WORD MAPS, ISOTOPY AND ENTROPY

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ABSTRACT. We find diffeomorphisms of low entropy in each isotopy class on $S^3 \times S^3$. These arise as word maps, a nonabelian analogue of toral automorphisms. Hyperbolic examples of equal entropy are also found. The group $\pi_0 \text{Diff}(S^3 \times S^3)$ is computed.

The rule $x_{n+1} = x_n + x_{n-1}$ that generates the Fibonacci sequence can be applied to any initial pair of integers x_1, x_2 : Fibonacci used 1, 1, Lucas used 1, 3, etc. One can use the corresponding multiplicative rule $x_{n+1} = x_n x_{n-1}$ for any group H and any initial pair of group elements. This defines a sort of discrete delay equation, or a "second order" transformation on H , that can be viewed as a "first-order" transformation σ on pairs of elements: $\sigma(x, y) = (y, yx)$, $\sigma: H \times H \rightarrow H \times H$. The Fibonacci and Lucas series are two orbits of σ for $H = \mathbb{Z}$.

This map σ is an instance of what we call a word map on two letters, since it assigns to the letters x, y a new pair of words in x, y and their inverses. Note σ is invertible and σ^{-1} is also a word map, $\sigma^{-1}(x, y) = (x^{-1}y, x)$. Such invertible word maps on two letters are determined by automorphisms of the free group $F_2 = F(x_1, x_2)$ on two generators. For example, σ corresponds to the automorphism sending x_1 to x_2 and x_2 to $x_2 x_1$. Similar remarks hold for word maps on n letters, $n \geq 1$.

The case $H = S^1$ is well known, since here an invertible word map on n letters is just an automorphism of the torus T^n of dimension n . These toral automorphisms are among the best-understood dynamical systems. Our interest is in the case H a compact connected Lie group and comparing its invertible word maps with toral automorphisms.

Suppose, for example, that H is the group $SO(d)$ of rotations of Euclidean space of dimension d . Then σ defines a transformation σ_d for each d . The case $d = 2$ gives the toral automorphism $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ with topological entropy $h(\sigma_2) = \log \rho$, ρ the Golden Ratio $(1 + \sqrt{5})/2$. In §1, we will show that, surprisingly, $h(\sigma_3) = h(\sigma_2)$. In this sense, the Fibonacci transformation on $SO(3)$ is no more complicated than that for dimension 2. One has $h(\sigma_4) = 2h(\sigma_2)$, $h(\sigma_d) \geq [d/2]h(\sigma_2)$. The same holds true for any invertible word map. We do not know $h(\sigma_d)$ for $d \geq 5$.

Our main interest is in invertible word maps on $H = S^3$ on two letters. Besides computing their entropy we show in §2 that they furnish representatives for all isotopy classes of the 6-manifold $M = S^3 \times S^3$. This determines the group structure of $\pi = \pi_0 \text{Diff } M$, the group of isotopy classes of diffeomorphisms of M . Previously one knew a certain three-step filtration of π without knowing what the extensions

Received by the editors August 26, 1985.

1980 *Mathematics Subject Classification* (1985 Revision) Primary 58F15, 57R52, 28D20.

¹Partially supported by the Sloan Foundation and the National Science Foundation.

were: indeed, Kreck gave such a description for a large class of smooth manifolds [Kr].

This new information on π has a pleasant dynamical consequence. The natural homomorphism $\text{Diff } M \rightarrow Gl_2\mathbf{Z}$ that assigns to diffeomorphism its action on $H_3(M; \mathbf{Z}) \cong \mathbf{Z}^2$ is onto, just like the homomorphism $\text{Diff } T^2 \rightarrow Gl_2\mathbf{Z}$. But whereas the linear action of $Gl_2\mathbf{Z}$ on T^2 splits the latter map, there is no splitting of the former. In fact we show there is not even a splitting modulo isotopy; i.e., the extension $\pi \rightarrow Gl_2\mathbf{Z}$ does not split. This depends critically on the existence of exotic 7-spheres.

