

EQUIVARIANT BUNDLES AND COHOMOLOGY

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ABSTRACT. Let G be a topological group, A an abelian topological group on which G acts continuously and X a G -space. We define "equivariant cohomology groups" of X with coefficients in A , $H_G^i(X; A)$, for $i \geq 0$ which generalize Graeme Segal's continuous cohomology of the topological group G with coefficients in A . In particular we have $H_G^1(X; A) \simeq \{\text{equivalence classes of principal } (G, A)\text{-bundles over } X\}$. We show that when G is a compact Lie group and A an abelian Lie group we have for $i > 1$ $H_G^i(X; A) \simeq H^i(EG \times_G X; \tau A)$ where τA is the sheaf of germs of sections of the bundle $(X \times EG \times A)/G \rightarrow (X \times EG)/G$. For $i = 1$ and the trivial action of G on A this is a theorem of Lashof, May and Segal.

0. Introduction. Let G be a topological group, A an abelian topological group (both G and A compactly generated Hausdorff spaces) and let $\alpha: G \rightarrow \text{Aut}(A)$ be a homomorphism from G to the group of automorphisms of A such that the action $G \times A \rightarrow A$ (the adjoint of α) is continuous. By a (G, α, A) -bundle over a G -space X we mean a principal A -bundle $p: E \rightarrow X$ such that

- (1) E is a G -space and p is an equivariant map.
- (2) For all $g \in G$, $a \in A$ and $e \in E$, $g(ea) = (ge)\alpha_g(a)$, where we write the action of G on the left and that of A on the right. (For more discussion of (G, A) -bundles see [13].)

For a (paracompact Hausdorff) G -space X let $\beta(G, \alpha, A)(X)$ denote the set of equivalence classes of (G, α, A) -bundles over X . For a paracompact Hausdorff space Y let $\beta(A)(Y)$ denote the set of equivalence classes of principal A -bundles over Y . Both $\beta(G, \alpha, A)(X)$ and $\beta(A)(Y)$ have a natural group structure. In the case when A is an abelian compact Lie group, G a compact Lie group, α the trivial homomorphism and X has the homotopy type of a G -CW complex, Lashof, May and Segal [9] showed that the natural transformation $\beta(G, 1, A)(X) \rightarrow \beta(A)(X_G)$ is an isomorphism, where $X_G = EG \times_G X$ and EG denotes a contractible G -free space. Using this theorem they obtained simple proofs of the main results of Hattori and Yoshida [8] on lifting compact Lie group actions in fibre bundles, in fact generalizing them from the case of bundles with tori as structure groups to the case of bundles whose structural group is an arbitrary compact abelian Lie group (a product of a torus and a finite group). The argument in [9] depends primarily on equivariant homotopy theory but it also makes use of the cohomology of topological groups with coefficients in topological abelian groups developed by Graeme Segal in [12]. Our main purpose here is to prove a result more general than the theorem of Lashof, May and Segal by generalizing Segal's cohomology of topological groups

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to “equivariant cohomology” of a G -space and using only (more or less) techniques used in [12]. Our main result can be stated as follows. Let G and A be as above (from now on we shall ignore α in our notation). We shall define “equivariant cohomology groups” $H_G^i(X; A)$ for $i \geq 0$ such that

$$H_G^0(X; A) = \text{Map}_G(X; A), \quad H_G^1(X; A) = \beta(G, A)(X)$$

and in general $H_G^i(X; A)$ is the i th derived factor of the functor $A \mapsto \text{Map}_G(X; A)$ from Segal’s quasi-abelian category $G\text{-Topab}$ to abelian groups.

THEOREM 1. *If G is a compact Lie group and A a Lie group we have*

$$H_G^i(X; A) \simeq H^i(X \times_G EG; \tau A)$$

for all $i > 0$, where τA is the sheaf of germs of sections of the bundle

$$(X \times EG \times A)/G \rightarrow (X \times EG)/G. \quad \square$$

Taking $i = 1$ (with the trivial G -action on A) we obtain the theorem of Lashof, May and Segal. The “equivariant cohomology groups” $H_G^i(X; A)$ can be defined by means of a “soft resolution” of A (in the terminology of [12]) and most of our arguments could be made to parallel those in [12]. We shall, however, adopt a different approach (inspired by [14]) which, besides being more general, leads to certain related questions considered at the end of our article. Briefly, while the equivariant cohomology groups $H_G^i(X; A)$ are defined for all compactly generated Hausdorff topological abelian groups A , they are G -homotopy functors (for $i > 0$) only when A is a Lie group. We shall show, however, that by means of a method analogous to that used in defining $H_G^*(X; A)$ we can define G -homotopy invariant functors $\overline{H}_G^*(X; A)$, which indeed form an “equivariant cohomology theory” in the usual sense of the term.

