ABOVE AND BELOW SUBGROUPS OF A LATTICE-ORDERED GROUP

RICHARD N. BALL,¹ PAUL CONRAD AND MICHAEL DARNEL

ABSTRACT. In an l -group G , this paper defines an l -subgroup A to be *above* an l -subgroup B (or B to be *below* A) if for every integer n , $a \in A$, and $b \in B$, $n(|a| \wedge |b|) \leq |a|$. It is shown that for every l -subgroup A , there exists an l -subgroup B maximal below A which is closed, convex, and, if the l -group G is normal-valued, unique, and that for every l -subgroup B there exists an l -subgroup A maximal above B which is *saturated*: if $0 = x \wedge y$ and $x + y \in A$, then $x \in A$.

Given l -groups A and B , it is shown that every lattice ordering of the splitting extension $G = A \times B$, which extends the lattice orders of A and B and makes A lie above B , is uniquely determined by a lattice homomorphism π from the lattice of principal convex l -subgroups of A into the cardinal summands of B . This extension is sufficiently general to encompass the cardinal sum of two l -groups, the lex extension of an l -group by an o -group, and the permutation wreath product of two l -groups.

Finally, a characterization is given of those abelian l -groups G that *split off below*: whenever G is a convex l -subgroup of an l -group H , H is then a splitting extension of G by A for any l -subgroup A maximal above G in H .

0. Introduction. Though the groups are not in general abelian, the notation is additive with the lattice operations taking priority: $a + b \wedge c$ is $a + (b \wedge c)$, $-b \wedge c$ is $(-b) \vee (-c)$, but $na \wedge b$ is $(na) \wedge b$. g^+ , g^- , and $|g|$ denote as usual $g \vee 0$, $-g \vee 0$, and $g \vee (-g)$, respectively. g is a *component* of h if $|g| \wedge |h - g| = 0$; we remind the reader that components commute and that $x + y = x \vee y$ whenever x and y are *disjoint*, that is, whenever $x \wedge y = 0$. $\mathcal{C}(G)$, $\mathcal{C}_h(G)$, $\mathcal{X}(G)$, $\mathcal{S}(G)$, and $\mathcal{P}(G)$ denote respectively the lattices of convex l -subgroups, principal convex l -subgroups, closed convex l -subgroups, cardinal summands, and polars. (The reader unfamiliar with these basic concepts can find them in [6].) $\mathcal{P}(G)$ is a complete Boolean algebra with $\mathcal{S}(G)$ as a subalgebra. We write $A \leq B$ to indicate that A is an l -subgroup of G , and $G(a)$ for $\{g \in G: |g| \leq n|a| \text{ for some integer } n\}$, the principal convex l -subgroup of G generated by a . If A is any subset of G we use $\langle A \rangle$ to designate the l -subgroup generated by A . Note that the elements of $\langle A \rangle$ have the form $\bigvee_{i=1}^n \bigwedge_{j=1}^m x_{ij}$, where the x_{ij} 's are from the subgroup generated by A . Finally, $b \ll a$ means that $n|b| \leq |a|$ for all integers n .

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A subgroup P of G is *regular* if it is maximal in $\mathcal{C}(G)$ with respect to omitting some element g , and we say that P is a *value* of g in this case. Such a subgroup is *prime* inasmuch as it contains at least one element from each disjoint pair. A set Γ of regular subgroups is *plenary* provided that $\bigcap \Gamma = 0$ and that $P \in \Gamma$ and $P \subseteq Q$ for a regular subgroup Q imply $Q \in \Gamma$. Such a set forms a *root system* in the inclusion order; that is, a partially ordered set such that for any $\gamma \in \Gamma$ the set $\{\delta \in \Gamma: \delta \geq \gamma\}$ is a chain. Most often we view Γ as an index set and use small Greek letters to indicate its elements; this abuse of notation allows us to write the corresponding regular prime as G_γ for $\gamma \in \Gamma$ and to use $G^\gamma = \bigcap \{G_\delta: \delta \in \Gamma \text{ and } G_\gamma \subset G_\delta\}$ for its *cover*. Note that G^γ is prime but not in general regular, and so G^γ need not correspond to a member of Γ . G is said to be *normal-valued* if each G_γ is normal in G^γ . If B is any subset of G , then $\Gamma(B)$ designates $\{\gamma \in \Gamma: G_\gamma \text{ is a value of some } b \in B\}$. An element g is *special* if $\Gamma(g)$ is a singleton, and in this case we term its value special also.

Given l -groups A and B their *cardinal sum* $A \boxplus B$ is their group direct sum with cardinal order: $a + b \geq 0$ if and only if $a \geq 0$ and $b \geq 0$. If $\{G_\lambda: \lambda \in \Lambda\}$ is a set of l -groups, then their *cardinal product* (it should be called the large sum but is not) is $\Pi G_\lambda = \{f: \Lambda \rightarrow \bigcup G_\lambda: f(\lambda) \in G_\lambda \text{ for all } \lambda \in \Lambda\}$ with componentwise group and lattice operations. The *cardinal sum* (or small sum) is $\{f \in \Pi G_\lambda: \{\lambda \in \Lambda: f(\lambda) \neq 0\} \text{ is finite}\}$. Now suppose Γ is any root system and S any subgroup of the real numbers $\mathbf{R}$. $V(\Gamma, S)$ designates $\{f: \Lambda \rightarrow S: \text{supp}(f) \text{ admits no infinite ascending chains}\}$, where $\text{supp}(f) = \{\lambda \in \Lambda: f(\lambda) \neq 0\}$. The group operations in $V(\Gamma, S)$ are componentwise and the order is determined by declaring $f \geq 0$ if and only if $f(\lambda) > 0$ for all maximal elements λ of $\text{supp}(f)$. The l -groups $V(\Gamma, S)$ are universal for abelian l -groups [12]. $\Sigma(\Gamma, S) = \{g \in V(\Gamma, S): \text{supp}(g) \text{ is finite}\}$.

Now let $\mathcal{B}$ be a Boolean algebra. $1_{\mathcal{B}}$ will denote the greatest element of $\mathcal{B}$ and $0_{\mathcal{B}}$ the least element. If $\mathcal{B} = \mathcal{P}(G)$ for an l -group G , then, $1_{\mathcal{B}} = G$ and $0_{\mathcal{B}} = 0$.

1. Above and below subgroups. In this section we characterize in various ways what it means for an l -subgroup A to lie above another l -subgroup B in an l -group G . We show that for any l -subgroup A there is an l -subgroup B maximal below A which is convex, closed, and unique in the normal valued case, and that for any l -subgroup B there is an l -subgroup A maximal above B which is closed and saturated. These results are fundamental for the rest of the paper.

PROPOSITION 1.1. *The following are equivalent for elements a and b in G .*

- (a) $|a| \wedge |b| \ll |a|$.
- (b) $n|b| \wedge |a|$ is a component of $n|b|$ for all positive integers n .
- (c) There is no prime P and no integer n for which $P < P + |a| < P + nb$.

If these conditions hold, then $G(b) = (G(b) \cap G(a)) \boxplus (G(b) \cap a^\perp)$, and we designate the projection l -homomorphisms $\nu a: G(b) \rightarrow G(a)$ and $\mu a: G(b) \rightarrow a^\perp$; that is, $x\nu a \in G(a)$, $x\mu a \in a^\perp$, and $x = x\nu a + x\mu a$ for all $x \in G(b)$. In particular, $x\nu a = x^+ \wedge a^+ + x^+ \wedge a^- - x^- \wedge a^+ - x^- \wedge a^-$ for all $x \in G(b)$.

PROOF. Suppose $0 \leq a \wedge b \ll a$ and let $b_0 = b \wedge a$ and $b_1 = b - b_0 = (b - a)^+$. We claim $b_0 \wedge b_1 = 0$, for which it is sufficient to show $b_1 \wedge a = 0$. For if P is any

prime such that $P + b_1 \wedge a > P$, then $P + b > P + a > P$, whence $P + a + b \geq P + 2a > P + a$ and $P + 2b \geq P + b + a \geq P + b > P + a$, meaning

$$P + 2a \wedge (a + b) \wedge (b + a) \wedge 2b = P + 2(a \wedge b) > P + a,$$

contradicting $a \wedge b \ll a$. The claim shows that b_0 and b_1 are complementary components of b , hence nb_0 and nb_1 are complementary components of nb for any positive integer n . But $nb_1 \in a^\perp$, hence $nb \wedge a = nb_0 \wedge a = nb_0$ since $b_0 \ll a$. This shows that (a) implies (b). Now suppose (b) holds and that P is a prime for which $P < P + |a|$. Since $n|b| = n|b| \wedge |a| + z$ for some $z \in a^\perp \subseteq P$, $P + nb \leq P + n|b| = P + n|b| \wedge |a| \leq P + |a|$, which establishes (c). Finally suppose $|a| \wedge |b| \ll |a|$ fails—say $n(|a| \wedge |b|) \not\leq |a|$ for some positive integer n . Then there must be some prime P for which $P + n(|a| \wedge |b|) > P + |a|$. Observe that

$$P < P + |a| \wedge |b| \leq P + |a| < P + n(|a| \wedge |b|) \leq P + n|b| = P + nb \vee (-n)b.$$

This proves that (c) implies (a).

Consider arbitrary a and b satisfying the conditions, and fix $x \in G(b)$. Then there is some integer n for which $|x| \leq n|b|$, and the argument that (a) implies (b) above shows that

$$k(x^+ \wedge |a|) \vee k(x^- \wedge |a|) \leq k(|x| \wedge |a|) \leq k(n|b| \wedge |a|) = kn(|b| \wedge |a|) \leq |a|$$

for all positive integers k . That is, the properties above hold when b is replaced by either x^+ or x^- . Therefore there are unique elements p and q in $a^\perp$ such that $x^+ = x^+ \wedge |a| + p = x^+ \wedge a^+ + x^+ \wedge a^- + p$ and $x^- = x^- \wedge |a| + q = x^- \wedge a^+ + x^- \wedge a^- + q$. By substituting these expressions into $x = x^+ - x^-$ one obtains $x = x^+ \vee a + (p + q) \in (G(b) \cap G(a)) \boxplus (G(b) \cap a^\perp)$. $\square$

If a and b satisfy the conditions of Proposition 1.1 we shall say that a lies above b or that b lies below a . More generally, we shall say that a subset $A \subseteq G$ lies above a subset $B \subseteq G$, or that B lies below A , if each element of A lies above each element of B .

PROPOSITION 1.2. *Suppose G is a normal-valued l -group having positive elements a and b and plenary set Δ of regular primes. Then $b \ll a$ if and only if $a \notin G^\delta$ for all $\delta \in \Delta(b)$, and b lies below a if and only if $b \in G_\delta$ for all $\delta \in \Delta(a)$.*

PROOF. If $b \ll a$ and $\delta \in \Delta(b)$, then $G_\delta + nb < G_\delta + a$ for all positive integers n , hence $a \notin \{g \in G: G_\delta + |g| \leq G_\delta + nb \text{ for some } n\} = G^\delta$. Conversely, if it is not true that $b \ll a$ then there is some positive integer n and $\delta \in \Delta$ with $G_\delta + nb > G_\delta + a$. Clearly a lies in the cover of the unique value of b containing G_δ . Now suppose b is below a and consider $\delta \in \Delta(a)$. Then $a \in G^\delta$ implies $a \wedge b \in G_\delta$, so $G_\delta = G + a \wedge b = G + b$, meaning $b \in G_\delta$. If b is not below a then there is some $\delta \in \Delta(a \wedge b)$ for which $a \in G^\delta$. But then $a \notin G_\delta$ implies $\delta \in \Delta(a)$, and $b \notin G_\delta$. $\square$

The failure of Proposition 1.2 in $A(\mathbf{R})$, the l -group of order-preserving automorphisms of the real numbers $\mathbf{R}$, shows that the hypothesis of normal-valuedness cannot be omitted from Proposition 1.2, or indeed from the other results of this section in which it appears.

PROPOSITION 1.3. *For each positive element a in the normal valued l -group G , $D = \{b: |b| \ll a\}$ is the unique largest subgroup of G such that $D \leq a$, and $D \in \mathcal{X}(G)$. Furthermore, $C = a^\perp \boxplus D = \bigcap \{G_\delta: \delta \in \Delta(a)\} = \{b: b \text{ lies below } a\}$ is the unique largest subgroup of G which lies below a , and $C \in \mathcal{X}(G)$.*

PROOF. Let Δ be any plenary set of regular primes of G . Consider $0 < b_1, b_2 \in D$ and $\delta \in \Delta(b_1 + b_2)$. Since $\Delta(b_1 + b_2) \subseteq \Delta(b_1) + \Delta(b_2)$, $a \notin G^\delta$ by Proposition 1.2, hence $b_1 + b_2 \ll a$. It follows that $D \in \mathcal{C}(G)$, and is therefore clearly maximal with respect to $D \leq a$. To show D closed suppose $\bigvee E = e$, where $E = \{b \in D: b \leq e\}$. By Lemma 1.17 of [4], $ne = \bigvee \{nb: b \in E\}$ for each positive integer n . Now $\{nb: b \in E\} \subseteq D \leq a$, so $ne \leq a$, proving $a \gg e \in D \in \mathcal{X}(G)$.

Propositions 1.1 and 1.2 establish $\bigcap \{G_\delta: \delta \in \Delta(a)\} = \{b: b \text{ lies below } a\} \subseteq a^\perp \boxplus D$. But for any $\delta \in \Delta(a)$, $a \notin G_\delta$ implies $a^\perp \subseteq G_\delta$, and $D \subseteq G_\delta$ by Proposition 1.2. This shows all those sets are equal. The closure of C is a straightforward consequence of the closure of $a^\perp$ and of D . $\square$

PROPOSITION 1.4. *The following are equivalent for subgroups A and B of G .*

- (a) A lies above B .
- (b) bva is a component of b for each $b \in B$ and $a \in A$.
- (c) There are no elements $a \in A$ and $b \in B$ and no prime P satisfying $P < P + a < P + b$. That is, $a^+ \wedge (b - a)^+ = 0$ for all $a \in A$ and $b \in B$.

If G is normal-valued with plenary set Δ of regular primes, then the conditions above are equivalent to the condition that $\delta_1 \not\geq \delta_2$ for all $\delta_1 \in \Delta(B)$ and $\delta_2 \in \Delta(A)$, or that $B \subseteq \bigcap \{G_\delta: \delta \in \Delta(A)\}$. If, in addition, B is a convex l -subgroup, then the conditions above are equivalent to $\Delta(A) \cap \Delta(B) = \emptyset$. Under any circumstances, if A is a convex l -subgroup, then the conditions above are equivalent to $B \subseteq A^\perp$.

PROOF. The equivalence of conditions (a), (b), and (c) results from Proposition 1.1, the second assertion is a consequence of Proposition 1.3, and the second assertion readily implies the third. This last observation is also straightforward. $\square$

COROLLARY 1.5. *If subgroup A lies above subgroup B of G , then $A \cap B = 0$. B lies both above and below A if and only if $B \subseteq A^\perp$.*

PROOF. Suppose B lies both above and below A , and consider $a \in A$, $b \in B$. Then $|a| \wedge |b| \ll |a|$ and $|a| \wedge |b| \ll |b|$, meaning $|a| \wedge |b| = 0$ or $B \subseteq A^\perp$. $\square$

Suppose C is a subgroup of G . A subgroup B which is maximal among those subgroups lying below C is said to be *maximal below C* , or simply *maximal below*, and likewise a subgroup A which is maximal among those subgroups lying above C is said to be *maximal above C* , or simply *maximal above*. C is said to be *saturated* if C contains each component of each of its elements. Observe that a saturated subgroup C is a sublattice since $c \vee 0$ is a component of each $c \in C$.

THEOREM 1.6. *Suppose subgroup A lies above subgroup B in G . Then A is contained in some subgroup maximal above B , and B is contained in some subgroup maximal below A . If G is normal-valued, then the unique subgroup maximal below A is $\bigcap \{G_\delta: \delta \in \Delta(A)\}$. In any case a maximal below subgroup is a closed convex l -subgroup and a maximal above subgroup is a saturated and closed l -subgroup.*

PROOF. The first assertion is a consequence of Zorn's Lemma, and the second assertion has already been established. Suppose now that B is maximal below A in G . B must be a sublattice, for if not then there exist elements $a \in A$ and $t \in \langle B \rangle$ and prime P such that $P < P + |a| < P + t$. But $t = \bigvee_I \bigwedge_J b_{ij}$ for some $b_{ij} \in B$ and finite I and J , hence $P + t = P + b_{ij}$ for some $i \in I$ and $j \in J$. But then $P < P + |a| < P + b_{ij}$, contradicting the assumption that B lies below A and showing B to be a sublattice. To show B convex consider an arbitrary t such that $b_1 \leq t \leq b_2$ for $b_1, b_2 \in B$. Then $|t| \leq |b_1| \vee |b_2| \in B$, hence t lies below A . Since $\{t: b_1 \leq t \leq b_2, b_i \in B\}$ is a subgroup containing B , $t \in B$. To show B closed consider $z = \bigvee S$, where $S = \{b \in B^+: b \leq z\}$, and let B' be the group generated by $B \cup \{z\}$. We claim B' lies below A , for if not then by Proposition 1.4(c) $|a| \wedge (t - |a|)^+ \neq 0$ for some $t \in B'$. Now t has the form $\sum_{i=1}^n (b_i + \varepsilon_i z)$ where $b_i \in B$ and ε_i is ± 1 . By replacing t by $\sum_{i=1}^n (b_i + y_i)$, where $y_i \in S$ if $\varepsilon_i = -1$ and $y_i = z$ if $\varepsilon_i = \pm 1$, we may assume $t = \sum_{i=1}^n (b_i + z)$. Now $t = \bigvee U$, where $U = \{\sum_{i=1}^n (b_i + s): s \in S\}$, hence $|a| \wedge (t - |a|)^+ = \bigvee \{|a| \wedge (u - |a|)^+: u \in U\}$, so there is some $u \in U$ for which $|a| \wedge (u - |a|)^+ > 0$. Since $u \in B$, this contradicts the assumption that B lies below A . We can only conclude that B is closed.

Suppose that A is maximal above B . To prove A saturated consider the element $a \in A$ with component z , and let A' be the group generated by $A \cup \{z\}$. To prove that A' lies above B it is enough to show that for each $t \in A'$ and prime P there is some $a_0 \in A$ with $P + t = P + a_0$. Since t has the form $\sum_{i=1}^n (a_i + \varepsilon_i z)$, where $a_i \in A$ and $\varepsilon_i = \pm 1$, we induct on n . Consider the case when $n = 1$ and $\varepsilon_i = 1$. Because the prime $-a_1 + P + a_1$ contains at least one of the disjoint elements $|z|$ and $|a - z|$, $P + a_1 + z$ is either $P + a_1$ or $P + a_1 + a$. The argument for $\varepsilon_i = -1$ is analogous, and the induction step yields to a simple argument using the above.

It remains to show that if A is maximal above B then A is closed. Suppose $z = \bigvee S$ for $S \subseteq A$ and let A' be the group generated by $A \cup \{z\}$. For the sake of contradiction assume that there are elements $t \in A'$ and $b \in B$ for which $t^+ \wedge (b - t)^+ > 0$. Let us say t is $\sum_{i=1}^n (a_i + \varepsilon_i z)$ for some $a_i \in A$ and $\varepsilon_i = \pm 1$. Then $\bigvee S = z$ implies $\bigvee R = t$, where $R = \{\sum_{i=1}^n (a_i + \varepsilon_i y_i): y_i \in S \text{ if } \varepsilon_i = \pm 1 \text{ and } y_i = z \text{ if } \varepsilon_i = -1\}$, and it follows that

$$t^+ \wedge (b - t)^+ = \bigvee \{r^+ \wedge (b - r)^+: r \in R\}.$$

Choose a specific $r = \sum_{i=1}^n (a_i + \varepsilon_i y_i) \in R$ such that $r^+ \wedge (b - r)^+ > 0$. Observe that $r^+ \wedge (b - r)^+ > 0$ also. Again, $\bigvee S = z$ implies $r = \bigwedge U$, where $U = \{\sum (a_i + \varepsilon_i w_i): w_i = y_i \text{ if } \varepsilon_i = 1 \text{ and } w_i \in S \text{ if } \varepsilon_i = -1\} \subseteq A$, and it follows that

$$r^+ \wedge (b - r)^+ = \bigvee \{r^+ \wedge (b - u)^+: u \in U\}.$$

Choose a specific $u = \sum_{i=1}^n (a_i + \varepsilon_i w_i) \in U$ such that $r^+ \wedge (b - u)^+ > 0$. Then $u^+ \wedge (b - u)^+ > 0$ and $u \in A$, contradicting the assumption that A lies above B . $\square$

If subgroup B of G is maximal below and A is maximal above B , then it need not follow that B is maximal below A . Consider $G = \mathbf{Z} \times (\mathbf{Z} \boxplus \mathbf{Z})$, the splitting extension of $\mathbf{Z} \boxplus \mathbf{Z}$ by $\mathbf{Z}$ lexicographically ordered, where

$$-(m; 0, 0) + (0; a, b) + (m; 0, 0) = \begin{cases} (0; a, b) & \text{if } m \text{ is even,} \\ (0; b, a) & \text{if } m \text{ is odd.} \end{cases}$$

This is a simple example of a Scrimger group [15], studied in connection with l -group varieties; it is also a simple instance of the general construction of §3 and of Theorem 3.21 in particular. Let $B = 0 \bar{\times} (\mathbf{Z} \boxplus 0)$, a subgroup maximal below $\{(m; 0, n): m \text{ even}\}$, and let $A = \mathbf{Z} \bar{\times} (0 \boxplus 0)$, another subgroup maximal above B . The point is that B is proper in $0 \bar{\times} (\mathbf{Z} \boxplus \mathbf{Z})$ which is maximal below A .