Given a diffeomorphism f of a compact manifold X , we let $s(f)$ be the spectral radius of the homology map $f_*: H_*(X; \mathbf{R}) \leftarrow$. Shub's entropy conjecture [S] is that $h(f) \geq \log s(f)$. On a simply connected manifold, $\log s(f)$ is the only topological constant available that might so bound entropy. This conjecture is not known for $X = M$, but our results show that each isotopy class on M has a representative f with $h(f) = \log s(f)$. So, modulo this difficult conjecture, such f 's should be representatives of greatest "efficiency".²

We go further (in §3) and show that f can be chosen structurally stable. We do this by an isentropic deformation of a word map, i.e., one that does not change entropy. This f can even be chosen fitted, which is compatible with conjectures of Shub that the lowest entropy occurs for fitted diffeomorphisms, which are hyperbolic transformations built using handlebodies [S]. Some counterexamples for X not simply connected are known, e.g., for $X = T^4$; these also use word maps [F2]. It is conceivable that these give counterexamples on $M \times M = (S^3)^4$.

Thus, in passing from T^2 to M , the behavior of entropy in an isotopy class seems to persist. But we do not even know that word maps are points of lower semicontinuity for entropy on $\text{Diff } M$, let alone that they are minima on their components. It would be interesting to show they are local minima.

In the course of our computations, we find a surprising deformation of the action of $PGl_2\mathbf{Z} = \Gamma$ on $T^2/\pm 1$ induced from the linear action of $Gl_2\mathbf{Z}$ on T^2 . The deformation passes through analytic actions of Γ on S^2 that preserve area, and it ends with a finite group of orthogonal motions of S^2 . We do not know whether the intermediate actions are ergodic.

We thank Dan Asimov and Bill Goldman for their interesting observations.

1. Word maps. Let F_n be the free group on generators $x_1, \dots, x_n$, and let Φ_n be the automorphism group of F_n . Given any group H , there is a natural bijection of H^n with the collection $\text{Hom}(F_n, H)$ of homomorphisms from F_n to H given by evaluating the homomorphism on the generators. This gives an action of Φ_n on H^n . If $\alpha \in \Phi_n$, then we denote the corresponding bijection of H^n by α_H . We call α_H a word map on n letters since it assigns to an n -tuple in H an n -tuple of words in the entries.

For example, suppose $n = 2$, $\alpha(x_1) = x_2$ and $\alpha(x_2) = x_2x_1$. Then $\alpha_H(h_1, h_2) = (h_2, h_2h_1)$ for $h_1, h_2 \in H$, so $\alpha_H = \sigma$ is the Fibonacci map discussed above.

When $H = \mathbf{R}$ or $\mathbf{C}$, the α_H are linear maps. Indeed, when H is abelian, Φ_n acts by group automorphisms and the action factors through $Gl_n\mathbf{Z}$, the automorphism group of the abelianization of F_n .

²Yomdin has recently proven the conjecture for $C^\infty f$.

For general H , H acts on H^n by conjugation:

$$h \cdot (h_1, \dots, h_n) = (hh_1h^{-1}, \dots, hh_nh^{-1}).$$

Let $C_n = C_n(H)$ be the orbit space. Then the actions of Φ_n and H on H^n commute, so Φ_n acts on C_n . In this action, inner automorphisms act trivially, and so there is an induced action of the outer automorphism group $\text{Out } F_n (= \Phi_n/F_n \text{ for } n \geq 2)$ on C_n . We denote by $\bar{\alpha}_H$ the map on C_n induced by α_H .

Now we fix $n = 2$. We need some results due to Nielsen [N, MKS]. First, each $\alpha \in \Phi_2$ “preserves” the commutator of $x_1^{-1}x_2^{-1}x_1x_2 = c(x_1, x_2)$; i.e., $\alpha(c(x_1, x_2)) = gc(x_1, x_2)^\varepsilon g^{-1}$, for some $\varepsilon = \pm 1$, $g \in F_2$. This implies that the commutator map $c: H^2 \rightarrow H$, $c(x, y) = x^{-1}y^{-1}xy$, induces a map $\bar{c}: C_2 \rightarrow C_1$ that is equivariant for the actions of $\text{Out } F_i$, $i = 1, 2$, under the homomorphism $\varepsilon: F_2 \rightarrow \text{Out } F_1$. Second, the natural map $\text{Out } F_2 \rightarrow \text{Gl}_2\mathbf{Z}$ is an isomorphism. So $\ker(\varepsilon) = \text{Sl}_2\mathbf{Z}$ acts on C_2 , and $\bar{c}$ is a $\text{Sl}_2\mathbf{Z}$ -invariant function. Note $\bar{c}$ is constant iff H is abelian.

