WEIGHTED INEQUALITIES FOR THE ONE-SIDED HARDY-LITTLEWOOD MAXIMAL FUNCTIONS

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ABSTRACT. Let $M^+f(x) = \sup_{h>0} (1/h) \int_x^{x+h} |f(t)| dt$ denote the one-sided maximal function of Hardy and Littlewood. For $w(x) \geq 0$ on R and $1 < p < \infty$, we show that M^+ is bounded on $L^p(w)$ if and only if w satisfies the one-sided A_p condition:

$$(A_p^+) \quad \left[\frac{1}{h} \int_{a-h}^a w(x) dx \right] \left[\frac{1}{h} \int_a^{a+h} w(x)^{-1/(p-1)} dx \right]^{p-1} \leq C$$

for all real a and positive h . If in addition $v(x) \geq 0$ and $\sigma = v^{-1/(p-1)}$, then M^+ is bounded from $L^p(v)$ to $L^p(w)$ if and only if

$$\int_I [M^+(\chi_I \sigma)]^p w \leq C \int_I \sigma < \infty$$

for all intervals $I = (a, b)$ such that $\int_{-\infty}^a w > 0$. The corresponding weak type inequality is also characterized. Further properties of A_p^+ weights, such as $A_{p-\epsilon}^+ \Rightarrow A_p^+$ and $A_p^+ = (A_1^+)(A_1^-)^{1-p}$, are established.

1. Introduction. For f locally integrable on the real line R , define the maximal function Mf at x by

$$Mf(x) = \sup_{a < x < b} \frac{1}{b-a} \int_a^b |f(t)| dt.$$

In [9], B. Muckenhoupt characterized, for $1 < p < \infty$, the nonnegative functions, or weights, $w(x)$ on R satisfying the weighted norm inequality

$$(N_p) \quad \int_{-\infty}^{\infty} [Mf(x)]^p w(x) dx \leq C \int_{-\infty}^{\infty} |f(x)|^p w(x) dx, \quad \text{for all } f,$$

as those weights w satisfying the A_p condition

$$(A_p) \quad \left[\frac{1}{h} \int_a^{a+h} w(x) dx \right] \left[\frac{1}{h} \int_a^{a+h} w(x)^{-1/(p-1)} dx \right]^{p-1} \leq C', \quad a \text{ in } R, h > 0.$$

This leaves open, however, the characterization of the corresponding norm inequalities for the original maximal function of Hardy and Littlewood [5],

$$M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^x |f(t)| dt,$$

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and its counterpart

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)| dt.$$

(M^+ arises in the ergodic maximal function discussed below.) Denote by (N_p^+) and (N_p^-) the weighted inequalities corresponding to (N_p) with M replaced by M^+ and M^- respectively. Observe that A_p is not a necessary condition for either N_p^+ or N_p^- . In fact, the product of any nondecreasing function with an A_p weight satisfies N_p^+ . More generally, if $\{w_\alpha(x)\}_{\alpha \in R}$ is a family of weights uniformly in A_p , and if μ is a positive measure on R , then $w(x) = \int \chi_{(\alpha, \infty)}(x) w_\alpha(x) d\mu(\alpha)$ satisfies N_p^+ .

THEOREM 1. *Suppose $w(x) \geq 0$ on R and $1 < p < \infty$. Then the weighted inequality (N_p^+) holds if and only if w satisfies the one-sided A_p condition (A_p^+)*

$$\left[\frac{1}{h} \int_{a-h}^a w(x) dx \right] \left[\frac{1}{h} \int_a^{a+h} w(x)^{-1/(p-1)} dx \right]^{p-1} \leq C' \quad \text{for all real } a, \text{ and } h > 0.$$

Similarly, (N_p^-) holds if and only if

$$(A_p^-) \quad \left[\frac{1}{h} \int_a^{a+h} w(x) dx \right] \left[\frac{1}{h} \int_{a-h}^a w(x)^{-1/(p-1)} dx \right]^{p-1} \leq C' \quad \text{for all real } a, \text{ and } h > 0.$$

REMARKS. (A) The following “duality” relationship holds: w satisfies A_p^+ if and only if $w^{1-p'}$ satisfies $A_{p'}^-$ where $1/p + 1/p' = 1$.

