techniques will enable us to construct explicitly a dense $*$-subalgebra of analytic elements for $D(\delta)$ (Proposition 3.2). Moreover, a direct application of our results shows that the set of closed $*$-derivations annihilating $\mathcal{C}$ forms a family of strongly commuting generators (Corollary to Theorem 3.3).

LEMMA 3.1. *Let δ be a closed $*$-derivation vanishing on $\mathcal{C}$. For $n \in \mathbf{N} \cup \{0\}$, let Φ_n be the conditional expectation onto $\langle \mathcal{A}_n, \mathcal{C} \rangle$ constructed in §2. Then for any $x \in D(\delta)$, $\Phi_n(x)$ is also in $D(\delta)$ and $\delta(\Phi_n(x)) = \Phi_n(\delta x)$.*

PROOF. For $m > n$, recall that $\Phi_{m,n}(x) = \sum_{j \in J} p_j x p_j$, where $\{p_j : j \in J\}$ is the set of minimal projections of $\mathcal{A}'_n \cap \mathcal{C}_m$. Hence if $x \in D(\delta)$, then $\Phi_{m,n}(x) \in D(\delta)$, and

$$\delta(\Phi_{m,n}(x)) = \sum_{j \in J} \delta(p_j x p_j) = \sum_{j \in J} p_j (\delta x) p_j = \Phi_{m,n}(\delta x). \tag{1}$$

By Proposition 2.1, $\Phi_n(x)$ and $\Phi_n(\delta x)$ are the uniform limits of $\{\Phi_{m,n}(x) : m > n\}$ and $\{\Phi_{m,n}(\delta x) : m > n\}$. Combining this result with (1), we conclude from the closedness of δ that $\Phi_n(x) \in D(\delta)$ and that $\delta(\Phi_n(x)) = \Phi_n(\delta x)$. □

PROPOSITION 3.2. *Let δ be a closed $*$-derivation vanishing on $\mathcal{C}$. Then $D(\delta)$ contains a dense $*$-subalgebra of analytic elements. In fact, if $x \in \bigcup_n \mathcal{A}_n$, then x is an analytic element for δ.*

PROOF. Fix $n \in \mathbf{N} \cup \{0\}$. Since $\mathcal{A}_n$ is finite-dimensional, it is isomorphic to a direct sum $\sum_{k=1}^{p_n} \mathcal{M}_{r_k}$ of $r_k \times r_k$ matrix algebras $\mathcal{M}_{r_k}$ over $\mathbf{C}$. Hence, for $1 \leq k \leq p_n$, one may choose matrix units $e^k_{ij} \in \mathcal{A}_n$ $(1 \leq k \leq p_n,\ 1 \leq i,j \leq r_k)$ satisfying the identities (i) $e^k_{ij} e^k_{pq} = \delta_{jp} e^k_{iq}$, and (ii) for $k \neq l$, $e^k_{ij} e^l_{pq} = 0$. Furthermore, we may assume that the matrix units have been chosen so that the diagonal elements e^k_{ii} lie in $\mathcal{C}$. To show that $\mathcal{A}_n$ consists of analytic elements for δ it clearly suffices to show that the e^k_{ij} are analytic.

We begin by showing that each matrix unit e^k_{ij} lies in $D(\delta)$. For, if $1 > \varepsilon > 0$, there exists $x \in D(\delta)$ such that $\|x - e^k_{ij}\| < \varepsilon$. But then $e^k_{ii} x e^k_{jj} \in D(\delta)$ and we have

$$\|e^k_{ii} x e^k_{jj} - e^k_{ij}\| = \|e^k_{ii}(x - e^k_{ij}) e^k_{jj}\| \le \|x - e^k_{ij}\| < \varepsilon.$$

Hence we may assume, without loss of generality, that $x = e^k_{ii} x e^k_{jj}$. Therefore, x admits a decomposition $x = e^k_{ij} y$, where $y \in \mathcal{A}'_n$ (see [**5**]); whence $\Phi_n(x) = \Phi_n(e^k_{ij} y) = e^k_{ij} \Phi_n(y)$. Let $d_1 = \Phi_n(y)$; then $d_1 \in (\mathcal{A}'_n \cap \mathcal{C})$, from Proposition 2.1, so that

$$\begin{aligned}\|e^k_{jj} d_1 - e^k_{jj}\| &= \|e^k_{ji}(e^k_{ij} d_1 - e^k_{ij})\| \le \|e^k_{ij} d_1 - e^k_{ij}\| \\ &= \|\Phi_n(x) - \Phi_n(e^k_{ij})\| \le \|x - e^k_{ij}\| < \varepsilon.\end{aligned}$$