Finally I would like to thank Graeme Segal in whose published and unpublished work many of the results of this paper can be found and for letting me take the responsibility for their presentation. I would also like to thank Peter May for bringing to my attention [8] of which I was unaware when I wrote the first version of this article.

1. Definition and main properties of $H_G^*(X; A)$. Let G be a compactly generated Hausdorff paracompact topological group. Let $G\text{-Topab}$ denote the category of compactly generated abelian topological groups on which G acts continuously as above. (As in [12] by a topological group we mean one in the category of k -spaces; i.e., the group operation $A \times A \rightarrow A$ is continuous when the product is given the k -topology.) A “distinguished” (or “proper”) short exact sequence in $G\text{-Topab}$ is a sequence $0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$ which is exact as a sequence of discrete groups and such that $i(A)$ is a closed subgroup of B , $B/i(A) \simeq C$ is a topological isomorphism and $p: B \rightarrow C$ is a locally trivial principal A -bundle, hence a (G, A) -bundle (i.e., p has a local section). Note that, unlike in [12], we do not, at this stage anyway, require the objects of $G\text{-Topab}$ to be locally contractible. With the above choice of distinguished short exact sequences $G\text{-Topab}$ satisfies the axioms of Yoneda [15] for a quasi-abelian category. Alternatively, we can define a class J of G -monomorphisms $i: A \rightarrow B$ in $G\text{-Topab}$ such that $i(A)$ is closed in B and each $b \in B$ has a neighbourhood U and a map $s: U \rightarrow A$ such

that $s \circ i = 1$: $i^{-1}(U) \rightarrow i^{-1}(U)$. It is then easy to check that J is an h.f. class in the sense of Buchsbaum [4] (see also [11, p. 367]). In either case we can define the derived functors of $\text{Hom}_G(A, B)$ (continuous equivariant homomorphism), $\text{Ext}_G^i(A, B)$, for $i > 0$ as certain equivalence classes of distinguished long exact sequences

$$0 \rightarrow B \rightarrow E_1 \rightarrow \cdots \rightarrow E_n \rightarrow A \rightarrow 0$$

where a distinguished long exact sequence is one which breaks up into distinguished short exact ones. The equivalence relation is generated by 'similarity', where two long exact sequences are similar if there exists a map of one into the other. The functors $\text{Ext}_G^i(A, B)$ possess all the usual properties of derived functors of $\text{Hom}(_, _)$ in an abelian category; i.e., $\{\text{Ext}_G^i(A, B)\}_{i \geq 0}$ (where $\text{Ext}_G^0(A, B) = \text{Hom}_G(A, B)$) is an exact universal connected sequence of functors (see [4, Proposition 4.3]). Now let X be a compactly generated paracompact Hausdorff space on which G acts continuously with a G -fixed base point.

Let NX denote the free abelian group generated by X with the base point identified with the zero element. NX has a natural topology when viewed as a quotient of $\bigsqcup_{n \geq 0} (X \times \mathbb{Z})^n$ in the obvious way and is an object of $G\text{-Topab}$, where the action of G on NX naturally extends that on X . Now let X be a G -space (without a base point). Let $X_+ = X \cup \{\text{disjoint base point}\}$ where G acts trivially on the base point. It is easy to prove

PROPOSITION 1. $\text{Hom}_G(NX_+, A) \approx \text{Map}_G(X; A)$ where $\text{Map}_G(X; A)$ denotes continuous G -maps from X to A . \square

We can now define equivariant cohomology groups of X with coefficients in A by $H_G^i(X; A) = \text{Ext}_G^i(NX_+, A)$ for $i \geq 0$. Thus $H_G^0(X; A) = \text{Map}_G(X; A)$. We also have

PROPOSITION 2. $H_G^1(X; A) \approx \{\text{equivalence classes of principal } (G, A)\text{-bundles on } X\} (= \beta(G, A)(X))$. \square

