LOWER BOUNDS OF THE GAP BETWEEN THE FIRST AND SECOND EIGENVALUES OF THE SCHRÖDINGER OPERATOR

BY

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ABSTRACT. In this paper the authors prove the following theorem:

Let Ω be a smooth strictly convex bounded domain in R^n and $V: \Omega \rightarrow R$ a nonnegative convex function. Suppose λ_1 and λ_2 are the first and second nonzero eigenvalues of the equation

$$-\Delta f + Vf = \lambda f, \quad f|_{\partial\Omega} \equiv 0.$$

Then $\lambda_2 - \lambda_1 \geq \pi^2/d^2$, where d is the diameter of Ω .

Let $\Omega \subset R^n$ be a smooth strictly convex bounded domain and $W: \Omega \rightarrow R$ a nonnegative convex smooth function. The eigenvalues of the equation

$$(1) \quad -\Delta f + Wf = \lambda f, \quad f = 0, \quad \text{on } \partial\Omega$$

can be arranged in nondecreasing order as follows:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

B. Wong, S.-T. Yau and Stephen S.-T. Yau [3] proved that

$$(2) \quad \lambda_2 - \lambda_1 \geq \pi^2/4d^2,$$

where d is the diameter of Ω . In this paper the authors will use the method of [3 and 4] to prove the following theorem:

THEOREM. *Let Ω be a smooth strictly convex bounded domain in R^n and $W: \Omega \rightarrow R$ a nonnegative convex function. Suppose λ_1 and λ_2 are the first and second nonzero eigenvalues of (1). Then*

$$(3) \quad \lambda_2 - \lambda_1 \geq \pi^2/d^2,$$

where d is the diameter of Ω .

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In this paper the assumptions of all the lemmas are the same as those of the Theorem. We will not state them again.

Let f_1 and f_2 be the first and second eigenfunctions of (1); then $f_1(x) > 0$, $x \in \Omega$ [2], and $u = f_2/f_1$ is smooth to the boundary of Ω [3]. Suppose that

$$A = \max_{x \in \bar{\Omega}} u(x); \quad -k = \min_{x \in \bar{\Omega}} u(x).$$

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We may assume that $A \geq k$, otherwise, we can use $-f_2$ instead of f_2 .

Since $\int_{\Omega} f_1 f_2 = 0$ and $f_1 > 0$, $k > 0$. Setting $\tilde{u} = u/A$, we have $1 \geq \tilde{u} \geq -k/A = -\tilde{k}$ and $1 \geq \tilde{k} > 0$;

$$(4) \quad v = \left(\tilde{u} - \frac{1 - \tilde{k}}{2} \right) / \left(\frac{1 + \tilde{k}}{2} \right)$$

and

$$(5) \quad a = \frac{1 - \tilde{k}}{1 + \tilde{k}}; \quad 1 > a \geq 0.$$

Then v is a smooth function on $\bar{\Omega}$. By computing, we have

$$(6) \quad \Delta v = -\lambda(v + a) - 2(\nabla v \cdot \nabla \log f_1),$$

where λ is $\lambda_2 - \lambda_1 > 0$.

LEMMA 1. *Let $z(v)$ be a smooth function defined on $\bar{\Omega}$ and $m > 0$ a constant. Suppose that*

$$(7) \quad G(x) = m|\nabla v|^2 \neq z(v),$$

$P \in \partial\Omega$ and $G(P) = \max_{x \in \bar{\Omega}} G(x)$. Then $\nabla v(P) = 0$.

PROOF. We can choose an orthonormal frame $l_1, l_2, \dots, l_n$ around P such that l_1 is perpendicular to $\partial\Omega$ and pointing outward. We also use the notation $\partial/\partial x_1$ to denote the restriction of l_1 on $\partial\Omega$. Since $G(P)$ is the maximum of $G(x)$,

$$(8) \quad 0 \leq \frac{\partial G}{\partial x_1}(P) = 2m \sum_{i=1}^n v_i v_{i1} + z' v_1.$$