(B) If $M^- w_1 \leq C w_1$ and $M^+ w_2 \leq C w_2$, it is trivially verified that $w = w_1(w_2)^{1-p}$ satisfies A_p^+ . Theorem 1 together with Remark A and the argument of Coifman, Jones and Rubio in [4] yields the converse: If w satisfies A_p^+ , then there are w_1, w_2 such that $M^- w_1 \leq C w_1$, $M^+ w_2 \leq C w_2$ and $w = w_1(w_2)^{1-p}$. In future we say that w satisfies the $A_1^+(A_1^-)$ condition if $M^- w(M^+ w) \leq C w$.

(C) A modification of B. Jawerth’s proof of the reverse Hölder inequality for A_1 weights [7] shows that if w satisfies A_1^+ , then $w^{1+\delta}$ also satisfies A_1^+ for some $\delta > 0$. (Note however that w cannot in general satisfy a reverse Hölder inequality.) Combining this with the factorization of A_p^+ weights discussed in the previous remark, we obtain the implication $A_p^+ \Rightarrow A_{p-\varepsilon}^+$: More precisely, if w satisfies A_p^+ for a given p , $1 < p < \infty$, then w satisfies $A_{p-\varepsilon}^+$ for some $\varepsilon > 0$. Details can be found in §3.

(D) Suppose T is a measure preserving (not necessarily invertible) ergodic transformation on a probability space $(\Omega, \mathcal{M}, \mu)$. Let

$$f^*(x) = \sup_{k \geq 0} \frac{1}{k+1} \sum_{j=0}^k |f(T^j x)|$$

denote the ergodic maximal function of f . It can be shown (see Atencia and De La Torre [3]) that

$$\int_{\Omega} |f^*(x)|^p w(x) d\mu(x) \leq C \int_{\Omega} |f(x)|^p w(x) d\mu(x)$$

for all f if and only if for μ -almost every x in Ω ,

$$(1.1) \quad \sum_{n=0}^{\infty} \left[\sup_{k \geq 0} \frac{1}{k+1} \sum_{j=0}^k |f(T^{n+j}x)| \right]^p w(T^n x) \leq C \sum_{n=0}^{\infty} |f(T^n x)|^p w(T^n x)$$

for all f defined on $\{T^n x\}_{n=0}^{\infty}$. (The ergodic property is needed only for the necessity of (1.1).) A discrete analogue of Theorem 1 together with elementary measure theory shows that (1.1) holds if and only if there is C' such that for μ -almost every x in Ω and every $k \geq 0$

$$\left[\frac{1}{k+1} \sum_{j=0}^k w(T^j x) \right] \left[\frac{1}{k+1} \sum_{j=k}^{2k} w(T^j x)^{-1/(p-1)} \right]^{p-1} \leq C'.$$

A similar characterization for the two-sided ergodic maximal function corresponding to an invertible measure preserving ergodic transformation T was obtained in [3].

We turn now to the two-weight norm inequality for M^+ :

$$(1.2) \quad \int_{-\infty}^{\infty} |M^+ f|^p w \leq C \int_{-\infty}^{\infty} |f|^p v \quad \text{for all } f.$$

It is convenient to set $\sigma = v^{1-p'} = v^{-1/(p-1)}$ and replace f with $f\sigma$ in (1.2). The resulting equivalent inequality reads

$$(1.3) \quad \int_{-\infty}^{\infty} |M^+(f\sigma)|^p w \leq C \int_{-\infty}^{\infty} |f|^p \sigma \quad \text{for all } f.$$