Now we fix a compact connected Lie group H . Let T be a maximal torus and $W = N(T)/T$ the Weyl group, where $N(T)$ is the normalizer of T in H . For the obvious action of W on T , it is well known that the map $T/W \rightarrow C_1$ is a homeomorphism, where T/W is the orbit space of T under the finite group W . This identifies the image of $\bar{c}$ with the orbifold T/W of dimension $r = \dim T$.

We trivialize the tangent space of H^2 by left translation and fix an ad-invariant inner product on the Lie algebra $\mathcal{H}$ of H . This gives H , H^2 bi-invariant Riemannian metrics. Then the permutation $\alpha(x_1) = x_2$, $\alpha(x_2) = x_1$ determines an isometry α_H of H^2 , and the shear $\beta(x_1) = x_1x_2$, $\beta(x_2) = x_2$ has differential $T\beta_H = \begin{pmatrix} \Delta & 0 \\ I & I \end{pmatrix}$ at (h_1, h_2) , where $\Delta = \text{ad } h_2 \in O(\mathcal{H})$. Taking $\gamma \in \Phi_2$ to be any positive word in α, β , we find

$$T\gamma_H = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where the operator norms of A_{ij} satisfy $\|A_{ij}\| \leq a_{ij}$, where the a_{ij} are the entries of the image $\gamma_{\mathbf{R}}$ of γ in $\text{Gl}_2\mathbf{Z} \subset \text{Gl}_2\mathbf{R}$. This implies that $\|T\gamma_H\| \leq \|\gamma_{\mathbf{R}}\|$ at all points of H^2 .

Because the action of H on H^2 is isometric, there is an induced Riemannian metric on the dense open set $\mathcal{O}$ of points in C_2 with least isotropy. This gives a Riemannian metric on a regular level of $\bar{c}_2$ in $\mathcal{O}$ on which $\|T\bar{\gamma}_H\| \leq \|\gamma_{\mathbf{R}}\|$.

But any α in $\text{Sl}_2\mathbf{Z}$ of infinite order is conjugate to $\pm\gamma$, γ of the above form. We compute in $\text{PSl}_2\mathbf{Z} = \text{Sl}_2\mathbf{Z}/\pm 1 \cong \langle a, b | a^2 = b^3 = 1 \rangle$, where $a = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $b = \pm \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. Then $ab = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $ab^{-1} = \pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. One can conjugate any element of infinite order to $\prod_i (ab)^{c_i} (ab^{-1})^{d_i}$. Now write $ab^{-1} = \pm\beta_{\mathbf{Z}}$ and $ab = \pm\alpha_{\mathbf{Z}}\beta_{\mathbf{Z}}\alpha_{\mathbf{Z}}$.

It follows that for any α , $\|T\alpha_H^n\|$ is of the order $\|\alpha_{\mathbf{R}}^n\|$, for all $n > 0$, hence of the order r_α^n , where r_α is the spectral radius of $\alpha_{\mathbf{R}}$.

We specialize to the case $H = S^3$, the group of unit quaternions. In this case we can identify C_2 with a compact region P in $\mathbf{R}^3$ via the real part of the graph of the group law

$$\tau[x, y] = (\text{Re } x, \text{Re } y, \text{Re } xy),$$

where $[x, y]$ denotes the conjugacy class in C_2 of the pair $(x, y) \in S^3 \times S^3$. Clearly τ is well defined because the real part of a quaternion is invariant under conjugation.

One can check that τ is 1-1 by conjugating (x, y) to normal form with $x \in \mathbf{C}$ and $y \in \mathbf{C} + \mathbf{R}j$.

We identify C_1 with $[0, 1]$ by the map $N[x] = \|1 - x\|^2/4$. Then $\bar{c}: C_2 \rightarrow C_1$ can be factored through the cubic polynomial

$$p(r, s, t) = 1 - (r^2 + s^2 + t^2) + 2rst;$$

i.e., $p\tau[x, y] = N\bar{c}[x, y]$. Again this is easily checked in normal form. One sees that p has critical points $(0, 0, 0)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, where $|\varepsilon_i| = 1$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 = 1$. It is clear that P lies in $[-1, 1]^3 \cap p^{-1}[0, 1]$, and one checks these are equal. This was noted independently by Dan Asimov, who observed that P is the region spanned by the three possible pairs of correlations of three independent random variables. He coined the term “tetrahedral pillow” for P . This captures the fact that P is obtained from a tetrahedron inscribed in $[-1, 1]^3$ by rounding the edges so that only the four vertices are singularities of ∂P . Note that P meets each side of this cube in a diagonal segment.