PROOF. An element of $H_G^1(X; A)$ is represented by a proper short exact sequence $0 \rightarrow A \xrightarrow{i} E \rightarrow NX_+ \rightarrow 0$. The group NX_+ contains a copy of X consisting of elements of the form $[x, 1] \in NX_+ (= \bigsqcup_n (X_+ \times \mathbb{Z})^n / \sim)$. Let $E_1 = p^{-1}X$. Then $p|_{E_1}: E_1 \rightarrow X$ is a (G, A) -bundle, and it is clear that equivalent sequences correspond to equivalent bundles. This defines a map $H_G^1(X; A) \rightarrow \beta(G, A)(X)$. To define its inverse, consider a (G, A) -bundle $p: E \rightarrow X$. Let $\tilde{p}: E \cup A \rightarrow X_+$ be p on E , take A to the disjoint base point, and let $Np: N(E \cup A) \rightarrow NX_+$ be the induced group homomorphism. For $e \in E \cup A$ and $a \in A$ let ae denote the action of a on e , where A acts on E by the principal action and on A by group addition. Let M denote the closed subgroup of $N(E \cup A)$ generated by elements of the form $ae - i(a) - e$, $a \in A$, $e \in E \cup A$ where $i: A \rightarrow E \cup A$ is the inclusion. Since Np is 0 on M , it induces a homomorphism $N(E \cup A)/M \rightarrow N(X_+)$ and it is easy to verify that the sequence $0 \rightarrow A \rightarrow N(E \cup A)/M \rightarrow N(X_+) \rightarrow 0$ is distinguished. \square

Let now $\widetilde{G\text{-Topab}}$ denote the full subcategory of $G\text{-Topab}$ consisting of locally contractible G -modules (denoted by $G\text{-Topab}$ in [12]).

PROPOSITION 3. The connected sequence of functions $\{H_G^i(X; _)\}_{i \geq 0}$ is universal on $\widetilde{G\text{-Topab}}$. \square

PROOF. We know that the sequence is universal on $\widetilde{G\text{-Topab}}$. To prove that it is universal as a connected sequence of functors on $\widetilde{G\text{-Topab}}$, we use Proposition 4.2 of [4] ("effaceability"). So let A be an object in $\widetilde{G\text{-Topab}}$, and let $\alpha \in H_G^n(X; A)$ where $n > 0$. We wish to find a distinguished short exact sequence $0 \rightarrow A \rightarrow A' \rightarrow A'' \rightarrow 0$ in $\widetilde{G\text{-Topab}}$ such that $i_*(\alpha) = 0$ in $H_G^n(X; A)$. Let $0 \rightarrow A \xrightarrow{k} B \xrightarrow{i} C \rightarrow 0$ be such a sequence in $G\text{-Topab}$. Let EB be as in [12], i.e. a contractible abelian group which contains B as a closed subgroup. Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{k} & B \\ i_1 \downarrow & & \downarrow i_2 \\ EA & \xrightarrow{Ek} & EB \end{array}$$

Clearly the homomorphism $(i_2 k)^* = (Ek i_1)^*: H_G^n(X; A) \rightarrow H_G^n(X; EB)$ takes α to 0. Since EB is contractible, it is locally contractible; hence it belongs to $\widetilde{G\text{-Topab}}$; and since i_1 and Ek admit a local section (see [12]) $Ek \circ i_1: A \rightarrow EB$ is distinguished. \square

From now on all coefficient groups will be assumed to lie in $\widetilde{G\text{-Topab}}$.

PROPOSITION 4. When $G = 1$ (the trivial group) $H_G^i(X; A) = H^i(X; \tau A)$, where A is the sheaf of germs of continuous maps $X \rightarrow A$. \square

PROOF. Since the connected sequence of functors $\{H_G^i(X; A)\}_{i \geq 0}$ is universal, it is only necessary to observe that so is $\{H^i(X; \tau A)\}_{i \geq 0}$ since their zeroth terms coincide, both being $\text{Map}(X; A)$. To see that $\{H^i(X; \tau A)\}_{i \geq 0}$ is a connected sequence of functors, observe that a distinguished short exact sequence of abelian groups $0 \rightarrow A \rightarrow A' \rightarrow A'' \rightarrow 0$ gives rise to an exact sequence of sheaves $0 \rightarrow \tau A \rightarrow \tau A' \rightarrow \tau A'' \rightarrow 0$. Also, given $A \in \widetilde{G\text{-Topab}}$ the sequence $0 \rightarrow A \rightarrow EA \rightarrow BA \rightarrow 0$ is distinguished, and thus the sequence of sheaves $0 \rightarrow \tau A \rightarrow \tau EA \rightarrow \tau BA \rightarrow 0$ is short exact. However, as is well known, τEA is a soft sheaf; hence $H_G^i(X; \tau EA) = 0$ for $i > 0$, and thus the sequence $H^i(X; \tau A)$ is universal.

REMARK. As in [12] we could also show that $H^n(X; \tau A) = [X; B^n A]$ for $n > 0$, where $B^n A$ is the n -fold classifying space of A (which is again an abelian group).