G is *projectable* if $\mathcal{P}(G) = \mathcal{S}(G)$. This concept is also termed strong projectability in the literature. A subset $Z \subseteq G^+$ is called *type $\mathcal{U}$* if

$$\bigcap \{((Z - z) \vee 0)^{\perp\perp} : z \in Z\} = \bigcap \{((-z + Z) \vee 0)^{\perp\perp} : z \in Z\} = 0_{\mathcal{P}}$$

[1]. For example, a pairwise disjoint subset of G^+ is type $\mathcal{U}$. G is said to be *sup $\mathcal{U}$ complete* if every type $\mathcal{U}$ subset of G^+ has a supremum in G , and G is said to be *laterally complete* if every pairwise disjoint subset of G^+ has a supremum in G^+ . The following proposition relates these concepts, and impinges on the results of §6, but is otherwise unrelated to the subject of this section.

PROPOSITION 1.7. *A projectable l -group G is sup $\mathcal{U}$ complete if and only if it is laterally complete.*

PROOF. Any sup $\mathcal{U}$ complete l -group is laterally complete. Now suppose G is laterally complete and projectable, and consider a type $\mathcal{U}$ subset $Z \subseteq G^+$. Let $\mathcal{U}$ be $\{\mathcal{R} \subseteq \mathcal{P}(G): \mathcal{R} \text{ is pairwise disjoint, and for each } R \in \mathcal{R} \text{ there is some } z \in Z \text{ with } R \subseteq ((Z - z) \vee 0)^{\perp}\}$. Now $\mathcal{U}$ is closed with respect to unions of chains, and so $\mathcal{U}$ contains a maximal element $\mathcal{R}$. Since $\bigvee \{((Z - z) \vee 0)^{\perp} : z \in Z\} = 1_{\mathcal{P}}$ it follows that $\bigvee \mathcal{R} = 1_{\mathcal{P}}$. For each $R \in \mathcal{R}$ let a_R and z_R satisfy $z_R \in Z$, $R \subseteq ((Z - z_R) \vee 0)^{\perp}$, $a_R \in R$, and $z_R - a_R \in R^{\perp}$. Then $a = \bigvee \{a_R : R \in \mathcal{R}\}$ exists by the lateral completeness of G . We shall prove that $\bigvee Z = a$, the first step being to show $Z \leq a$. Suppose for contradiction that $a \not\leq z$ for some $z \in Z$; find $T \in \mathcal{R}$ such that $T \cap ((z - a)^+)^{\perp\perp} \neq 0_{\mathcal{P}}$, and let P be some prime subgroup satisfying $P < P + t$, where $t \in T^+$ and $t \leq (z - a)^+$. Then $P \leq P + a < P + z$. Now $a - a_T \in T^{\perp}$, for otherwise $\{a_R : R \in \mathcal{R}\} \leq -s + a < a$ for any $s \in T$ with $0 < s \leq a - a_T$, so $a - z_T = (a - a_T) - (z_T - a_T) \in T^{\perp} \subseteq P$, meaning $P + z_T < P + z$. But $T \subseteq ((Z - z_T) \vee 0)^{\perp}$ implies $(z - z_T) \vee 0 \in T^{\perp} \subseteq P$, which in turn implies $P + z \leq P + z_T$. This contradiction proves $Z \leq a$. $\bigvee Z = a$ by Lemma 5.6 of [1] since, as has just been pointed out, $T \subseteq (a - z_T)^{\perp}$ for all $T \in \mathcal{R}$, and so $\bigvee \{(a - z)^{\perp} : z \in Z\} = 1_{\mathcal{P}}$. $\square$

If $A \subseteq G$ we say that the *polars of $\mathcal{A}$ and $\mathcal{G}$ correspond* whenever the intersection map is one-one from $\mathcal{P}(G)$ onto $\mathcal{P}(A)$. This is equivalent to having each nontrivial polar of G intersect A nontrivially.

COROLLARY 1.8. *Suppose A is a maximal above l -subgroup of G . If G is projectable then so is A . If G is sup $\mathcal{U}$ complete and polars in G and A correspond, then A is sup $\mathcal{U}$ complete.*

PROOF. Suppose G is projectable, P is a polar of A and $Q = P^{\perp\perp}$ in G . For $a \in A$ let $a = a_1 + a_2$, where $a_1 \in Q$ and $a_2 \in Q^{\perp}$. Now a_1 lies in A by virtue of the saturation of A , and so $a_1 \in Q \cap A = P$. Next suppose G to be sup $\mathcal{U}$ complete

and consider a type $\mathcal{U}$ subset $Z \subseteq A^+$. Because polars correspond, Z is also of type $\mathcal{U}$ considered as a subset of G , and so has a supremum $z \in G$. z must lie in A by the closure of A . $\square$

The relation of one subgroup being maximal above another has a tidy characterization in an abelian l -group. This characterization is important for §§5 and 6.

PROPOSITION 1.9. *Let G be an abelian l -group with plenary set Δ of regular subgroups, and let l -subgroup A lie above l -subgroup B of G . Then A is maximal above B if and only if for each $g \in G \setminus A$ there is some $a \in A$ and positive integer n such that $ng - a$ has a value in $\Delta(\bar{B})$.*

PROOF. If A is maximal above B , then it is maximal above $\bar{B}$ by Theorem 1.6, and if $g \in G \setminus A$, then by Proposition 1.4 there is an element $ng - a$ of the subgroup generated by $A \cup \{g\}$ with $\Delta(ng - a) \cap \Delta(\bar{B}) \neq \emptyset$. The converse also follows from Proposition 1.4. $\square$

COROLLARY 1.10. *Let G be an abelian l -group with maximal above subgroup A . Then $ng \in A$ implies $g \in A$ for any $g \in G$ and nonzero integer n . Therefore A is divisible if G is.*

PROOF. A is maximal above $B = \bigcap \{G_\delta : \delta \in \Gamma(A)\}$. If $g \notin A$, then there is some $a \in A$ and positive integer k for which $kg - a$ has a value in $\Gamma(B)$. But then $n(kg - a) = k(ng) - na \in A$ has a value in $\Gamma(B)$ also, a contradiction to the fact that A lies above B . $\square$

COROLLARY 1.11. *Let G be a divisible abelian l -group with plenary set Δ of regular subgroups, and let l -subgroup A lie above l -subgroup B of G . Then A is maximal above B if and only if for each $g \in G \setminus A$ there is some $a \in A$ such that $g - a$ has a value in $\Delta(\bar{B})$.*

PROOF. Suppose A is maximal above B and that $g \in G \setminus A$, and find $a_1 \in A$ and positive integer n such that $ng - a_1$ has value $\delta \in \Delta(\bar{B})$. Since G is divisible there is an element $a \in G$ such that $na = a_1$, and $a \in A$ by the previous corollary. Since $n(g - a) = ng - a_1$, $g - a$ has value δ . $\square$

COROLLARY 1.12. *A maximal above subgroup A is convex if and only if A is a polar if and only if A is also maximal below.*

PROOF. If A is maximal above B and convex, then $A \subseteq B^\perp$ by Proposition 1.4, and since $B^\perp$ lies above B by Corollary 1.5, $A = B^\perp$ and so A is a polar. But any polar is maximal below its complementary polar by Corollary 1.5. And any maximal below subgroup is convex by Theorem 1.6. $\square$

Given subgroup B of G let $\mathcal{A}(B)$ designate the collection of all subgroups of G maximal above B . Note that $\mathcal{A}(B) \neq \emptyset$ since the o -subgroup lies above B and is therefore contained in some subgroup maximal above B by Theorem 1.6.

PROPOSITION 1.13. $\bigcap \mathcal{A}(B) \subseteq B^\perp$ for any subgroup B of G . Therefore if B has only one subgroup maximal above it, then this subgroup is $B^\perp$. G is representable if and only if $\bigcap \mathcal{A}(B) = B^\perp$ for all subgroups B of G .

PROOF. Consider $0 < a \in \cap \mathcal{A}(B)$. By Theorem 1.6 we may assume $B \in \mathcal{C}(G)$. Then for any $0 < b \in B$, $\langle a + a \wedge b \rangle$ lies above B , for if $P < P + a + a \wedge b < P + b_1$ for some $b_1 \in B$ and prime P , then $P \leq P + a \leq P + a + a \wedge b < P + b_1$ implies $a \in P$, hence $a \wedge b \in P$, contradicting $P < P + a + a \wedge b$. Therefore there is some $A \in \mathcal{A}(B)$ such that $a + a \wedge b \in A$. But then $a \in A$ implies $a \wedge b \in A \cap B = 0$. This argument shows that $\cap \mathcal{A}(B) \subseteq B^\perp$. Now suppose B is a subgroup of the representable l -group G having plenary set Δ , and consider $A \in \mathcal{A}(B)$. Because $B^\perp$ is normal in G , the subgroup generated by $B^\perp \cup A$ is $B^\perp + A$; consider an arbitrary $c + a \in B^\perp + A$ and $\delta \in \Delta(B)$. Since $B \not\subseteq G_\delta$ it follows that $B^\perp \subseteq G_\delta$, hence $G_\delta + (c + a) = G_\delta + a$, so $a \notin G_\delta$. This shows that $B^\perp + A$ lies above B , and by maximality that $B^\perp \subseteq A$. Thus $\cap \mathcal{A}(B) = B^\perp$. Finally, if G is not representable then it is only because it contains elements $b, g > 0$ for which $(g + b - g) \wedge b = 0$. We claim $\langle g \rangle$ lies above $\langle b \rangle$, for if $P < P + g < P + nb$ for some positive integer n , then $P + g - nb - g < P - g < P$, hence $P < P + (g + nb - g) \wedge b$, a contradiction. Therefore there is some $A \in \mathcal{A}(\langle b \rangle)$ such that $g \in A$. But $b^\perp \subseteq A$ would give $g + b - g \in A$, which leads to the contradiction $b \in A \cap \langle b \rangle = 0$. That is, $\cap \mathcal{A}(\langle b \rangle) \neq b^\perp$. $\square$

COROLLARY 1.14. *For an l -group G the following are equivalent.*

- (a) *G is archimedean.*
- (b) *Each maximal above subgroup is convex.*
- (c) *Each maximal above subgroup is a polar.*
- (d) *Each subgroup B has a unique subgroup maximal above it.*
- (e) *Each closed convex l -subgroup is maximal above and below.*

PROOF. If G is archimedean and $a, b > 0$, then $a \wedge b \ll a$ if and only if $a \wedge b = 0$. Thus (a) implies (b), and we have proven the equivalence of (b), (c), and (d) in the last two results. To show that (d) implies (a) suppose that G is not archimedean, say $0 < b \ll a$. Then $\langle a \rangle$ and $\langle a + b \rangle$ both lie above $\langle b \rangle$, and yet a and $a + b$ cannot both belong to the same subgroup A maximal above $\langle b \rangle$, for then $b \in \langle b \rangle \cap A = 0$, a contradiction. Finally, to prove the equivalence of (a) and (e) it is enough to recall a theorem of Bigard [5]: G is archimedean if and only if each closed convex l -subgroup is a polar. $\square$

The following example from [7] sheds light on some of the ideas we have been developing. It also nicely illustrates Corollary 1.14 by barely violating each of its conditions. Let $\Gamma = \{1, 2, 3, \dots, \omega, \alpha\}$, partially ordered by declaring only that $\omega > \alpha$. Let $G = \{g \in V(\Gamma, \mathbf{R}) : \lim_{n \rightarrow \infty} g(n) = g(\omega)\}$. Let $H = \{g \in G : g(\omega) = 0\}$. One readily checks that G is a nonarchimedean l -group all of whose convex l -subgroups are closed. The maximal below subgroups are precisely the polars, however, and in particular the only subgroup above H is (0) . The polar $G_1 = \{g \in G : g(n) = 0 \text{ for all } n = 1, 2, 3, \dots\}$ has numerous subgroups maximal above it, no one of which is convex. Also observe that H is the join in $\mathcal{K}(G)$ of the polars $E = \{g \in G : g(n) = 0 \text{ for all odd integers } n\}$ and $O = \{g \in G : g(n) = 0 \text{ for all even integers } n\}$. Thus the maximal below subgroups do not constitute a sublattice of $\mathcal{K}(G)$.

In some instances an l -subgroup A will lie above an l -subgroup B provided only that $A \cap B = 0$. The remainder of this section is devoted to an investigation of some of these instances.

LEMMA 1.15. *Suppose $t, g \in G^+$ with $g \wedge (t - g)^+ > 0$ and t special. Then $2t \wedge g$ is a nonzero component of g .*

PROOF. Consider a prime subgroup R omitting $g \wedge (t - g)^+$. Then $R < R + g < R + t$, hence $R \subseteq Q$, where Q is the unique value of t , and this implies $Q + g \leq Q + t < Q + 2t$. If $u = (g - (2t \wedge g)) \wedge (2t \wedge g) > 0$, then for any prime P omitting u we would have $P < P + 2t < P + g$. Again, the fact that $t \notin P$ implies $P \subseteq Q$, which in turn implies $Q + 2t \leq Q + g$, a contradiction. Therefore $u = 0$, proving $2t \wedge g$ to be a component of g . $2t \wedge g > 0$ because $R + (2t \wedge g) = R + g > R$, where R has the meaning above. $\square$

LEMMA 1.16. *Suppose $a, b \in G^+$ with $r = a \wedge (b - a)^+ > 0$. If t is a nonzero component of b such that $t^{\perp\perp} \subseteq r^{\perp\perp}$, then $t \wedge a$ is a nonzero component of a .*

PROOF. Let us show first that $u = (a - (a \wedge t)) \wedge (a \wedge t) = 0$. If $u > 0$, then since $u \leq t \in r^{\perp\perp}$ we may find a prime P omitting $u \wedge r$, from which follows $P < P + t < P + a$. Now since t is a component of b we have $P + t = P + b < P + a$, hence $(b - a)^+ \in P$. But this contradicts the assumption that $P < P + r \leq P + (b - a)^+$. We conclude $u = 0$, meaning $t \wedge a$ is a component of a . $0 \neq t^{\perp\perp} \subseteq r^{\perp\perp}$ implies $0 \neq t \wedge r \leq t \wedge a$. $\square$

PROPOSITION 1.17. *Suppose that A and B are l -subgroups of G such that B is convex and A is saturated. Suppose further that B satisfies either (a) or (b) below. Then A lies above B if and only if $A \cap B = 0$.*

(a) $b = \vee(T \wedge b)$ for all $b \in B^+$, where $T = \{t \in B^+ : t \text{ special}\}$.

(b) $\mathcal{S}(B)$ is dense in $\mathcal{P}(B)$. That is, for each $0 \neq X \in \mathcal{P}(B)$ there is some $0 \neq Y \in \mathcal{S}(B)$ such that $Y \subseteq X$.

PROOF. Suppose that B satisfies (a) and that $r = a \wedge (b - a)^+ > 0$ for some $a \in A^+$ and $b \in B^+$. Since $r = \vee\{(t \wedge (b - a))^+ \wedge a : t \in T\}$, we may assume $b \in T$. By Lemma 1.15, $2b \wedge a$ is a nonzero component of A . $2b \wedge a \in A$ by saturation and $2b \wedge a \in B$ by convexity, so $A \cap B \neq 0$. Now suppose that B satisfies (b) and that $r = a \wedge (b - a)^+ > 0$ for some $a \in A^+$ and $b \in B^+$. Let $X = r^{\perp\perp}$ and find $0 \neq Y \in \mathcal{S}(B)$ with $Y \subseteq X$. Let $b = t + (b - t)$, where $t \in Y$ and $b - t \in Y^\perp$. By Lemma 1.16, $t \wedge a$ is a nonzero component of a , hence $t \wedge a \in A \cap B \neq 0$. $\square$

Let $\mathbf{S}$ designate the class of l -groups G whose every positive element is a disjoint supremum of special elements, or equivalently, those l -groups G having a plenary set of special subgroups. It is known (Theorem 3.18 of [3]) that G is normal-valued and completely distributive if and only if G is a large l -subgroup of a member of $\mathbf{S}$.

COROLLARY 1.18. *Suppose that A and B are l -subgroups of G such that B is convex and that A is saturated. Suppose further that either $B \in \mathbf{S}$ or B is projectable. Then A lies above B if and only if $A \cap B = 0$.*

We next show that G is normal valued if and only if G is an l -subgroup of a member of $\mathbf{S}$. The proof modifies that of Proposition 3.17 of [3], and requires some auxiliary ideas.

Suppose Δ is a plenary set of regular subgroups of G which is *normal*, that is, $-g + G_\delta + g \in \Delta$ whenever $G_\delta \in \Delta$ and $g \in G$. Then $\mathcal{C}(\Delta) = \{Y \subseteq G: Y \supseteq \bigcap \Delta_0 \text{ for finite } \Delta_0 \subseteq \Delta\}$ is the neighborhood filter of 0 of a topology (also designated $\mathcal{C}(\Delta)$) with respect to which all group and lattice operations are continuous [4]. The subbasic open sets have the form $G_\delta + g$ for $G_\delta \in \Delta$ and $g \in G$. We write $G \leq_\Delta H$ to mean that H is an l -group having a normal plenary set Δ' of regular subgroups such that G is an l -subgroup of H dense with respect to $\mathcal{C}(\Delta')$ and such that Δ' and Δ are in one-one correspondence by intersection. The density condition may be reformulated as follows: any finite collection of cosets of elements in Δ' which has nonempty intersection in H must have nonempty intersection in G also.

PROPOSITION 1.19. *Suppose Δ is a normal plenary set of regular subgroups of an l -group G . Then there is an l -group G^Δ which is unique up to l -isomorphism over G with respect to the following property. For any l -group H , $G \leq_\Delta H$ if and only if there is an l -monomorphism $\theta: H \rightarrow G^\Delta$ over G .*

PROOF. Let G^Δ be the completion of G with respect to the l -Cauchy structure $\mathcal{D}(\Delta) = \{\mathcal{F}: \mathcal{F} \text{ is a filter on } G \text{ such that } (\mathcal{F} - \mathcal{F}) \cap (-\mathcal{F} + \mathcal{F}) \supseteq \mathcal{C}(\Delta)\}$ induced by $\mathcal{C}(\Delta)$ [1]. For each $G_\delta \in \Delta$ let $\text{cl}(G_\delta) = \{[\mathcal{F}]: G_\delta \in \mathcal{F}\}$, a prime subgroup of G^Δ , and let $\Delta' = \{\text{cl}(G_\delta): \delta \in \Delta\}$. It can readily be verified that $G \leq_\Delta G^\Delta$. Now suppose $G \leq_\Delta H$ and consider $h \in H$. Let $\mathcal{F}$ be the trace of the neighborhood filter (with respect to $\mathcal{C}(\Delta')$) of h on G , and let $\rightarrow$ designate convergence in $\mathcal{C}(\Delta')$. Then $\mathcal{F} - \mathcal{F} \rightarrow h - h = 0$ and $-\mathcal{F} + \mathcal{F} \rightarrow -h + h = 0$, so $\mathcal{F} \in \mathcal{D}(\Delta)$. Define $h\theta = [\mathcal{F}]$. One easily verifies that θ is an l -monomorphism. $\square$

PROPOSITION 1.20. *G is normal valued if and only if $G^\Delta \in \mathbf{S}$, and in this case Δ' is the minimal plenary set of special subgroups of G^Δ .*

PROOF. Let H be G^Δ , and consider $0 < h \in H$. Then $h = [\mathcal{H}]$ for some Cauchy filter $\mathcal{H}$ on G . Fix $\delta \in \Delta$ for which H_δ is a value of h in Δ' . Let $\mathcal{F}$ be $\mathcal{M} \cup \mathcal{N}$, where $\mathcal{M} = \{G_\gamma + g_\gamma \in \mathcal{H}: G_\gamma \subseteq G_\delta\}$ and $\mathcal{N} = \{G_\gamma: G_\gamma \text{ unrelated to } G_\delta\}$. We claim that every finite subset of $\mathcal{F}$ has nonempty intersection. Given finite $\mathcal{N}_1 \subseteq \mathcal{N}$ find $0 < x_\gamma \in G_\gamma \setminus G_\delta$ for each $G_\gamma \in \mathcal{N}_1$ and let $x = \bigwedge x_\gamma \in (\bigcap G_\gamma) \setminus G_\delta$. Given finite $\mathcal{M}_1 \subseteq \mathcal{M}$ choose $g \in \bigcap \mathcal{M}_1$, possible since $\mathcal{M}_1 \subseteq \mathcal{H}$. Because G has normal values there is some positive integer n such that $G_\delta + g < G_\delta + nx$. Therefore $nx \wedge g \in (\bigcap \mathcal{M}_1) \cap (\bigcap \mathcal{N}_1) \neq \emptyset$. This completes the proof of the claim that $\mathcal{F}$ generates a proper filter on G , one which is readily seen to be in $\mathcal{D}(\Delta)$. Let $[\mathcal{F}] = f \in H$. Now for $\gamma \in \Delta$, $G_\gamma \subseteq G_\delta$ implies the existence of some $g_\gamma \in G$ for which $G_\gamma + g_\gamma \in \mathcal{H} \cap \mathcal{F}$, hence $G_\gamma \in \mathcal{H} - \mathcal{F}$, while G_γ unrelated to G_δ implies $G_\gamma \in \mathcal{F}$. Therefore $(\mathcal{H} - \mathcal{F}) \wedge \mathcal{F} \supseteq \mathcal{C}(\Delta)$, meaning that $(h - f) \wedge f = 0$ so that f is a component of h , and $h \in H_\gamma$ for all $\gamma \in \Delta$ such that G_γ is unrelated to G_δ , while $h - f \in H_\gamma$ for all $\gamma \in \Delta$ such that $G_\gamma \subseteq G_\delta$. Therefore h is special with sole value H_δ , which must therefore be closed. Clearly h must be the supremum of its special components. $\square$

To be sure, not all members of $\mathbf{S}$ have the form G^Δ for some l -group G . It is not difficult to show, for example, that G^Δ must be $\sup \mathcal{U}$ complete and therefore laterally complete. But G^Δ need not be Dedekind MacNeille complete; in particular, if G is the l -group of rational numbers, then $G^\Delta = G$. Is it possible to characterize l -groups of the form G^Δ ?