We denote the level set $p^{-1}(v) \cap P$ by L_v . For $v = 0$, $L_0 = \partial P \cong S^1 \times S^1/\pm 1$ under τ , $S^1 \subset S^3$ any one parameter subgroup. For $0 < v < 1$, L_t is an analytic S^2 . For $v = 1$, L_1 is the origin.

The action of $Gl_2\mathbf{Z}$ preserves these levels. The action on $\partial P = L_0$ is the usual action on $T^2/\pm 1$ (note the vertex singularities in ∂P correspond to the fixed points of the involution of T^2).

The action of $Gl_2\mathbf{Z}$ on P extends to a polynomial action on $\mathbf{R}^3$ that preserves volume. This can be checked on generators using the normal form: e.g., for the shear β ,

$$\bar{\beta}_H(r, s, t) = (t, s, 2st - r).$$

It follows that each level L_v , $0 < v < 1$, inherits an analytic area form invariant by Γ .

Now we use S. Katok’s theorem that the entropy of a diffeomorphism f of a compact manifold is bounded by $\log s(f^\#)$, where $f^\#$ is the action of f on differential forms and s denotes spectral radius $[\mathbf{K}]$. Using the invariant area, we see that the action $f_2^\#$ of $f = \bar{\alpha}_{S^3}|_{L_v}$, $0 < v < 1$, on 2-forms has spectral radius zero. Because the same holds trivially for 0-forms, $s(f^\#) = s(f_1^\#) \leq \log r_\alpha$, by our estimates on $\|T\alpha_{S^3}^n\|$, $n > 0$. Altogether $h(f) \leq \log r_\alpha$.

For $v = 0$, $h(\bar{\alpha}_{S^3}|_{S_0}) = h(\bar{\alpha}_{S^1}) = \log r_\alpha$. Thus by $[\mathbf{B}, \text{Corollary 18}]$, $h(\bar{\alpha}_{S^3}) = \sup_v h(\bar{\alpha}_{S^3}|_{L_v}) = \log r_\alpha$. Since $\bar{\alpha}_{S^3}$ is the factor of α_{S^3} by the action of the compact group H and the entropy of α_{S^3} on each H -orbit is clearly zero, $[\mathbf{B}, \text{Theorem 17}]$ implies $h(\alpha_{S^3}) = h(\bar{\alpha}_{S^3}) = \log r_\alpha$.

It follows that for all $\alpha \in \Phi_2$ (even with $\varepsilon = -1$) $h(\alpha_{S^3}) = \log r_\alpha$, since both sides double when α is replaced by α^2 . Note that Φ_1 acts trivially on C_1 , so the action of $Gl_2\mathbf{Z}$ on C_2 leaves $\bar{c}$ invariant.

For $f = \alpha_{S^3}$, the action of f_* on $H^3(S^3 \times S^3; \mathbf{Z}) \cong \mathbf{Z}^2$ is just $\alpha_{\mathbf{Z}}$. Hence r_α is the spectral radius $s(f)$ of the action of f on real homology. We have

THEOREM 1. *For $\alpha \in \Phi_2$, $h(\alpha_{S^3}) = h(\bar{\alpha}_{S^3}) = h(\alpha_{S^1}) = \log r_\alpha$, where r_α is the spectral radius of $\alpha_{\mathbf{R}}$ or, equally, the spectral radius of the map induced by α_{S^3} (or α_{S^1}) on real homology.*

The induced action of $Gl_2\mathbf{Z}$ on $C_2(S^3)$ preserves the function $\|xy - yx\|^2$. The level sets of this function define a deformation of the standard action of $Gl_2\mathbf{Z}/\pm 1$

on $T^2/\pm 1$ to the linear action on S^2 defined by the orthogonal representation

$$\begin{aligned} \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &\rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ \pm \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} &\rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\ \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &\rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

For the last statement, one checks that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ corresponds to $(r, s, t) \rightarrow (s, r, 2rs - t)$, which has order two; hence the action of Φ_2 factors through $Gl_2\mathbf{Z}/\pm 1$. Then one blows up the origin, i.e. replaces it by the sphere of directions using spherical coordinates, to obtain the time $v = 1$ action for our deformation. In this deformation the curvature of the quotient metric goes from a delta measure at the four vertices to a constant function.