Since $\varepsilon < 1$, the inequality above implies that d_1 is invertible in the commutative C^*-subalgebra $e^k_{jj}\mathcal{C}$, i.e., there exists $d_2 \in e^k_{jj}\mathcal{C}$ such that $d_1 d_2 = e^k_{jj}$. But $d_2 \in D(\delta)$, $\Phi_n(x) \in D(\delta)$, so that $e^k_{ij} = \Phi_n(x)\, d_2 \in D(\delta)$.

From Lemma 3.1,

$$\delta(\Phi_n(x)) = \delta(\Phi_n(e^k_{ii} x e^k_{jj})) = \delta(e^k_{ii} \Phi_n(x) e^k_{jj}) = e^k_{ii} \Phi_n(\delta x) e^k_{jj}.$$

Hence $\delta(\Phi_n(x))$ has the form $\delta(\Phi_n(x)) = e^k_{ij} d_3$, $d_3 \in \mathcal{A}'_n \cap \mathcal{C}$. Then writing $d = d_3 d_2$ $(\in \mathcal{A}'_n \cap \mathcal{C})$,

$$\delta(e^k_{ij}) = \delta[\Phi_n(x) d_2] = [\delta(\Phi_n(x))]\, d_2 = e^k_{ij} d.$$

Iterating, we have, for any $r \in \mathbf{N}$, $\delta^r(e^k_{ij}) = e^k_{ij}(d^r)$, so that e^k_{ij} is easily seen to be an analytic element of $D(\delta)$. $\square$

REMARK. We note that a somewhat similar analysis to the foregoing is carried out in [**8**], where δ is a closed $*$-derivation on a UHF algebra $\mathcal{A}$ of Glimm type n^∞. There it is shown that if δ vanishes on the natural embedding of $S(\infty)$ into $\mathcal{A}$ ($S(\infty)$ is the group of finite permutations on countably many symbols) and satisfies $\tau \circ \delta = 0$, where τ is the unique trace on $\mathcal{A}$, then δ admits an extension to a generator on $\mathcal{A}$.

From the proof of the proposition it is clear that $\langle \mathcal{A}_n, \mathcal{C} \rangle \subseteq D(\delta)$, all $n \in \mathbf{N}$, and that $\delta \colon \langle \mathcal{A}_n, \mathcal{C} \rangle \to \langle \mathcal{A}_n, \mathcal{C} \rangle$. Indeed, for $1 \le k \le p_n$, let d^k_j, $1 \le j \le r_k$, be the diagonal elements satisfying $\delta(e^k_{1j}) = e^k_{1j} d^k_j$. Consider the (skew-hermitian) element

$$ih_n = \sum_{k=1}^{p_n} \sum_{j=1}^{r_k} (e^k_{j1}) \delta(e^k_{1j}).$$

Then we have the well-known identity $\delta|_{\mathcal{A}_n} = \mathrm{Ad}(ih_n)|_{\mathcal{A}_n}$, i.e., for $x \in \mathcal{A}_n$, $x = [ih_n, x] = x(ih_n) - (ih_n)x$ (see [**5**]). But

$$\begin{aligned} ih_n &= \sum_{k=1}^{p_n} \sum_{j=1}^{r_k} (e^k_{j1}) \delta(e^k_{1j}) \\ &= \sum_{k=1}^{p_n} \sum_{j=1}^{r_k} e^k_{j1} e^k_{1j} (d^k_j) = \sum_{k=1}^{p_n} \sum_{j=1}^{r_k} e^k_{jj} d^k_j, \end{aligned}$$

so that $h_n \in \mathcal{C}$. Extending to $\langle \mathcal{A}_n, \mathcal{C}\rangle$, it is straightforward to show that $\delta|_{\langle \mathcal{A}_n,\mathcal{C}\rangle} = \mathrm{Ad}(ih_n)|_{\langle \mathcal{A}_n,\mathcal{C}\rangle}$ as well. From this identity it now follows easily that $\delta|_{\langle \mathcal{A}_n,\mathcal{C}\rangle}$ is a generator, with corresponding one-parameter group $\{\alpha_t^{(n)}: t \in \mathbf{R}\}$ given by

$$\alpha_t^{(n)}(x) = \exp(-ith_n)(x)\exp(ith_n), \qquad x \in \langle \mathcal{A}_n, \mathcal{C}\rangle.$$