In the next result, $A^\sim$ designates the convexification and $[A]$ designates $\bigcap \{C : A \leq C \leq G \text{ and } C \text{ saturated}\}$ for an l -subgroup A of G .

THEOREM 1.21. *Suppose A and B are l -subgroups of the normal-valued l -group G . Then A lies above B if and only if $[A] \cap B^\sim = 0$ in G^Δ . In particular, if $G \in \mathbf{S}$, then a saturated subgroup A lies above $B \in \mathcal{C}(G)$ if and only if $A \cap B = 0$.*

PROOF. Suppose $G \in \mathbf{S}$ and that Δ is the minimal plenary set of special subgroups of G . Suppose that the saturated subgroup A fails to lie above the convex l -subgroup B . By Proposition 1.4 there is some $a \in A^+$, $b \in B^+$, and $\delta \in \Delta$ such that $\delta \in \Delta(a) \cap \Delta(b)$. Because $G \in \mathbf{S}$ we may replace a by its special component having value δ , which also lies in A because A is saturated. Because G^δ/G_δ is archimedean, $a \leq nb$ for some positive integer n . That is, $a \in A \cap B \neq \emptyset$. $\square$

2. Maximal below subgroups. In this section we present conditions sufficient for an l -group B to be maximal below in G . Given another l -subgroup A the notation $B \vee A^\perp$ refers to $\{g \in G : |g| \leq n(b \vee x) \text{ for some } 0 < b \in B, 0 < x \in A^\perp, \text{ and positive integer } n\}$, the supremum of B and $A^\perp$ in $\mathcal{C}(G)$.

PROPOSITION 2.1. *For l -subgroups A and B with B convex, (a) implies (b), and (b) and (c) are equivalent. If $G = A^{\perp\perp} \boxplus A^\perp$ then (b) implies (a).*

- (a) $B \vee A^\perp$ is maximal below A in G .
- (b) $B \cap A^{\perp\perp}$ is maximal below A in $A^{\perp\perp}$.
- (c) $B \cap A^{\perp\perp}$ is maximal below $A \boxplus A^\perp$ in G .

PROOF. Assuming (a) holds, $B \cap A^{\perp\perp}$ surely lies below A in $A^{\perp\perp}$. If $B \cap A^{\perp\perp} \subsetneq C \in \mathcal{C}(A^{\perp\perp})$, then $C \vee A^\perp \supsetneq B \vee A^\perp$ by virtue of the distributivity of $\mathcal{C}(G)$; hence there are elements $a \in A^+$, $c \in C^+$, $0 < x \in A^\perp$, a prime subgroup P , and a positive integer n such that $P < P + a < P + n(c \vee x) = P + nc \vee nx = P + nc$. ($n(c \vee x) = n(c + x) = nc + nx = nc \vee nx$ because $c \wedge x = 0$, and $nx \in P$ because $a \wedge x = 0$ and $a \notin P$.) This proves that (b) holds. If (b) holds and $B \cap A^{\perp\perp} \subsetneq C \in \mathcal{C}(G)$ such that C lies below $A \boxplus A^\perp$, then $C \subseteq A^{\perp\perp}$ by the last assertion of Proposition 1.4, and so $C = B \cap A^{\perp\perp}$, meaning (c) holds, and (c) clearly implies (b). Finally, suppose that $G = A^{\perp\perp} \boxplus A^\perp$ and that (b) holds. Given $0 < a \in A$, $0 < b \in B$, and $0 < x \in A^\perp$ we may write $b = bva + b\mu a$, where $bva \ll a$. Therefore

$$a \wedge (n(b \vee x)) \leq a \wedge (n(b + x)) = a \wedge (nbva + n(b\mu a + x)) = nbva \ll a,$$

proving that $B \vee A^\perp$ lies below A . If $B \vee A^\perp \subsetneq C \in \mathcal{C}(G)$ such that C lies below A , then

$$C = C \cap G = (C \cap A^{\perp\perp}) \boxplus (C \cap A^\perp) = (B \cap A^{\perp\perp}) \boxplus A^\perp = B \vee A^\perp,$$

where the penultimate equality holds by (b). $\square$

The following example shows that (b) does not generally imply (a) in Proposition 2.1. Let $\Gamma = \{0, 1, 2, \dots\}$ be ordered by declaring $0 > 2n + 1$ for $n = 1, 2, 3, \dots$. Let $G = \{g \in V(\Gamma, \mathbb{Z}): \{n \in \Gamma: (n)g \neq 0\} \text{ is finite for some } k \in \mathbb{Z}\}$, let A be $\langle a \rangle$ where $(0)a = 1$ and $(n)a = 0$ for $n > 0$, and let $B = \{g \in G: (2n)g = 0 \text{ for } n = 0, 1, 2, \dots\}$. Then B is maximal below A in $A^{\perp\perp} = A$, yet $B \vee A^{\perp} = \{g \in G: 0 \notin \{n \in \Gamma: (n)g \neq 0\} \text{ finite}\}$ is properly contained in $B_1 = \{g \in G: (0)g = 0\}$, and B_1 is (maximal) below A in G .

COROLLARY 2.2. *Suppose A and B are l -subgroups of G . Then B is maximal below A in $A^{\perp\perp}$ if and only if B is maximal below $A \boxplus A^{\perp}$.*

We refer to (A, B) as a *maximal above and below pair* if A is an l -subgroup of G maximal above B and B is an l -subgroup of G maximal below A .

PROPOSITION 2.3. *Suppose (A, B) is a maximal above and below pair in G . If B is prime, then A must be totally ordered. In a normal-valued l -group G , B is prime if and only if A is a totally ordered set of special elements. In particular, if a prime subgroup of a normal-valued l -group is maximal below, then it is an intersection of special subgroups. Therefore a regular subgroup of a normal-valued l -group is maximal below if and only if it is special.*

PROOF. If a_1 and a_2 are disjoint elements of A , then one must lie in B , hence one is 0. That is, A is totally ordered. If G is normal valued, then $B = \bigcap \{G_{\delta}: \delta \in \Delta(A)\}$, hence $\Delta(A)$ must be totally ordered and all elements of A must be special. $\square$

To see that the normal-valued hypothesis cannot be omitted from the preceding result take G to be $\{g \in A(\mathbb{R}): g + p = p + g\}$, where $(r)p = r + 1$ for all $r \in \mathbb{R}$. Let $A = \langle a \rangle$, where $(r)a = r + 2$, and let B be $\{g \in G: (0)g = 0\}$.

COROLLARY 2.4. *A normal-valued l -group is finite-valued if and only if every regular subgroup is maximal below.*

As in §1, let $\mathbf{S}$ designate the class of all l -groups G whose every positive element is a disjoint supremum of special elements.

COROLLARY 2.5. *If G is normal-valued, completely distributive, and if every closed convex l -subgroup of G is maximal below, then $G \in \mathbf{S}$.*

PROOF. Let Δ be the minimal plenary set of closed regular subgroups of G . Since each G_{δ} is maximal below, it is also special. $\square$

PROPOSITION 2.6. *A subgroup B of an abelian l -group G is prime and maximal below if and only if B is an intersection of a chain of special subgroups.*

PROOF. Suppose $B = \bigcap \{G_{\lambda}: \lambda \in \Lambda\}$ for some chain Λ of special subgroups. For each $\lambda \in \Lambda$ choose special $a_{\lambda} \in G^{\lambda} \setminus G_{\lambda}$, and let A be $\langle a_{\lambda}: \lambda \in \Lambda \rangle$. Then $B = \bigcap \{G_{\lambda}: \lambda \in \Delta(A)\}$ is maximal below A . $\square$

COROLLARY 2.7. *An abelian l -group is finite-valued if and only if each prime subgroup is maximal below.*

COROLLARY 2.8. *If $0 < a$ is special with value G_γ , then $G_\gamma \cap a^{\perp\perp}$ is maximal below $\langle a \rangle \boxplus a^\perp$ and also maximal below $\langle a \rangle$ in $a^{\perp\perp}$.*

PROOF. Take A to be $\langle a \rangle$ and B to be G_γ in Proposition 2.1. $\square$

The convex l -subgroup M of the following lemma has been well studied. A proof can be found in [11], for example, of the final assertion of the following lemma.

LEMMA 2.9. *Suppose P is a prime subgroup of G and let $M = \{g \in G: |g| \wedge q = 0 \text{ for some } q \in G \setminus P\}$. Then $M \in \mathcal{C}(G)$, and any convex l -subgroup C which contains M must be related to P . Thus $M = \bigcap \{Q \in \mathcal{C}(G): Q \subseteq P \text{ and } Q \text{ prime}\}$.*

PROOF. If $0 < g_1, g_2 \in M$ with $g_i \wedge q_i = 0$ for $q_1, q_2 \in G \setminus P$, then $(g_1 + g_2) \wedge (q_1 \wedge q_2) = 0$, and $q_1 \wedge q_2 \in G \setminus P$ by the primeness of P . Thus $M \in \mathcal{C}(G)$. Now suppose $M \subseteq C \not\subseteq P$ for some $C \in \mathcal{C}(G)$; say $c \in C \setminus P$. Consider $p \in P^+$. Then $c \wedge p \in P \cap C$, hence $c' = c(c \wedge p)^{-1} \notin P$. Because $c' \wedge p' = 0$, where $p' = p(c \wedge p)^{-1}$, we conclude that $p' \in M \subseteq C$, whence $p \in C$. That is, $P \subseteq C$. Finally, let $\mathcal{M} = \{Q \in \mathcal{C}(G): Q \subseteq P, Q \text{ prime}\}$. Given $p \in P \setminus M$ there is some prime subgroup Q such that $M \subseteq Q$ and $p \notin Q$, and the preceding argument shows $Q \in \mathcal{M}$. Therefore $\bigcap \mathcal{M} \subseteq M$. On the other hand, for any $Q \in \mathcal{M}$, $g \in G$, and $q \in G \setminus P$ such that $g \wedge q = 0$ we must have $g \in Q$ by primeness. That is, $M \subseteq \bigcap \mathcal{M}$. $\square$

PROPOSITION 2.10. *If G is abelian and $0 < a \in G$ is special with value G_γ , then each $N \in \mathcal{C}(G)$ between $G_\gamma \cap a^{\perp\perp}$ and $a^{\perp\perp}$ is maximal below.*

PROOF. If $N = a^{\perp\perp}$, then N is maximal below $a^\perp$ by Proposition 1.4, and if $N = G_\gamma \cap a^{\perp\perp}$, then N is maximal below $\langle a \rangle \boxplus a^{\perp\perp}$ by Corollary 2.8, so assume $G_\gamma \cap a^{\perp\perp} < N < a^{\perp\perp}$. Observe that any $h \in a^{\perp\perp} \setminus G_\gamma$ must be special. (If h had a value G_δ unrelated to G_γ , then by Lemma 2.9 there would be disjoint elements $x \in G_\delta \setminus G_\gamma$ and $y \in G_\gamma \setminus G_\delta$, and by replacing x by some multiple we could assume $x > g$. Then $y \in g^\perp \setminus h^\perp$, contrary to assumption.) Let $\Gamma = \{G_\gamma: G_\gamma \text{ is the value of some } 0 < h \in a^{\perp\perp} \setminus N\}$ and for each choose $h_\gamma \in a^{\perp\perp} \setminus N$ with value G_γ . Let $A = \langle h_\gamma: \gamma \in \Gamma \rangle$. Since N is maximal below A in $a^{\perp\perp}$ by construction, N is maximal below $A \boxplus a^\perp$ by Proposition 2.1. $\square$

PROPOSITION 2.11. *Suppose $\{G_\lambda: \lambda \in \Lambda\}$ is a set of special subgroups. If G is abelian, or if each element of Λ exceeds an element minimal in Λ , then $B = \bigcap \{G_\gamma: \lambda \in \Lambda\}$ is maximal below.*

PROOF. For each $\lambda \in \Lambda$ choose a_λ to be a positive special element in $G^\lambda \setminus G_\lambda$. If G is abelian, then the subgroup A generated by the a_λ 's is the direct sum of the cyclic subgroups $\langle a_\lambda \rangle$, hence $\Gamma(A) = \Lambda$, where Γ here represents the plenary set of all regular subgroups of G . Therefore B is maximal below A by Theorem 1.6. If each element of Λ exceeds an element in $\Lambda_0 = \{\lambda \in \Lambda: \lambda \text{ is minimal in } \Lambda\}$, then let $A = \langle a_\lambda: \lambda \in \Lambda_0 \rangle$. Observe that since $a_\lambda \wedge a_\delta = 0$ for $\lambda \neq \delta$ in Λ_0 , A is the cardinal sum of the cyclic subgroups $\langle a_\lambda \rangle$ for $\lambda \in \Lambda_0$. If $0 < a \in A$ and $0 < b \in \bigcap \{G_\lambda: \lambda \in \Lambda_0\}$ satisfy $P < P + a < P + b$, then the value of a containing P must lie in Λ_0 and therefore contains b , a contradiction. This shows that $\bigcap G_\lambda$ lies below

A. If $G_\lambda \subset B \in \mathcal{C}(G)$, then there is some $\lambda \in \Lambda_0$ such that $B \not\subseteq G_\lambda$. Because G_λ is normal in G^λ (Corollary 4.3.5 and Proposition 4.4.4 of [6]), $G_\lambda + a_\lambda < G_\lambda + b$ for some $b \in B$, hence $a_\lambda \in B$, a contradiction. $\square$

COROLLARY 2.12. *If $\Gamma(G)$ is finite, then every convex l -subgroup is maximal below.*

LEMMA 2.13. *Suppose B is a convex l -subgroup of an abelian l -group G . If there is a plenary set Δ of regular primes such that for each $\delta \in \Delta \setminus \Delta(B)$ there is some $0 < g(\delta) \in G$ with the property that $\{\gamma \in \Delta(g(\delta)) : \gamma \text{ is related to an element of } \Delta(B)\} \subseteq \{\delta\} \subseteq \Delta(g(\delta))$, then $B' = \{g \in G : \Delta(g) \subseteq \Delta(B)\}$ is maximal below.*

PROOF. An element of the group generated by $\{g(\delta) : \delta \in \Delta \setminus \Delta(B)\}$ has the form $x = \sum_{i=1}^k n_i g(\delta_i)$, where $\delta_i \in \Delta \setminus \Delta(B)$ and $n_i \in \mathbf{Z}$. Suppose x has value γ related to an element $\lambda \in \Delta(B)$. By relabelling, assume $g(\delta_i) \notin G_\gamma$ for $1 \leq i \leq m$ and $g(\delta_i) \in G_\gamma$ for $m+1 \leq i \leq k$, so that $G_\gamma + x = G_\gamma + \sum_{i=1}^m n_i g(\delta_i)$. For each i , $1 \leq i \leq m$, $g(\delta_i)$ has a value ρ_i containing γ , and since ρ_i is also related to λ , $\rho_i = \delta_i$ in each case. Therefore γ is the greatest element in the chain $\{\delta_i : 1 \leq i \leq m\}$, and in particular $\gamma \in \Delta \setminus \Delta(B)$. Since $\Delta(A) = \Delta \setminus \Delta(B)$ it follows that B' is maximal below A . $\square$

PROPOSITION 2.14. *If g is a finite-valued element of an abelian l -group G , then $G(g)$ is maximal below.*

PROOF. Write g as $g_1 + g_2 + \cdots + g_n$, where the g_i 's are disjoint and special with respective values $\gamma_1, \gamma_2, \dots, \gamma_n$. Given $\lambda \in \Gamma \setminus \Gamma(g)$ let $I = \{i : \lambda \not\asymp \gamma_i\}$, and for each $i \in I$ choose $0 < x_i \in (G^\lambda \setminus G_\lambda) \cap G_{\gamma_i}$. Let $g(\lambda) = \bigwedge \{(x_i - |g_i|)^+ : i \in I\}$ if $I \neq \emptyset$, otherwise let $g(\lambda)$ be any positive element of $G^\lambda \setminus G_\lambda$. Now take $B = G(g)$ in Lemma 2.13, and note that $B' = B$. $\square$

The descending chain condition (every descending chain is finite) is abbreviated DCC.

PROPOSITION 2.15. *Suppose $G \in \mathbf{S}$, and suppose G is either abelian or that the minimal plenary set Δ satisfies the DCC. Then every closed convex l -subgroup is maximal below.*

PROOF. For $B \in \mathcal{X}(G)$ we claim $B = \bigcap \{G_\delta : \delta \in \Delta, B \subseteq G_\delta\}$. For if $0 < x \notin B$ then $x = \vee T$ for some disjoint set T of special elements. Since B is closed it cannot contain T , say $0 < t \in T \setminus B$ with value $\delta \in \Delta$. Then $B \subseteq G_\delta$, since otherwise $t \in B$, and this proves the claim. To prove the proposition take $\Gamma = \{\delta \in \Delta : B \subseteq G_\delta\}$ in Proposition 2.11. $\square$

COROLLARY 2.16. *If G is finite-valued and its plenary set of regular subgroups enjoys the DCC, then every convex l -subgroup of G is maximal below.*

The example following Theorem 2.19 shows that the DCC cannot be omitted from the hypotheses of the previous corollary. In the following proposition, $\bar{B} = \bigcap \{C \in \mathcal{X}(G) : B \subseteq C\}$ denotes the *closure* of B . It is known that $\bar{B} = \{g \in G : |g| = \vee \{b \in B^+ : b \leq |g|\}\}$.

LEMMA 2.17. *Suppose G is normal-valued and completely distributive with minimal plenary set Δ . Then $\bar{B} = \bigcap \{G_\delta : B \subseteq G_\delta \text{ and } \delta \in \Delta\} = \{g \in G : \Delta(g) \subseteq \Delta(B)\}$.*

PROOF. The first equality follows from the fact that any closed convex l -subgroup of a normal-valued completely distributive l -group is an intersection of closed prime subgroups [14]. The second equality can be routinely verified. $\square$

PROPOSITION 2.18. *Suppose G is an abelian l -group in $\mathbf{S}$. If A is a subgroup maximal above $B \in \mathcal{C}(G)$, then $\bar{B}$ is maximal below A .*

PROOF. $\bar{B}$ lies below A by Theorem 1.6. Suppose $\bar{B} \subsetneq H \in \mathcal{C}(G)$ and that H lies below A . Fix $0 < h \in H \setminus \bar{B}$, and write $h = \vee T$ for some disjoint set T of special elements. Now $T \not\subseteq \bar{B}$, for otherwise the closure of $\bar{B}$ would imply $h \in \bar{B}$. Fix $t \in T \setminus \bar{B}$ with value γ . We claim that the subgroup A' generated by $A \cup \{t\}$ lies above B . Consider a value δ of a typical element $a + nt \in A'$, where $a \in A$ and n is an integer. If $t \in G_\delta$, then δ is a value of a and so $G_\delta \supseteq B$. If $t \notin G_\delta$, then $\delta \leq \gamma$, and by Proposition 1.2 $\gamma < \alpha$ for some value α of a . Therefore $\delta = \alpha$, and again $G_\delta \supseteq B$. This proves the claim. But then the maximality of A forces $t \in A$, contradicting the assumption that H lies below A . We can only conclude that $\bar{B} = H$. $\square$

The following theorem sums up the preceding results.