Note that the entropy calculations of Theorem 1 are valid for $H = SO(3) \cong S^3/\pm 1$ as well. For $H = (S^3)^k$, one has $h(\alpha_H) = h(\bar{\alpha}_H) = k \log r_\alpha$ and the same holds for H covered by $(S^3)^k$.

To try to generalize Theorem 1 further, one might consider various unitary embeddings of the compact connected Lie group and use the resulting functions $N(x, y) = \|xy - yx\|^2$ to lower the dimension. This fails even for $H = SU(3)$: since its irreducible complex representations are tensor products of the usual representation and its dual, all representations give essentially the same N even though $\dim T = 2$. One can pass to level sets of $\bar{c}$ instead. Then $S = \bar{c}^{-1}[1]$ is the image of $T \times T$, since commuting elements of H lie in some conjugate of T , and so on this level set one has entropy $(\dim T) \log r_\alpha = h(\alpha_H|S)$. It is not clear any longer that other level sets have entropy bounded by this: They may have larger dimension than S , for instance, hence more expansion. Namely, if H is semisimple (one easily reduces to this case), then generic level sets of $\bar{c}$ have dimension $\dim H - \dim T$, S has dimension $2 \dim T$ and $\dim H > 3 \dim T$ unless H is covered by $(S^3)^k$. We do not know the precise entropy in these cases: perhaps it is $\log r_\alpha \cdot \frac{1}{2}(\dim H - \dim T)$.

It would also be interesting to know whether the action of $PSl_2\mathbf{Z}$ on each sphere L_v , $0 < v < 1$, is ergodic.

W. Goldman informed us that the invariance of $p(r, s, t)$ under the given action of $Sl_2\mathbf{Z}$ was known to Fricke and Klein. Their interest lay in a certain noncompact level surface disjoint from P which serves as Teichmüller space for the punctured torus. They viewed (r, s, t) in terms of traces of elements in $Sl_2\mathbf{R}$. Our application is to S^3 , the compact form of the Lie algebra of $Sl_2\mathbf{R}$, and some of the algebra carries over.

Note that for any connected compact Lie group H with maximal torus K and any $\alpha \in \Phi_n$, $n \geq 1$, the submanifold $K^n \subset H^n$ is invariant for α_H . This gives $h(\alpha_H) \geq h(\alpha_K) = k \log r_\alpha$, where $k = \dim K$. The real cohomology ring of H is a k -fold tensor product of cohomology rings of odd-dimensional spheres. The

action of α_H on cohomology is the k -fold tensor product of the cohomology action α_{S^1} (aside from differences in grading), and so $s(\alpha_H) = \tau_\alpha^k$. Thus Shub's entropy conjecture holds for word maps. Taking $H = SO(d)$, one obtains the estimates mentioned in the introduction.

2. Isotopy classes on $S^3 \times S^3$. We show that every isotopy class of $M = S^3 \times S^3$ contains word maps. It suffices to show this for isotopy classes that fix $H_3(M; \mathbf{Z})$ (i.e., are homotopic to the identity) and word maps arising from inner automorphisms, $\alpha = i(g): a \rightarrow gag^{-1}$ ($a, g \in F_2$). For the group K of such isotopy classes, Kreck gives an exact sequence [Kr, Theorem 2, Lemma 3b]

$$0 \rightarrow \theta_7 \rightarrow K \xrightarrow{\chi} H^3(M; \mathbf{Z}) \rightarrow 0,$$

where $\theta_7 \cong \mathbf{Z}_{28}$ is the group of smooth homotopy 7-spheres. The map χ is defined as follows. Choose an $S^3 \subset M$ with trivial normal bundle and a diffeomorphism f that fixes $H_3(M; \mathbf{Z})$ and leaves S^3 pointwise fixed. Then the normal part of the derivative of f on S^3 defines a map $S^3 \rightarrow Gl_3^+ \mathbf{R}$. Passing to the class of this map in $\pi_3(Gl_3^+ \mathbf{R}) = \pi_3(SO(3)) = \mathbf{Z}$, we obtain a pairing $K \otimes H_3(M; \mathbf{Z}) \rightarrow \mathbf{Z}$, and hence a homomorphism χ .