Applying [**4**, Theorem 3.2.51] one shows that the closure δ_0 of the derivation $\delta|_{[\bigcup_n \langle \mathcal{A}_n,\mathcal{C}\rangle]}$ is an (approximately inner) derivation on $\mathcal{A}$ with corresponding one-parameter group $\{\alpha_t: t \in \mathbf{R}\}$ satisfying

$$\alpha_t(x) = \lim_n \exp(-ith_n)(x)\exp(ith_n), \qquad x \in \mathcal{A}.$$

We show that δ_0 coincides with δ. Clearly $D(\delta_0) \subseteq D(\delta)$. To verify the reverse inclusion let $x \in D(\delta)$. Then $x = \lim_n \Phi_n(x)$ and $\delta(x) = \lim_n \Phi_n(\delta x) = \lim_n \delta(\Phi_n(x))$. Since $\Phi_n(x) \in \langle \mathcal{A}_n, \mathcal{C}\rangle \subseteq D(\delta_0)$, the result holds. Hence we have the following.

THEOREM 3.3. *Let δ be a closed $*$-derivation vanishing on the diagonal m.a.s.a. $\mathcal{C}$ of $\mathcal{A}$. Then δ is a generator. In fact there exists a sequence $(h_n) \subseteq \mathcal{C}$ of hermitian operators such that δ generates the approximately inner dynamics $\{\alpha_t: t \in \mathbf{R}\}$, where $\alpha_t(x) = \lim_n \exp(-ith_n)(x)\exp(ith_n)$, $x \in \mathcal{A}$.*

REMARK. Since the sequence $(h_n)_{n\in\mathbf{N}}$ lies in $\mathcal{C}$, the hermitian operators are mutually commuting. Hence δ is a commutative $*$-derivation in the sense of Sakai (see [**9**]).

We recall the following notion.

DEFINITION 3.1. Let δ, δ' be generators on a C^*-algebra $\mathcal{A}$. Then δ, δ' are said to be strongly commuting if their corresponding one-parameter groups $\{\alpha_t: t \in \mathbf{R}\}$, $\{\alpha'_t: t \in \mathbf{R}\}$ satisfy $\alpha_t \circ \alpha'_{t_1} = \alpha'_{t_1} \circ \alpha_t$, $t, t_1 \in \mathbf{R}$.

COROLLARY. *The set of generators which vanish on the diagonal m.a.s.a. $\mathcal{C}$ of $\mathcal{A}$ form a family of strongly commuting derivations.*

PROOF. Let δ, δ' be two such derivations, and let $\{\alpha_t\}$, $\{\alpha'_t\}$ be their corresponding one-parameter groups. Then from the preceding theorem there exist operators $h_n, h'_n \in \mathcal{C}$ for $n \in \mathbf{N}$ such that

$$\alpha_t|_{\langle \mathcal{A}_n,\mathcal{C}\rangle} = \mathrm{Ad}(\exp(-ith_n)), \quad \alpha'_t|_{\langle \mathcal{A}_n,\mathcal{C}\rangle} = \mathrm{Ad}(\exp(-ith'_n)).$$

Since h_n, h'_n commute it is straightforward to verify that $\alpha_t(\alpha'_{t_1}(x)) = \alpha'_{t_1}(\alpha_t(x))$ for $x \in \langle \mathcal{A}_n, \mathcal{C}\rangle$ and $t, t_1 \in \mathbf{R}$. The result now follows by continuity. □

4. Applications to semiderivations. Let $\mathcal{A}$ be a C^*-algebra. A linear operator $\delta: D(\delta) \to \mathcal{A}$ is said to be a *semiderivation* (or, alternatively, a *dissipation*) if it satisfies the following properties:

(a) $D(\delta)$ is a uniformly dense $*$-subalgebra of $\mathcal{A}$,
(b) $\delta(x)^* = \delta(x^*)$, all $x \in D(\delta)$, and
(c) $\delta(x^*x) \geq \delta(x^*)x + x^*(\delta x)$ for $x \in D(\delta)$.