Furthermore, in $\Delta v = -\lambda(v + a) - 2(\nabla v \cdot \nabla \log f_1)$, v and Δv are all smooth on $\bar{\Omega}$; hence

$$\nabla v \cdot \nabla \log f_1 = \frac{1}{f_1} \left[v_1(f_1)_1 + \sum_{i=2}^n v_i(f_1)_i \right]$$

achieves finite value on $\partial\Omega$. But $f_1|_{\partial\Omega} \equiv 0$, thus

$$\left[v_1(f_1)_1 + \sum_{i=2}^n v_i(f_1)_i \right] \Big|_{\partial\Omega} \equiv 0.$$

Since $f_1 \equiv 0$ and $\partial\Omega$ and l_i , $2 \leq i \leq n$, are the tangent vectors of $\partial\Omega$, $(f_1)_i|_{\partial\Omega} \equiv 0$, $2 \leq i \leq n$. Hence,

$$v_1(f_1)_1 \equiv 0 \quad \text{on } \partial\Omega.$$

By Hopf's lemma, $\partial f_1/\partial x_1 \neq 0$. Therefore,

$$(9) \quad v_1 \equiv 0 \quad \text{on } \partial\Omega.$$

Putting (9) into (8), we have

$$(10) \quad 0 \leq m \sum_{i=2}^n v_i v_{i1}(P) = \frac{\partial G}{\partial x_1}(P).$$

From the definition of the second fundamental form in R^n , we have (note $v_1 = 0$)

$$(11) \quad v_{1i} = - \sum_{j=2}^n h_{ij} v_j,$$

where (h_{ij}) is the second fundamental form. Putting (11) into (10), we obtain

$$0 \leq - \sum_{i,j=2}^n m h_{ij} v_i v_j(P).$$

Since Ω is strictly convex, (h_{ij}) is positive definite; thus

$$0 \leq -m \sum_{i,j=2}^n h_{ij} v_i v_j(P) \leq 0.$$

Hence, $v_i(P) = 0$, $2 \leq i \leq n$, i.e., $\nabla v(P) = 0$.

LEMMA 2. For any given $b > 1$,

$$\frac{|\nabla v|^2}{b^2 - v^2} \leq \lambda(1 + a).$$

PROOF. For $\varepsilon > 0$, consider the function defined on $\bar{\Omega}$

$$G(x) = |\nabla v|^2 + \lambda(1 + \varepsilon + a)v^2.$$

Suppose $G(P) = \max_{x \in \bar{\Omega}} G(x)$. If $P \in \partial\Omega$, by Lemma 1, we have $\nabla v(P) = 0$ and

$$(12) \quad G(x) \leq G(P) = \lambda(1 + \varepsilon + a)v^2(P) \leq \lambda(1 + \varepsilon + a).$$

Now we assume that $P \in \Omega$, and from the maximum principle we have that at P

$$0 = \nabla |\nabla v|^2 + \lambda(1 + \varepsilon + a)\nabla v^2$$

i.e.,

$$(13) \quad v_i v_{ij} = -\lambda(1 + \varepsilon + a)v v_j, \quad 1 \leq j \leq n.$$

Also at P

$$(14) \quad \begin{aligned} 0 &\geq \frac{1}{2} \Delta G(P) = \sum_{i,j} v_{ij}^2 + \nabla v \cdot \nabla(\Delta v) + \lambda(1 + \varepsilon + a)v \Delta v + \lambda(1 + \varepsilon + a)|\nabla v|^2 \\ &= \sum_{i,j} v_{ij}^2 - |\nabla v|^2 - 2\nabla v \cdot \nabla(\nabla v \cdot \nabla \log f_1) - \lambda^2(1 + \varepsilon + a)v(v + a) \\ &\quad - 2\lambda(1 + \varepsilon + a)v(\nabla v \cdot \nabla \log f_1) + \lambda(1 + \varepsilon + a)|\nabla v|^2. \end{aligned}$$

If $\nabla v(P) = 0$, then (12) is valid. If $\nabla v(P) \neq 0$, we can choose an orthonormal frame such that $v_i(P) = 0$, $2 \leq i \leq n$, and $v_1(P) \neq 0$. (13) gives

$$(15) \quad v_{11} = -\lambda(1 + \varepsilon + a)v, \quad v_{1i} = 0, \quad 2 \leq i \leq n.$$

