SPHERICAL POLYNOMIALS AND THE PERIODS OF A CERTAIN MODULAR FORM

BY

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ABSTRACT. The space of cusp forms on $SL_2(\mathbf{Z})$ of weight $2k$ is spanned by certain modular forms with rational periods.

Kohnen and Zagier [5] have investigated those modular forms on $SL_2(\mathbf{Z})$ whose periods (in the sense of Eichler-Shimura) are rational. As an example they consider the function

$$f_{k,D}(z) = C \sum_{(a,b,c)} \frac{1}{(az^2 + bz + c)^k}$$

where D is a positive integer, k an integer greater than one, C a constant, and the summation over all triples of integers (a, b, c) with $b^2 - 4ac = D$. This function was introduced by Zagier in [13], where it was shown to be a cusp form of weight $2k$ on $SL_2(\mathbf{Z})$, and was later shown by Kohnen [4] to be the D th Fourier coefficient of the holomorphic kernel function for the Shimura-Shintani correspondence between modular forms of half-integral and integral weight. In [6] it is shown that the $f_{k,D}$ span S_{2k} , the space of cusp forms on $SL_2(\mathbf{Z})$ of weight $2k$, if and only if $L(f, k)$, the L -series associated to the modular form f , does not vanish at $s = k$, the center of the critical strip, for every Hecke eigenform f . It is at present only a conjecture, supported by some numerical evidence, that the $f_{k,D}$ span S_{2k} for all even k (for k odd, $f_{k,D} = 0$). We study the related functions $f_{k,D,A}(z)$ where now the summation is restricted to quadratic forms (a, b, c) belonging to the $SL_2(\mathbf{Z})$ equivalence class A . We prove that certain finite sets of the $f_{k,D,A}$ span S_{2k} . The method of proof relies on the special structure of certain ideal classes and does not indicate how the conjecture that the $f_{k,D}$ span S_{2k} might be settled.

A similar result for modular forms on an arbitrary discrete cocompact subgroup of $SL_2(\mathbf{R})$ has been obtained by Svetlana Katok [3].

Let $\mathcal{A}$ be the set of all narrow equivalence classes of primitive binary quadratic forms of positive discriminant. Throughout k will denote a positive integer, and we set $w = 2k - 2$. Let A be an $SL_2(\mathbf{Z})$ equivalence class of quadratic forms. We define the following classes associated to A :

$$\theta A = \{(-c, -b, -a) \mid (a, b, c) \in A\}, \quad A' = \{(a, -b, c) \mid (a, b, c) \in A\}.$$

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Let P_k denote the $\mathbf{C}$ -vector space of homogeneous polynomials in two variables of degree w . For an equivalence class $A \in \mathcal{A}$ of forms of discriminant D , let

$$P_{k,D,A}(x, y) = \sum_{Q \in A} Q(x, y)^{k-1},$$

with the summation over the reduced forms of A (a binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ is reduced if $a > 0$, $c > 0$, $b > a + c$), and then set

$$P_{k,D,A}^{\pm} = P_{k,D,A} \pm P_{k,D,\theta A}.$$

We shall study the span in P_k of $P_{k,D,A}^{\pm}$, $A \in \mathcal{A}$, and prove a theorem about the space of modular forms on $\mathrm{SL}_2(\mathbf{Z})$ of weight $2k$.

DEFINITION 1. A polynomial $\pi(a, b, c)$ is called *spherical* with respect to $b^2 - 4ac$ if it satisfies

$$\left(\frac{\partial^2}{\partial b^2} - \frac{\partial^2}{\partial a \partial c} \right) \pi(a, b, c) = 0.$$

PROPOSITION 1. Let $d_{n,k}(a, b, c)$ be defined by

$$\sum_{n=0}^{2k-2} d_{n,k}(a, b, c) x^n y^{2k-2-n} = (ax^2 + bxy + cy^2)^{k-1}.$$

The set $\{d_{n,k}(a, b, c), 0 \leq n \leq w\}$ is a basis for the space of homogeneous spherical polynomials of degree $k - 1$.

For a proof see [8, the corollary to Theorem 18].