THEOREM 2.19. *Suppose G is normal-valued and completely distributive with minimal plenary set Δ . If G is either abelian or Δ has the DCC, then each closed convex l -subgroup of G is maximal below if and only if G is in $\mathbf{S}$. If G is abelian and in $\mathbf{S}$ and if A is a subgroup maximal above $B \in \mathcal{C}(G)$, then $\bar{B} = \{g \in G : \Delta(g) \subseteq \Delta(B)\} = \bigcap \{G_\delta : \delta \in \Delta \text{ and } B \subseteq G_\delta\}$ is maximal below A .*

We close this section with an example which shows that the hypothesis that G be either abelian or have a root system with the DCC cannot be dropped from the last several results. This example also shows that a normal convex l -subgroup of a totally ordered group need not be maximal below. Let G be the free group on a countably infinite set of generators and let $B = [G, G]$ be its commutator subgroup. Then G/B is group isomorphic to $\sum_{i=1}^{\infty} \mathbb{Z}_i$ and G can be totally ordered so that B is convex and G/B is o -isomorphic to $\Sigma(\omega^*, \mathbb{Z})$, where ω^* is the set of integers under reverse order. The point here is that G/B has no smallest nontrivial convex l -subgroup.

Now suppose B is maximal below A and pick $u, v \in A$. Then $u + v - u - v \in A \cap B = 0$, meaning A is abelian. Since A is also free, $A = \langle a \rangle$. If G_a is the value of a in G we get $B \subsetneq G_a$ below A . Thus B is not maximal below. Observe that this example shows that the set of maximal below subgroups of G is not closed under intersection.

The construction above can be modified to produce a totally ordered group H whose every convex l -subgroup is normal, but whose only maximal below subgroups are the regular ones. Let H be the free group on countably many generators as before, and let $H = H^{(0)} \geq H^{(1)} \geq \dots$ be its descending central series. Then H can be totally ordered so that each $H^{(n)}$ is convex and so that each quotient $H^{(n)}/H^{(n+1)}$

is order isomorphic to $\Sigma(\omega^*, \mathbf{Z})$. An argument similar to the one above shows that H has the stated properties.

3. The construction of the splitting extension. Suppose l -groups A and B are given along with a group homomorphism σ from A into the group of l -automorphisms of B . Can the resulting group splitting extension of B by A , denoted $A \times B$, be endowed with a lattice order extending those on A and B and with respect to which A lies above B ? We prove that the answer to this question is affirmative, and that every such order is uniquely determined by a lattice homomorphism $\pi: \mathcal{C}_\ell(A) \rightarrow \mathcal{S}(B)$ such that $\pi(0) = 0$. The construction of $G = A \times B$ from A , B , σ , and π is shown to be sufficiently general to encompass cardinal product, lex extension, and l -permutation wreath product.

LEMMA 3.1. *Suppose a lies above b in G . Then $(a + b)^+ = a^+ + a^+ \wedge b^+ - a^+ \wedge b^- + (b\mu a)^+$ and $(a + b)^- = a^- \wedge b^- - a^- \wedge b^+ + a^- + (b\mu a)^-$. Thus $a + b \geq 0$ if and only if $a \geq 0$ and $b\mu a \geq 0$.*

PROOF. By substituting the expression for $b\nu a + b\mu a$ in Proposition 1.1 for b in $a + b$ and grouping, one obtains $a + b = c - d + b\mu a$, where $c = a^+ + a^+ \wedge b^+ - a^+ \wedge b^-$ and $d = a^- \wedge b^- - a^- \wedge b^+ + a^+$. Now $a^+ = 0$ implies $c = 0$ and $a^+ > 0$ implies $c = a^+ + a^+ \wedge b^+ - a^+ \wedge b^- \geq a^+ - a^+ \wedge b^- > 0$ because b^- lies below a^+ . Likewise $a^- = 0$ implies $d = 0$ and $a^- > 0$ implies $d > 0$. Since c and d are disjoint, we get $(a + b)^+ = c + (b\mu a)^+$ and $(a + b)^- = d + (b\mu a)^-$. Clearly $(a + b)^- = 0$ if and only if $a^- = (b\mu a)^- = 0$. $\square$

THEOREM 3.2. *Suppose A and B are l -subgroups of G such that A is above B and let H be the subgroup generated by A and B . If B is normal in H , then H is a splitting extension of B by A , and the following are equivalent.*

- (a) H is an l -subgroup of G .
- (b) $a \wedge b \in B$ for all $a \in A^+$ and $b \in B^+$.
- (c) $B = B \cap H(a) \boxplus B \cap a^\perp$ for all $a \in A$.

Furthermore, B is maximal below A and A is maximal above B in H . In particular, if G is normal-valued and if B is maximal below A in G , then B is normal in H and the conditions above hold.

PROOF. Suppose B is normal in H . Then H is a splitting extension of B by A because $B \cap A = 0$. Now assume (a) and consider $a \in A^+$ and $b \in B^+$. Then $a \wedge b = a_1 + b_1$ for some $a_1 \in A$ and $b_1 \in B$. We claim that $a_1 = 0$, for if not there is some prime P omitting a_1 . P also omits $a_1 + b_1$, for otherwise we would have $P < P + a_1 = P - b_1 < P - 2b_1$ or perhaps $P < P - a$, meaning $Q < Q - a_1 = Q + b_1 < Q + 2b_1$ where $Q = -a_1 + P + a_1$, in either case a violation of Proposition 1.1(c). So $P < P + a_1 + b_1 = P + a \wedge b = P + b$, hence $P < P + a_1 = P + (b - b_1) < P + 2(b - b_1)$, again contradicting Proposition 1.1(c). This proves that $a \wedge b = b_1 \in B$. If (b) holds then $b\nu a \in B$ by Proposition 1.1 and so (c) holds. If $a \in A^+$ and $b \in B^+$ and (c) holds, then $b = b_1 + b_2$ where $b_1 \in B \cap H(a)$ and $b_2 \in B \cap a^\perp$, but by Proposition 1.1, $b_1 = b\nu a = b \wedge a \in B$, hence (b) holds.

And (b) implies (a) by Lemma 3.1. Finally, observe that A is maximal above B in H in the very strong sense that any subgroup of H properly containing A has nontrivial intersection with B , and B is likewise maximal below A in H .

If B is maximal below A in the normal-valued l -group G and if $a \in A^+$ and $b \in B^+$, then $b \wedge (a + a_1 - a) \ll a + a_1 - a$ implies $(-a + b + a) \wedge a_1 \ll a_1$. This shows that B is normal in H , and the convexity of B in G implies that conditions (a), (b), and (c) hold in this case. $\square$

Suppose A , B , and H satisfy the conditions of Theorem 3.2. With each $a \in A$ associate the l -automorphism $\sigma(a)$ of B defined by $b\sigma(a) = -a + b + a$. Then the addition in $H = A + B$ is given by

$$(a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1\sigma(a_2) + b_2).$$

With each $a \in A$ also associate $[a] = B \cap H(a) = \{b \in B : |b| \ll |a|\} \in \mathcal{S}(B)$. Denote the complementary summand $[a]' = B \cap a^\perp$. Let $\pi: \mathcal{C}\ell(A) \rightarrow \mathcal{S}(B)$ be defined by $\pi(A(a)) = [a]$. Note that π is well defined since $A(a_1) = A(a_2)$ implies $H(a_1) = H(a_2)$. Associated with the summand $[a]$ are the projection l -epimorphisms $\nu_a: B \rightarrow [a]$ and $\mu_a: B \rightarrow [a]'$ satisfying $b = b\nu_a + b\mu_a$ for all $b \in B$. The following proposition, whose proof is straightforward, sets down the relevant attributes of σ and π .

PROPOSITION 3.3. *Suppose A and B are l -subgroups of G such that B lies below A , B is normal in the group H generated by $A \cup B$, and H is an l -subgroup of G . Then σ is a group homomorphism from A into the group of l -automorphisms of B , π is a lattice homomorphism from $\mathcal{C}\ell(A)$ into $\mathcal{S}(B)$, and the following properties hold.*

- (a) $\pi(0) = 0$.
- (b) $\sigma(a)$ is the identity on $[a]'$ for all $a \in A$.
- (c) $[a_1]\sigma(a_2) = [-a_2 + a_1 + a_2]$ for all $a_1, a_2 \in A$.

The following theorem, which is a converse of Proposition 3.3, is one of the central results of this paper. It asserts that the properties of σ and π pointed out above are the important ones, for they are sufficient to construct H from A , B , σ , and π .

THEOREM 3.4. *Suppose A and B are l -groups, σ is a group homomorphism from A into the group of l -automorphisms of B , and π is a lattice homomorphism from $\mathcal{C}\ell(A)$ into $\mathcal{S}(B)$, where $\pi[A(a)]$ is denoted by $[a]$. Suppose further that properties (a), (b), and (c) of Proposition 3.3 hold. Let $G = A \times B$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1\sigma(a_2) + b_2)$, and define $(a, b) \geq 0$ if and only if $a \geq 0$ and $b\mu_a \geq 0$. Then G is an l -group in which A and B are embedded as l -subgroups such that A is maximal above B and B is maximal below A . Moreover $b\sigma(a) = -a + b + a$ and $\pi[A(a)] = [a] = G(a) \cap B$ for all $a \in A$ and $b \in B$.*

References to A and B as subgroups of G in Theorem 3.4 presume the usual identification of each $a \in A$ with $(a, 0) \in G$ and each $b \in B$ with $(0, b) \in G$. We shall prove this theorem by a sequence of lemmas, each of which assumes the notation and terminology of Theorem 3.4. The first lemma in the sequence proves

the antisymmetry of the order; it makes use of the fact that $[0]' = B$ because $[0] = \pi(0) = 0$, hence $b\nu 0 = 0$ and $b\mu 0 = b$ for all $b \in B$. Note also that $-(a, b) = (-a, -b\sigma(-a))$ in G .

LEMMA 3.5. *If $(a, b) \geq 0$ and $-(a, b) \geq 0$, then $a = 0$ and $b = 0$.*

PROOF. $(a, b) \geq 0$ implies $a \geq 0$, and $-(a, b) = (-a, -b\sigma(-a)) \geq 0$ implies $a \leq 0$, so $a = 0$. And $(a, b) \geq 0$ also implies $b\mu a = b\mu 0 = b = 0$. $\square$

The next lemma proves the closure of the positive cone under addition. It makes use of the fact that for $a_1, a_2 \in A$ and $b \in B$, $b\nu a_1\sigma(a_2)$ is a component of $b\sigma(a_2)$ because $b\nu a_1$ is a component of b . Since $[a_1]\sigma(a_2) = [-a_2 + a_1 + a_2]$, we see $b\nu a_1\sigma(a_2) = b\sigma(a_2)\nu(-a_2 + a_1 + a_2)$ and $b\mu a_1\sigma(a_2) = b\sigma(a_2)\mu(-a_2 + a_1 + a_2)$. In addition if $A(a_1) \subseteq A(a_2)$, then $[a_1] = \pi(A(a_1)) \subseteq \pi(A(a_2)) = [a_2]$. So under these circumstances $b\nu a_2\nu a_1 = b\nu a_1\nu a_2 = b\nu a_1$ and $b\mu a_2\mu a_1 = b\mu a_1\mu a_2 = b\mu a_2$, hence $b\nu a_2 \geq 0$ implies $b\nu a_1 = 0$ and $b\mu a_1 = 0$ implies $b\mu a_2 = 0$.

LEMMA 3.6. *If $(a_1, b_1), (a_2, b_2) \geq 0$, then $(a_1, b_1) + (a_2, b_2) \geq 0$.*

PROOF. Clearly $a_1 + a_2 \geq 0$ because $a_1, a_2 \geq 0$, and

$$\begin{aligned} (b_1\sigma(a_2) + b_2)\mu(a_1 + a_2) &= b_1\sigma(a_2)\mu(a_1 + a_2) + b_2\mu(a_1 + a_2) \\ &= b_1\mu(a_2 + a_1)\sigma(a_2) + b_2\mu(a_1 + a_2) \\ &\geq 0 \end{aligned}$$

because $b_1\mu a_1, b_2\mu a_2 \geq 0$ imply $b_1\mu(a_1 + a_2)$ and $b_2\mu(a_1 + a_2) \geq 0$. $\square$

The next lemma establishes the normality of the positive cone.

LEMMA 3.7. *If $(a_2, b_2) \geq 0$, then $-(a_1, b_1) + (a_2, b_2) + (a_1, b_1) \geq 0$ for any $(a_1, b_1) \in G$.*

PROOF.

$$-(a_1, b_1) + (a_2, b_2) + (a_1, b_1) = (a, -b_1\sigma(a) + b_2\sigma(a_1) + b_1),$$

where $a = -a_1 + a_2 + a_1 \geq 0$, and

$$\begin{aligned} -b_1\sigma(a)\mu a + b_2\sigma(a_1)\mu a + b_1\mu a &= -b_1\mu a\sigma(a) + b_2\mu a_2 + b_1\mu a \\ &= -b_1\mu a + b_2\mu a_2 + b_1\mu a \geq 0, \end{aligned}$$

where the last equality holds because $\sigma(a)$ is the identity on $[a]'$. $\square$

Thus far we have shown G to be a partially ordered group. We shall show outright that G is a lattice in Lemma 3.9, which is made easier by the following result.

LEMMA 3.8. *$(a_1, b_1) \geq (a_2, b_2)$ if and only if $a_1 \geq a_2$ and $b_1\mu(-a_2 + a_1) \geq b_2\mu(-a_2 + a_1)$.*

PROOF.

$$(a_1, b_1) - (a_2, b_2) = (a_1 - a_2, (b_1 - b_2)\sigma(-a_2)) \geq 0$$

if and only if $a_1 - a_2 \geq 0$ and

$$(b_1 - b_2)\sigma(-a_2)\mu(a_1 - a_2) = (b_1 - b_2)\mu(-a_2 + a_1)\sigma(-a_2) \geq 0.$$

But this is equivalent to $a_1 \geq a_2$ and $(b_1 - b_2)\mu(-a_2 + a_1) \geq 0$, which is the condition above. $\square$

LEMMA 3.9.

$$\begin{aligned} (a_1, b_1) \vee (a_2, b_2) &= (a_1 \vee a_2, b_1 \vee (-a_2 + a_1)^+ + b_2 \vee (-a_1 + a_2)^+ + (b_1 \vee b_2)\mu(-a_2 + a_1)), \\ (a_1, b_1) \wedge (a_2, b_2) &= (a_1 \wedge a_2, b_2 \vee (-a_2 + a_1)^+ + b_1 \vee (-a_1 + a_2)^+ + (b_1 \wedge b_2)\mu(-a_2 + a_1)). \end{aligned}$$

PROOF. Abbreviate by $(a_1 \vee a_2, b)$ the expression for $(a_1, b_1) \vee (a_2, b_2)$ above. We claim $(a_1 \vee a_2, b) \geq (a_1, b_1)$ for $a_1 \vee a_2 \geq a_1$ and

$$\begin{aligned} b\mu(-a_1 + a_1 \vee a_2) &= b\mu(-a_1 + a_2)^+ = b_1 \vee (-a_2 + a_1)^+ + (b_1 \vee b_2)\mu(-a_2 + a_1) \\ &\geq b_1 \vee (-a_2 + a_1)^+ + b_1 \mu(-a_2 + a_1). \end{aligned}$$

Now $A((-a_2 + a_1)^+) \vee A((-a_1 + a_2)^+) = A(-a_1 + a_2)$ and $A((-a_2 + a_1)^+) \cap A((-a_1 + a_2)^+) = 0$, and since π is a lattice homomorphism we have $[(-a_2 + a_1)^+] \boxplus [(-a_1 + a_2)^+] = [-a_1 + a_2]$ and $[(-a_2 + a_1)^+] \cap [(-a_1 + a_2)^+] = 0$, from which follows $[(-a_2 + a_1)^+] \boxplus [-a_1 + a_2]' = [(-a_1 + a_2)^+]'$. Therefore $b_1 \vee (-a_2 + a_1)^+ + b_1 \mu(-a_2 + a_1) = b_1 \mu(-a_1 + a_2)^+$, which completes the verification of the claim. A similar argument shows $(a_1 \vee a_2, b) \geq (a_2, b_2)$.

Suppose $(a_3, b_3) \geq (a_1, b_1), (a_2, b_2)$. Clearly $a_3 \geq a_1 \vee a_2$, and we know $b_3 \mu(-a_1 + a_3) \geq b_1 \mu(-a_1 + a_3)$ and $b_3 \mu(-a_2 + a_3) \geq b_2 \mu(-a_2 + a_3)$. We must show $b_3 \mu a \geq b \mu a$, where $a = -a_1 \vee a_2 + a_3 = (-a_1 + a_3) \wedge (-a_2 + a_3)$. Since $[(-a_2 + a_1)^+] \boxplus [(-a_1 + a_2)^+] \boxplus [-a_2 + a_1]' = B$, this amounts to showing that each term of $b_3 \mu a = b_3 \vee (-a_2 + a_1)^+ \mu a + b_3 \vee (-a_1 + a_2)^+ \mu a + b_3 \mu(-a_2 + a_1) \mu a$ exceeds (≥ 0) the corresponding term of $b \mu a$. We begin by showing

$$b_3 \vee (-a_2 + a_1)^+ \mu a \geq b_1 \vee (-a_2 + a_1)^+ \mu a.$$

Now $0 \leq (-a_2 + a_1)^+ = -a_2 + (a_1 \vee a_2) \leq -a_2 + a_3$, so $[(-a_2 + a_1)^+] \subseteq [-a_2 + a_3]$ or $[(-a_2 + a_1)^+] \cap [-a_2 + a_3]' = 0$, which together with $[-a_1 + a_3]' \vee [-a_2 + a_3]' = [a]'$ yields $[(-a_2 + a_1)^+] \cap [a]' \subseteq [-a_1 + a_3]'$. Because

$$b_3 \mu(-a_1 + a_3) \geq b_1 \mu(-a_1 + a_3)$$

it follows that $b_3 \vee (-a_2 + a_1)^+ \mu a \geq b_1 \vee (-a_2 + a_1)^+ \mu a$, and a similar argument shows $b_3 \vee (-a_1 + a_2)^+ \mu a \geq b_2 \vee (-a_1 + a_2)^+ \mu a$. It remains to show

$$b_3 \mu(-a_2 + a_1) \mu a \geq (b_1 \vee b_2) \mu(-a_2 + a_1) \mu a.$$

Now $A(-a_1 + a_3) \subseteq A(-a_1 + a_2) \vee A(-a_2 + a_3)$ and $A(-a_1 + a_3) \subseteq A(-a_1 + a_2) \vee A(-a_1 + a_3)$ imply $A(-a_1 + a_3) \subseteq A(-a_1 + a_2) \vee A(a)$, and similarly

$$A(-a_2 + a_3) \subseteq A(-a_1 + a_2) \vee A(a).$$

Therefore

$$[-a_1 + a_3] \vee [-a_2 + a_3] \subseteq [-a_1 + a_2] \vee [a],$$

or

$$[-a_1 + a_2]' \cap [a]' \subseteq [-a_1 + a_3]' \cap [-a_2 + a_3]'$$

Then $b_3\mu(-a_1 + a_3) \geq b_1\mu(-a_1 + a_3)$ and $b_3\mu(-a_2 + a_3) \geq b_2\mu(-a_2 + a_3)$ imply

$$b_3\mu(-a_2 + a_1)\mu a \geq (b_1 \vee b_2)\mu(-a_2 + a_1)\mu a. \quad \square$$

LEMMA 3.10. *B lies below A in G.*

PROOF. Given $0 < (0, b) \in B$ and $0 < (a, 0) \in A$, $(a, 0) \wedge (0, b) = (0, b\nu a)$, hence $n(a \wedge b) = (0, nb\nu a) \leq (a, 0)$ because $a \geq 0$ and $(nb\nu a)\mu a = 0 \leq 0\mu a$. $\square$

The reader may readily verify that $b\sigma(a) = -a + b + a$ and that $[a] = G(a) \cap B$ for each $a \in A$ and $b \in B$. Thus Lemma 3.10 completes the proof of Theorem 3.4. We shall refer to G as the splitting extension of B by A determined by σ and π . Because A is obviously maximal among subgroups of G lying above B and likewise B maximal below A , we know that A and B are closed l -subgroups of G , that B is convex, and that A is saturated.

The next few results concern $\mathcal{C}(G)$ as determined by $\mathcal{C}(A)$ and $\mathcal{C}(B)$. We continue to assume the terminology of Theorem 3.4.

PROPOSITION 3.11. *Let C and D be l-subgroups of A and B, respectively. Then $C \times D$ is an l-subgroup of G if and only if $D\sigma(c) = D$ and $D = (D \cap [c]) \boxplus (D \cap [c]')$ for all $c \in C$. In this case $C \times D$ is the splitting extension of D by C determined by (the restrictions of) σ and π .*

PROOF. Clearly $C \times D$ is closed under addition if and only if $D\sigma(c) = D$ for all $c \in C$. Given $0 < c \in C$ and $0 < d \in D$, $c \wedge d = (0, d\nu c)$. Thus in order for $C \times D$ to be a sublattice it is necessary that $D = (D \cap [c]) \boxplus (D \cap [c]')$ for all $c \in C$; Lemma 3.9 shows that this condition is sufficient as well. $\square$

Given an l -subgroup C of A let $B_C = \bigcup\{[c]: c \in C\}$. Notice that $B_C \in \mathcal{C}(B)$ because $\{[c]: c \in C\}$ is a directed set. Given the l -subgroup D of B let $A_D = \{a \in A: [a] \subseteq D\} \in \mathcal{C}(A)$.

PROPOSITION 3.12. *The following are equivalent for l-subgroups $C \leq A$ and $D \leq B$.*

- (a) $C \times D \in \mathcal{C}(G)$.
- (b) $C \in \mathcal{C}(A)$ and $B_C \subseteq D \in \mathcal{C}(B)$.
- (c) $D \in \mathcal{C}(B)$ and $A_D \supseteq C \in \mathcal{C}(A)$.