Putting (15) into (14), we obtain

$$\begin{aligned}
 0 &\geq v_{11}^2 + (\varepsilon + a)\lambda|\nabla v|^2 - 2v_1(\nabla v \cdot \nabla \log f_1)_1 - \lambda^2(1 + \varepsilon + a)v^2 \\
 &\quad - \lambda^2(1 + \varepsilon + a)av - 2\lambda(1 + \varepsilon + a)v(\nabla v \cdot \nabla \log f_1) \\
 &= \lambda^2(1 + \varepsilon + a)^2v^2 - \lambda^2(1 + \varepsilon + a)v^2 + (\varepsilon + a)\lambda|\nabla v|^2 - \lambda^2(1 + \varepsilon + a)av \\
 &\quad - 2\sum_{i=1}^n v_1v_{i1}(\log f_1)_i - 2\sum_{i=1}^n v_1v_i(\log f_1)_{i1} \\
 &\quad - 2\lambda(1 + \varepsilon + a)vv_1(\log f_1)_1 \\
 &= \lambda(\varepsilon + a)[|\nabla v|^2 + \lambda(1 + \varepsilon + a)v^2] - \lambda^2(1 + \varepsilon + a)av - 2v_1v_{11}(\log f_1)_1 \\
 &\quad - 2v^2(\log f_1)_{11} - 2(1 + \varepsilon + a)vv_1(\log f_1)_1 \\
 &= \lambda(\varepsilon + a)G(P) - \lambda^2(1 + \varepsilon + a)av + 2v_1\lambda(1 + \varepsilon + a)v(\log f_1)_1 \\
 &\quad - 2v_1^2(\log f_1)_{11} - 2\lambda(1 + \varepsilon + a)vv_1(\log f_1)_1.
 \end{aligned}$$

Hence

$$(16) \quad 0 \geq \lambda(\varepsilon + a)G(P) - \lambda^2(1 + \varepsilon + a)av - 2v_1^2(\log f_1)_{11}.$$

Since W and Ω are all convex, $\log f_1$ is concave [1], in particular, $-(\log f_1)_{11}(P) \geq 0$. Noting that $v \leq 1$, we have

$$(17) \quad \lambda(1 + \varepsilon + a)a \geq (\varepsilon + a)G(P), \quad G(x) \leq \lambda(1 + \varepsilon + 1)\frac{a}{\varepsilon + a} \leq \lambda(1 + \varepsilon + a).$$

From (12) and (17) we can obtain that $G(x) \leq (1 + \varepsilon + a)\lambda$, $x \in \bar{\Omega}$. This is

$$|\nabla v|^2 \leq \lambda(1 + \varepsilon + a)(1 - v^2) \leq \lambda(1 + \varepsilon + a)(b^2 - v^2).$$

Letting $\varepsilon \rightarrow 0$, we complete the proof of the lemma. Q.E.D.

LEMMA 3. $\lambda \geq 1/(1 + a) \cdot \pi^2/d^2$.

PROOF. By Lemma 2, we have

$$(18) \quad \frac{|\nabla(v/b)|}{\sqrt{1 - (v/d)^2}} \leq \lambda^{1/2}(1 + a)^{1/2}.$$

Suppose that q_1 and $q_2 \in \bar{\Omega}$ such that $v(q_1) = 1$, $v(q_2) = -1$, and let L be the line segment between q_1 and q_2 . L lies on $\bar{\Omega}$ completely, because it is convex. We integrate both sides of (18) along L from q_2 to q_1 and obtain

$$(19) \quad \lambda^{1/2}(1 + a)^{1/2}d \geq \lambda^{1/2}(1 + a)^{1/2}\text{length of } L \geq \arcsin \frac{1}{b} - \arcsin \frac{-1}{b}.$$

For any $b > 1$, (19) is valid and, letting $b \rightarrow 1$, the lemma is completely proved. Q.E.D.