THEOREM 1. Let $\mathcal{W}_k^{\pm}$ be the subspaces of P_k spanned by the collections $\{P_{k,D,A}^{\pm} + P_{k,D,A'}^{\pm}\}$, $A \in \mathcal{A}$.

I. Then the dimensions of $\mathcal{W}_k^{\pm}$ are equal to $\eta_k^{\pm}$, where

$$\eta_k^+ = \begin{cases} k/2 + 1, & k \text{ even}, \\ (k+1)/2, & k \text{ odd}, \end{cases}$$

and

$$\eta_k^- = \begin{cases} k/2, & k \text{ even}, \\ (k+1)/2, & k \text{ odd}. \end{cases}$$

II. There is an infinite subset of the set $\mathcal{A}$ of equivalence classes of forms, which we shall construct explicitly, that has the following property: any $2k$ classes in this set have corresponding $P_{k,D,A}^{\pm} + P_{k,D,A'}^{\pm}$ which span $\mathcal{W}_k^{\pm}$. Moreover, given any positive discriminant D , we can choose these classes such that each has discriminant $f^2 D$ for some integer f ; i.e., all of the equivalence classes are associated to module classes in the same real quadratic field.

NOTE. We consider $P_{k,D,A}^{\pm} + P_{k,D,A'}^{\pm}$ rather than $P_{k,D,A}^{\pm}$ alone, because in the course of the proof we need to assume that our summation includes the form

(c, b, a) whenever it contains (a, b, c) . Let

$$\Sigma_A^\pm \quad \text{mean} \quad \sum_A + \sum_{A'} \pm \sum_{\theta A} \pm \sum_{\theta A'},$$

where the summation is over the reduced forms in the given class. Since the coefficients of $Q(x, y)^{k-1}$ are the $d_{n,k}(a, b, c)$, it follows from Proposition 1 that to establish the dimension of $\mathscr{W}_k^\pm$ it suffices to find $w + 1 - \eta_k^\pm$ $\mathbf{Q}$ -linearly independent linear combinations of the $d_{n,k}(a, b, c)$, for each of which $\Sigma_A^\pm = 0$ for all $A \in \mathscr{A}$, and $\eta_k^\pm$ $\mathbf{Q}$ -linearly independent linear combinations of the $d_{n,k}(a, b, c)$, of which no nontrivial linear combination sums $(\Sigma_A^\pm)$ to zero for all cycles $A \in \mathscr{A}$.

We first establish the requisite number of relations. Let

$$(1) \quad \phi_{n,k}(a, b, c) = d_{n,k}(a, b, c) - d_{w-n,k}(a, b, c), \quad 0 \leq n \leq w,$$

and

$$(2) \quad \begin{aligned} \psi_{n,k}^\pm(a, b, c) &= d_{n,k}(a, b - 2a, a - b + c) - d_{n,k}(c, 2c - b, a - b + c) \\ &\quad \mp (-1)^k [d_{n,k}(a - b + c, b - 2a, a) - d_{n,k}(c - b + a, 2c - b, c)], \\ &\quad 0 \leq n \leq w. \end{aligned}$$

We assert that for any $A \in \mathscr{A}$

$$(i) \quad 0 = \Sigma_A^\pm \phi_{n,k}(a, b, c),$$

$$(ii) \quad 0 = \Sigma_A^\pm \psi_{n,k}^\pm(a, b, c).$$

That (i) is true follows immediately from the facts that $d_{w-n,k}(a, b, c) = d_{n,k}(c, b, a)$ and $(a, b, c) \in A \Leftrightarrow (c, b, a) \in A'$. To establish (ii), let $R_{n,k}^\pm(a, b, c) = d_{n,k}(a, b, c) \mp (-1)^k d_{n,k}(c, b, a)$. Then $R_{n,k}^\pm(a, b, c) = \pm R_{n,k}^\pm(-c, -b, -a)$, and the result follows from [7, Proposition 3]. From among the polynomials (1) and (2) we select $w + 1 - \eta_k^\pm$ that are independent.