The corresponding weak type inequality for M^+ is

$$(1.4) \quad |\{M^+(f\sigma) > \lambda\}|_w \leq \frac{C}{\lambda^p} \int_{-\infty}^{\infty} |f|^p \sigma \quad \text{for all } f,$$

where the notation $|E|_w$ stands for $\int_E w(x) dx$. In analogy with results in [11 and 9], we have

THEOREM 2. *Suppose $w(x)$, $\sigma(x)$ are nonnegative measurable functions on $\mathbb{R}$ and $1 < p < \infty$. Then the strong type inequality, (1.3), holds if and only if*

$$(1.5) \quad \int_I |M^+(\chi_I \sigma)|^p w \leq B \int_I \sigma < \infty$$

for all intervals $I = (a, b)$ such that $\int_{-\infty}^a w > 0$. If C and B are the best constants in (1.3) and (1.5) then their ratio is bounded between two positive constants depending only on p .

The weak type inequality, (1.4), holds if and only if²

$$(1.6) \quad \left[\frac{1}{h} \int_{a-h}^a w \right] \left[\frac{1}{h} \int_a^{a+h} \sigma \right]^{p-1} \leq A \quad \text{for all } a \in \mathbb{R}, h > 0.$$

Again, the best constants in (1.4) and (1.6) are comparable. Corresponding results hold for M^- .

² This characterization of the weak type inequality is due to K. Andersen.

Theorem 1 follows easily from Theorem 2 using a clever argument of Hunt, Kurtz and Neugebauer [6] as follows. First, N_p^+ implies A_p^+ by a standard argument in [9] that involves testing N_p^+ with $f = \chi_{(a, a+h)} w^{1-p'}$. For the converse, it suffices to show that A_p^+ implies (1.5) with $\sigma = w^{1-p'}$. Fix an interval $I = (a, b)$ with $\int_{-\infty}^a w > 0$ and a point x in I . Choose $h > 0$ so that $x + h$ is in I and $h^{-1} \int_x^{x+h} \sigma \geq \frac{3}{4} M^+(\chi_I \sigma)(x)$. Since $h^{-1} \int_x^{x+h/2} \sigma \leq \frac{1}{2} M^+(\chi_I \sigma)(x)$ by definition, we conclude

$$\begin{aligned} M^+(\chi_I \sigma)(x) &\leq \frac{4}{h} \int_{x+h/2}^{x+h} \sigma \leq C \left[\frac{h}{2} / \int_x^{x+h/2} w \right]^{p'-1} \text{ by } A_p^+ \\ &= C \left[\int_x^{x+h/2} w^{-1} w / \int_x^{x+h/2} w \right]^{p'-1} \leq C M_w^+(\chi_I w^{-1})(x)^{p'-1} \end{aligned}$$

where

$$M_w^+ f(x) = \sup_{h>0} \left[\int_x^{x+h} |f| w / \int_x^{x+h} w \right].$$

Now M_w^+ is bounded on $L^q(w)$ for any $w \geq 0$ on R and $1 < q < \infty$, and thus

$$\int_I |M^+(\chi_I \sigma)|^p w \leq C^p \int_I |M_w^+(\chi_I w^{-1})|^{p'} w \leq C \int_I \sigma.$$

Finally, we consider reverse weighted inequalities for M^+ . For v, w nonnegative and locally integrable on R and $1 < p < \infty$, the reverse weighted strong type inequality

$$(1.7) \quad \int_{-\infty}^{\infty} |f|^p v \leq C \int_{-\infty}^{\infty} |M^+ f|^p w \quad \text{for all } f$$

holds only in two trivial cases: either $\int_{-\infty}^{\infty} w(x)/|x|^p dx = \infty$ or $v(x) \leq C' w(x)$ for a.e. x . The reverse weighted weak type (1, 1) inequality,

$$(1.8) \quad |\{M^+ f > \lambda\}|_w \geq \frac{C}{\lambda} \int_{\{f > \lambda\}} f v \quad \text{for all } f \geq 0,$$

holds if and only if for almost every a in R ,

$$(1.9) \quad \inf_{h>0} \frac{1}{h} \int_{a-h}^a w(x) dx \geq C' v(a).$$

Proofs of these assertions can be found in §3. See [2 and 10] concerning the reverse weighted weak-type (1, 1) inequality for M and its applications.