If $a = 0$, then the Theorem is proved. Now suppose $a > 0$. From Lemma 2

$$\frac{|\nabla(v/b)|^2}{1 - (v/b)^2} \leq \lambda(1 + a), \quad b > 1.$$

Set $\theta: \bar{\Omega} \rightarrow R$, $\theta = \arcsin(v/b)$, $\arcsin(-1/b) \leq \theta \leq \arcsin(1/b)$. Then

$$\frac{|\nabla(v/b)|^2}{1 - (v/b)^2} = |\nabla\theta|^2 \leq \lambda(1 + a).$$

Obviously, $\nabla\theta = 0$ if $v = 0$. Define $F: [\arcsin -1/b, \arcsin 1/b] \rightarrow R$ as

$$(20) \quad F(\theta_0) = \max_{\substack{x \in \bar{\Omega} \\ \theta(x) = \theta_0}} \frac{|\nabla(v/b)|^2}{1 - (v/b)^2}.$$

$F(\theta_0(x))$ is continuous on $\bar{\Omega}$ and

$$F(\theta_0) \leq \lambda(1 + a), \quad \theta_0 \in [\arcsin -1/b, \arcsin 1/b].$$

For any $\theta_0 \in [\arcsin -1/b, \arcsin 1/b]$ there must be an $x_0 \in \bar{\Omega}$ such that

$$\theta(x_0) = \theta_0 \quad \text{and} \quad F(\theta_0) = \frac{|\nabla(v/b)|^2}{1 - (v/b)^2}(x_0)$$

are valid. Since $a > 0$, we can define a continuous function φ on $\bar{\Omega}$ which satisfies

$$(21) \quad F(\theta) \equiv \lambda \left(1 + \frac{a}{b} \varphi(\theta) \right), \quad \varphi(\theta) \leq b.$$

LEMMA 4. *The C^∞ function $y: [\arcsin -1/b, \arcsin 1/b] \rightarrow R$ satisfies*

- (i) $y(\theta) \geq \varphi(\theta)$, $\theta \in [\arcsin -1/b, \arcsin 1/b]$;
- (ii) *there is an $x_0 \in \bar{\Omega}$ such that $\theta(x_0) = \theta_0$ and $y(\theta_0) = \varphi(\theta_0)$;*
- (iii) $y(\theta) \geq -1$ for any $\theta \in [\arcsin -1/b, \arcsin 1/b]$;
- (iv) $y'(\theta_0) \geq 0$.

Then the following inequality is valid:

$$\varphi(\theta_0) \leq \sin \theta_0 - y'(\theta_0) \sin \theta_0 \cos \theta_0 + \frac{1}{2} y''(\theta_0) \cos^2 \theta_0.$$

PROOF. Consider the function $\Phi(x): \bar{\Omega} \rightarrow R$,

$$\Phi(x) = \left\{ \frac{|\nabla v|^2}{b^2 - v^2} - \lambda(1 + cy) \right\} \cos^2 \theta,$$

where $b > 1$ and $c = a/b$. Obviously, $\Phi(x) \leq 0$ for $x \in \bar{\Omega}$ and $\Phi(x_0) = 0$, i.e., $\Phi(x)$ attains its maximum at x_0 , since

$$\Phi(x) = \frac{1}{b^2} |\nabla v|^2 - \lambda \left(1 - \frac{v^2}{b^2} \right) (1 + cy).$$

If $\nabla v(x_0) = 0$, then

$$0 = \Phi(x_0) = -\lambda(1 - v^2/b^2)(1 + cy)|_{x_0}$$

and

$$y(x_0) = -1/c = -a/b < -1.$$

This contradicts the assumption (iii). Thus $\nabla v(x_0) \neq 0$. By Lemma 1, $x_0 \notin \partial\Omega$, i.e., $x_0 \in \Omega$. According to the maximum principle, we have

$$(22) \quad \nabla \Phi(x_0) = 0,$$

$$(23) \quad \Delta \Phi(x_0) \leq 0.$$

For convenience we write $\Phi(x)$ as

$$\Phi(x) = \frac{1}{b^2} |\nabla v|^2 - \cos^2 \theta (1 + cy).$$

Then

$$\Phi_j = \frac{1}{b^2} \sum_i v_i v_{ij} - \lambda(1 + cy)(-2 \cos \theta \sin \theta) \theta_j - c \lambda \cos^2 \theta y' \theta_j.$$