A central problem in the theory of semiderivations is to determine when δ is the generator of a strongly continuous one-parameter contraction semigroup $\{\alpha_t: t \in \mathbf{R}_+\}$ of positivity-preserving maps. For semiderivations vanishing on $\mathcal{C}$ we have the following analogues to Theorem 3.3 and its corollary. (We thank P. E. T. Jørgensen for suggesting this extension of our original results.)

THEOREM 4.1. *Let δ be a closed semiderivation annihilating the m.a.s.a. $\mathcal{C}$ of $\mathcal{A}$. Then δ is the generator of a strongly continuous one-parameter contraction semigroup $\{\alpha_t : t \in \mathbf{R}_+\}$ of symmetric, strongly positive maps, i.e., $\alpha_t(x^*) = \alpha_t(x)^*$ and $\alpha_t(x^*x) \geq \alpha_t(x)^*\alpha_t(x)$ for all $x \in \mathcal{A}$, $t \in \mathbf{R}_+$.*

PROOF. We preserve the notation of Proposition 3.2. Since $\mathcal{C} \subseteq D(\delta)$ and $\delta|_{\mathcal{C}} \equiv 0$, we have, by [**3**, Lemma 1.1],

$$\delta(xy) = (\delta x)y, \quad \delta(yx) = y(\delta x), \qquad y \in \mathcal{C},\ x \in D(\delta). \tag{2}$$

Now using (2) we may employ the "averaging" argument of Lemma 3.1 to conclude that for all $n \in \mathbf{N} \cup \{0\}$ and $x \in D(\delta)$, $\Phi_n(x) \in D(\delta)$ and $\delta[\Phi_n(x)] = \Phi_n(\delta x)$. This implies that $\langle \mathcal{A}_n, \mathcal{C} \rangle \subseteq D(\delta)$ for all n, and

$$\delta : \langle \mathcal{A}_n, \mathcal{C} \rangle \to \langle \mathcal{A}_n, \mathcal{C} \rangle.$$

We consider $\delta|_{\langle \mathcal{A}_n, \mathcal{C} \rangle}$. Since δ is everywhere defined on $\langle \mathcal{A}_n, \mathcal{C} \rangle$ it follows from [**6**, Theorem 1] that $\delta|_{\langle \mathcal{A}_n, \mathcal{C} \rangle}$ is both dissipative and bounded. In particular, $\langle \mathcal{A}_n, \mathcal{C} \rangle$ consists of analytic elements for δ. Now, let $x \in \langle \mathcal{A}_n, \mathcal{C} \rangle$. Then x may be decomposed as

$$x = \sum_{k=1}^{p_n} \sum_{i,j=1}^{r_k} e_{ij}^k c_{ij}^k, \qquad c_{ij}^k \in \mathcal{C}.$$

Following the proof of Proposition 3.2, there exists for each matrix unit $e_{ij}^k \in \mathcal{A}_n$ an element $d_{ij}^k \in \mathcal{C}$ such that $\delta(e_{ij}^k) = e_{ij}^k d_{ij}^k$. Then using (2) repeatedly we have, for $c_{ij}^k \in \mathcal{C}$ and $r \in \mathbf{N}$,

$$\delta^r(x) = \sum_{k=1}^{p_n} \sum_{i,j=1}^{r_k} (e_{ij}^k c_{ij}^k)(d_{ij}^k)^r. \tag{3}$$

Define δ_0 to be the semiderivation $\delta_0 = \delta|_{D(\delta_0)}$, where $D(\delta_0) = \bigcup_n \mathcal{A}_n$. Then δ_0 is a dissipative semiderivation whose domain contains a dense $*$-subalgebra of analytic elements. Hence δ_0 is closable, and its closure $\overline{\delta_0}$ is a generator of a contractive semigroup $\{\alpha_t : t \in \mathbf{R}_+\}$ [**2**, Theorem 5]. Now applying an argument identical to the one in the last paragraph of the proof of Theorem 3.3, we conclude that $\overline{\delta_0}$ coincides with δ.