Throughout this paper the letter C will denote a positive constant that may vary from line to line but will remain independent of the relevant quantities.

2. Proof of Theorem 2. In proving the analogue of Theorem 2 for the two-sided maximal function, the following key property of M is used: $Mf(x) \geq (1/|I|) \int_I |f|$ for x in I . This fails for both M^+ and M^- and accounts for the bulk of difficulty in dealing with one-sided maximal operators. We circumvent this obstacle with the aid of the next lemma and some known results on Hardy operators.

LEMMA 2.1. *Suppose $g \geq 0$ is integrable with compact support on R . If $I = (a, b)$ is a component interval of the open set $\{M^+ g > \lambda\}$, $\lambda > 0$, then*

$$(2.1) \quad \frac{1}{b-x} \int_x^b g \geq \lambda \quad \text{for } a \leq x < b.$$

To prove the lemma, fix $a < x < b$ and let r be the largest number such that $(r - x)^{-1} \int_x^r g \geq \lambda$. If $r < b$, then $(s - r)^{-1} \int_r^s g > \lambda$ for some $s > r$ by definition and this yields $(s - x)^{-1} \int_x^s g \geq \lambda$, contradicting the definition of r . Thus $r \geq b$. If $r > b$, then $(r - b)^{-1} \int_b^r g \leq \lambda$ since b is not in $\{M^+g > \lambda\}$. Together with $(r - x)^{-1} \int_x^r g \geq \lambda$, we obtain $(b - x)^{-1} \int_x^b g \geq \lambda$ as required.

To deal with the weak type inequality (1.4) we need the following ([12]: see Theorem 4 and the subsequent note; see also [1]).

LEMMA 2.2. *Let σ, w be nonnegative weights on $(0, \infty)$, $1 < p < \infty$ and $T_1g(x) = x^{-1} \int_0^x g(t) dt$ for locally integrable g . Then*

$$|\{T_1(f\sigma) > \lambda\}|_w \leq C \left[\sup_{0 < x \leq s < \infty} s^{-p} \left(\int_x^s w \right) \left(\int_0^x \sigma \right)^{p-1} \right] \frac{1}{\lambda^p} \int_0^\infty |f|^p \sigma.$$

We now prove the equivalence of (1.4) and (1.6). First, (1.4) implies (1.6) by a standard argument (see e.g. [9]) that involves testing (1.4) with $f = \chi_I$ and $\lambda = \frac{1}{2} \int_I \sigma$. Conversely, suppose (1.6) holds. It suffices to prove (1.4) for functions $f \geq 0$ such that $f\sigma$ is bounded with compact support. So fix such an f and a $\lambda > 0$. Let $\{I_j\}_j$ be the component intervals of $\{M^+(f\sigma) > \lambda\}$. Applying Lemmas 2.1 and 2.2 (with a linear change of variable) to a fixed interval $I_j = (a, b)$, we obtain

$$\begin{aligned} |I_j|_w &\leq \left| \left\{ x: \frac{1}{b-x} \int_x^b \chi_{I_j} f\sigma \geq \lambda \right\} \right|_w \\ &\leq C \left[\sup_{a \leq s \leq x < b} (b-s)^{-p} \left(\int_s^x w \right) \left(\int_x^b \sigma \right)^{p-1} \right] \frac{1}{\lambda^p} \int_{I_j} |f|^p \sigma \\ &\leq \frac{CA}{\lambda^p} \int_{I_j} |f|^p \sigma \quad \text{by (1.6) with } a = x \text{ and } h = b - s. \end{aligned}$$

Summing over j yields (1.4).