(22) gives that at x_0

$$(22') \quad \frac{2}{b^2} \sum_i v_i v_{ij} = \lambda[cy' \cos^2 \theta - 2(1 + cy) \cos \theta \sin \theta] \theta_j, \quad 1 \leq j \leq n.$$

And also

$$\begin{aligned} \Delta \Phi &= \frac{2}{b^2} \sum_{i,j} v_{ij}^2 + \frac{2}{b^2} \nabla v \cdot \nabla(\Delta v) - \lambda c \cos^2 \theta \Delta y \\ &\quad - \lambda(1 + cy) \Delta \cos^2 \theta - 2 \lambda c \nabla \cos^2 \theta \cdot \nabla y \\ &= \frac{2}{b^2} \sum_{i,j} v_{ij}^2 + \frac{2}{b^2} \nabla v \cdot \nabla(\Delta v) - \lambda c \cos^2 \theta (y'' |\nabla \theta|^2 + y' \Delta \theta) \\ &\quad + 4 \lambda c y' \cos \theta \sin \theta |\nabla \theta|^2 - \lambda(1 + cy) \Delta \cos^2 \theta. \end{aligned}$$

From (23) we have that at x_0

$$(23') \quad \begin{aligned} 0 &\geq \frac{2}{b^2} \sum_{i,j} v_{ij}^2 + \frac{2}{b^2} \nabla v \cdot \nabla(\Delta v) - \lambda c \cos^2 \theta (y'' |\nabla \theta|^2 + y' \Delta \theta) \\ &\quad + 4 \lambda c y' \cos \theta \sin \theta |\nabla \theta|^2 - \lambda(1 + cy) \Delta \cos^2 \theta. \end{aligned}$$

Since $\nabla v(x_0) \neq 0$, we can choose an orthonormal frame such that $v_1(x_0) \neq 0$ and $v_i(x_0) = 0$, $2 \leq i \leq n$. Then by (22') (note $\sin \theta = v/b$, $\theta_j = v_j/b \cos \theta$)

$$(24) \quad \begin{aligned} v_{i1} &= 0, \quad 2 \leq i \leq n, \\ v_{11} &= (b/2) \lambda [cy' \cos \theta - 2(1 + cy) \sin \theta]. \end{aligned}$$

Now we compute the terms in (23') at the particular frame

$$\begin{aligned} \nabla v \cdot \nabla(\Delta v)_{x_0} &= \sum_{i,j} v_i v_{jji} = \sum_j v_1 (v_{jj})_1 \\ (25) \quad &= v_1 \left[-\lambda(v + a) - 2 \sum_i v_i (\log f_1)_i \right]_1 \\ &= -\lambda v_1^2 - 2v_1 \sum_i v_{i1} (\log f_1)_i - 2v \sum_i v_i (\log f_1)_{i1} \\ &= -\lambda v_1^2 + 2v_1 v_{11} (\log f_1)_1 = 2v_1^2 (\log f_1)_{11} \quad (\because (24)). \end{aligned}$$

From $\Delta v/b = \Delta \sin \theta = \cos \theta \Delta \theta - \sin \theta |\nabla \theta|^2$, we have

$$(26) \quad \Delta \theta = \frac{1}{\cos \theta} \left[\frac{\Delta v}{b} + \sin \theta |\nabla \theta|^2 \right].$$

And

$$\begin{aligned} (27) \quad \Delta \cos \theta &= \Delta \left(1 - \frac{v^2}{b^2} \right) = -\frac{1}{b^2} \Delta v^2 = -\frac{2}{b^2} (v \Delta v + |\nabla V|^2) \\ &= -\frac{2}{b^2} v \Delta v - 2 \cos^2 \theta |\nabla \theta|^2. \end{aligned}$$

From $\Phi(x_0) = 0$, we have at x_0

$$(28) \quad |\nabla\theta|^2 = \frac{1}{b^2} \frac{|\nabla v|^2}{\cos^2 \theta} = \frac{1}{b^2} \frac{|\nabla v|^2}{1 - (v/b)^2} = \frac{|\nabla v|^2}{b^2 - v^2} = \lambda(1 + cy).$$