LEMMA 1. *The collections*

$$\mathscr{R}_k^+ = \{ \phi_{n,k}, 0 \leq n < k - 1; \psi_{n,k}^+, 1 \leq n < k - 1, n \text{ odd} \}$$

and

$$\mathscr{R}_k^- = \{ \phi_{n,k}, 0 \leq n < k - 1; \psi_{n,k}^-, 1 \leq n \leq k - 1, n \text{ odd} \}$$

are each composed of linearly independent polynomials.

PROOF. From Proposition 1 it is clear that the $\phi_{n,k}$, $0 \leq n < k - 1$, are linearly independent. Moreover, $\phi_{n,k}(a, b, c) = -\phi_{n,k}(c, b, a)$, while for n odd, $\psi_{n,k}^\pm(a, b, c) = +\psi_{n,k}^\pm(c, b, a)$. Thus it suffices now to show that the $\psi_{n,k}^\pm(a, b, c)$ in $\mathscr{R}_k^\pm$ are independent among themselves.

To establish this we introduce a change of basis for the space of spherical polynomials, namely the transformation

$$(a, b, c) \mapsto (A, B + 2A, A + B + C).$$

Then

$$\begin{aligned}
 \psi_{n,k}^{\pm}(a, b, c) &= \psi_{n,k}^{\pm}(A, B + 2A, A + B + C) \\
 &= d_{n,k}(A, B, C) - d_{n,k}(A + B + C, B + 2C, C) \\
 &\quad \mp (-1)^k d_{n,k}(A, B, C) - d_{n,k}(C, B + 2C, A + B + C) \\
 &= d_{n,k}(A, B, C) - \sum_{j=0}^n \binom{w-j}{n-j} d_{j,k}(A, B, C) \\
 &\quad \mp (-1)^k \left(d_{w-n,k}(A, B, C) - \sum_{j=0}^{w-n} \binom{w-j}{n} d_{j,k}(A, B, C) \right),
 \end{aligned}$$

which for $1 \leq n \leq k-1$, n odd, is a linear combination of the $d_{j,k}(A, B, C)$ with $0 \leq j \leq w-n-1$ and the coefficient of $d_{w-n-1,j}$ not equal to zero (except for $\psi_{k-1,k}^+$, k even, which is zero). This triangularization of the $\psi_{n,k}^{\pm}$ in $\mathcal{R}_k^{\pm}$ demonstrates their linear independence.

The number of $\phi_{n,k}$ in $\mathcal{R}_k^{\pm}$ is $k-1$ and the number of $\psi_n^{\pm}$ in $\mathcal{R}_k^{\pm}$ is given in Table 1.

TABLE 1. The number of $\psi_{n,k}^{\pm}$ in $\mathcal{R}_k^{\pm}$

	$\mathcal{R}^+$	$\mathcal{R}^-$
k even	$k/2 - 1$	$k/2$
k odd	$(k-1)/2$	$(k-1)/2$

Thus in all cases we have, as asserted, $w+1-\eta_k$ independent relations.

We complete the proof of Theorem 1 by proving the following: Consider the family of discriminants $D_t = t^2 + 4t$, $t = 2, 3, 4, \dots$. Belonging to D_t is a wide equivalence class of quadratic forms associated to the reduced (in the wide sense) number

$$[[1, t]] = 1 + \frac{1}{t + \frac{1}{1 + \dots}}.$$

(For a discussion of the relationship between continued fractions and reduced quadratic forms see [12].) Since this “+”-continued fraction is of even length, the wide equivalence class consists of two distinct narrow equivalence classes, A_t and θA_t , whose cycles of reduced forms are generated by the “-”-continued fractions $\mu_0 = ((3, 2, 2, \dots, 2))$ and $\omega_0 = ((t+2))$, respectively.

PROPOSITION 2. Set

$$(3) \quad g_{n,k}^{\pm}(t) = \frac{1}{2} \sum_{A_t}^{\pm} d_{n,k}(a, b - 2a, a - b + c).$$

Then any nontrivial linear combination of elements of the set (of cardinality $\eta_k^{\pm}$)

$$\left\{ g_{n,k}^{\pm}(t), n \text{ odd}, 1 \leq n < k-1 \left(\leq k-1 \text{ if } (-1)^k = \pm 1 \right); g_{w,k}^{\pm}(t) \right\}$$

is zero for at most $2k$ values of $t > 2$.