PROPOSITION 5. There exists a spectral sequence converging to $H_G^*(X; A)$ with

$$E_1^{p,q} = H^q(X \times \underbrace{G \times \cdots \times G}_p; A)$$

(cf. Proposition 3.2 of [12]).

PROOF. Let

$$X_n = X \times \underbrace{G \times \cdots \times G}_n.$$

X_n is a G -space, where G acts on each factor. Let $A_m = EB^m A$. Let $E_0^{p,q} = \text{Map}(X_p, A_q)$. Let $\delta_0^{p,q}: E_0^{p,q} \rightarrow E_0^{p+1,q}$ be given by

$$\begin{aligned} (\delta_0^{p,q} f)(x, g_1, g_2, \dots, g_{p+1}) &= g_{p+1} f(g_{p+1}^{-1} x, g_{p+1}^{-1} g_1, \dots, g_{p+1}^{-1} g_p) \\ &\quad + \sum_{i=0}^p (-1)^{i+1} f(x, g_1, \dots, \hat{g}_{p+1-i}, \dots, g_{p+1}). \end{aligned}$$

Let $\delta_1^{p,q}: \text{Map}(X_p; A_q) \rightarrow \text{Map}(X_p; A_{q+1})$ be induced by the composite $A_q = EB^q A \rightarrow B^{q+1} A \rightarrow EB^{q+1} A = A_{q+1}$. Let $d_0^{p,q} = \delta_0^{p,q} + \delta_1^{p,q}$. Then $d_0^2 = \delta_0^2 = \delta_1^2 = 0$. As usual we obtain two spectral sequences associated with the bicomplex $\{E_0^{p,q}, d_0^{p,q}\}$ and converging to the total cohomology of the bicomplex, $\mathcal{H}_G^{p+q}(X; A)$. A simple argument shows that the second spectral sequence has $E_1^{p,q} = H^q(X_p; \tau A)$, with differential $d_1^{p,q}$ induced by $\delta_0^{p,q}$. We wish to show that $\mathcal{H}_G^n(X; A) = H_G^n(X; A)$ for each n . Since $E_1^{0,0} = \text{Map}(X; A)$, $E_1^{1,0} = \text{Map}(X \times G; A)$ and $d_1^{0,0}$ takes f to the function $x \mapsto gf(g^{-1}x) - f(x)$ we see that $\mathcal{H}_G^0(X; A) = \text{Map}_G(X; A) = H_G^0(X; A)$. Since a distinguished short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ clearly induces a long exact sequence

$$0 \rightarrow \mathcal{H}_G^0(X; A) \rightarrow \mathcal{H}_G^0(X; B) \rightarrow \mathcal{H}_G^0(X; C) \rightarrow \mathcal{H}_G^1(X; A) \rightarrow \dots$$

it remains to verify that the exact connected sequence of functors $\{\mathcal{H}_G^n(X; A)\}_{n \geq 0}$ is universal. Consider the distinguished short exact sequence $0 \rightarrow A \rightarrow EA \rightarrow BA \rightarrow 0$. Since EA is contractible, we have $H^m(X_n; \tau EA) = 0$ for $m > 0$, and the E_1 term of the spectral sequence converging to $\mathcal{H}_G^*(X; EA)$ reduces to the complex $\{\text{Map}(X_n; EA), \delta_0^{n,0}\}$. We can easily verify that for every A the map $A \rightarrow \text{Map}(G, A)$ effaces the cohomology of $\{\text{Map}(X_n; A), \delta_0^{n,0}\}$; in fact $f \in \text{Map}(X_n; A)$ is taken to the boundary of $\text{adj } f \in \text{Map}(X_{n-1}, A)$. Hence $\mathcal{H}_G^n(X; A)$ is effaced by $0 \rightarrow A \rightarrow \text{Map}(G; E) \rightarrow \text{Map}(G; EA)/A \rightarrow 0$, which is clearly a distinguished short exact sequence. \square

COROLLARY 1. *For each $n > 0$, $H_G^n(X; \text{Map}(G; EA)) = 0$.*

REMARK. G -modules of the form $\text{Map}(G; EA)$ are called ‘soft’ in [12]. The groups $H_G^*(X; A)$ could be defined by applying the functor $\text{Map}_G(X, _)$ to ‘soft resolutions’ and taking cohomology as was indeed done by G. Segal (unpublished).