Furthermore, every $E \in \mathcal{C}(G)$ is of the form $E = C \times D$ for unique l -subgroups $C = E \cap A$ and $D = E \cap B$ satisfying the conditions above.

PROOF. Suppose $C \times D \in \mathcal{C}(G)$. Then $0 < x < c \in C$ implies $0 < (x, 0) < (c, 0) \in C \times D$, hence $(x, 0) \in C \times D$ or $x \in C \in \mathcal{C}(A)$. Likewise $D \in \mathcal{C}(B)$. For any $c \in C$ and $0 < b \in [c]$, $(0, 0) < (0, b) < ([c], 0) \in C \times D$, hence $(0, b) \in C \times D$ by convexity, or $b \in D$. This proves that (a) implies (b); now assume (b) to prove (a). For $c \in C$, $D = [c] \boxplus (D \cap [c]')$ and $D\sigma(c) = [c]\sigma(c) \boxplus (D \cap [c]') = D$, so $C \times D$ is an l -subgroup by Proposition 3.11. Consider $0 < (x, y) < (c, d) \in C \times D$. We know from Lemma 3.8 that $0 \leq x \leq c$, $y\mu x \geq 0$, and $d\mu(-x + c) \geq y\mu(-x + c)$. It follows that $x \in C$ and that $d\mu c \geq y\mu c \geq 0$, and since $d\nu c \in [c] \subseteq D$, $d\mu c \in D$ and $y\mu c \in D$ by convexity. That is, $y = y\nu c + y\mu c \in D$, proving (a). (b) and (c) are clearly equivalent.

Suppose $E \in \mathcal{C}(G)$ and let $C = E \cap A$ and $D = E \cap B$. Observe that $0 < (x, y) \in G$ implies $0 \leq (x, 0) \leq 2(x, y) = (2x, y\sigma(x) + y)$ because $2x \geq x$ and

$$(y\sigma(x) + y)\mu(-x + 2x) = y\mu x \geq 0 = 0\mu(-x + 2x).$$

Therefore $0 < (c, d) \in E$ implies $(c, 0) \in E$ by convexity. If (c, d) is an arbitrary element of E , then $(c, d)^+ = (c^+, x) \in E$ and $(c, d)^- = (c^-, y) \in E$ for some $x, y \in B$, hence $(c^+, 0), (c^-, 0) \in E$ as above, and $(c, 0) = (c^+, 0) - (c^-, 0) \in E$ and $-(c, 0) + (c, d) = (0, d) \in E$, proving $E = C \times D$. $\square$

PROPOSITION 3.13. *The map $C \rightarrow C \times B$ is a lattice isomorphism from $\mathcal{C}(A)$ onto $\{E \in \mathcal{C}(G): B \subseteq E\}$. Thus C is a prime subgroup of A if and only if $C \times B$ is a prime subgroup of G . In particular, C is a value of $a \in A$ if and only if $C \times B$ is a value of $(a, b) \in G$ for any $b \in G$. In this case C is normal in its cover C^* if and only if $C \times B$ is normal in its cover $C^* \times B$.*

PROOF. The first assertion follows from Proposition 3.12, the second follows from the fact that a convex l -subgroup is prime if and only if the convex l -subgroups containing it form a chain, and the third and fourth assertions can be verified in straightforward fashion. $\square$

The following lemma is well known and quite general. We prove it here for completeness.

LEMMA 3.14. *Suppose B and D are convex l -subgroups of an l -group G and that $D \leq B$. Then $X = \{g \in G: |g| \wedge b \in D \text{ for all } b \in B^+\}$ is the maximum element of $\{E \in \mathcal{C}(G): E \cap B \subseteq D\}$. X is prime in G if and only if D is prime in B .*

PROOF. If $0 < x_1, x_2 \in X$ and $0 < b \in B$, then

$$\begin{aligned} 0 \leq b \wedge (x_1 + x_2) &\leq (2b) \wedge (b + x_2) \wedge (x_1 + b) \wedge (x_1 + x_2) \\ &= b \wedge x_1 + b \wedge x_2 \in D. \end{aligned}$$

It follows that X is a convex l -subgroup, and hence the maximum element of $\{E \in \mathcal{C}(G): E \cap B \subseteq D\}$. Suppose now that D is prime in B , and that $u, v \in G$ with $u \wedge v = 0$. If $u \notin X$, then there is some $b_0 \in B^+$ such that $u \wedge b_0 \notin D$. Thus for any $b \in B^+$, $u \wedge (b \vee b_0) \geq u \wedge b_0 \notin D$ and $[u \wedge (b \vee b_0)] \wedge [v \wedge (b \vee b_0)] = (u \wedge v) \wedge (b \vee b_0) = 0$ imply $v \wedge (b \vee b_0) \in D$, meaning $v \wedge b \in D$ or $v \in X$. $\square$

Given $B \in \mathcal{C}(G)$ let $\mathcal{C}_B(G) = \{X \in \mathcal{C}(G): X \subsetneq Y \in \mathcal{C}(G) \text{ implies } X \cap B \subsetneq Y \cap B\}$.

PROPOSITION 3.15. *The map $D \rightarrow A_D \times D$ is a one-one order preserving map from $\mathcal{C}(B)$ onto $\mathcal{C}_B(G)$. Thus D is a prime subgroup of B if and only if $A_D \times D$ is a prime subgroup of G , and every prime subgroup E of G such that $B \not\subseteq E$ has the form $E = A_D \times D$ for a (unique) proper prime subgroup D of B . For such a D , $A_D \times B$ is a value of (a, b) in G if and only if D is a value of $b\mu a$ in B .*

PROOF. The fact that $A_D \times D$ is the maximum element of $\{E \in \mathcal{C}(G): E \cap B \subseteq D\}$ follows from Proposition 3.12, and the second assertion results from Lemma 3.14. Suppose now that E is a prime subgroup of G for which $B \not\subseteq E$. By

Proposition 3.12 $E = C \times D$ for D a proper prime subgroup of B and $C \in \mathcal{C}(A)$ such that $C \leq A_D$. If $0 < a \in A_D \setminus C$, then $a \notin E$, so $[a]' \subseteq a^\perp \subseteq E$, hence $[a]' \subseteq E \cap B = D$. But together with the fact that $[a] \subseteq D$ since $a \in A_D$, this violates the assumption that $B \neq D$. We conclude that $C = A_D$ and $E = A_D \times D$.

Suppose that D is a proper prime subgroup of B , and fix $a \in A$ and $b \in B$. If D is a value of $b\mu a$ in B with cover D^* , then $(a, b) \notin A_D \times D$, and by the preceding paragraph every $E \in \mathcal{C}(G)$ satisfying $A_D \times D \subsetneq E$ also satisfies $D^* \subseteq E$, hence $A_D \times D$ is a value of $(0, b\mu a)$ in G . Now $(0, b\mu a)$ is a component of (a, b) (because $(a, b) = (a, b\nu a) + (0, b\mu a)$), so $A_D \times D$ is a value of (a, b) as well. Now suppose $A_D \times D$ is a value of (a, b) . It follows that $a \in A_D$, for otherwise $A_D \times B$ is a prime subgroup properly containing $A_D \times D$ which nevertheless does not contain (a, b) . By convexity, $(a, b\nu a) \in A_D \times D$; hence $(0, b\mu a) \notin D$, and if $D \subseteq E \in \mathcal{C}(B)$, then $(0, b\mu a) \in A_E \times E \supsetneq A_D \times D$, meaning that D is a value of $b\mu a$. $\square$

PROPOSITION 3.16. *If D and E are prime subgroups of B such that $D \vee E \neq B$ in $\mathcal{C}(B)$, then $A_D = A_E$.*

PROOF. It is sufficient to establish the case in which $D \subsetneq E \subsetneq B$. Now $A_E \times E$ and $A_D \times B$ are members of $\mathcal{C}(G)$ containing the prime subgroup $A_D \times D$, and so are related. Since $B \not\subseteq E$ we must have $A_E \times E \subseteq A_D \times B$, or $A_E \subseteq A_D$. $\square$

Proposition 3.16 completes the description of $\Gamma(G)$ from $\Gamma(A)$ and $\Gamma(B)$. Call a pair of prime subgroups D and E of B *distantly related* if $D \vee E \neq B$ in $\mathcal{C}(B)$ and let $\langle\langle E \rangle\rangle = \{D: E \text{ and } D \text{ are distantly related prime subgroups of } B\}$. Start with $\Gamma(A)$, and attach each equivalence class $\langle\langle E \rangle\rangle$ below $A_E \in \Gamma(A)$. (Actually, we are cheating a bit here. In this paragraph only are we considering $\Gamma(A)$ to be the partially ordered set of all (not only regular) primes of A .) The resulting partially ordered set is $\Gamma(G)$.

COROLLARY 3.17. *G is normal-valued if and only if both A and B are normal-valued.*

PROOF. Suppose that A and B are normal valued, that P is a regular prime of G with cover P^* , and that $(a, b) \in P^* \setminus P$. If $B \subseteq P$, then $P = C \times B$ for some value C of a in A having cover C^* . For any $(c, b_1) \in P$,

$$-(a, b) + (c, b_1) + (a, b) = (-a + c + a, b_2) \in C^* \times B = P^*.$$

If $B \not\subseteq P$, then $P = A_D \times D$ for some value D of $b\mu a$ in B having value D^* . Since $b\mu a$ is a component of (a, b) , $-(a, b) + P + (a, b) = -b\mu a + P + b\mu a$. So consider arbitrary $(c, d) \in P = A_D \times D$. Then

$$\begin{aligned} -(0, b\mu a) + (c, d) + (0, b\mu a) &= (c, -b\mu a\sigma(c) + d + b\mu a) \\ &= (c, -b\mu a\nu c\sigma(c) + (-b\mu a\mu c + d + b\mu a\mu c) + b\mu a\nu c), \end{aligned}$$

and $b\mu a\nu c\sigma(c)$ and $b\mu a\nu c$ lie in D because $[c] \subseteq D$ since $c \in A_D$, while $(-b\mu a\mu c + d + b\mu a\mu c) \in D$ because $b\mu a\mu c \in D^*$ since $b\mu a \in D^*$. Therefore $-(a, b) + P + (a, b) \subseteq P^*$. $\square$

Here is an example which shows that G need not lie in $\mathbf{S}$ when A and B do. Let N be the trivially ordered set $\{0, 1, 2, \dots\}$, let $B = \Sigma(N, \mathbf{Z})$, let $A = \{g \in V(N, \mathbf{Z}): \text{there is some } m \in N \text{ such that } (n)g = (0)g \text{ for all } n \geq m\}$, and for each $a \in A$ let

$\sigma(a)$ be the identity l -automorphism of B and $\pi(a) = [a] = \{b \in B: (n)b \neq 0 \text{ implies } (n)a \neq 0 \text{ for all } n \in N\} \in \mathcal{S}(B)$. Let $a \in A$ be the element such that $(n)a = 1$ for all $n \in N$. Then no disjoint set of special elements in G can have supremum a .

COROLLARY 3.18. *G is finite-valued if and only if both A and B are.*

PROOF. The set of all values of $(a, b) \in G$ is $\{P \times B: P \text{ is a value of } a \text{ in } A\} \cup \{A_D \times D: D \text{ is a value of } b\mu a \text{ in } B\}$, a finite set. $\square$

We close this section by pointing out that the splitting extension developed in this section is general enough to encompass cardinal product, lex extension, and permutation wreath product.

THEOREM 3.19. *Given l -groups A and B define $\pi(A(a)) = [a] = 0 \in \mathcal{S}(B)$ for each $a \in A$ and define $\sigma(a)$ to be the identity automorphism of B . Then the splitting extension of B by A determined by π and σ is (l -isomorphic to) the cardinal product $A \boxplus B$.*

The notion of *lex extension* is developed in [10].

THEOREM 3.20. *Suppose σ is any group homomorphism from the totally ordered group A into the l -automorphisms of the l -group B , and suppose $\pi(A(a)) = [a] = B \in \mathcal{S}(B)$ for all $a \in A$. Then the splitting extension of B by A determined by π and σ is a lex extension of B by A . Conversely, if G is any l -group whose lex kernel B splits, that is for which $G = A + B$ for some subgroup A , then A is totally ordered and there is some homomorphism σ such that G is obtained from A and B as above.*

We employ the permutation wreath product notation of Chapter 5 of [13]. $\Pi_T G = \{f: f: T \rightarrow G\}$ designates the cardinal product of copies of G indexed by T . For $f \in \Pi_T G$ we designate the component of f at index t by f_t .

THEOREM 3.21. *Let (G, Ω) and (H, T) be transitive l -permutation groups and let $B = \Pi_T G$. For $h \in H$ define $[h] = \pi(H(h)) = \{f \in B: \alpha f_t = \alpha \text{ for all } \alpha \in \Omega \text{ and } t \in T \text{ such that } th = t\} \in \mathcal{S}(B)$, and define $f\sigma(h) \in B$ by $\alpha(f\sigma(h))_t = \alpha f_s$, where $s = th^{-1} \in T$. Then the splitting extension of B by H determined by σ and π is l -isomorphic to $G \text{ Wr } H$.*

4. Characterizing the lattice homomorphism $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(B)$. In some important instances it is possible to completely characterize the maps $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(B)$ of Proposition 3.3. The basic insight is lattice theoretical in nature.

PROPOSITION 4.1. *Suppose that D is a distributive lattice and E is a complete Boolean algebra. If for each prime ideal P of D there is an element $e_P \in E$ satisfying property (a) below, then the map $\pi: D \rightarrow E$ defined by $\pi(d) = \bigvee \{e_P: d \notin P\}$ enjoys properties (b), (c), and (d) below. Conversely, suppose $\pi: D \rightarrow E$ satisfies (b), (c), and (d) and define $e_P = (\bigwedge \{\pi(d): d \notin P\}) \wedge (\bigwedge \{\pi(d)^\perp: d \in P\})$ for P a prime ideal of D . Then (a) holds, and $\pi(d) = \bigvee \{e_P: d \notin P\}$ for all $d \in D$.*

- (a) $e_P \wedge e_Q = 0$ for distinct prime ideals P and Q .
- (b) π is a lattice homomorphism.

(c) $\pi(0) = 0$.

(d) *The complete subalgebra $F \subseteq E$ generated by $\pi(D)$ is atomic.*

PROOF. Suppose that for each prime ideal $P \subseteq D$ there is an element $e_P \in E$ satisfying (a), and define $\pi(d) = \bigvee \{e_P : d \notin P\}$. Let $A = \{e_P : P \text{ a prime ideal of } D\} \cup \{\bigwedge \{e_P^\perp : P \text{ a prime ideal of } D\}\}$, a maximal pairwise disjoint subset of E , and let $F = \{\bigvee A_0 : A_0 \subseteq A\}$, the complete subalgebra of E generated by A . Observe that F is isomorphic to the Boolean algebra of subsets of A , and that A is the set of atoms of F . Properties (b) and (c) are immediate consequences of these observations; to establish (d) we must show that $\pi(D)$ generates F . This is done by showing that $e_P = e$ for each prime ideal P of D , where e stands for $(\bigwedge \{\pi(d) : d \notin P\}) \wedge (\bigwedge \{\pi(d)^\perp : d \in P\})$. Now $e_P \leq e$ because e_P is an atom of F , and if $e_P < e$ it could only be because $e_Q \leq e$ for some prime ideal $Q \neq P$. But if $d \in Q \setminus P$, then $e_Q \leq \pi(d)^\perp$ while $e_P \leq \pi(d)$, and similarly for $d \in P \setminus Q$. This contradiction establishes that $e_P \in F$.

Suppose that $\pi: D \rightarrow E$ satisfies (b), (c), and (d) and define

$$e_P = \left(\bigwedge \{ \pi(d) : d \notin P \} \right) \wedge \left(\bigwedge \{ \pi(d)^\perp : d \in P \} \right)$$

for each prime ideal P of D . If P and Q are distinct prime ideals, then there is some $d \in D$ such that $d \in P \setminus Q$, say, and so $e_P \leq \pi(d)^\perp$ and $e_Q \leq \pi(d)$, proving (a). Now consider a particular $d \in D$ and let $e = \bigvee \{e_P : d \notin P\}$. Clearly $\pi(d) \geq e$; to show equality, consider an atom $a \in F$ such that $a \leq \pi(d)$. Let $Q = \{d' \in D : \pi(d') \not\geq a\} = \{d' \in D : \pi(d')^\perp \geq a\}$, a prime ideal of D . It is straightforward to verify that $a \leq e_Q \leq e$, from which follows $\pi(d) = e$. $\square$

The prime subgroups of an l -group A correspond to the prime ideals of the distributive lattice $\mathcal{C}\mathcal{H}(A)$. Therefore the following result is an application of the preceding one. In the following we use $[a]$ to designate $\pi(A(a))$.

THEOREM 4.2. *Let A and B be l -groups. If to each prime subgroup P of A there corresponds a polar $B_P \in \mathcal{P}(B)$ satisfying (a) and (b) below, then the map $\pi: \mathcal{C}\mathcal{H}(A) \rightarrow \mathcal{S}(B)$ defined by $[a] = \bigvee \{B_P : a \notin P\}$ enjoys properties (c), (d), and (e). Conversely, suppose $\pi: \mathcal{C}\mathcal{H}(A) \rightarrow \mathcal{S}(B)$ satisfies (c), (d), and (e), and for each prime subgroup P of A define $B_P = (\bigcap \{[a] : a \notin P\}) \cap (\bigcap \{[a]' : a \in P\})$. Then (a) and (b) hold, and $\pi(a) = \bigvee \{B_P : a \notin P\}$.*

(a) $B_P \cap B_Q = 0$ for distinct prime subgroups P and Q .

(b) $\bigvee \{B_P : a \notin P\} \in \mathcal{S}(B)$ for all $a \in A$. (The supremum here is taken in $\mathcal{P}(B)$.)

(c) π is a lattice homomorphism.

(d) $\pi(0) = 0$.

(e) *The complete subalgebra of $\mathcal{P}(B)$ generated by $\pi(\mathcal{C}\mathcal{H}(A))$ is atomic.*

LEMMA 4.3. *A complete subalgebra S of a complete atomic Boolean algebra T is itself atomic.*

PROOF. Let A be the set of atoms of T , and for each $a \in A$ let $(a) = \bigwedge \{s \in S : a \leq s\} \in S$. Then $B = \{(a) : a \in A\}$ is a maximal pairwise disjoint subset of T , hence $\{\bigvee B_0 : B_0 \subseteq B\}$ is a complete Boolean subalgebra of T contained in S . Since

$s = \bigvee \{a \in A : a \leq s\}$, it follows that $s = \bigvee \{(a) \in B : a \leq s\}$ for all $s \in S$. Therefore S is $\{\bigvee B_0 : B_0 \subseteq B\}$ and B is the set of atoms of S . $\square$

COROLLARY 4.4. *Suppose A and B are l -groups and that B has a basis. Then any lattice homomorphism $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(B)$ such that $\pi(0) = 0$ is of the type described in Theorem 4.2.*

PROOF. B has a basis if and only if $\mathcal{P}(B)$ is atomic. $\square$

COROLLARY 4.5. *Suppose A and B are l -groups such that $\Gamma(A)$ is finite. Then any lattice homomorphism $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(B)$ such that $\pi(0) = 0$ has the form described in Theorem 4.2.*

PROOF. $\Gamma(A)$ is finite if and only if $\mathcal{C}_\#(A)$ is finite, in which case $\pi(\mathcal{C}_\#(A))$ generates a complete subalgebra of $\mathcal{P}(B)$ which is finite and so atomic. $\square$

COROLLARY 4.6. *Suppose B is any l -group and $A = \sum \mathbf{R}_\lambda$ is a small cardinal sum of archimedean o -groups. Then any lattice homomorphism $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(B)$ has the form described in Theorem 4.2.*

PROOF. Each $\mathbf{R}_\gamma$ is an atom of $\mathcal{C}_\#(A)$ in the sense that it is minimal in $\mathcal{C}_\#(A) \setminus \{0\}$, and each element $A(a) \in \mathcal{C}_\#(A)$ is a finite join of such atoms. Therefore $\{\pi(\mathbf{R}_\gamma) : \pi(\mathbf{R}_\gamma) > 0\}$ is the set of atoms of the complete subalgebra of $\mathcal{P}(B)$ generated by $\pi(\mathcal{C}_\#(A))$. $\square$

COROLLARY 4.7. *Suppose A and B are l -groups such that each positive element of B exceeds a positive special element. Then every lattice homomorphism $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(B)$ such that $\pi(0) = 0$ has the form described in Theorem 4.2.*

PROOF. Observe that for any special $b \in B^+$ and $a \in A$, either $b \in [a]$ or $b \in [a]'$. Let $(b) = (\bigcap \{[a] : a \in A \text{ and } b \in [a]\}) \cap (\bigcap \{[a]' : a \in A \text{ and } b \in [a]'\})$, a nonzero element of $\mathcal{P}(B)$, and let $C = \{(b) : 0 < b \in B, b \text{ special}\}$. Since each element of B exceeds a special element, $[a] = \bigvee \{(b) \in C : b \in [a]\}$ for each $a \in A$. Therefore the complete subalgebra of $\mathcal{P}(B)$ generated by $\pi(\mathcal{C}_\#(A))$ is atomic, and C is the set of its atoms. $\square$

If B is finite-valued, then the polars B_p of Theorem 4.2 are themselves cardinal summands. To prove this it is necessary only to observe that $\mathcal{S}(B)$ is a complete subalgebra of $\mathcal{P}(B)$ in this case.