PROOF. The cycle of reduced forms of A_t is $\{Q_p = (a_p, b_p, c_p)\}_{0 \leq p \leq t-1}$ where $a_p = (p+1)t - p^2$, $b_p = (2p+3)t - 2(p^2 + p)$, $c_p = (p+2)t - (p+1)^2$. θA_t contains the single reduced form $\bar{Q} = (1, t+2, 1)$. (Note that $A_t = A'_t$, so $\Sigma_{A_t}^\pm = 2(\Sigma_{A_t} \pm \Sigma_{\theta A_t})$.) Taking n odd, and recalling that

$$d_{n,k}(a, b, c) = (-1)^n d_{n,k}(a, -b, c),$$

we have

$$(4) \quad \sum_{A_t} d_{n,k}(a, b - 2a, a - b + c) = \sum_{p=0}^{t-1} d_{n,k}((p+1)t - p^2, t - 2p, -1).$$

Under the transformation $p \mapsto t - p$ the right-hand side of (4) becomes

$$\begin{aligned} \sum_{p=1}^t d_{n,k}((p+1)t - p^2, 2p - t, -1) &= - \sum_{p=1}^t d_{n,k}((p+1)t - p^2, t - 2p, -1) \\ &= - \sum_{p=0}^{t-1} d_{n,k}((p+1)t - p^2, t - 2p, -1) + d_{n,k}(t, t, -1) - d_{n,k}(t, -t, -1), \end{aligned}$$

from which we conclude that

$$\begin{aligned} \sum_{A_t} d_{n,k}(a, b - 2a, a - b + c) &= \frac{1}{2}(d_{n,k}(t, t, -1) - d_{n,k}(t, -t, -1)) \\ &= d_{n,k}(t, t, -1). \end{aligned}$$

We also have

$$\sum_{\theta A_t} d_{n,k}(a, b - 2a, a - b + c) = d_{n,k}(1, t, -t).$$

Thus for n odd,

$$\left(\sum_A \pm \sum_{\theta A} \right) d_{n,k}(a, b - 2a, a - b + c) = d_{n,k}(t, t, -1) \pm d_{n,k}(1, t, -t) = g_{n,k}^\pm(t).$$

Writing

$$d_{n,k}(a, b, c) = \sum_{\substack{2\alpha+\beta=n \\ \alpha+\beta \leq k+1}} \binom{k-1}{\beta} \binom{k-1-\beta}{\alpha} a^\alpha b^\beta c^{k-(1+\alpha+\beta)},$$

we have

$$(5) \quad g_{n,k}^\pm(t) = \sum_{\substack{2\alpha+\beta=n \\ \alpha+\beta \leq k-1}} \binom{k-1}{\beta} \binom{k-1-\beta}{\alpha} (-1)^{k-(1+\alpha+\beta)} (t^{\alpha+\beta} \pm t^{k-1-\alpha}).$$

If $n = k - 1$, then $\alpha + \beta = k - 1 - \alpha$, and so $g_{n,k}^\pm(t) = 0$. Otherwise, for $n \leq k - 1$, n odd, we claim that the right-hand side of (5) is a polynomial in t of degree $\leq k - 1$ whose $t^{(n+1)/2}$ coefficient is nonzero but all of whose lower powers of t vanish.

To see this, observe that $\alpha + \beta = n - \alpha$, and so $\alpha + \beta$ and $k - 1 - \alpha$ are each minimal for the largest possible choice of α , namely for $\beta = 1$, $\alpha = (n - 1)/2$. Then $\alpha + \beta = (n + 1)/2$, and $k - 1 - \alpha = k - ((n + 1)/2) \geq (n + 1)/2$, since $n \leq k - 1$.