COROLLARY 2. *Let A be a discrete G -module. Then the functors $H_G^i(X; A)$ for $i > 0$ possess the following “strong G -homotopy invariance property”: a G -map $f: X \rightarrow Y$ between two G -spaces which is a (nonequivariant) homotopy equivalence induces an isomorphism $f^*: H_G^i(Y; A) \xrightarrow{\sim} H_G^i(X; A)$.*

PROOF. If A is discrete the groups $H^i(X_p; \tau A) = [X_p; B^i A]$ are homotopy invariant. The result follows from Proposition 5.

COROLLARY 3. *If V is a real vector space (and G is a compact group) $H_G^n(X; V) = 0$ for all $n > 0$.*

PROOF. Since V is contractible $H^n(X_p; \tau V) = 0$, so the spectral sequence reduces to the cohomology of the complex $\{\text{Map}(X_p; V), \delta_0^{p,0}\}$ which is acyclic in positive degrees. Indeed, consider the complex $\{\text{Map}(X_{p+1}; V), \delta^p\}$ where

$$\delta^p f(x, g_1, \dots, g_{p+1}) = \sum (-1)^i f(x, g_1, \dots, \hat{g}_i, \dots, g_{p+1}).$$

The complex is obviously acyclic with contracting homotopy $h(f)(x, g_1, \dots, g_p) = f(x, g_1, \dots, g_p, e)$.

Now it is easy to see that we have a homomorphism of chain complexes $T: \{\text{Map}(X_p; V), \delta_0^{p,0}\} \rightarrow \{\text{Map}(X_{p+1}; V), \delta^p\}$ where

$$T(f)(x, g_1, g_2, \dots, g_{p+1}) = g_{p+1} f(g_{p+1}^{-1} g_1, g_{p+1}^{-1} g_2, \dots, g_{p+1}^{-1} g_p).$$

Clearly T maps $\{\text{Map}(X_p; V), \delta^p\}$ isomorphically onto the subcomplex

$$\{\text{Map}(X_{p+1}; V), \delta^p\}^G$$

of G -invariant maps, where G acts on $\text{Map}(X_{p+1}; V)$ by

$$(gf)(x, g_1, \dots, g_{p+1}) = gf(g^{-1}x, g^{-1}g_1, \dots, g^{-1}g_{p+1}).$$

However, since G is compact and V is a real vector space, integration over G shows that the complex of invariant maps is a direct factor of the acyclic complex $\{\text{Map}(X_{p+1}, V), \delta^p\}$ and hence is also acyclic. \square

We next prove

LEMMA 1. *Let G be a compact Lie group and suppose G acts freely on X . Then for every $A \in G\text{-}\widetilde{\text{Topab}}$ we have*

$$H_G^i(X; A) \simeq H^i(X/G; \tau A) \quad \text{for } i > 0$$

where τA is the sheaf of germs of sections of the bundle $\tau(A): X_G \times A \rightarrow X/G$.

PROOF. By Theorem 5.8 of [2] $\pi: X \rightarrow X/G$ is a locally trivial bundle and so is $X \times_G A \rightarrow X/G$. $\{H_G^i(X; A)\}_{i \geq 0}$ and $\{H^i(X/G; \tau A)\}_{i \geq 0}$ are connected exact sequences of functors on $G\text{-}\widetilde{\text{Topab}}$ and the former is universal. Since the sequences have isomorphic 0th term $H_G^0(X; A) = \text{Map}_G(X; A) = H^0(X/G; \tau A)$, it is enough to show that the second sequence is universal. To prove this it suffices to show that $H^i(X/G; \tau \text{Map}(G; EA)) = 0$ for $i > 0$. Let $K \subset X/G$ be a closed set, and let s be a section of $\tau \text{Map}(G; EA)$ over K . Let $\tilde{s}: \pi^{-1}(K) \rightarrow \text{Map}(G; EA)$ be the corresponding G -map, and let $\tilde{\tilde{s}}: \pi^{-1}(K) \rightarrow EA$ be given by $\tilde{\tilde{s}}(k) = \tilde{s}(k)(e)$. Since EA is contractible, $\tilde{\tilde{s}}$ can be extended to X ; hence \tilde{s} can be extended equivariantly to X and s to X/G . This shows that $\tau \text{Map}(G; EA)$ is a soft sheaf [1], and thus $H^i(X/G; \tau \text{Map}(G; EA)) = 0$ for $i > 0$. \square