Putting (25)–(28) into (23'), we obtain that at x_0

$$(29) \quad \begin{aligned} 0 \geq & \frac{2}{b^2} \sum_{i,j} v_{ij}^2 + \frac{2}{b^2} [-\lambda v_1^2 - 2v_1 v_{11} (\log f_1)_1 - 2v_1^2 (\log f_1)_{11}] \\ & - \lambda c \left[\lambda y''(1 + cy) + y' \left(\frac{\Delta v}{b \cos \theta} + \frac{\sin \theta}{\cos \theta} \right) \lambda(1 + cy) \right] \cos^2 \theta \\ & + 4(1 + cy) \lambda^2 c y' \cos \theta \sin \theta - \lambda(1 + cy) \left[-\frac{2}{b^2} v \Delta v - 2\lambda \cos^2 \theta (1 + cy) \right]. \end{aligned}$$

Since $|\nabla v|^2 = v_1^2 = b^2 \cos^2 \theta |\nabla\theta|^2 = \lambda^2 \cos^2 \theta (1 + cy)$ ($\because$ (28)) and $\Delta v = -\lambda(v + a) - 2\nabla v \cdot \nabla \log f_1$, (29) can be written as

$$\begin{aligned} 0 \geq & \frac{2}{b^2} \sum_{i,j} v_{ij}^2 - 2\lambda^2(1 + cy) \cos^2 \theta - \lambda^2 c y''(1 + cy) \cos^2 \theta \\ & + 2\lambda^2 c y' \cos \theta (\sin \theta + c) + 3\lambda^3 c y'(1 + cy) \sin \theta \cos \theta \\ & + 2\lambda(1 + cy)^2 \cos^2 \theta - 2\lambda^2(1 + cy) \sin \theta (\sin \theta + c) \\ & - \frac{4}{b^2} v_1^2 (\log f_1)_{11} - \frac{4}{b^2} v_1 v_{11} (\log f_1)_1 \\ & + \frac{2}{b} \lambda [c y' \cos \theta - 2(1 + cy) \sin \theta] v_1 (\log f_1)_1. \end{aligned}$$

Putting the second formula of (24) into the above inequality and noting that $(\log f_1)_{11} \leq 0$, we have

$$\begin{aligned} 0 \geq & \frac{1}{2} \lambda^2 c^2 (y')^2 \cos^2 \theta + 2\lambda^2(1 + cy)^2 - 2\lambda^2(1 + cy) \cos^2 \theta \\ & + \lambda^2 c y' [(1 + cy) \sin \theta \cos \theta + \cos \theta (\sin \theta + c)] \\ & - \lambda^2 c (1 + cy) y'' \cos^2 \theta - 2\lambda^2(1 + cy) \sin \theta (\sin \theta + c). \end{aligned}$$

Dividing both sides of the above inequality by $\lambda^2(1 + cy) > 0$, we have

$$\begin{aligned} 0 \geq & y' \left(\sin \theta \cos \theta + \cos \theta \frac{\sin \theta + c}{1 + c} \right) - y'' \cos^2 \theta + \frac{2}{c} (1 + cy) - \frac{2}{c} - 2 \sin \theta, \\ 2y - 2 \sin \theta \leq & y'' \cos^2 \theta - y' \left(\sin \theta \cos \theta + \cos \theta \frac{\sin \theta + c}{1 + cy} \right). \end{aligned}$$

Since $-1 \leq y(\theta_0) = \varphi(\theta_0) \leq b$, thus,

$$|y(\theta_0)| \leq b, \quad y(\theta_0) \sin \theta_0 \leq |y(\theta_0)| |\sin \theta_0| \leq b v(\theta_0)/b \leq 1$$

and

$$c \geq c y \sin \theta, \quad c + \sin \theta \geq (1 + cy) \sin \theta, \quad \frac{c + \sin \theta}{1 + cy} \geq \sin \theta.$$