We now consider

$$\begin{aligned} g_{w,k}^{\pm}(t) &= \sum_{p=0}^{t-1} d_{w,k}((p+1)t - p^2, t - 2p, -1) \pm d_{w,k}(1, t, -t) \\ &= (\pm 1) + \sum_{p=0}^{t-1} ((p+1)t - p^2)^{k-1}. \end{aligned}$$

We claim that the sum in this expression is a polynomial in t with no constant term, whence $g_{w,k}^{\pm}(t)$ does have a constant term, namely ± 1 .

$$\sum_{p=0}^{t-1} ((p+1)t - p^2)^{k-1} = \sum_{p=0}^{t-1} (\pm p^w + \text{higher powers of } t).$$

But

$$\sum_{p=0}^{t-1} p^w = \frac{B_{w+1}(t) - B_{w+1}}{w+1},$$

where $B_s(t)$ is the s th Bernoulli polynomial. Since $w+1$ is odd, this is a polynomial in t with no constant term, and our claim has been proved.

This triangularization of the polynomials in (3) shows them to be linearly independent.

The $g_{n,k}^{\pm}(t)$ for n odd have degree at most $k-1$, while the degree of $g_{w,k}^{\pm}(t)$ is $2k-1$. Thus any nontrivial linear combination of the $g_{n,k}^{\pm}$ is a polynomial in t of degree at most $w+1 = 2k-1$ and so can be zero for at most $2k-1$ values of t . Thus for any $2k$ values of $t \geq 2$, the corresponding set of the $P_{k,D,A_t}^{\pm}$ must span $\mathcal{W}_k^{\pm}$.

To show that we can choose the A_t to be associated to the same real quadratic field, fix a discriminant $D > 0$. The class A_t has discriminant $t^2 + 4t$, and since $t^2 + 4t = f^2 D$ is equivalent to $(t+2)^2 - f^2 D = 4$, which is just Pell's equation, we can find infinitely many values of t for which A_t has discriminant $f^2 D$ (for different values of f).

We note the following corollary, which we shall need in the proof of a later theorem.

COROLLARY. *The sets $\{\psi_{n,k}^+(a, b, c)\}$, n odd, $1 \leq n < k-1$, and $\{\psi_{n,k}^-(a, b, c)\}$, n odd, $1 \leq n \leq k-1$, are bases for the spaces of those spherical polynomials P of degree $k-1$ for which $\sum_A^{\pm} P = 0$ for all $A \in \mathcal{A}$.*

We now consider the function

$$f_{k,D}(z) = \frac{(-1)^{(k-1)/2} 2^{3k-2} D^{k-1/2}}{\pi^{(2k-2k-1)}} \sum_{(a,b,c)} \frac{1}{(az^2 + bz + c)^k},$$

where D is a positive integer, k an integer ≥ 2 , and where the summation is over all triples of integers (a, b, c) with $b^2 - 4ac = D$. This function was introduced by Zagier in [13], where it was shown to be a cusp form of weight $2k$ on $\mathrm{SL}_2(\mathbf{Z})$. Then Kohnen showed in [4] that $f_{k,D}$ is the D th Fourier coefficient of the holomorphic kernel function for the Shimura-Shintani correspondence between modular forms of weight $k + \frac{1}{2}$ and weight $2k$. We shall be interested in the functions $f_{k,D,A}(z)$,

defined in the same way as $f_{k,D}$, except that here the summation is over only those forms (a, b, c) belonging to the equivalence class A . The periods of the functions $f_{k,D,A}^{\pm} = f_{k,D,A} \pm f_{k,D,A'}$ were studied by Kohnen and Zagier in [5]. Let us here recall the basic facts about periods of modular forms.

Let f be a cusp form of weight $2k$. Then for any integer $0 \leq n \leq 2k - 2 = w$, we can define the n th period of f ,

$$r_n(f) = \int_0^{i\infty} f(z) z^n dz.$$

The fundamental theorems on periods are the following, our exposition being essentially that in Lang [9].