To deal with the strong type inequality (1.3) we need an apparent strengthening of the usual weighted inequality for the adjoint Hardy operator.

LEMMA 2.3. *Suppose σ, u are nonnegative weights on R and $1 < p < \infty$. Then for all f ,*

$$\int_{-\infty}^{\infty} \left| \int_x^{\infty} f\sigma \right|^p u(x) dx \leq C_p \int_{-\infty}^{\infty} \left[\sup_{r \leq x} \left(\int_{-\infty}^r u \right) \left(\int_r^{\infty} \sigma \right)^{p-1} \right] |f(x)|^p \sigma(x) dx.$$

To see this, rewrite (2.2) as $\int_{-\infty}^{\infty} |f_x^\infty g|^p u(x) dx \leq C_p \int_{-\infty}^{\infty} |g|^p \mu v$ where $v = \sigma^{1-p}$, $g = f\sigma$ and $\mu(x)$ denotes the supremum in square brackets on the right side of (2.2). This latter inequality holds since $(\int_{-\infty}^r u)(\int_r^{\infty} (\mu v)^{1-p'})^{p-1} \leq 1$ for all $r > 0$ (see [8]).

We will also need

LEMMA 2.4. *For $1 < p < \infty$ and $\sigma, w \geq 0$ on R , condition (1.5) implies*

$$(2.3) \quad \left[\int_{-\infty}^r \frac{w(x)}{(b-x)^p} dx \right] \left[\int_r^b \sigma \right]^{p-1} \leq CB \quad \text{for all } -\infty < r \leq b < \infty.$$

To prove this, let $x_0 = r > x_1 > x_2 \cdots$ satisfy $\int_{x_k}^b \sigma = 2^k \int_r^b \sigma$ for $k = 0, 1, 2, \dots$. Then

$$\begin{aligned} \left[\int_{-\infty}^r \frac{w(x)}{(b-x)^p} dx \right] \left[\int_r^b \sigma \right]^p &= \sum_{k=0}^{\infty} \left[\int_{x_{k+1}}^{x_k} \frac{w(x)}{(b-x)^p} dx \right] 2^{-kp} \left[\int_{x_k}^b \sigma \right]^p \\ &\leq \sum_{k=0}^{\infty} 2^{-kp} \int_{x_{k+1}}^b |M^+(\chi_{(x_{k+1}, b)} \sigma)|^p w \\ &\leq \sum_{k=0}^{\infty} 2^{-kp} B \int_{x_{k+1}}^b \sigma = B \sum_{k=0}^{\infty} 2^{-kp+k+1} \int_r^b \sigma = CB \int_r^b \sigma \end{aligned}$$

by (1.5) which yields (2.3).

We now prove the equivalence of (1.3) and (1.5). Once again, a standard argument (see e.g. [11]) shows that (1.3) implies (1.5). Conversely, suppose (1.5) holds. It suffices to prove (1.3) for functions $f \geq 0$ such that $f\sigma$ is bounded with compact support. So fix such an f and for k in $\mathbb{Z}$, let $I_j^k = (a_j^k, b_j^k)$, j an integer, be the component intervals of the open set $\Omega_k = \{M^+ f\sigma > 2^k\}$. With $E_j^k = I_j^k - \Omega_{k+1}$ we have

$$(2.4) \quad \int_{-\infty}^{\infty} |M^+(f\sigma)|^p w \leq 2^p \sum_k 2^{kp} |\Omega_k - \Omega_{k+1}|_w \leq C \sum_{k,j} 2^{kp} |E_j^k|_w.$$

For future reference, let

$$\mu_j^k(x) = \sup_{a_j^k \leq r \leq x} \left[\int_{a_j^k}^r \frac{\chi_{E_j^k} w(t) dt}{(b_j^k - t)^p} \right] \left[\int_r^{b_j^k} \sigma \right]^{p-1} \quad \text{for } x \text{ in } I_j^k.$$