We first compute the map $\bar{\chi}: F_2 \rightarrow H^3(M; \mathbf{Z})$ that assigns to $g \in F_2$ the class $\chi(f(g))$, where $f(g) = i(g)_{S^3}$. Choose $S^3 = S^3 \times 1 \subset M$. Then $f(g)$ fixes S^3 pointwise. The normal bundle of S^3 is trivialized by left translation, so that the normal differential of $f(g)$ at $(x, 1)$ corresponds to conjugation by x^a , where $a = a(g)$ is the sum of the exponents of x_1 in g , i.e., the x_1 component of the abelianization map $F_2 \rightarrow \mathbf{Z}^2$. For $a = 1$ this conjugation is the usual action of S^3 on quaternions with zero real part that gives the generator for $\pi_3(SO(3))$. Reasoning similarly for the other coordinate 3-sphere $1 \times S^3 \subset M$, we see $\bar{\chi}$ is onto with kernel the commutator subgroup $F_2' = [F_2, F_2]$.

It now suffices to show that the isotopy class $\xi = [f(g)] \in K$, $g = c(x_1, x_2)$, corresponds to a generator of θ_7 . Note $i(g) = c(i(x_1), i(x_2))$, so that if d_j , $j = 1, 2$, is isotopic to $f(x_j)$ then $d = c(d_1, d_2)$ represents ξ . Choose a map $u: S^3 \rightarrow S^3$ homotopic to the identity so that $u^{-1}(1)$ is a neighborhood of 1 and set $d_1(x, y) = (x, u(x)yu(x)^{-1})$, $d_2(x, y) = (u(y)xu(y)^{-1}, y)$. Note that d_1 fixes a tubular neighborhood of the coordinate sphere $1 \times S^3$ and preserves a tubular neighborhood of the other coordinate sphere $S^3 \times 1$, and vice versa for d_2 . Thus, d_1 and d_2 commute near $S^3 \vee S^3$, so d acts on $\mathbf{R}^6 = M - S^3 \vee S^3$ with compact support. Let ϕ be the corresponding diffeomorphism of $S^6 = \mathbf{R}^6 \cup \{\infty\}$. Using ϕ to glue two copies of the 7-disc along their boundaries gives the homotopy 7-sphere δ corresponding to ξ under Kreck's exact sequence.

Let $v \in \pi_3 SO(3)$ be the generator mentioned above, given by the conjugation action of unit quaternions. The commutator construction above was generalized by Milnor [M, pp. 17–18] to a bilinear pairing $\pi_3 SO(3) \otimes \pi_3 SO(3) \rightarrow \theta_7$, so (up to sign) $v \otimes v$ goes to δ . Milnor notes that δ generates θ_7 , and this can be checked by showing that the Eells-Kuiper homomorphism $\mu: \theta_7 \rightarrow \mathbf{Q}/\mathbf{Z}$ takes the value $\pm 1/28$ on δ ; cf. [L].

Let $\pi = \pi_0 \text{Diff } M$. Let $W: \Phi_2 \rightarrow \pi$ be the homomorphism sending $\alpha \in \Phi_2$ to the isotopy class of the word map α_{S^3} . We have shown W is onto and $\text{Ker } W \subset F_2'$ with $F_2'/\text{Ker } W \cong \mathbf{Z}_{28}$. We now compute this kernel $S = \text{Ker } W$.

Let $c_{ij} = c(x_1^i, x_2^j) \in F_2'$. Then using the bilinearity of Milnor's pairing, one sees that $W(c_{ij}) = ij\delta \in \theta_7 \subset \pi$. Now identify F_2 with $\pi_1 X$, X a figure eight, so that $F_2' = \pi_1 \tilde{X}$ where the grid $\tilde{X} = \mathbf{R} \times \mathbf{Z} \cup \mathbf{Z} \times \mathbf{R} \subset \mathbf{R}^2$ is the maximal abelian covering of X . Then $W: F_2' \rightarrow \theta_7$ induces $\bar{W}: H_1(\tilde{X}; \mathbf{Z}) \rightarrow \mathbf{Z}_{28}$. The class c_{ij} , $i, j \in \mathbf{Z} - \{0\}$, corresponds to a rectangular loop in $\tilde{X}$ based at $(0, 0)$. These loops clearly generate $H_1 \tilde{X}$. Thus $\bar{W}$ is the homomorphism which assigns to a loop γ its enclosed area $A(\gamma)$ reduced modulo 28. Note that $\ker A = [F_2, F_2']$ and that $F_2/[F_2, F_2']$ is the nilpotent group