THEOREM 2 (EICHLER - SHIMURA). *There are certain relations among the periods of modular forms:*

$$(i) \quad r_s + (-1)^s \sum_{\substack{j=0 \\ j \text{ even}}}^s \binom{s}{j} r_{w-s+j} + (-1)^s \sum_{\substack{j=0 \\ j \equiv s \pmod{2}}}^{w-s} \binom{w-s}{j} r_j = 0;$$

$$(ii) \quad \sum_{\substack{j=1 \\ j \text{ odd}}}^s \binom{s}{j} r_{w-s+j} + \sum_{\substack{j=0 \\ j \not\equiv s \pmod{2}}}^{w-s} \binom{w-s}{j} r_j = 0.$$

Since $r_j(f)$ is real if j is odd, and pure imaginary if j is even, these relations can be separated into two sets of relations: one involving only the even periods, the other only the odd. These relations then define two subspaces, V^+ of $\mathcal{P}_k^+$ and V^- of $\mathcal{P}_k^-$, where $\mathcal{P}_k^+$ (resp. $\mathcal{P}_k^-$) is the subspace of $\mathcal{P}_k$ consisting of polynomials even (resp. odd) in both variables.

THEOREM 3 (EICHLER - SHIMURA). *Let S_{2k} be the space of cusp forms of weight $2k$ on $\mathrm{SL}_2(\mathbf{Z})$. Define the mappings $r^{\pm}$ of S_{2k} into $\mathcal{P}_k^{\pm}$ by*

$$(6) \quad r^{\pm}(f)(x, y) = \varepsilon_{\pm} \sum_{\substack{0 \leq n \leq w \\ (-1)^n = \pm 1}} \binom{w}{n} r_n(f) x^n y^{w-n},$$

where $\varepsilon_+ = -i$, $\varepsilon_- = 1$.

Then r^- is an isomorphism of S_{2k} and V^- , while r^+ is an isomorphism of S_{2k} with a subspace of V^+ of codimension 1, not containing $x^w - y^w$.

Recall that $P_{k,D,A}(x, y) = \sum_A Q(x, y)^{k-1}$.

In [5] the following theorem is proved.

THEOREM 4. *Let $k \geq 2$, and let A be an $\mathrm{SL}_2(\mathbf{Z})$ class of quadratic forms of discriminant $D > 0$, D not a square. Then*

$$\begin{aligned} & \frac{(-1)^{(k+1)/2}}{2^{3k+1}} r^{\pm}(f_{k,D,A}) \\ &= P_{k,D,A}(x+y, -x) - P_{k,D,A}(x-y, y) + (-1)^k P_{k,D,\theta A}(x, -x+y) \\ & \quad - P_{k,D,\theta A}(-y, x+y) \frac{1+(-1)^n}{B_{2k}} \zeta(A, 1-k)(x^w - y^w), \end{aligned}$$

or, equivalently, for $(-1)^n = \pm 1$,

$$\begin{aligned} & \frac{(-1)^{(k+1)/2}}{2^{3k-1}} \binom{w}{n} r_n(f_{k,D,A}^\pm) \\ &= \sum_{\substack{(a,b,c) \in A \\ \text{reduced}}} d_{n,k}(c-b+a, 2a-b, a) - d_{n,k}(a, b-2a, a-b+c) \\ &+ (-1)^k \sum_{\substack{(a,b,c) \in \theta A \\ \text{reduced}}} (d_{n,k}(a-b+c, b-2c, c) - d_{n,k}(c, 2c-b, c-b+a)) \\ &+ (\delta_{w,n} - \delta_{0,n}) \frac{2k}{B_{2k}} \zeta(A, 1-k), \end{aligned}$$

where B_s is the s th Bernoulli number, $\delta_{j,k}$ the Kronecker delta, and $\zeta(A, s)$ the Dedekind zeta function of the ideal class associated to A .

We can now state and prove our second result.

THEOREM 5. Let $f_{k,D,A}$ be defined as above. For k an integer greater than 1, the collection of the $f_{k,D,A}$, $A \in \mathcal{A}$, spans S_{2k} , the space of cusp forms on $\text{SL}_2(\mathbb{Z})$ of weight $2k$.

PROOF. By Theorems 3 and 4, it suffices to show that the polynomials $r^+(f_{k,D,A}^+)$ span the image of $r^+(S_{2k})$ as defined in Theorem 3.

LEMMA 2. $f_{k,D,A}^\pm = (-1)^k f_{k,D,\theta A}^\pm$.