Since $y'(\theta_0) \geq 0$, we have

$$\varphi(\theta_0) = y(\theta_0) \leq \sin \theta_0 - y'(\theta_0) \sin \theta_0 \cos \theta_0 + \frac{1}{2} y''(\theta_0) \cos^2 \theta_0. \quad \text{Q.E.D.}$$

LEMMA 5. Define a function $\psi: [-\pi/2, \pi/2] \rightarrow \mathbb{R}$ as

$$(30) \quad \begin{cases} \psi(\theta) = \frac{(4/\pi)(\theta + \cos \theta \sin \theta) - 2 \sin \theta}{\cos^2 \theta}, & \theta \in (-\pi/2, \pi/2), \\ \psi(-\pi/2) = -1, & \psi(\pi/2) = 1. \end{cases}$$

Then ψ is a C^∞ function in $(-\pi/2, \pi/2)$ and is continuous on $[-\pi/2, \pi/2]$ and also $y = \psi(\theta)$ satisfies the following equation:

$$(31) \quad y - \sin \theta + y' \sin \theta \cos \theta - \frac{1}{2} y'' \cos^2 \theta = 0,$$

and $y'(\theta) \geq 0$, $\theta \in (-\pi/2, \pi/2)$.

PROOF. See reference [4].

LEMMA 6. Let $\varphi(\theta)$ be the function defined by (21). Then

$$\varphi(\theta) \leq \psi(\theta), \quad \theta \in \left[\arcsin \frac{-1}{b}, \arcsin \frac{1}{b} \right],$$

where $\psi(\theta)$ is defined by (30).

PROOF. We will use the reduction to absurdity. If

$$(32) \quad \sigma = \varphi(\theta_0) - \psi(\theta_0) = \max_{\theta} \{ \varphi(\theta) - \psi(\theta) \} > 0$$

we could choose $\psi(\theta) + \sigma = \tilde{y}$ as y in Lemma 4. Therefore,

$$\varphi(\theta_0) = \tilde{y}(\theta_0) = \psi(\theta_0) + \sigma \leq \sin \theta_0 - \psi'(\theta_0) \sin \theta_0 \cos \theta_0 + \frac{1}{2} \psi''(\theta_0) \cos^2 \theta_0 = \psi(\theta_0).$$

This contradicts (32). Q.E.D.

PROOF OF THE THEOREM. By Lemma 6, we have

$$|\nabla \theta|^2 = \frac{|\nabla v|^2}{b^2 - v^2} \leq \lambda \left(1 + \frac{a}{b} \psi(\theta) \right),$$

where $\psi(\theta)$ is the function defined by (30). Hence,

$$(33) \quad \lambda^{1/2} \geq \frac{|\nabla \theta|}{\sqrt{1 + (a/b)\psi(\theta)}}.$$

Obviously,

$$(34) \quad \psi(0) = 0, \quad \psi(-\theta) = -\psi(\theta).$$

Integrating both sides of (33) as in Lemma 3, we obtain

$$\begin{aligned} \lambda^{1/2} d &\geq \int_{\arcsin -1/b}^{\arcsin 1/b} \frac{d\theta}{\sqrt{1 + (a/b)\psi(\theta)}} \\ &= \int_0^{\arcsin 1/b} \left(\frac{1}{\sqrt{1 + (a/b)\psi(\theta)}} + \frac{1}{\sqrt{1 - (a/b)\psi(\theta)}} \right) d\theta. \end{aligned}$$

Since $|\pm (a/b)\psi(\theta)| \leq 1$,

$$\frac{1}{\sqrt{1 + (a/b)\psi(\theta)}} + \frac{1}{\sqrt{1 - (a/b)\psi(\theta)}} = 2 \left[1 + \sum_{p=1}^{\infty} \frac{1 \cdot 3 \cdots (4p-1)}{2 \cdot 4 \cdots (4p)} \left(\frac{a}{b} \right)^{2p} \psi^{2p} \right] \geq 2.$$

Thus

$$\lambda^{1/2} d \geq 2 \arcsin \frac{1}{b}, \quad \lambda \geq \frac{4}{d^2} \left(\arcsin \frac{1}{b} \right)^2.$$

Letting $b \rightarrow 1$, we obtain $\lambda \geq \pi^2/d^2$. Q.E.D.

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