We now fix k, j momentarily and estimate $2^{kp} |E_j^k|_w$. For convenience in writing let $I = (a, b) = I_j^k$, $E = E_j^k$, $\mu = \mu_j^k$ and for those I_i^{k+1} contained in I_j^k , let $J_i = I_i^{k+1}$. Define $g = \chi_{E \setminus f}$ and $h = \sum_i (|J_i|_{\sigma}^{-1} \int_{J_i} f\sigma) \chi_{J_i}$. For x in E we have

$$2^k \leq \frac{1}{b-x} \int_x^b f\sigma = \frac{1}{b-x} \int_x^b (g+h)\sigma$$

by Lemma 2.1 and so

$$\begin{aligned} (2.5) \quad 2^{kp} |E|_w &\leq \int_E \frac{w(x)}{(b-x)^p} \left(\int_x^b (g+h)\sigma \right)^p dx \\ &\leq C \int \mu(x) [g(x)^p + h(x)^p] \sigma(x) dx \quad \text{by Lemma 2.3} \\ &\leq CB \int g(x)^p \sigma(x) dx + C \int \mu(x) h(x)^p \sigma(x) dx \end{aligned}$$

by Lemma 2.4. Reverting to our previous notation (2.5) becomes

$$\begin{aligned} (2.6) \quad 2^{kp} |E_j^k|_w &\leq CB \int_{E_j^k} f^p \sigma \\ &\quad + C \sum_{I_i^{k+1} \subset I_j^k} \mu_j^k(a_i^{k+1}) |I_i^{k+1}|_{\sigma} \left(\frac{1}{|I_i^{k+1}|_{\sigma}} \int_{I_i^{k+1}} f\sigma \right)^p. \end{aligned}$$

Plugging (2.6) into (2.4) we obtain

$$(2.7) \quad \int_{-\infty}^{\infty} |M^+(f\sigma)|^p w \leq CB \int f^p \sigma + C \sum_{k,j} \sum_{I_i^{k+1} \subset I_j^k} \mu_j^k(a_i^{k+1}) |I_i^{k+1}|_{\sigma} \left(\frac{1}{|I_i^{k+1}|_{\sigma}} \int_{I_i^{k+1}} f \sigma \right)^p$$

since the E_j^k are pairwise disjoint.

Since the I_j^k are nested ($k < l \Rightarrow$ either $I_i^l \subset I_j^k$ or $I_i^l \cap I_j^k = \emptyset$ for all i, j), a standard interpolation argument (see [11]) shows that the second term on the right side of (2.7) is dominated by $CB \int f^p \sigma$ provided the following Carleson condition holds:

$$(2.8) \quad \sum_{I_j^k \subset I_s^l} \gamma_j^k |I_j^k|_{\sigma} \leq CB |I_s^l|_{\sigma}, \quad \text{for all } t, s$$

where $\gamma_j^k = \mu_l^{k-1}(a_j^k)$ and where l is such that $I_j^k \subset I_l^{k-1}$. It will be convenient to denote this “predecessor” of I_j^k , namely I_l^{k-1} , by (c_j^k, d_j^k) . Since $\gamma_s^t \leq CB |I_s^t|_{\sigma}$ by Lemma 2.4, we need only estimate the sum in (2.8) over intervals I_j^k properly contained in I_s^t . For each k, j let r_j^k satisfy $c_j^k \leq r_j^k \leq a_j^k$ and

$$\left[\int_{c_j^k}^{r_j^k} \frac{\chi_{E_l^{k-1}} w(t)}{(d_j^k - t)^p} dt \right] \left[\int_{r_j^k}^{d_j^k} \sigma \right]^{p-1} \geq \frac{\gamma_j^k}{2}.$$