PROOF. If $(a, b, c) \in A$, then $(-a, -b, -c) \in \theta A'$. Thus $f_{k,D,A} = (-1)^k f_{k,D,\theta A'}$, and the result follows at once from the definition of $f_{k,D,A}^\pm$.

From Lemma 2 and Theorem 4, as well as the fact that $(a, b, c) \in A \Rightarrow (c, b, a) \in A'$, we immediately deduce

LEMMA 3. Let

$$\begin{aligned} S_{n,k}(a, b, c) &= -d_{n,k}(a-b+c, b-2a, a) \\ &\quad -d_{n,k}(c-b+a, b-2c, c) + d_{n,k}(c, b-2c, c-b+a). \end{aligned}$$

The even periods of $f_{k,D,A}^+$ are given for $2 \leq n \leq w-2$ by

$$\begin{aligned} & \frac{-(-1)^{(k+1)/2} \binom{w}{n}}{2^{3k-3}} r_n(f_{k,D,A}^+) \\ &= \left(\sum_A + (-1)^k \sum_{\theta A} \right) (-d_{n,k}(a-b+c, b-2a, a) - d_{n,k}(c-b+a, b-2c, c) \\ &\quad + d_{n,k}(c, b-2c, c-b+a)) \\ &=: \left(\sum_A + (-1)^k \sum_{\theta A} \right) S_{n,k}(a, b, c). \end{aligned}$$

Now, by the corollary to Theorem 1 and Lemma 3, there is a relation among these even periods of $f_{k,D,A}^+$ if and only if some linear combinations of the $S_{n,k}$, n even, equals a linear combination of the $S_{n,k}$, n odd. Thus to prove Theorem 5, it suffices

to show that the dimension of the subspace of the spherical polynomials of degree $k - 1$ spanned by the $S_{n,k}$ has dimension equal to the span of the $S_{n,k}$, n odd (which is $[(k - 1)/2]$) plus the dimension of S_{2k} (which is $[k/6]$, unless $2k \equiv 2 \pmod{12}$, in which case it is $[k/6] - 1$), plus 1 (for the zeroth period), which totals $[2k/3]$ in all cases. We proceed now to calculate this dimension. For $0 \leq n \leq w$ define polynomials $\kappa_{n,k}(x, y)$ in $\mathcal{P}_k$ by

$$\begin{aligned} \kappa_{n,k}(x, y) = & (x - y)^n (-x)^{w-n} + (-x)^n (x - y)^{w-n} \\ & - (x - y)^n y^{w-n} - y^n (x - y)^{w-n}. \end{aligned}$$

LEMMA 4. $\sum_{n=0}^w S_{n,k}(a, b, c) = \sum_{n=0}^w d_{n,k}(a, b, c) \kappa_{n,k}(x, y)$.

PROOF.

$$\begin{aligned} & \sum_{n=0}^w S_{n,k}(a, b, c) x^n y^{w-n} \\ &= \sum_{n=0}^w (d_{n,k}(a, b - 2a, a - b + c) + d_{n,k}(c, b - 2c, c - b + a) \\ & \quad - d_{n,k}(a - b + c, b - 2a, a) - d_{n,k}(c - b + a, b - 2c, c)) x^n y^{w-n} \\ &= ((a - b + c)x^2 + (b - 2a)xy + ay^2)^{k-1} \\ & \quad + ((c - b + a)x^2 + (b - 2c)xy + cy^2)^{k-1} \\ & \quad - (ax^2 + (b - 2a)xy + (a - b + c)y^2)^{k-1} \\ & \quad - (cx^2 + (b - 2c)xy + (c - b + a)y^2)^{k-1} \\ &= \sum_{n=0}^w (d_{n,k}(a, b, c)) ((x - y)^n (-x)^{w-n} + (-x)^n (x - y)^{w-n} \\ & \quad - (x - y)^n y^{w-n} - y^n (x - y)^{w-n}) \\ &= \sum_{n=0}^w d_{n,k}(a, b, c) \kappa_{n,k}(x, y). \end{aligned}$$

We now assert that the dimension of the span of the $\mathcal{P}_{n,k}$ in the space of spherical polynomials of degree $k - 1$ is equal to the dimension of the subspace of $\mathcal{P}_k$ spanned by the $\kappa_{n,k}(x, y)$.