$$H_{\mathbf{Z}} = \langle x, y, z | z = c(y, z), c(x, z) = 1 = c(y) \rangle.$$

Clearly $H_{\mathbf{Z}}$ is isomorphic to the integral Heisenberg group, i.e., the integral 3×3 matrices of form $I + N$ with N strictly upper triangular. We have shown

THEOREM 2. *There is an exact sequence*

$$1 \rightarrow S \rightarrow \Phi_2 \xrightarrow{W} \pi_0 \text{Diff}(S^3 \times S^3) \rightarrow 1,$$

where $[F_2, F_2'] \subset S \subset F_2' \subset F_2 \subset \Phi_2$. In the quotient $F_2'/[F_2, F_2'] \cong \mathbf{Z}$, $S/[F_2, F_2']$ corresponds to the subgroup $28\mathbf{Z}$.

In particular, $K = H_{\mathbf{Z}}/z^{28}$ is nilpotent and χ is a central extension. These properties of K (and their analogues for other manifolds of high connectivity) are due to Bernhard Schmidt [Sc].

Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in Gl_2 \mathbf{Z}$, $J^6 = I$. Then we show for the natural map $\Phi_2 \rightarrow Gl_2 \mathbf{Z}$ induced by $F_2 \rightarrow \mathbf{Z}^2$:

LEMMA. *If $B \in \Phi_2$ covers J , then $B^6 \notin S$.*

The next corollary follows immediately from the lemma.

COROLLARY. *The natural map $\pi \rightarrow Gl_2 \mathbf{Z}$ from isotopy classes to homotopy classes does not split.*

Let $A \in \Phi_2$ with $Ax = y$, $Ay = yx^{-1}$. Then A covers J and $A^3 = i(yx^{-1}) \circ \phi$, where $\phi x = x^{-1}$, $\phi y = y^{-1}$. By Nielsen's result any B covering J is of the form $B = i(g) \circ A$ for some $g \in F_2$. Then $B^3 = i(h) \circ \phi$ for $h = g \cdot Ag \cdot A^2g \cdot yx^{-1}$.

Since $I + J + J^2$ has even entries, $h \equiv x^i y^j z^k \pmod{S}$ with i, j odd. Thus

$$B^6 = (B^3)^2 = i(h \cdot \phi(h)) \circ \phi^2 = i(x^i y^j z^k x^{-i} y^{-j} z^k) \equiv i(z^l) \pmod{S},$$

for l odd. Since z has even order mod S (namely $28 = |\theta_7|$), $B^6 \notin S$, as desired. This proves the lemma and corollary.

Note the essential use of the nontriviality of the subgroup θ_7 of homotopy 7-spheres. Indeed for the quotient group π/θ_7 , the natural map $\pi/\theta_7 \rightarrow Gl_2 \mathbf{Z}$ does split, meaning that the usual action of $Gl_2 \mathbf{Z}$ on $\mathbf{Z}^2$ lifts to an action by automorphisms of $H_{\mathbf{Z}}$. This can easily be seen by noting that the real Heisenberg group is the free two-step nilpotent Lie group generated by $\mathbf{R}^2$ and that $H_{\mathbf{Z}}$ is generated by $\mathbf{Z}^2 \subset \mathbf{R}^2$.

Also note that B^6 has infinite order in Φ_2 (since F_2 is torsionfree and centerless), so the word map B_{S^3} has infinite order for all B (the only trivial word map on S^3 comes from $1 \in \Phi_2$). We do not know if there is a periodic map of M inducing J on $H_3 M$; if it is smooth, then we see its period must be divisible by 24.

3. Isentropic approximation. By the results of §1, we see that every isotopy class on M contains diffeomorphisms of entropy $\log s$, where s is the spectral radius of the action on real homology. Thus if Shub's entropy conjecture holds on M , these word maps minimize entropy in their isotopy class. But it is not even known whether the entropy function on $\text{Diff } M$ is lower semicontinuous at a word map, aside from the trivial case of zero entropy.