We now consider the semigroup $\{\alpha_t\}$. Since $\delta(x)^* = \delta(x^*)$ for $x \in D(\delta)$, it follows immediately that $\alpha_t(x^*) = \alpha_t(x)^*$, all $x \in \mathcal{A}$. We show that the maps α_t are strongly positive. For $n \in \mathbf{N}$, consider any $x \in \langle \mathcal{A}_n, \mathcal{C} \rangle$. Then since δ is a bounded dissipative semiderivation on $\langle \mathcal{A}_n, \mathcal{C} \rangle$ with $\delta(1) = 0$, it follows from [**7**, Corollary 3] that $\{\alpha_t|_{\langle \mathcal{A}_n, \mathcal{C} \rangle} : t \in \mathbf{R}_+\}$ is strongly continuous and that $\alpha_t(x^*)\alpha_t(x) \leq \alpha_t(x^*x)$. Using the strong continuity on $\langle \mathcal{A}_n, \mathcal{C} \rangle$ for each $n \in \mathbf{N}$, it follows immediately that the semigroup of contractions is strongly continuous on $\mathcal{A}$. Now, using continuity, it is straightforward to show that $\alpha_t(x^*)\alpha_t(x) \leq \alpha_t(x^*x)$, all $x \in \mathcal{A}$, so that the semigroup is strongly positive. □

COROLLARY. *Let δ, δ' be any two semiderivations on $\mathcal{A}$ which annihilate $\mathcal{C}$. Then δ and δ' are strongly commuting generators of contraction semigroups.*

PROOF. From the theorem we deduce that δ (respectively, δ') are the generators of the strongly continuous contractive semigroups $\{\alpha_t : t \in \mathbf{R}_+\}$ (respectively,

$\{\alpha_t' : t \in \mathbf{R}_+\})$. Fix $n \in \mathbf{N}$, and consider $x \in \langle \mathcal{A}_n, \mathcal{C} \rangle$. Then, by the theorem, x is an analytic element for both δ and δ', i.e.,

$$\alpha_t(x) = \sum_{n=0}^{\infty} [(t\delta)^r(x)]/(n!) \quad \left(\text{respectively, } \alpha_t'(x) = \sum_{n=0}^{\infty} [(t\delta')^r(x)]/(n!)\right).$$

Using the formula in (3) for $\delta^r(x)$, $r \in \mathbf{N}$ (and a similar formula for $(\delta')^s(x)$, $s \in \mathbf{N}$), it is trivial to verify that $[\delta^r \circ (\delta')^s](x) = [(\delta')^s \circ (\delta)^r](x)$. But from this identity it follows immediately that for $t, t_1 \in \mathbf{R}$, $\alpha_t(\alpha_{t_1}'(x)) = \alpha_{t_1}'(\alpha_t(x))$. By continuity this equation holds for all $x \in \mathcal{A}$, so that δ, δ' are strongly commuting semiderivations. □

REFERENCES

1. O. Bratteli and P. E. T. Jørgensen, *Unbounded derivations tangential to compact groups of automorphisms*, J. Funct. Anal. **48** (1982), 107–133.
2. ——, *Unbounded $*$-derivations and infinitesimal generators on operator algebras*, Proc. Sympos. Pure Math., vol. 38, part 2, Amer. Math. Soc., Providence, R.I., 1980, pp. 353–365.
3. O. Bratteli, P. E. T. Jørgensen, A. Kishimoto and D. W. Robinson, *A C^*-algebraic Schoenberg theorem*, Australian National University, 1983, preprint.
4. O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics*, vol. I, Springer-Verlag, New York, 1979.
5. G. Elliott, *Derivations of matroid C^*-algebras*, Invent. Math. **9** (1970), 253–269.
6. A. Kishimoto, *Dissipations and derivations*, Comm. Math. Phys. **47** (1976), 25–32.
7. D. Evans and H. Hanche-Olsen, *The generators of positive semigroups*, J. Funct. Anal. **32** (1979), 207–212.
8. R. T. Powers and G. Price, *Derivations vanishing on $S(\infty)$*, Comm. Math. Phys. **84** (1982), 439–447.
9. S. Sakai, *On commutative normal $*$-derivations*, Comm. Math. Phys. **43** (1975), 39–40.
10. S. Stratila and D. Voiculescu, *Representations of AF-algebras and of the group $U(\infty)$*, Lecture Notes in Math., vol. 486, Springer-Verlag, Berlin and New York, 1975.
11. A. Kumjian, *Preliminary algebras arising from local homeomorphisms*, Math. Scand. **52** (1983).

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