Then, for fixed I_l^{k-1} contained in I_s^t we have

$$(2.9) \quad \begin{aligned} \sum_{I_j^k \subset I_l^{k-1}} \gamma_j^k |I_j^k|_{\sigma} &\leq \sum_{I_j^k \subset I_l^{k-1}} \int \frac{\chi_{E_l^{k-1}} w(t)}{(d_j^k - t)^p} \left[\int_{r_j^k}^{d_j^k} \sigma \right]^{p-1} \chi_{(c_j^k, r_j^k)}(t) |I_j^k|_{\sigma} dt \\ &= \int_{E_l^{k-1}} \frac{w(t)}{(b_l^{k-1} - t)^p} \left\{ \sum_{I_j^k \subset I_l^{k-1}} \chi_{(c_j^k, r_j^k)}(t) \left(\int_{r_j^k}^{b_l^{k-1}} \sigma \right)^{p-1} |I_j^k|_{\sigma} \right\} dt \\ &\leq \int_{E_l^{k-1}} \frac{w(t)}{(b_l^{k-1} - t)^p} \left(\int_t^{b_l^{k-1}} \sigma \right)^p dt \\ &\leq \int_{E_l^{k-1}} |M^+(\chi_{I_s^t} \sigma)|^p w. \end{aligned}$$

Summing (2.9) over all I_l^{k-1} contained in a fixed I_s^t , we obtain

$$\begin{aligned} \sum_{I_j^k \not\subset I_s^t} \gamma_j^k |I_j^k|_{\sigma} &\leq \sum_{I_l^{k-1} \subset I_s^t} \int_{E_l^{k-1}} |M^+(\chi_{I_s^t} \sigma)|^p w \\ &\leq \int_{I_s^t} |M^+(\chi_{I_s^t} \sigma)|^p w \leq B |I_s^t|_{\sigma} \end{aligned}$$

by (1.5). This establishes (2.8) and completes the proof of Theorem 2.

3. Appendix. We now complete the proof of Remark (C). Suppose w satisfies A_1^+ , i.e. $M^-w \leq Cw$. Fix an interval $I = (a, b)$. If $\lambda > M^-w(b)$, then $\Omega_\lambda = \{M^-(\chi_I w) > \lambda\}$ is contained in I . If $\{I_j\}_j$ are the component intervals of Ω_λ , then $(1/|I_j|) \int_{I_j} w \geq \lambda$ for all j by Lemma 2.1. But $(1/|I_j|) \int_{I_j} w \leq \lambda$ since the right endpoint of I_j is not in Ω_λ . Thus we have

$$(3.1) \quad |\Omega_\lambda|_w = \sum_j \int_{I_j} w = \lambda \sum_j |I_j| = \lambda |\Omega_\lambda|$$

$$\leq \lambda |I \cap \{w > \lambda/C\}| \quad \text{since } M^-w \leq Cw.$$

The argument of B. Jawerth in §5 of [7] now applies as follows.

$$\begin{aligned} \int_{I \cap \{w > M^-w(b)\}} w^{1+\delta} &= |I \cap \{w > M^-w(b)\}| (M^-w(b))^{1+\delta} \\ &\quad + \delta \int_{M^-w(b)}^\infty \lambda^{\delta-1} |I \cap \{w > \lambda\}|_w d\lambda \\ &\leq |I| (M^-w(b))^{1+\delta} + \delta \int_{M^-w(b)}^\infty \lambda^\delta \left| I \cap \left\{ w > \frac{\lambda}{C} \right\} \right| d\lambda \end{aligned}$$

(using $|I \cap \{w > \lambda\}|_w \leq |\Omega_\lambda|_w$ and then (3.1))

$$\leq |I| (M^-w(b))^{1+\delta} + C \frac{\delta}{1+\delta} \int_{I \cap \{w > M^-w(b)\}} w^{1+\delta}.$$

Choosing $\delta > 0$ sufficiently small we get

$$\int_I w^{1+\delta} \leq C |I| (M^-w(b))^{1+\delta} \leq C |I| w^{1+\delta}(b)$$

since $M^-w \leq Cw$, and this shows that $M^-(w^{1+\delta}) \leq Cw^{1+\delta}$ as required.