2. Proof of Theorem 1. Since any abelian Lie group $A = A' \times T^n \times V$ (for some n), where A' is a discrete group, T^n is an n -torus and V is a real vector space, and since the action of G must preserve this decomposition we can consider each of the three cases: a discrete group, a torus and a real vector space, separately. The case $A = V$ is the simplest since $H_G^n(H; V) = 0$ for $n > 0$ by Corollary 3, while $H^n(X \times_G EG; \tau V) = 0$ for $n > 0$ since τV is a soft sheaf over $X \times_G EG$. (This can be seen from an argument analogous to the one in the proof of Lemma 1, the key point being that an equivariant map $Y \rightarrow V$, where Y is a closed G -subspace of X can be extended equivariantly to X by extending it first arbitrarily and then integrating over G .) Suppose now that A is discrete. Then, by Corollary 1, $H_G^n(X; A)$ has the strong G -homotopy invariance property; hence $H_G^n(X; A) \simeq H_G^n(X \times EG; A)$. However, by Lemma 1 $H_G^n(X \times EG; A) = H^n(X \times_G EG; \tau A)$ (since G acts freely on $X \times EG$). Finally let $A = T^n$. Consider the sequence

$$(*) \quad 0 \rightarrow Z^n \rightarrow \mathbf{R}^n \xrightarrow{\exp} T^n \rightarrow 0.$$

The action of G on T^n lifts to an action on the sequence $(*)$ which is clearly distinguished in $G\text{-}\widetilde{\text{Topab}}$. We also have a short exact sequence of sheaves over $X \times_G EG$:

$$(**) \quad 0 \rightarrow \tau Z^n \rightarrow \tau \mathbf{R}^n \rightarrow \tau T^n \rightarrow 0.$$

From (*) and (**) we obtain the diagram

$$\begin{array}{ccccccc}
 0 = & H_G^i(X; \mathbf{R}^n) & \rightarrow & H_G^i(X; T^n) & & & \\
 & \wr & & \downarrow & & & \\
 0 = & H^i(X \times_G EG; \tau \mathbf{R}^n) & \rightarrow & H^i(X \times_G EG; \tau T^n) & & & \\
 & & \rightarrow & H_G^{i+1}(X; \mathbf{Z}^n) & \rightarrow & H^{i+1}(X; \mathbf{R}^n) & = 0 \\
 & & & \downarrow \wr & & \parallel & \\
 & & \rightarrow & H^{i+1}(X \times_G EG; \tau \mathbf{Z}^n) & \rightarrow & H^{i+1}(X \times_G EG; \tau \mathbf{R}^n) & = 0
 \end{array}$$

from which $H_G^i(X; T^n) = H^i(X \times_G EG; \tau T^n)$. \square

In particular we obtain

$$\text{COROLLARY. } H_G^1(X; T^n) \simeq H_G^2(X; \mathbf{Z}^n) \simeq H^2(X \times_G EG; \tau \mathbf{Z}^n). \quad \square$$

This can be viewed as a generalization of two well-known results: Letting $X = \text{pt}$ we have $\text{Hom}(G; T^n) \simeq H^2(G; \mathbf{Z}^n)$ which for discrete G is a well-known isomorphism (e.g. see [11, p. 117]). Letting X be nontrivial, $G = 1$ and $n = 1$, we obtain the well-known characterization of line bundles by their first Chern class. In general we can think of the above isomorphism as taking an equivariant torus bundle to its first equivariant Chern class.

3. G -homotopy invariance. Observe that the functors $H_G^i(X; A)$ were defined for arbitrary topological G -modules A , not necessarily Lie groups. It is easy to see that in general Theorem 1 is not true. For example, let A be an abelian group model of the Eilenberg-Mac Lane space $K(\mathbf{Z}/2; 1)$ where every element has order 2, $G = \mathbf{Z}/2$ with trivial action on A and $X = \text{pt}$. Then $H_G^1(X; A) = \text{Hom}(G; A) \simeq A$ (viewed as a discrete group) while $H^1(X \times_G EG; \tau A) = H^1(B\mathbf{Z}/2; \tau A) = H^2(\mathbf{R}P^\infty; \mathbf{Z}/2) = \mathbf{Z}/2$. Thus we can deduce that the functors $H_G^i(X; A)$ for $i > 0$ do not, in general, possess the strong homotopy invariance property of Corollary 2. Furthermore, we can show that they are not G -homotopy invariant even in the weak sense when A is not a Lie group. Indeed, take A and G as above, and let X be a contractible space with trivial G -action. From the spectral sequence of Proposition 5 we deduce that

$$\begin{aligned}
 H_G^1(X; A) &= H^1(G; \text{Map}(X; A)) \quad (\text{group cohomology}) \\
 &\simeq \text{Hom}(G; \text{Map}(X; A)) = \text{Map}(X; A)
 \end{aligned}$$