LEMMA 4.8. *If G is finite-valued, then $\mathcal{S}(G)$ is a complete sublattice of $\mathcal{P}(G)$.*

PROOF. Observe that for special $t \in G^+$ and $C \in \mathcal{S}(G)$, $t \in C$ or $t \in C^\perp$. ($t = c + d$ for $c \in C$ and $d \in C^\perp$, and the value P of t contains exactly one of c and d , say d . For every prime subgroup $Q \subseteq P$, $c \notin P$ implies $(t - c) = d \in Q$, or $Q + t = Q + c$. For any prime $Q \not\subseteq P$ we get $t \in Q$, since some multiple of $q \in Q \setminus P$ will exceed t . Therefore $Q \leq Q + c \leq Q + t = Q$ in that case also. This proves $t \in c$.) Suppose now that $\bigvee \mathcal{C} = D$ in $\mathcal{P}(B)$ for some $\mathcal{C} \subseteq \mathcal{S}(B)$, and consider $g \in G^+$. Then $g = g_1 + g_2 + \cdots + g_n$ for disjoint special elements g_i . By relabelling we may assume $g_i \in \bigcup \mathcal{C}$ for $1 \leq i \leq k$ and $g_i \notin \bigcup \mathcal{C}$ for $k < i \leq n$.

Then $(g_1 + g_2 + \cdots + g_k) \in D$ and $(g_{k+1} + g_{k+2} + \cdots + g_n) \in \bigcap \{C^\perp : C \in \mathcal{C}\} = D^\perp$. $\square$

PROPOSITION 4.9. *Suppose A and B are l -groups such that B is finite-valued, and that $\pi: \mathcal{C}_\ell(A) \rightarrow \mathcal{S}(B)$ is a lattice homomorphism with $\pi(0) = 0$. Then for each prime P of A there is a summand $B_P \in \mathcal{S}(B)$ such that $B_P \neq B_Q$ for distinct primes P and Q , and such that $[a] = \sum \{B_P : a \notin P\}$ for each $a \in A$.*

The following theorem summarizes the results of this investigation as applied to lattice orderings of semidirect products of l -groups in which the factors appear above and below one another.

THEOREM 4.10. *Suppose A and B are l -groups, and suppose the group semidirect product $G = A \times B$ is endowed with a lattice order extending the orders on A and B . If A lies above B , then A is maximal above B and hence is saturated, and B is maximal below A and hence convex. Conversely, if A is saturated, B is convex, and if B satisfies (a) or (b) below, then A lies above B . Furthermore, if A lies above B and any one of (c), (d), (e), or (f) holds, then the lattice order on G is determined as per Theorem 4.2.*

- (a) $B = \vee(T \wedge b)$ for all $b \in B^+$, where $T = \{t \in B^+ : t \text{ special}\}$.
- (b) $\mathcal{S}(B)$ is dense in $\mathcal{P}(B)$.
- (c) B has a basis.
- (d) $\Gamma(A)$ is finite.
- (e) A is a cardinal sum of archimedean o -groups.
- (f) Each positive element of B exceeds a positive special element.

PROOF. The subgroup A is maximal with respect to $A \cap B = 0$, and so is maximal above B whenever it lies above B . In this case A must be saturated and B convex by Theorem 1.6. The converse is Proposition 1.12. The last assertion has been established in the preceding corollaries. $\square$

As an example of Theorem 4.10 consider an l -group G which is a splitting extension of the l -ideal B by the saturated l -subgroup A , where B is l -isomorphic to $\Sigma_{i=1}^n \mathbb{Z}_i$ and A is divisible. Since the group of l -automorphisms of B is the group of permutations on n elements and since any homomorphic image of A is divisible, G is the group direct product of A and B . Furthermore, B lies below A , and the order on G is given by assigning a cardinal summand of B to each prime of A as in Theorem 4.2.

5. l -groups in which each maximal below subgroup splits. Let $S_1(S_2)$ be the class of l -groups G such that if B is a maximal below subgroup of G , then $G = A + B$ for some subgroup (every subgroup) A maximal above B . We shall show that these classes are closed with respect to convex l -subgroups, cardinal sums, and cardinal products.

LEMMA 5.1. *Suppose (A, B) is a maximal above and below pair of $G = G_1 \boxplus G_2$. Then $B = (B \cap G_1) \boxplus (B \cap G_2)$ and $A = (A \cap G_1) \boxplus (A \cap G_2)$, so $(A \cap G_i, B \cap G_i)$ is a maximal above and below pair in G_i for $i = 1, 2$.*

PROOF. $0 < b \in B$ implies $b = b_1 + b_2$ for unique $b_i \in G_i$, and $b_1, b_2 \geq 0$ because the order on G is the lattice product of the orders on the G_i 's. Therefore $b_1, b_2 \in B$ by convexity, meaning $B = (B \cap G_1) \boxplus (B \cap G_2)$. Likewise $0 < a \in A$ implies $a = a_1 + a_2$ for unique $a_i \in G_i$, and the a_i 's are components of a , again by virtue of the order on G . Then $a_1, a_2 \in A$ because A is saturated by Theorem 1.6, so that $A = (A \cap G_1) \boxplus (A \cap G_2)$.

It is clear from Proposition 1.4 that $A \cap G_1$ lies below $B \cap G_1$ in G_1 . If C is maximal among those convex l -subgroups of G_1 which contain $A \cap G_1$ and lie below $B \cap G_1$, then $C \boxplus (B \cap G_2)$ lies below A , hence $C = A \cap G_1$. A similar argument shows that $A \cap G_1$ is maximal among l -subgroups of G_1 above $B \cap G_1$. $\square$

PROPOSITION 5.2. *If $G \in \mathbf{S}_n$ and $C \in \mathcal{C}(G)$, then $C \in \mathbf{S}_n$, $n = 1, 2$.*

PROOF. Suppose $G \in \mathbf{S}_1$, $C \in \mathcal{C}(G)$, and B is maximal below A in C . Find $B_1 \in \mathcal{C}(G)$ maximal among those convex l -subgroups of G containing B and lying below A . Note that $B_1 \cap C$ lies below A in C , hence $B_1 \cap C = B$. Now find A_1 maximal among l -subgroups of G above B_1 such that $G = A_1 + B_1$. By Proposition 3.12 $C = (A_1 \cap C) + B$, proving $C \in \mathbf{S}_1$. The argument for $G \in \mathbf{S}_2$ is similar. $\square$

PROPOSITION 5.3. *The following are equivalent for any set $\{G_\lambda: \lambda \in \Lambda\}$ of l -groups.*

- (a) $G_\lambda \in \mathbf{S}_1$ for all $\lambda \in \Lambda$.
- (b) $\Sigma G_\lambda \in \mathbf{S}_1$.
- (c) $\Pi G_\lambda \in \mathbf{S}_1$.

These conditions remain equivalent if $\mathbf{S}_1$ is replaced by $\mathbf{S}_2$.

PROOF. Proposition 5.2 provides the implication from (c) to (b) to (a). Suppose that $G_\lambda \in \mathbf{S}_1$ for all $\lambda \in \Lambda$ and let B be maximal below A in ΠG_λ . We may without loss of generality assume that A is maximal above B , so that by Lemma 5.1 $B \cap G_\lambda$ is maximal below $A \cap G_\lambda$ in G_λ for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ choose A_λ maximal above $B \cap G_\lambda$ such that $G_\lambda = (B \cap G_\lambda) + A_\lambda$, and let $A_1 = \{f \in \Pi A_\lambda: \lambda f \in A_\lambda \text{ for all } \lambda \in \Lambda\}$. The componentwise nature of the operations on ΠG_λ makes it clear that A_1 lies above B and that $A_1 + B = \Pi G_\lambda$. The argument for $\mathbf{S}_2$ is similar. $\square$

PROPOSITION 5.4. *If G is an archimedean l -group, then $G \in \mathbf{S}_2$ if and only if $G \in \mathbf{S}_1$ if and only if G is projectable.*

PROOF. In an archimedean l -group the only maximal above and below pairs are complementary polars. $\square$

The most general condition sufficient for membership in $\mathbf{S}_2$ is given by Theorem 6.18 below, which we state here. Its proof in §6 depends on none of the intervening material.

THEOREM 5.5. *A divisible abelian sup $\mathcal{V}$ complete l -group lies in $\mathbf{S}_2$.*

In the following proposition let $F(\Gamma, \mathbf{R})$ denote $\{g \in V(\Gamma, \mathbf{R}): \{\gamma \in \Gamma: (\gamma)g \neq 0\} \text{ admits only finite disjoint subsets}\}$ [9]. Observe that $\Sigma \leq F \leq V$.

PROPOSITION 5.6. *For any root system Γ , $\Sigma(\Gamma, \mathbf{R}) \in \mathbf{S}_1$, $F(\Gamma, \mathbf{R}) \in \mathbf{S}_1$, and $V(\Gamma, \mathbf{R}) \in \mathbf{S}_2$; furthermore $\Sigma(\Gamma, \mathbf{R}) \in \mathbf{S}_2$ and $F(\Gamma, \mathbf{R}) \in \mathbf{S}_2$ whenever Γ satisfies the DCC.*

PROOF. The fact that $V(\Gamma, \mathbf{R}) \in \mathbf{S}_2$ is a consequence of Theorem 5.5. To see that $\Sigma(\Gamma, \mathbf{R}) \in \mathbf{S}_1$ consider $B \in \mathcal{C}(\Sigma)$. Let $A = \{g \in \Sigma: \gamma g = 0 \text{ for all } \gamma \in \Gamma(B)\}$, an l -subgroup which lies above B by Proposition 1.4. Since $B = \{g \in \Sigma: \gamma g = 0 \text{ for all } \gamma \notin \Gamma(B)\}$, $A + B = \Sigma$. The proof that $F \in \mathbf{S}_1$ is analogous. Now suppose Γ has the DCC and that (A, B) is a maximal above and below pair in F . (The argument that $\Sigma \in \mathbf{S}_2$ is analogous.) Suppose for contradiction that $g \in F \setminus (A + B)$. At least one of the (finitely many) special components of g —say g_1 —fails to lie in $A + B$. Let γ_1 be the value of g_1 . Corollary 1.11 provides $a \in A$ such that $g_1 - a$ has a value in $\Gamma(B)$. This implies that one of the special components of a —say a_1 —has value γ_1 and that $(\gamma_1)a_1 = (\gamma_1)g_1$. Now $a_1 \in A$ because A is saturated, so $g_1 - a_1 \notin A + B$. Let g_2 be one of the special components of $g_1 - a_1$ such that $g_2 \notin A + B$, and let γ_2 be its value. Observe that $\gamma_2 < \gamma_1$. But a continuation of this argument produces an infinite descending chain in Γ , contrary to the DCC. $\square$

The following example makes several points, the first being that $\Sigma(\Gamma, \mathbf{R})$ need not lie in $\mathbf{S}_2$ in general. Let Γ be $\Delta \cup \Lambda$, where $\Delta(\Lambda) = \{0^n (0^n 1): n = 1, 2, 3, \dots\}$. Here $0^n (0^n 1)$ refers to the string of n zeros (followed by a single 1). Order Γ by declaring $\alpha > \beta$ whenever $\beta = \alpha\gamma$ for some $\gamma \in \Gamma$ (where $\alpha\gamma$ is the string formed by concatenating the α and γ strings). Let $B = \{g \in \Sigma(\Gamma, \mathbf{R}): \Gamma(g) \subseteq \Lambda\}$ and define $a_n \in \Sigma$ by

$$(\gamma)a_n = \begin{cases} 1 & \text{if } \gamma = 0^n, 0^n 1, \text{ or } 0^{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Let A be the l -subspace of the vector lattice Σ generated by $\{a_n: n = 1, 2, 3, \dots\}$. Then A lies above B by Proposition 1.4 since $\Gamma(A) \subseteq \Delta$, and in fact (A, B) is a maximal above and below pair by Corollary 1.11. Yet $A + B$ is proper in Σ , since the latter fails to contain the element $g \in \Sigma$ defined by

$$(\gamma)g = \begin{cases} 1 & \text{if } \gamma = 0, \\ 0 & \text{otherwise.} \end{cases}$$

A finite-valued vector lattice need not belong to $\mathbf{S}_1$. Let Γ have the meaning above. In his thesis [8], Byrd showed that $\Sigma(\Gamma, \mathbf{R})$ has an uncountable number of pairwise non- l -isomorphic a -closures in the class of abelian l -groups, each of which is a vector lattice G which may be taken to satisfy $\Sigma(\Gamma, \mathbf{R}) \leq G \leq V(\Gamma, \mathbf{R})$. (See Chapter 4 of [9] for additional background.) We will show that exactly one of these a -closures lies in $\mathbf{S}_1$. Now if B has the meaning above then it is still maximal below A in G ; therefore if $G \in \mathbf{S}_1$ then there must be some l -subgroup A_1 above B in G such that $G = A_1 + B$. The map $\theta: A_1 \rightarrow V(\Gamma, \mathbf{R})$ defined by

$$(\gamma)\theta(a) = \begin{cases} (\gamma)a & \text{if } \gamma \in \Delta, \\ 0 & \text{if } \gamma \in \Lambda, \end{cases}$$

is an l -monomorphism and so it can be extended to an l -monomorphism $\hat{\theta}: G \rightarrow V$, which must be the identity on B . (See Theorem 3.3 of [12].) Since $G\hat{\theta} \leq F(\Gamma, \mathbf{R}) = \{g \in V: \{\gamma \in \Lambda: (\gamma)g \neq 0\} \text{ finite}\}$ and since F is an a -extension of Σ , it follows that G is l -isomorphic to F .

COROLLARY 5.7. *A finite-valued vector lattice G of countable dimension belongs to S_1 .*

PROOF. G is l -isomorphic to $\Sigma(\Gamma, \mathbf{R})$ for a suitable choice of Γ [9]. $\square$

PROPOSITION 5.8. *A totally ordered group G belongs to S_1 if and only if each convex subgroup splits. That is, for each $B \in \mathcal{C}(G)$ there is some subgroup A such that $A + B = G$.*

PROOF. Suppose $G \in S_1$. Each regular subgroup of G is maximal below by Proposition 2.3 and so splits. In particular, each regular subgroup and hence each convex subgroup is normal. Given $B \in \mathcal{C}(G)$ choose $a_\gamma \in G^\gamma \setminus G_\gamma$ for each regular G_γ such that $G_\gamma \supseteq B$. Then B is maximal below the subgroup A generated by the a_γ 's. The converse is obvious. $\square$

PROPOSITION 5.9. *Suppose $G \in S_2$. If (A_1, B) and (A_2, B) are maximal above and below pairs in G , then A_1 is l -isomorphic to A_2 .*

PROOF. Both A_1 and A_2 are l -isomorphic to G/B . $\square$

Does Proposition 5.9 hold even when G does not lie in S_2 ? The authors know of no counterexample to this conjecture. However, the following example provides counterexamples to several closely related conjectures. Let $\Gamma = \{a_1, a_2, b_1, b_2\}$, ordered by declaring $a_i > b_i$, and let $B = V(\Gamma, \mathbf{Z})$. Let A be $\mathbf{Z}$, $\pi(A(a)) = [a] = B \in \mathcal{S}(B)$ for all $0 \neq a \in A$, and let $\sigma(1)$ be the l -automorphism of B defined by $(x)(g\sigma(1)) = (\psi(x))g$, where $\psi(a_1) = a_2$, $\psi(a_2) = a_1$, $\psi(b_1) = b_2$, and $\psi(b_2) = b_1$. Let G be the splitting extension of B by A determined by π and σ . G is a Scrimger group [15] and so normal-valued, and its elements can be naturally represented in the form

$$\begin{pmatrix} & m & \\ n_1 & & n_2 \\ p_1 & & p_2 \end{pmatrix}$$

for integers m , n_i , and p_i . Using this notation let $A_1 = \{g \in G: m = n_1 = p_1 = p_2 = 0\}$, $A_2 = \{g \in G: m \text{ even, } n_1 = p_1 = p_2 = 0\}$, $A_3 = \{g \in G: p_1 = p_2 = 0\}$, $B_1 = \{g \in G: m = n_1 = n_2 = p_2 = 0\}$, $B_2 = \{g \in G: m = n_2 = 0\}$, and $B_3 = \{g \in G: m = n_1 = n_2 = 0\}$. Observe that B_1 lies below A_1 , that B_2 is maximal with respect to containing B_1 and lying below A_1 , that A_2 is maximal with respect to containing A_1 and lying above B_2 , that A_3 is maximal with respect to containing A_1 and lying above B_1 , and that B_3 is maximal with respect to containing B_1 and lying below A_3 . The point is that A_2 is not l -isomorphic to A_3 , B_2 is not l -isomorphic to B_3 , and that $A_2 + B_2$ is not l -isomorphic to $A_3 + B_3$.

6. Abelian l -groups G which split off from below in every abelian extension in which G is a convex l -subgroup. Suppose G is a convex l -subgroup of the l -group H . We say G *splits off from below* in H if $H = U + G$ for every l -subgroup $U \leq H$ such that U is maximal above G . The subject of this section is a characterization of those abelian l -groups G which split off from below in every abelian l -group having G as a convex l -subgroup (Theorem 6.16). If G has this property we shall sometimes simply say that G splits off from below. The following lemma is useful; it is Lemma 5.6 of [2].

LEMMA 6.1. *Suppose X is a subset and g an element of G such that $g \geq X$. Then X is type $\mathcal{Y}$ and $\bigvee X = g$ if and only if $\bigvee \{(y - x)^\perp : x \in X\} = 1_\emptyset$.*

An l -group G which splits off from below is closed in every l -group H in which G is a convex l -subgroup. The latter property depends on the possession by G of a supremum for each subset of the following type. Let us call a subset $X \subseteq G^+$ *type $\mathcal{Z}$* if X is type $\mathcal{Y}$ and if $\bigvee (X \wedge g)$ exists in G for all $g \in G$. G is *sup $\mathcal{Z}$ complete* if every type $\mathcal{Z}$ subset of G possesses a supremum in G .

PROPOSITION 6.2. *If G is sup $\mathcal{Z}$ complete, then G is closed in every l -group H in which G is a convex l -subgroup.*

PROOF. Suppose $\bigvee X = h \in H$ for $X = G^+ \wedge h \subseteq G^+$. We first claim that X is a type $\mathcal{Y}$ subset of H^+ . Lemma 6.1 is enough to show $\bigvee_{x \in X} (h - x)^\perp = 1_\emptyset$ in the Boolean algebra $\mathcal{P}$ of polars of H . For this purpose consider $0 < g \in H$; we shall prove that $g \notin (h - x)^\perp$ for some $x \in X$. Since $G^{\perp\perp}$ is closed in H it follows that $h \in G^{\perp\perp}$, which allows us to consider only the case in which $g \in G$. Because $\bigvee X = h$, there must be some $x \in X$ with $x \not\leq -g + h$ or $g + x \not\leq h$. Let $x' = (g + x) \wedge h \in X$; we claim $g \notin (h - x')^\perp$. For $h \vee (g + x) > h$ implies $0 < (h \vee (g + x)) - h = 0 \vee (g + x - h) = g' \leq g$, and $(h - x') \wedge g' = [0 \vee (h - x - g)] \wedge [0 \vee (g + x - h)] = 0 \vee (-|h - x - g|) = 0$. This completes the proof that X is a type $\mathcal{Y}$ subset of H . Since G is convex in H , intersection provides a one-one correspondence from $\{P \in \mathcal{P} \mid P \subseteq G^{\perp\perp}\}$ onto the set of polars of G , from which easily follows that X is a type $\mathcal{Y}$ subset of G . X is actually type $\mathcal{Z}$, since by convexity $\bigvee (X \wedge g) = (\bigvee X) \wedge g = h \wedge g \in G$ for any $g \in G$. Since G is sup $\mathcal{Z}$ complete there is some t such that $\bigvee X = t \in G$. But then $0 \leq h \leq t$ implies by convexity that $h \in G$. $\square$

COROLLARY 6.3. *If G is sup $\mathcal{Y}$ complete, then it is closed in any l -group in which it is a convex l -subgroup.*

THEOREM 6.4. *An abelian l -group G is closed in every l -group in which it is a convex l -subgroup if and only if it is sup $\mathcal{Z}$ complete.*

PROOF. Let $G_{\mathcal{Y}}$ be the sup $\mathcal{Y}$ completion of G [1] and suppose $X \subseteq G^+$ is a type $\mathcal{Z}$ set with $\bigvee X = t \in G_{\mathcal{Y}} \setminus G$, and let W be the l -subgroup of $G_{\mathcal{Y}}$ generated by G and t . We shall prove that G is convex but not order closed in W . For this purpose we shall adopt the following terminology for this proof and the next one only. An element $s \in G$ will be *admissible* if $s\pi_0 g \in G$ for all $0 < g \in G$, where $\pi_{ab}: G_{\mathcal{Y}} \rightarrow [a, b]$ is

the lattice homomorphism defined by $x\pi ab = (x \vee a) \wedge b$. Observe that G is $\mathcal{Z}$ complete if and only if it contains each admissible element of $G_{\mathcal{Z}}$. However, the set of admissible elements is not closed under addition in general.