We now prove the assertions made in the introduction concerning reverse weighted inequalities for M^+ . Suppose $1 < p < \infty$ and v, w are nonnegative locally integrable weights satisfying the reverse weighted inequality (1.7). Suppose further that $\int_{-\infty}^x w(x)/|x|^p dx < \infty$. We must show that $v(x) \leq C'w(x)$ for a.e. x . Fix x , a Lebesgue point of both v and w , and let $\varepsilon > 0$ be given. Choose $R > 0$ so that $r^{-1} \int_{x-r}^x w \leq w(x) + \varepsilon$ for $0 < r \leq R$. For $k \geq 1$, set $r_k = 2^{-k}R$. With $f = \chi_{(x-r_k, x)}$ in (1.7) we obtain

$$\begin{aligned} \frac{1}{r_k} \int_{x-r_k}^x v &\leq C \sum_{j=0}^k \left(\frac{1}{2^j} \right)^{p-1} \left(\frac{1}{2^j r_k} \int_{x-2^j r_k}^x w \right) + C r_k^{p-1} \int_{-\infty}^{x-R} \frac{w(y)}{(r_k + |y|)^p} dy \\ &\leq C_p (w(x) + \varepsilon) + C \left(\frac{R}{2^k} \right)^{p-1} \int_{-\infty}^{x-R} \frac{w(y)}{|x-y|^p} dy. \end{aligned}$$

The integral in the second term on the right above is finite since $\int_{-\infty}^x w(x)/|x|^p dx < \infty$. Thus, if $k \rightarrow \infty$, we get $v(x) \leq C_p(w(x) + \varepsilon)$ and since $\varepsilon > 0$ is arbitrary, $v(x) \leq C_p w(x)$.

We now prove the equivalence of the reverse weighted weak type (1, 1) inequality, (1.8), and condition (1.9). Fix a , a Lebesgue point of v , and $h > 0$. For $0 < \varepsilon < h$,

let $f_\varepsilon = \varepsilon^{-1} \chi_{(a-\varepsilon, a)}$. Then $\{M^+ f_\varepsilon > 1/h\} = (a-h, a)$ and (1.8) yields $\int_{a-h}^a w \geq Ch \varepsilon^{-1} \int_{a-\varepsilon}^a v$. Letting $\varepsilon \rightarrow 0$ we obtain (1.9) with $C' = C$. Conversely, fix $f \geq 0$ bounded with compact support and $\lambda > 0$. Let $(I_j)_j$ be the component intervals of the open set $\Omega_\lambda = \{M^+ f > \lambda\}$. We claim

$$(3.2) \quad |I_j|_w \geq \frac{C}{\lambda} \int_{I_j} f v \quad \text{for all } j.$$

To see (3.2), suppose (for convenience) that $I_j = (0, 1)$. Then

$$\frac{1}{t} \int_t^{2t} f \leq 2 \frac{1}{2t} \int_0^{2t} f \leq 2M^+ f(0) \leq 2\lambda$$

for $0 < t < 1$. Thus

$$\begin{aligned} \int_0^1 w &\geq \frac{1}{2\lambda} \int_0^1 \left[\frac{1}{t} \int_t^{2t} f(x) dx \right] w(t) dt \\ &\geq \frac{1}{2\lambda} \int_0^1 f(x) \left[\int_{x/2}^x \frac{w(t)}{t} dt \right] dx \\ &\geq \frac{C'}{4\lambda} \int_0^1 f(x) v(x) dx \end{aligned}$$

by (1.9) as required. Summing (3.2) over j yields

$$|\Omega_\lambda|_w = \sum_j |I_j|_w \geq \frac{C}{\lambda} \int_{\{M^+ f > \lambda\}} f v \geq \frac{C}{\lambda} \int_{\{f > \lambda\}} f v.$$

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