viewed as a discrete group. The inclusion of a point in X induces the inclusion $A \rightarrow \text{Map}(X; A)$ which obviously is not an isomorphism in general. The fact that in general $H_G^1(X; A)$ is not G -homotopy invariant is not surprising, since in general (G, A) -bundles do not possess the G -homotopy lifting property. On the other hand it is well known (e.g. see [10 or 13]) that (G, A) -bundles have the G -homotopy lifting property when A is a Lie group and are thus G -homotopy invariant up to equivalence. It seems natural to try to modify our theory so as to obtain G -homotopy invariant functors on the category of G -spaces for arbitrary topological groups A . For this purpose we shall assume from now on that G is a finite group and all spaces are regular G -simplicial complexes (see [2]). Since the geometrical realization of the singular complex of a G -CW complex X is a regular G -simplicial complex G -homotopy equivalent to X , our functors can be defined on the category of G -spaces G -homotopy equivalent to G -CW complexes. These

functors $\overline{H}_G^i(X; A)$ ($i > 0$) will be defined analogously to $H_G^i(X; A)$ above, and in particular we shall have

$$\overline{H}_G^1(X; A) \simeq \{\text{equivalence classes of } G\text{-locally trivial } (G, A)\text{-bundles over } X\}.$$

Here, by a G -locally trivial (G, A) -bundle over X , we mean a (G, A) -bundle $p: E \rightarrow X$ such that each $x \in X$ has a G_x -invariant neighbourhood U_x (where G_x is the isotropy subgroup of G at x) and a G_x -equivariant section $s_x: U_x \rightarrow p^{-1}(U_x)$. This notion of G -local triviality is in general more restricted than that used in [13] but over G -locally contractible spaces they are equivalent. (A G -space X is said to be G -locally contractible if each $x \in X$ has a G_x -invariant G_x -contractible neighbourhood U_x .) In particular this is true for G -simplicial complexes (for each $x \in X$ $\text{star}(x)$ in some G -subdivision of X is a G_x -invariant G_x -contractible neighbourhood of x). Let now $G\text{-}\overline{\text{Topab}}$ denote the full subcategory of $G\text{-}\text{Topab}$ consisting of G -modules A such that the identity $O_A \in A$ has arbitrarily small open G -stable neighbourhoods (i.e. any open neighbourhood of $O_A \in A$ contains a G -stable one). We make $G\text{-}\overline{\text{Topab}}$ into a quasi-abelian category by defining distinguished short exact sequences to be sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ which are distinguished in $G\text{-}\text{Topab}$ and such that $O_C \in C$ has a G -invariant neighbourhood U_{O_C} and an equivariant section $s: U_{O_C} \rightarrow B$. It is clear that we obtain in this way a quasi-abelian category. We now have

PROPOSITION 6. *Let X be a G -simplicial complex with a G -fixed base point. Then NX is an object in $G\text{-}\overline{\text{Topab}}$. If (X, Y) is a (pointed) pair of G -simplicial complexes then $0 \rightarrow NY \rightarrow NX \rightarrow N(X/Y) \rightarrow 0$ is a distinguished short exact sequence in $G\text{-}\overline{\text{Topab}}$.*

PROOF. First note that a base of neighbourhoods of 0 in NX can be defined as follows: for each covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ of X by open sets let $N(\mathcal{U})$ consist of elements of NX which can be written in the form $\sum_{i=1}^k n_i(u_{\alpha_i}^1 - u_{\alpha_i}^2)$ where $\{\alpha_i\}_{i=1}^k$ is a finite subset of \mathcal{A} and $u_{\alpha_i}^1, u_{\alpha_i}^2 \in U_{\alpha_i}$. Then as \mathcal{U} runs over all open coverings of X , $N(\mathcal{U})$ runs over a base of open neighbourhoods of 0 in NX . Moreover, if \mathcal{U} is a G -stable covering (i.e., G permutes the elements of \mathcal{U}), $N(\mathcal{U})$ is an invariant neighbourhood of 0 in NX . Since G is compact and X is paracompact into every covering by open sets, one can inscribe a G -invariant covering, and thus NX contains arbitrarily small invariant neighbourhoods of 0. This completes the proof of the first part of the statement of Proposition 6. To prove the second part we need to show that $0 \in N(X/Y)$ has a G -invariant neighbourhood U_0 and a local G -section $s_0: U_0 \rightarrow NX$. The proof is exactly analogous to the nonequivariant case (see [6, Satz. 5.4, p. 264]).