PROPOSITION 3. *Let V and A be vector spaces over a field, of dimension N , with bases $v_1, \dots, v_N$ and $a_1, \dots, a_N$ respectively, and suppose that*

$$\sum_{i=1}^N w_i \otimes a_i = \sum_{j=1}^N v_j \otimes b_j,$$

for some w_i , $1 \leq j \leq N$, in V and b_j , $1 \leq j \leq N$, in A .

Then the dimensions of the subspace of V generated by the w_i , and of the subspace of A generated by the b_j are equal.

PROOF. Write, for $1 \leq i \leq N$,

$$w_i = \sum_{j=1}^N \alpha_{ij} v_j \quad \text{and} \quad b_i = \sum_{j=1}^N \beta_{ij} a_j.$$

Then $\sum_{i=1}^N w_i \otimes a_i = \sum_{i=1}^N v_i \otimes b_i$ implies $\alpha_{ij} = \beta_{ji}$. Hence the matrices (α_{ij}) and (β_{ij}) are equal, whence (α_{ij}) and (β_{ij}) have the same rank. Applying Proposition 3 to A , the space of spherical polynomials, $V = \mathcal{P}_k$, $a_n = S_{n,k}(a, b, c)$, and $v_n = \kappa_{n,k}(x, y)$ proves the assertion.

For a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbf{Z})$ we define an endomorphism $\Pi(\gamma)$ on $\mathcal{P}_k$ as follows: for any polynomial $P(x, y) \in \mathcal{P}_k$ let

$$\Pi(\gamma)P(x, y) = \phi_{n,k}((x, y)\gamma),$$

where

$$(x, y)\gamma = (x, y) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax + cy, bx + dy).$$

Now let

$$t = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathcal{U}$ be the image of $\mathcal{P}_k$ under $\Psi = (1 - \sigma)(t - t^2)$. Then a simple calculation shows that $\mathcal{U}$ is precisely the span of the $\kappa_{n,k}$. In calculating the dimension of $\mathcal{U}$, it makes no difference if we consider the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathcal{P}_k(\mathbf{C})$ instead of on $\mathcal{P}_k(\mathbf{Q})$. The operator $\Pi(t)$ has two eigenspaces on the homogeneous polynomials of degree 1: $\mathbf{C}(x + \rho y)$ with eigenvalue $-\rho$, and $\mathbf{C}(x + \bar{\rho}y)$ with eigenvalue $-\bar{\rho}$ ($\rho = e^{(2\pi i/3)}$), while $\Pi(\sigma)$ takes $(x + \rho y)$ to $\rho(x + \bar{\rho}y)$ and $(x + \bar{\rho}y)$ to $\bar{\rho}(x + \rho y)$.

Let $A = x + \rho y$, $B = x + \bar{\rho}y$. We have the direct sum decomposition

$$\mathcal{P}_w(\mathbf{C}) = \bigoplus_{j=0}^{k-1} \mathbf{C}(A^j B^{w-j} \oplus A^{w-j} B^j)$$

into ψ -invariant subspaces, with

$$\begin{aligned} \Pi(\Psi)(A^j B^{w-j} + A^{w-j} B^j) &= \rho^j \bar{\rho}^{w-j} (1 - \rho^j \bar{\rho}^{w-j}) (1 - \rho^j \bar{\rho}^{w-j}) A^j B^{w-j} \\ &\quad + \rho^{w-j} \bar{\rho}^j (1 - \rho^j \bar{\rho}^{w-j}) (1 - \rho^{w-j} \bar{\rho}^j) A^{w-j} B^j, \end{aligned}$$

which is zero if and only if $\rho^j \bar{\rho}^{w-j} = 1$. But

$$\rho^j \bar{\rho}^{w-j} = \rho^{2w-j} = 1 \Leftrightarrow j \equiv 2w \pmod{3}.$$

Thus the dimension of $\mathcal{U}$ is

$$\# \{j | 0 \leq j \leq k-1; 2w-j \equiv 1 \text{ or } 2 \pmod{3}\} = (2k/3),$$

as was to be shown.

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