We now show that a word map on two letters for S^3 or $SO(3)$ can be isotoped to an isentropic (equal entropy) fitted diffeomorphism. It suffices to work on $H = SO(3)$, because the isotopy can be lifted to M .

The function

$$(\text{Re } x)^2 + (\text{Re } y)^2 + (\text{Re } xy)^2 - 2(\text{Re } x)(\text{Re } y)(\text{Re } xy)$$

on M that (essentially) arose in §1 induces a function $g: H^2 \rightarrow [0, 1]$ regular over $(0, 1)$. The critical level $\tilde{L}_0 = g^{-1}0$ is a submanifold consisting of a single H -orbit corresponding to pairs of 180° rotations about perpendicular axes. The other critical level $\tilde{L}_1 = g^{-1}1$ corresponds to pairs of commuting rotations. $\tilde{L}_1 - (I, I)$ is a smooth submanifold. By assigning to a commuting pair $\neq (I, I)$ the common axis of rotation, one obtains a flat fibration of $\tilde{L}_1 - (I, I)$ over $\mathbf{R}P^2$, with fiber a punctured torus $T^2 - 0$ and monodromy ± 1 . There is a vector field X on H^2 such that

- (a) X is H -invariant.
- (b) $Xg \leq 0$ with $Xg = 0$ precisely on $\tilde{L}_0 \cup \tilde{L}_1$.
- (c) $X = 0$ on $\tilde{L}_0$ and on those points of $\tilde{L}_1$ outside an ε -neighborhood of (I, I) .
- (d) (I, I) is a source of X .
- (e) $\tilde{L}_1 - (I, I)$ is a normally hyperbolic repeller, and $\tilde{L}_0$ is a normally hyperbolic attractor.

Let ϕ_t be the corresponding flow. For $t \gg r_\alpha > 1$, $\beta = \phi_t \circ \alpha_H$ has a chain recurrent set $R(\beta)$ consisting of $\tilde{L}_0$, (I, I) and a family $M_1 \subset \tilde{L}_1 - (I, I)$ of DA attractors indexed by $\mathbf{R}P^2$. Moreover, $\beta|_{\tilde{L}_0} = \alpha_H|_{\tilde{L}_0}$ has finite order. Choose an equivariant Morse function on $\tilde{L}_0$ and extend its gradient vector field to a vector field Y_0 supported near $\tilde{L}_0$. Then choose a Morse function on $\mathbf{R}P^2$, lift its gradient vector field to M_1 using the flat connection and extend to a vector field Y_1 supported on a neighborhood of M_1 . Let ψ_t be the flow generated by $Y_0 + Y_1$. Then for $\varepsilon > 0$ and small enough, $\gamma = \psi_\varepsilon \circ \beta$ is Axiom A-No Cycles, with $R(\gamma)$ consisting of the source (I, I) , finitely many points in $\tilde{L}_0$ and finitely many DA attractors in M_1 . Finally, double DA these DA attractors to get an Axiom A-No Cycles map δ with $R(\delta)$ zero dimensional. The Artin-Mazur zeta functions satisfy

$$\zeta(\delta) = \zeta(\alpha_{S^1})^k \cdot R,$$

where k is the number of critical points of the Morse function on $\mathbf{R}P^2$ and R is a finite product of terms $(1 - t^n)^{\pm 1}$, so $h(\delta) = h(\alpha_{S^1}) = \log r_\alpha$. It is then easy to find a fitted diffeomorphism Ω -conjugate to δ and isotopic to δ .

The case $r_\alpha = 1$ requires an isotopy to Morse-Smale. The same procedure clearly works: no DA attractors arise, and one has to use instead a gradient flow on T^2 .

We have shown

THEOREM 3. *Every isotopy class of $S^3 \times S^3$ has a fitted representative f with entropy $\log s(f)$. The same holds for any isotopy class of $SO(3) \times SO(3)$ that contains a word map.*

See [F1] for results and a discussion of the corresponding question for word maps on S^1 , i.e., toral automorphisms. Even for T^3 some isotopy classes are not known to have a fitted representative with entropy $\log s$. Specifically for $\det(x - A) = x^3 - x - 1$, $A \in Sl(3, \mathbf{Z})$, it is not known whether the linear map A has an isentropic fitting. On T^4 there are examples where no isentropic Axiom A representative exists in the homotopy class of a certain linear map [F2].

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