The first claim is that if $0 < u, v \in G_{\mathcal{Z}}$ are admissible, then so is $u + v$. This is true because

$$\begin{aligned} (u + v)\pi 0g &= (u + v) \wedge g = (u + v) \wedge (g + v) \wedge (u + g) \wedge (g + g) \wedge g \\ &= (u \wedge g + v \wedge g) \wedge g = (u\pi 0g + v\pi 0g)\pi 0g \in G. \end{aligned}$$

Therefore nt is admissible for all positive integers n . The second claim is that if $0 < u \in G_{\mathcal{Z}}$ is admissible, then $u\pi ab \in G$ for all $a < b$ in G . This is true because

$$u\pi ab = (u\pi ab)^+ + (u\pi ab)^- = u\pi a^+b^+ + b^- = u\pi 0b^+ \vee a^+ + b^- \in G.$$

The third claim is that if $0 < u \in G_{\mathcal{Z}}$ is admissible, then $(k + u)\pi ab \in G$ for all $k, a, b \in G$ with $a < b$. This holds because

$$(k + u)\pi ab = k + [(u \vee (-k \vee a)) \wedge (-k \vee b)],$$

the latter term being in G by the second claim. The fourth and final claim is that if $0 < u \in G_{\mathcal{Z}}$ is admissible, then

$$(k - u)\pi ab = -(-k + u)\pi ab = -[(-k + u)\pi(-b)(-a)] \in G$$

for all $k, a, b \in G$ with $a < b$.

To complete the proof of the theorem consider an arbitrary $w \in W$. w has the form $\bigvee_{i=1}^n \bigwedge_{j=1}^m (k_{ij} + n_{ij}t)$ for $k_{ij} \in G$ and integers n_{ij} . The claims show each $k_{ij} + n_{ij}t$ to be admissible, and hence w is admissible. Therefore G is convex in W . However, G is clearly not order closed because $\bigvee X = t \in W \setminus G$. $\square$

COROLLARY 6.5. *Given any abelian l -group G there is an abelian l -group H in which G is an order dense convex l -subgroup and such that H is closed in any l -group containing H as a convex l -subgroup.*

PROOF. Let $G_0 = G$. Having defined G_α , define $G_{\alpha+1}$ to be G_α if G_α is $\mathcal{Z}$ complete, otherwise let $G_{\alpha+1}$ be an l -subgroup of $G_{\mathcal{Z}}$ generated by G_α and some admissible $0 < t \in G_{\mathcal{Z}} \setminus G$. Take unions at limit ordinals, and let $H = G_\alpha$ for the least ordinal α such that $G_\alpha = G_{\alpha+1}$. The proof of Theorem 6.4 shows that G is a convex l -subgroup of H . $\square$

It seems quite unlikely that the l -group H of the preceding corollary is unique. The general question is open: which l -groups admit unique $\mathcal{Z}$ hulls?

Those abelian l -groups G which split off from below in every abelian l -group having G as a convex l -subgroup can be characterized by the possession of three properties (Theorem 6.16). One of these properties is divisibility.

PROPOSITION 6.6. *If an abelian l -group G splits off from below in every lex extension, then G is divisible.*

PROOF. Let G^d be the divisible hull of G and suppose p is the least positive integer such that $px \in G$ for some $0 < x \in G^d \setminus G$. Consider the lexicographic product $\mathbf{Z} \vec{\times} G^d$ determined by letting $\sigma(n)$ be the identity l -automorphism on G^d

and $\pi(n) = [n] = G^d$ for each $n \in \mathbf{Z}$. Let H be the l -subgroup consisting of all pairs of the form $(pi + j, a + jx)$, where $i, j \in \mathbf{Z}$ and $a \in G$. Such pairs form an l -subgroup because $(pi + j, a + jx) \vee (0, 0)$ is either $(pi + j, a + jx)$ in case $pi + j > 0$, or is $(0, 0)$ in case $pi + j < 0$, or is $(0, (a - i(px)) \vee 0)$ in case $pi + j = 0$. H has (a copy of) G as the convex l -subgroup $0 + G$. Let $U = \{(pi, a) | i \in \mathbf{Z}\}$, an l -subgroup of H above G . To show that U is maximal above G consider an arbitrary $0 < h = (pi + j, a + jx) \in H \setminus U$ and let V be the l -subgroup of H generated by U and h . We may assume $0 \leq j < p$, and by subtracting $(pi, 0) \in U$ we may assume $h = (j, a + jx)$. If $j = 0$ then $a \neq 0$, so h itself has a value in $\Gamma(G)$, and if $j > 0$ then $ph - (pj, 0) = (0, pa + j(px)) \in V$ which also has a value in $\Gamma(G)$ since $0 \neq pa + j(px) \in G$. This proves U maximal above G , which proves the theorem since $U + G$ fails to contain $(1, x)$, among others. $\square$

Another characteristic of an abelian l -group G which splits off from below is the *summand sewing property* or SSP. Given a collection $\mathcal{S}$ of summands of G , a map $\theta: \mathcal{S} \rightarrow G$ is said to be *consistent* if $S\theta - R\theta \in (S \cap R)^\perp$ for all $S, R \in \mathcal{S}$. G has the summand sewing property if for every consistent $\theta: \mathcal{S} \rightarrow G$ there is some $g \in G$ such that $g - S\theta \in S^\perp$ for all $S \in \mathcal{S}$. Observe that such $g \in G$ must always exist for finite $\mathcal{S}$. Also note that g need not be unique. Finally, observe that it is enough to consider consistent maps $\theta: \mathcal{S} \rightarrow G^+$.

The definition of the SSP just given bears close resemblance to the definition of orthocompleteness given in [2], and in fact this resemblance can be converted with little difficulty to a proof that orthocompleteness implies the SSP. Thus every l -group is order dense in an l -group which enjoys the SSP.

PROPOSITION 6.7. *If G is sup $\mathcal{U}$ complete, then G has the SSP.*

PROOF. Suppose $\theta: \mathcal{S} \rightarrow G^+$ is a consistent map. For each $S \in \mathcal{S}$ let $g_S \in G$ be the projection of $S\theta$ on S , and let X be $\{g_S | S \in \mathcal{S}\}$. The fact that $(X - g_S)^+ \cup (-g_S + X)^+ \subseteq S^\perp$ for all $S \in \mathcal{S}$ demonstrates that X is a type $\mathcal{U}$ set and that $\bigvee X = g$ exists in G . But it can readily be shown that $g - S\theta \in S^\perp$ for all $S \in \mathcal{S}$. $\square$

PROPOSITION 6.8. *An abelian l -group G which splits off from below in every abelian l -group in which it is maximal below must have the SSP.*

PROOF. Let $\mathcal{S} = \{S_\lambda: \lambda \in \Lambda\}$ be a set of summands of G and $\theta: \mathcal{S} \rightarrow G^+$ a consistent map which denies the SSP. Let $\bar{G}$ be the orthocompletion of G and choose $t \in \bar{G}$ such that $t - S_\lambda\theta \in S_\lambda^\perp$ for all $\lambda \in \Lambda$. Let $\prod \mathbf{Z}_\lambda$ and $\sum \mathbf{Z}_\lambda$ designate as usual the large and small cardinal products of the integers $\mathbf{Z}$ over the index set Λ . Designate by $c \in \prod \mathbf{Z}$ the element defined by $\lambda c = 1$ for all $\lambda \in \Lambda$, and let $A = \{nc + s \in \prod \mathbf{Z}: n \in \mathbf{Z}, s \in \sum \mathbf{Z}\}$, an l -subgroup of $\prod \mathbf{Z}$. Define $\pi: \mathcal{C}_\#(A) \rightarrow \mathcal{S}(\bar{G})$ by declaring $\pi(A(a)) = [a] = \bigvee \{\bar{S}_\lambda: \lambda \in \text{supp}(a)\}$, where $\bar{S}_\lambda = S_\lambda^{\perp\perp} \in \mathcal{S}(\bar{G})$. Observe that for any $a \in A$, $\text{supp}(a)$ is a finite or cofinite subset of Λ and so $\bigvee \{\bar{S}_\lambda: \lambda \in \text{supp}(a)\}$ exists in $\mathcal{S}(\bar{G})$. Let σ be the map which assigns the identity l -automorphism of $\bar{G}$ to each $a \in A$, and let K be the splitting extension of $\bar{G}$ by A determined by σ and π . Finally, let H be $\{(nc + s, nt + g) \in K: n \in \mathbf{Z}, s \in \sum \mathbf{Z}, g \in G\}$.

We first claim that H is an l -subgroup of K . To verify this claim we must succeed in showing $(nc + s, nt + g) \vee (0, 0)$ lies in H . If $n = 0$ then $\text{supp}(nc + s) = \text{supp}(s)$ is finite, and $(s, g) \vee (0, 0) = (s \vee 0, g')$, where $g' = g \vee 0 + \sum\{g_\lambda^- : \lambda \in \text{supp}(s^+)\} - \sum\{g_\lambda : \lambda \in \text{supp}(s^-)\} \in G$. Here g_λ designates the element of S_λ such that $g - g_\lambda \in S_\lambda^\perp$. If $n > 0$ then $\text{supp}(nc + s)$ is cofinite, $\text{supp}((nc + s)^-)$ is finite, and

$$(nc + s, nt + g) \vee (0, 0) = ((nc + s) \vee 0, g'),$$

where

$$g' = g - \sum\{(nt_\lambda + g_\lambda)^- : \lambda \notin \text{supp}(nc + s)\} \\ - \sum\{(nt_\lambda + g_\lambda) : \lambda \in \text{supp}((nc + s)^-)\} \in G.$$

Here t_λ is the unique element of S_λ such that $S_\lambda\theta - t_\lambda \in S_\lambda^\perp$. The case in which $n < 0$ is handled similarly.

Let $U = \{(u, 0) : u \in \sum \mathbb{Z}\}$, an l -subgroup of H which lies above G . To show that U is maximal above G consider $h = (nc + u, nt + g) \in H \setminus U$ and let W be the l -subgroup of H generated by $U \cup \{h\}$. Now for each $\lambda \in \Lambda$ there is an element $u_\lambda \in \sum \mathbb{Z}$ such that $(\lambda)(nc + u + u_\lambda) = 0$. Therefore $nt_\lambda + g_\lambda \in G$ is a component of $h + (u_\lambda, 0) \in W$. Now if $nt_\lambda + g_\lambda = 0$ for all λ , then $nt + g \in S_\lambda^\perp$ for all λ , hence $t - f \in S_\lambda^\perp$ for all λ , where f is the element proven to exist in G by Proposition 6.6 such that $n(-f) = g$. This contradicts the assumption that θ violates the SSP and proves that $nt_\lambda + g_\lambda \neq 0$ for some $\lambda \in \Lambda$. But then $\Gamma(W) \cap \Gamma(G) \neq \emptyset$, proving that U is maximal above G . This proves the theorem since $U + G$ fails to contain $(c, t) \in H$, among others. $\square$

Suppose P is a polar of the l -group G . We shall say that a set $X \subseteq G^+$ is of type $\mathcal{W}(P)$ if X is type $\mathcal{U}$, $X \subseteq P^+$, and for every $0 \leq g \in G$ there is some element $g' \in G$ satisfying the following three conditions: $g - g' \in P$, $X \wedge g \leq g'$, and $P \subseteq \bigvee\{(g' - x \wedge g)^\perp : x \in X\}$. We shall call G *sup $\mathcal{W}$ complete* if $\forall X$ exists in G for every polar P and every $X \subseteq G^+$ of type $\mathcal{W}(P)$. Several comments deserve mention here. Though we shall not prove it, the element g' asserted to exist by the definition is unique for each $0 < g \in G$, and in fact $0 \leq g' \leq g$ in all cases. Also, not every type $\mathcal{W}(P)$ set X is type $\mathcal{Z}$, nor is every type $\mathcal{Z}$ set Y necessarily type $\mathcal{W}(P)$. For example, take $G = C[-1, 1]$, the l -group of continuous real-valued functions on the interval $[-1, 1]$, let $P = \{g \in G \mid \text{supp}(g) \subseteq (0, 1)\}$, let $X = \{g \in P^+ \mid (r)g \leq 1/r, r \in (0, 1)\}$, and let $Y = \{g \in P^+ \mid (r)g \leq r/(1 - r), r \in (0, 1)\}$. Nevertheless, the notions of type $\mathcal{Z}$ set and type $\mathcal{W}(P)$ set coincide for summands P .

PROPOSITION 6.9. *Suppose P is a summand of the l -group G . A set $X \subseteq P^+$ is of type $\mathcal{W}(P)$ if and only if X is type $\mathcal{Z}$ in P .*

PROOF. Suppose X is of type $\mathcal{W}(P)$. Then X is type $\mathcal{U}$ in G and therefore also in P . Consider $0 < p \in P$, and let p' be the element provided by the definition of X being type $\mathcal{W}(P)$. Then $p' \in P$ because $p - p' \in P$, $X \wedge p \leq p'$, and

$$\bigvee\{(p' - x \wedge p)^\perp : x \in X\} = 1_{\mathcal{Z}},$$

where $\mathcal{Q}$ is the polar algebra of P . By Lemma 6.1 $\vee(X \wedge p) = p' \in P$, meaning that X is type $\mathcal{X}$ in P .

Now suppose X is type $\mathcal{X}$ in P . Then $\vee\{((X - x) \vee 0)^\perp : x \in X\} = 1_{\mathcal{Q}}$ and $X \subseteq P$ imply $\vee((X - x) \vee 0)^\perp = 1_{\mathcal{P}}$, where $\mathcal{P}$ is the polar algebra of G , meaning X is type $\mathcal{Y}$ in G . Given $0 \leq g \in G$ let $g = y + z$, where $y \in P$ and $z \in P^\perp$, and let $g' = w + z$, where $w = \vee(X \wedge y)$ exists in P because X is type $\mathcal{X}$ in P . Then $g - g' = y - w \in P$, and $X \wedge g = X \wedge (y + z) \leq X \wedge y + z \leq w + z = g'$. To show $P \subseteq \vee\{(g' - x \wedge g)^\perp : x \in X\}$ in $\mathcal{P}$ first observe that $(x_1 \wedge y) - x \wedge y \leq (x_1 - x)^+$ for $x_1, x \in X$, from which follows $\vee\{((X \wedge y - x \wedge y) \vee 0)^\perp : x \in X\} = 1_{\mathcal{P}}$. A similar argument shows $\vee((-x \wedge y + X \wedge y) \vee 0)^\perp = 1_{\mathcal{P}}$, and so $X \wedge y$ is type $\mathcal{Y}$. By Lemma 6.1 $\vee(w - x \wedge y)^\perp = 1_{\mathcal{P}}$, from which follows $\vee(g' - x \wedge g)^\perp \supseteq P$. $\square$

COROLLARY 6.10. $X \subseteq G^+$ is of type $\mathcal{W}(G)$ if and only if X is type $\mathcal{X}$.

COROLLARY 6.11. If G is sup $\mathcal{W}$ complete, then every summand is sup $\mathcal{X}$ complete. If G is projectable and every polar is sup $\mathcal{X}$ complete, then G is sup $\mathcal{W}$ complete.

LEMMA 6.12. Suppose P is a polar of the l -group G , and that $X, Y \subseteq P^+$ are sets of type $\mathcal{W}(P)$. Then $X + Y$ is of type $\mathcal{W}(P)$.

PROOF. First we must show $X + Y$ to be type $\mathcal{Y}$. We know that $\vee X = x_0 \in G_{\mathcal{Q}}$ and $\vee Y = y_0 \in G_{\mathcal{Q}}$, hence by Lemma 6.1 and its analogue $\vee\{(y_0 - y)^\perp : y \in Y\} = \vee\{(-x + x_0)^\perp : x \in X\} = 1_{\mathcal{P}}$. It follows that

$$\begin{aligned} 1_{\mathcal{P}} &= \vee\{((y_0 - y)^\perp \cap (-x + x_0)^\perp) : x \in X, y \in Y\} \\ &= \vee((y_0 - y) \vee (-x + x_0))^\perp = \vee((y_0 - y) + (-x + x_0))^\perp. \end{aligned}$$

Upon conjugation by x_0 we obtain $\vee((x_0 + y_0) - (x + y))^\perp = 1_{\mathcal{P}}$, which proves $X + Y$ to be type $\mathcal{Y}$ in $G_{\mathcal{Q}}$ by Lemma 6.1. By Proposition 5.16 of [1], $X + Y$ is also type $\mathcal{Y}$ in the order dense l -subgroup G .

Now consider $0 \leq g \in G$ and let g_x and g_y be the elements of G provided by the type $\mathcal{W}(P)$ property of X and Y , respectively. Let $g' = (g_x + g_y) \wedge g$. Then $0 \leq g - g' = (g - g_y - g_x)^+ \leq (g - g_y)^+ \in P$, so $g - g' \in P$. And for $x \in X$ and $y \in Y$, $(x + y) \wedge g \leq (x \wedge g) + (y \wedge g) \leq g_x + g_y$, which implies $(X + Y) \wedge g \leq g'$. Finally, to show $P \subseteq \vee\{(g' - (x + y) \wedge g)^\perp : x \in X, y \in Y\}$ consider $0_{\mathcal{P}} \neq T_1 \subseteq P$. Choose $x \in X$ such that $0_{\mathcal{P}} \neq (g_x - x \wedge g)^\perp \cap T_1 = T_2$. The fact that $x \in P$ implies $0_{\mathcal{P}} \neq -x + T_2 + x \subseteq P$, hence there must be some $y \in Y$ such that $0_{\mathcal{P}} \neq (g_y - y \wedge g)^\perp \cap (-x + T_2 + x) = T_3$. Note that $0_{\mathcal{P}} \neq x + T_3 - x \subseteq T_2$. We claim $x + T_3 - x \subseteq (g' - (x + y) \wedge g)^\perp$. For if not then there is some $r \in x + T_3 - x$ such that $0 < r \leq g' - (x + y) \wedge g$. Let Q be a value of r , and observe that $Q + (x + y) \wedge g < Q + (g_x + g_y) \wedge g$. Since $r \in (g_x - x \wedge g)^\perp \setminus Q$, $Q + g_x = Q + x \wedge y$. If $Q + g \leq Q + x$, then $Q + g_x = Q + g$ and $Q + (x + y) \wedge g = Q + g = Q + (g_x + g_y) \wedge g$, a contradiction. Therefore $Q + g_x = Q + x < Q + g$. Now $-x + r + x \in (g_y - y \wedge g)^\perp \setminus R$ implies $R + g_y = R + y \wedge g$, where $R = -x + Q + x$. If $R + y \leq R + g$ we get $R + g_y = R + y$, which gives the contradiction

$Q + g_x + g_y = Q + x + y$. If $R + y > R + g$, then $R + g_y = R + g$, so $Q + x + y > Q + x + g \geq Q + g$ and $Q + g_x + g_y = Q + x + g_y = Q + x + g \geq Q + g$, and we again have the contradiction $Q + (x + y) \wedge g = Q + g = Q + (g_x + g_y) \wedge g$. $\square$

Theorem 6.14 is important and explains the significance of type $\mathcal{W}(P)$ subsets for our purposes. For its proof it is helpful to recast the definition of such sets slightly in Lemma 6.13, which in turn requires some terminology. Given an element g and polar P of G we call h the projection of g on P if $h \in P$ and $g - h \in P^\perp$. h need not exist, but when it does we write h as $g\nu P$. This terminology extends the use of ν in previous sections in the sense that *bva* there could also be written $b\nu[a]$ here.

LEMMA 6.13. Suppose $X \subseteq G^+$ and $P \in \mathcal{P}$. Then X is of type $\mathcal{W}(P)$ if and only if there is an element $t \in G_{\mathcal{W}}$ satisfying the following conditions.

- (a) $X \leq t \in P^{\perp\perp}$.
- (b) $\bigvee\{(t - x)^\perp : x \in X\} = 1_{\mathcal{P}}$.
- (c) $(g - t)^+\nu P \in G$ for all $g \in G^+$.