We can now define $\overline{H}_G^i(X; A) = \text{Ext}^i(NX; A)$ where Ext^i is meant in Yoneda's sense in the quasi-abelian category $G\text{-}\overline{\text{Topab}}$.

THEOREM 2. (i) $\overline{H}_G^i(X; A)$ for $i > 0$ is a G -homotopy functor on the category of G -complexes. In particular,

$$\overline{H}_G^1(X; A) \simeq \{\text{equivalence classes of } G\text{-locally trivial } (G, A)\text{-bundles over } X\}.$$

(ii) There is a natural long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Map}_G(X/Y; A) \rightarrow \text{Map}_G(X; A) \rightarrow \text{Map}_G(Y; A) \rightarrow \overline{H}_G^1(X/Y; A) \\ \rightarrow \overline{H}_G^1(X; A) \rightarrow \overline{H}_G^1(Y; A) \rightarrow \overline{H}_G^2(X/Y; A) \rightarrow \dots \end{aligned}$$

(where Map_G denotes base point preserving equivariant maps) defined for pairs of G -complexes (X, Y) .

PROOF. We first observe that an argument precisely analogous to that in Proposition 2 shows that

$$\overline{H}_G^1(X; A) \simeq \{\text{equivalence classes of } G\text{-locally trivial } (G, A)\text{-bundles on } X\}.$$

In the case of trivial G -action on A , Bierstone shows in [3] that G -locally trivial (G, A) -bundles satisfy the G -covering homotopy property, hence $H_G^1(X; A)$ is a G -homotopy functor—the general case can be handled analogously. Now, an element α of $\overline{H}_G^i(X; A)$ for $i > 1$ is represented by a distinguished sequence $0 \rightarrow A \rightarrow E_2 \rightarrow \cdots \rightarrow E_1 \rightarrow NX \rightarrow 0$ which can be split up into two distinguished sequences $0 \rightarrow A \rightarrow \cdots \rightarrow E_{i-1} \rightarrow F \rightarrow 0$ and $0 \rightarrow F \rightarrow E_i \rightarrow NX \rightarrow 0$. If now $f: Y \rightarrow X$ is a G -map and $Nf: NY \rightarrow NX$ the induced G -homomorphism, it follows from the definition that the image of the element α , $f^*\alpha \in \overline{H}_G^i(Y; A)$, is represented by the sequence $0 \rightarrow A \rightarrow E_1 \rightarrow \cdots \rightarrow E_{i-1} \rightarrow E'_i \rightarrow NY \rightarrow 0$ where $0 \rightarrow F \rightarrow E'_i \rightarrow NY \rightarrow 0$ is the pull-back of $0 \rightarrow F \rightarrow E_i \rightarrow NX \rightarrow 0$ by Nf . But the equivalence class of the latter depends, by the above, only on the G -homotopy class of f . Hence $\overline{H}_G^i(X; A)$ is a G -homotopy functor.

To prove the last statement of the theorem we observe that by Proposition 6 $0 \rightarrow NY \rightarrow NX \rightarrow N(X/Y) \rightarrow 0$ is a distinguished short exact sequence in $G\text{-Topab}$; hence the standard long exact sequence of Ext groups gives the required long exact sequence.

Theorem 2 together with the fact that $\overline{H}_G^i(\text{pt}; A) = 0$ for $i > 0$ shows that we can define an equivariant (reduced) integer graded cohomology theory $\overline{H}_G^*(X; A)$ by taking $\overline{H}_G^n(X; A)$ as above if $n > 0$ and defining $\overline{H}_G^n(X; A) = \overline{H}_G^1(\sum^{1-n} X; A)$ for $n < 0$, where $\sum^k X$ denotes the k -fold suspension of X . (This involves redefining $\overline{H}_G^0(X; A)$.) It is well known that the functor $\overline{H}_G^1(X; A)$ (equivalence classes of G -locally trivial (G, A) -bundles) is representable by a G -space $B_G A$ (in the case of trivial G action on A see, for example, [8]). Thus Theorem 2 shows that the space $B_G A$ can be given the structure of a G -infinite loop space. Of course in the case when A is a Lie group this fact follows trivially from Theorem 1, since $B_G A = \text{Map}(EG; BA)$ has equivariant delooping $\text{Map}(EG; B^i A)$, where $B^i A$ is the i th delooping of A .

Finally, one might conjecture that the analogue of Theorem 1 holds for $\overline{H}_G^n(X; A)$ for all A in $G\text{-Topab}$ and $n \geq 1$. At this time the author can neither prove nor disprove this conjecture even in the (most interesting) case $n = 1$.

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