PROOF. Suppose X is of type $\mathcal{W}(P)$; since X is type $\mathcal{U}$ there is some element $t \in G_{\mathcal{W}}$ such that $\bigvee X = t$, and $t \in P^{\perp\perp}$ because $P^{\perp\perp}$ is a closed convex l -subgroup of $G_{\mathcal{W}}$. By Lemma 6.1, $\bigvee(t - x)^\perp = 1_{\mathcal{P}}$. Given $g \in G^+$ find $g' \in G$ such that $g - g' \in P$ and $P \subseteq \bigvee(g' - x \wedge y)^\perp$. We claim $g - g' = (g - t)^+\nu P$, for which we must prove

$$\begin{aligned} (g - t)^+ - (g - g') &= (g - t) \vee 0 + g' - g \\ &= (g' - t) \vee (g' - g) = g' - t \wedge g \in P^\perp. \end{aligned}$$

If not then $P \cap (g' - t \wedge g)^\perp = R_1 \neq 0_{\mathcal{P}}$, and so $(g' - x_1 \wedge g)^\perp \cap R_1 = R_2 \neq 0_{\mathcal{P}}$ for some $x_1 \in X$, hence $(t - x_2)^\perp \cap R_2 \neq 0_{\mathcal{P}}$ for some $x_2 \in X$. Note that

$$t \wedge g - x_2 \wedge g = ((t - x_2) \wedge (g - x_2)) \vee ((t - g) \wedge 0) \leq t - x_2,$$

so $(t \wedge g - x_2 \wedge g)^\perp \cap R_2 \neq 0$. By replacing x_1 and x_2 by $x = x_1 \vee x_2$ we get $(g' - x \wedge g)^\perp \cap (t \wedge g - x \wedge g)^\perp \cap (g' - t \wedge g)^\perp \neq 0_{\mathcal{P}}$. But this is a contradiction since

$$\begin{aligned} (g' - x \wedge g)^\perp \cap (t \wedge g - x \wedge g)^\perp &\subseteq (g' - x \wedge g - (t \wedge g - x \wedge g))^\perp \\ &= (g' - t \wedge g)^\perp. \end{aligned}$$

Conversely suppose X and t satisfy conditions (a), (b), and (c). Then X is type $\mathcal{U}$, $\bigvee X = t$ by Lemma 6.1, and $X \subseteq P^{\perp\perp} \cap G = P$. Given $g \in G^+$ let $g' = -(g - t)^+\nu P + g \in G$. Then $g - g' = (g - t)^+\nu P \in P$. To show $X \wedge g \leq g'$ consider an $x \in X$ and arbitrary prime Q such that $Q < Q + x \wedge g$. Since $P \not\subseteq Q$ it follows that $(g - t)^+\nu P^\perp \in P^\perp \subseteq Q$, hence

$$\begin{aligned} Q + g' &= Q - (g - t)^+\nu P + g = Q - (g - t)^+ + g \\ &= Q = t \wedge g \geq Q + x + g. \end{aligned}$$

To show $P \subseteq \bigvee\{(g' - x \wedge g)^\perp : x \in X\}$ consider $0_{\mathcal{P}} \neq R \subseteq P$ and find $x \in X$ such that $(t - x)^\perp \cap R \neq 0_{\mathcal{P}}$. We claim $(t - x)^\perp \cap P \subseteq (g' - x \wedge g)^\perp \cap P$. For if not there is some $0 < p \in (t - x)^\perp \cap P$ such that $p \leq g' - x \wedge g$. Let Q be a

value of p . Again $P \not\subseteq Q$ implies $(g - t)^+ \nu P^\perp \in P^\perp \subseteq Q$, so

$$\begin{aligned} Q + x \wedge g &< Q + g' = Q - (g - t)^+ \nu P + g \\ &= Q - (g - t)^+ + g = Q + t \wedge g. \end{aligned}$$

But $p \wedge (t - x) = 0$ implies $t - x \in Q$ or $Q + t = Q + x$, hence $Q + x \wedge g = Q + t \wedge g$, a contradiction. $\square$

If $X \subseteq G^+$ is of type $\mathcal{W}(P)$ and $\forall X = t \in G_{\mathcal{Q}}$, then G need not be convex in the l -subgroup $\langle G \cup \{t\} \rangle \leq G_{\mathcal{Q}}$. The following theorem asserts, however, that t must be the projection on P of an element h in an extension H in which G is a convex l -subgroup. In the statement of this theorem $\bigvee\{(h - x)^\perp : x \in X\}$ refers to the supremum in the Boolean algebra $\mathcal{P}$ of polars of H .

THEOREM 6.14. *Suppose G is an abelian l -group with polar P and subset $X \subseteq G^+$. Then X is of type $\mathcal{W}(P)$ if and only if there is an l -group H and element $0 < h \in H$ such that G is a convex l -subgroup of H , $h \geq X \subseteq P$, and*

$$(G - H)^+ \subseteq P \subseteq \bigvee\{(h - x)^\perp : x \in X\}.$$

PROOF. Suppose G is a convex l -subgroup of an l -group H as above. To prove that X is of type $\mathcal{W}$ consider $T \neq 0_{\mathcal{P}}$; we seek $x \in X$ such that $((X - x) \vee 0)^\perp \cap T \neq 0_{\mathcal{P}}$. If $T \not\subseteq P$ we are done, for $((X - x) \vee 0)^\perp \supseteq P^\perp$ for every $x \in X$. Assuming $T \subseteq P$, find $x \in X$ such that $T \cap (h - x)^\perp \neq 0_{\mathcal{P}}$. Since $(x_1 - x) \vee 0 \leq h - x$ for all $x_1 \in X$, it follows that $((X - x) \vee 0)^\perp \cap T \neq 0_{\mathcal{P}}$. To prove that X is of type $\mathcal{W}(P)$ consider an arbitrary $0 \leq g \in G$ and let $g' = h \wedge g$, an element of G by virtue of the convexity of G in H . Then $g - g' = (g - h)^+ \in P$ by hypothesis, and $X \wedge g \leq h \wedge g = g'$. To demonstrate that $P \subseteq \bigvee\{(g' - x \wedge g)^\perp : x \in X\}$ consider $0_{\mathcal{P}} \neq T \subseteq P$ and find $x \in X$ such that $(h - x)^\perp \cap T \neq 0_{\mathcal{P}}$. Since

$$0 \leq h \wedge g - x \wedge g = ((h - g) \wedge (g - x)) \vee ((h - g) \wedge 0) \leq h - x,$$

it follows that $(g' - x \wedge g)^\perp \cap T \neq 0_{\mathcal{P}}$.

Suppose X is of type $\mathcal{W}(P)$. Let K be $\mathbf{Z} \times \bar{G}$, the splitting extension of the orthocompletion $\bar{G}$ of G by the integers $\mathbf{Z}$ determined by defining $\sigma(n)$ to be the identity l -automorphism on $\bar{G}$ and $\pi(n) = [n] = P^\perp \in \mathcal{S}(\bar{G})$ for all $0 \neq n \in \mathbf{Z}$. Let $\forall X = t \in \bar{G}$ and let H be the subgroup $\{(n, nt + g) : n \in \mathbf{Z}, g \in G\}$. We claim that H is an l -subgroup of K , the proof of which requires showing $(n, nt + g) \vee (0, 0) \in H$ for all $n \in \mathbf{Z}$ and $g \in G$. Consider first the case in which $n > 0$. By Lemma 3.9

$$\begin{aligned} (n, nt + g) \vee (0, 0) &= (n, (nt + g)\nu n + (nt + g)^+ \mu n) \\ &= (n, nt + g - (nt + g)^- \mu n) \\ &= (n, nt + g + (-g - nt)^+ \mu n) \\ &= (n, nt + g + ((-g)^+ - nt)^+ \nu P). \end{aligned}$$

The point is that by Lemma 6.12 $nX = \{\sum_{i=1}^n x_i : x_i \in X\}$ is type $\mathcal{W}(P)$ with supremum nt , and $((-g)^+ - nt)^+ \nu P$ exists in G by Lemma 6.13. Therefore $(n, nt + g) \vee (0, 0) \in H$ in this case. In case $n = 0$ we have

$$(0, g) \vee (0, 0) = (0, g \vee 0) \in H,$$

and in case $n < 0$ we have

$$\begin{aligned} (n, nt + g) \vee (0, 0) &= (0, (nt + g)^+ \mu n) = (0, (nt + g)^+ \nu P) \\ &= (0, (g^+ - (-n)t)^+ \nu P). \end{aligned}$$

As before, $kX = \{\sum_{i=1}^k x_i : x_i \in X\}$ is type $\mathcal{W}(P)$ with supremum kt , where $k = -n$, and so $(g^+ - kt)\nu P \in G$ by Lemma 6.13. This completes the proof that H is an l -subgroup of K , and G is clearly convex in H .

Let $h = (1, t) \in H$; we must verify that $(G - h)^+ \subseteq P \subseteq \bigvee \{(h - x)^\perp : x \in X\}$. Now for any $g \in G$,

$$(g - h)^+ = (-1, g - t) \vee (0, 0) = (0, (g - t)^+ \nu P) = (0, (g^+ - t)^+ \nu P) \in P.$$

To show $P \subseteq \bigvee (h - x)^\perp$ consider $0_\emptyset \neq R \subseteq P$, then find $x \in X$ such that $(t - x)^\perp \cap R \neq 0_\emptyset$. For $0 < p \in R \cap (t - x)^\perp$ we have

$$p \wedge (h - x) = (0, p) \wedge (1, t - x) = (0, (t - x) \wedge p) = (0, 0).$$

This completes the proof. $\square$

PROPOSITION 6.15. *An abelian l -group G which splits off from below in every abelian l -group in which it is a convex l -subgroup must be sup $\mathcal{W}$ complete. In this case any polar P which admits a type $\mathcal{W}(P)$ subset is a cardinal summand.*

PROOF. Suppose P is a polar and X a type $\mathcal{W}(P)$ subset of G and construct H with element h as in the proof of Theorem 6.14. Suppose for contradiction that $\bigvee X = t \in \overline{G} \setminus G$. We claim $\{0\}$ is maximal above G . Now an arbitrary element of $H^+ \setminus G$ has the form $nh + g = (n, nt + g)$ for $0 < n \in \mathbb{Z}$ and $g \in G$. Note that $(nt + g)\nu P = nt + g\nu P \neq 0$ in $\overline{G}$, for otherwise $0 < t \leq nt \leq |g|$ implies $t \in G$. By Proposition 3.15 $(n, nt + g)$ has a value in $\Gamma(G)$, hence G does not lie below $\langle (n, nt + g) \rangle$. This proves the claim, and since the claim directly violates the hypothesis of the proposition, we can only conclude that $t \in G$, or that G is sup $\mathcal{W}$ complete.

Observe that $h - t = (1, 0)$ lies above G , and that $H(h - t) \cap G = P^\perp$, so $P \in \mathcal{S}(G)$ by Proposition 1.1. $\square$

We have arrived at last at the main theorem.

THEOREM 6.16. *An abelian l -group G splits off from below in every abelian l -group in which it is a convex l -subgroup if and only if G is divisible, sup $\mathcal{W}$ complete, and has the SSP.*

PROOF. Suppose G has the three listed properties and is a convex l -subgroup of the abelian l -group H . Let U be an l -subgroup of H maximal above G , and consider an arbitrary $0 < h \in H$. We shall produce an element $g \in G$ such that $h - g \in U$. All polar computations will be done in $\mathcal{P}$, the Boolean algebra of polars of H .

For each $u \in U$ and positive integer n let

$$\begin{aligned} P(n, u) &= P = \bigvee \{((g - |nh - u|) \vee 0)^{\perp+} : g \in G^+\}, \\ X_1 &= P^+ \wedge (nh - u)^+, \quad \text{and} \quad X_2 = P^+ \wedge (u - nh)^+. \end{aligned}$$

We claim $P \subseteq \bigvee \{((nh - u)^+ - x_1)^\perp : x_1 \in X_1\}$. To verify this consider $0_\varnothing \neq T \subseteq P$ and find $0 < g \in G$ such that $((g - |nh - u|) \vee 0)^\perp \cap T \neq 0_\varnothing$, then find $0 < t \in T$ with $t \leq (g - |nh - u|)^+ \leq (g - (nh - u)^+)^+$. Let $x_1 = g \wedge (nh - u)^+ \in X_1$. Then

$$t \wedge ((nh - u)^+ - x_1) = t \wedge ((nh - u)^+ - g)^+ = 0,$$

proving the claim. By Theorem 6.14 X_1 , and analogously X_2 , are of type $\mathcal{W}(P)$, so by Proposition 6.15 there are elements $a, b \in G$ with $\bigvee X_1 = a$ and $\bigvee X_2 = b$, and $P(n, u)$ is a summand. By the divisibility of G there is some $g(n, u) \in G$ such that $ng(n, u) = a - b$.

We claim $n(h - g(n, u)) - u$ lies above G . To see this, first observe that $(nh - u)^+ - a \in P^\perp$ and $-(nh - u)^- - b \in P^\perp$, hence $n(h - g(n, u)) - u \in P^\perp$. If there were some prime Q and element $g \in G$ such that $Q < Q + |n(h - g(n, u)) - u| < Q + g$, then

$$Q + |nh - u| - |ng(n, u)| \leq Q + |n(h - g(n, u)) - u| < Q + g,$$

so $Q + |nh - u| < Q + g + |ng(n, u)|$, meaning $((g + |ng(n, u)|) - |nh - u|) \vee 0$ is an element of $P \setminus Q$. Therefore $nh - u - ng(n, u) \in P^\perp \subseteq Q$, contradicting $Q < Q + |n(h - g(n, u)) - u|$.

We claim the map θ which takes each $P(n, u)$ to $g(n, u)$ is consistent. That is, $g(n, u) - g(m, v) \in [P(n, u) \cap P(m, v)]^\perp$ for all $u, v \in U$ and positive integers m and n . For if not there is some prime Q of G such that $Q + g(n, u) \neq Q + g(m, v)$ and $Q \not\supseteq P(n, u) \cap P(m, v)$. We have seen in the previous paragraph that $n(h - g(n, u)) - u$ and $m(h - g(m, v)) - v$ lie in $P^\perp$. Therefore $\overline{Q} + nh - u = \overline{Q} + ng(n, u)$ and $\overline{Q} + mh - v = \overline{Q} + mg(m, v)$, where $\overline{Q}$ is the unique prime subgroup of H whose intersection with G is Q . Thus

$$\begin{aligned} \overline{Q} + mn(g(m, v) - g(n, u)) &= \overline{Q} + ((mnh - nv) - (mnh - mu)) \\ &= \overline{Q} + (mu - nv). \end{aligned}$$

It cannot be that $mu - nv \in U$ fails to lie in $\overline{Q}$, since in this case the value of $mu - nv$ containing $\overline{Q}$ would correspond to the value of $mn(g(m, v) - g(n, u))$ containing Q , and $mu - nv$ would have a value in $\Gamma(G)$ contrary to the assumption that U lies above G . But then $\overline{Q} + g(n, u) = \overline{Q} + g(m, v)$, contrary to the assumption that $Q + g(n, u) \neq Q + g(m, v)$. This completes the proof that θ is consistent.

Let $g \in G$ result from applying the SSP to θ ; explicitly, $g - g(n, u) \in P(n, u)^\perp$ for all u and all n . If $h - g \notin U$, then by Proposition 1.9 there is some $u \in U$ and positive integer n such that $n(h - g) - u$ has value Q in $\Gamma(G)$. Let $0 < k \in G$ satisfy $Q + k > Q + |n(h - g) - u|$. Since $Q + (k + ng) > Q + (nh + u)$ and $Q + (k - ng) > Q + (u - nh)$ it follows that $Q + (k + |ng|) > Q + |nh - u|$, that is $(k + |ng| - |nh - u|)^+ \in P(n, u) \setminus Q$. Since $Q \not\supseteq P(n, u)$, $g - g(n, u) \in P(n, u)^\perp \subseteq R$ for all prime subgroups $R \subseteq Q$. Therefore

$$R + (n(h - g) - u) = R + (n(h - g(n, u)) - u)$$

for all prime subgroups $R \subseteq Q$, and in particular, $n(h - g(n, u)) - u$ has value $Q \in \Gamma(G)$. This contradicts the fact that $n(h - g(n, u)) - u$ lies above G . $\square$

COROLLARY 6.17. *A divisible abelian l -group which is $\sup \mathcal{Y}$ complete splits off from below in every abelian l -group in which it is a convex l -subgroup.*

THEOREM 6.18. *Suppose G is a divisible abelian $\sup \mathcal{Y}$ complete l -group. Then a convex l -subgroup L splits off from below in G if and only if L is maximal below if and only if L is closed.*

PROOF. Given a closed convex l -subgroup L , we observe that each type $\mathcal{Y}$ subset $X \subseteq L^+$ remains type $\mathcal{Y}$ in G . Therefore L is $\sup \mathcal{Y}$ complete and divisible. $\square$

COROLLARY 6.19. *The maximal below l -subgroups of a divisible abelian $\sup \mathcal{Y}$ complete l -group form a lattice under the inclusion order.*

COROLLARY 6.20. *Suppose G is a divisible abelian $\sup \mathcal{Y}$ complete l -group, and that U_1 and U_2 are both maximal above the convex l -subgroup L . Then there is an l -automorphism θ of G such that θ is the identity on the closure of L and $U_1\theta = U_2$.*

LEMMA 6.21. *Suppose G is an abelian l -group. If G is divisible, then so is $G_{\mathcal{Y}}$.*

PROOF. Given X a type $\mathcal{Y}$ subset of G and positive integer n let $Y = \{z \in G \mid nz \in X\}$. Then Y must be type $\mathcal{Y}$, for if $x_1, x_2 \in X$ and if $ny_i = x_i$, then

$$((x_1 - x_2) \vee 0)^+ = ((ny_1 - ny_2) \vee 0)^+ = (n(y_1 - y_2) \vee 0)^+ = ((y_1 - y) \vee 0)^+,$$

meaning $((X - x_i) \vee 0)^+ = ((Y - y_i) \vee 0)^+$. To prove the lemma form $G_{\mathcal{Y}}$ inductively as follows. Let $G_0 = G$, and if $G_\alpha = G_{\mathcal{Y}}$ let $G_{\alpha+1} = G_\alpha$. If $G_\alpha \neq G_{\mathcal{Y}}$ choose $0 < x \in G_{\mathcal{Y}} \setminus G_\alpha$ and let $G_{\alpha+1}$ be the l -subgroup of $G_{\mathcal{Y}}$ generated by $G_\alpha \cup \{qx \mid q \text{ rational}\}$. Let $G_\beta = \bigcup_{\alpha < \beta} G_\alpha$ for limit ordinals β . A simple induction shows G_α divisible for all α , hence $G_{\mathcal{Y}}$, which is G_α for sufficiently large ordinals α , is also divisible. $\square$

In the next proposition G^d designates the l -group divisible hull of G .

PROPOSITION 6.22. *Any abelian l -group G is a large l -subgroup of some abelian l -group H which splits off from below in every abelian l -group in which it (H) is a convex l -subgroup. H may be taken to be $(G^d)_{\mathcal{Y}}$.*

We close this section by giving an example of an l -group G which is $\sup \mathcal{W}$ complete but lacks the SSP. Though G is simple in structure in the sense that $\sum_{n=1}^{\infty} \mathbb{Z}_n \leq G \leq \prod_{n=1}^{\infty} \mathbb{Z}_n$, the construction itself uses the Continuum Hypothesis: let $\{f_\alpha \mid \alpha < \aleph_1\}$ be an enumeration of all functions from N into N , where $N = \{1, 2, 3, \dots\}$. Define $G_0 = \sum_{n=1}^{\infty} \mathbb{Z}_n$. Given countable G_α define $G_{\alpha+1}$ as follows. Let $\{w_k(x) \mid k \in N\}$ be an enumeration of all l -group terms of the form $\bigvee_{i=1}^n \bigwedge_{j=1}^m (g_{ij} + n_{ij}x)$, where $g_{ij} \in G_\alpha$, $n_{ij} \in \mathbb{Z}$, and x is an indeterminate. Iteratively define a function $f: N \rightarrow N$ such that $(n)f \geq (n)f_\alpha$ for all $n \in N$, and such that $(n)(w_k(f)) \neq 1$ for all $1 \leq k \leq n \in N$. Let $G_{\alpha+1}$ be the l -subgroup of $\prod_{n=1}^{\infty} \mathbb{Z}_n$ generated by $G_\alpha \cup \{f\}$. Define $G_\gamma = \bigcup \{G_\alpha \mid \alpha < \gamma\}$ for limit ordinals $\gamma \leq \aleph_1$. Let G be the projectable hull of $G_{\aleph_1}$. That is, G consists of those elements $g \in \prod \mathbb{Z}_n$ for which there exist disjoint sets $X_1, X_2, \dots, X_n \subseteq N$ such that $\bigcup_{i=1}^n X_i = N$, and such that for each X_i there is some $g_i \in G_{\aleph_1}$ with $(n)g = (n)g_i$ for all $n \in X_i$. Observe

that for any $g \in G$ $\{n \in N: (n)g = 1\}$ is finite since this condition holds for all $g \in G_{\aleph_1}$. Since $\Sigma \mathbf{Z}_n \leq G \leq \prod \mathbf{Z}_n$ and since every polar of G is a cardinal summand, G cannot possess the SSP.

To demonstrate that G is $\sup \mathcal{W}$ complete we need only show that every polar is $\sup \mathcal{Z}$ complete. Consider a type $\mathcal{Z}$ subset X of polar P . Because X is type $\mathcal{U}$, $\{(n)p: p \in X\}$ is a bounded set for each index $n \in N$. Let $\alpha < \aleph_1$ be such that $(n)f_\alpha = V\{(n)p: p \in X\}$ for each $n \in N$. There is by construction some element $g \in G$ with $g \geq f_\alpha$, and by projecting it on P we may assume $g \in P$. By definition of type $\mathcal{Z}$, $\forall X = V(X \wedge g)$ exists in P . We have shown P to be $\sup \mathcal{Z}$ complete, and hence G to be $\sup \mathcal{W}$ complete.

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DEPARTMENT OF MATHEMATICS, BOISE STATE UNIVERSITY, BOISE, IDAHO 83725

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, SOUTH BEND, INDIANA 46634