DIMENSION-FREE QUASICONFORMAL DISTORTION IN n -SPACE

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ABSTRACT. Most distortion theorems for K -quasiconformal mappings in $\mathbf{R}^n$, $n \geq 2$, depend on both n and K in an essential way, with bounds that become infinite as n tends to ∞ . The present authors obtain dimension-free versions of four well-known distortion theorems for quasiconformal mappings—namely, bounds for the linear dilatation, the Schwarz lemma, the Θ -distortion theorem, and the η -quasisymmetry property of these mappings. They show that the upper estimates they have obtained in each of these four main results remain bounded as n tends to ∞ with K fixed. The proofs are based on a “dimension-cancellation” property of the function $t \mapsto \tau^{-1}(\tau(t)/K)$, $t > 0$, $K > 0$, where $\tau(t)$ is the capacity of a Teichmüller extremal ring in $\mathbf{R}^n$. The authors also prove a dimension-free distortion theorem for the absolute (cross) ratio under K -quasiconformal mappings of $\overline{\mathbf{R}}^n$, from which several other distortion theorems follow as special cases.

1. Introduction. For any integer $n \geq 2$ and for $1 \leq K < \infty$, let $\mathcal{F}(n, K)$ denote the class of K -quasiconformal mappings $f: \overline{\mathbf{R}}^n \rightarrow \overline{\mathbf{R}}^n$ normalized by the two conditions $f(0) = 0$ and $f(\infty) = \infty$. One can show that there exists a constant $A = A(n, K)$ such that

$$(1.1) \quad |f(x)| \leq A|f(y)| \quad \text{for } |x| \leq |y|$$

for all $f \in \mathcal{F}(n, K)$. Define

$$(1.2) \quad H_n(K) = \inf\{A: (1.1) \text{ holds for all } f \in \mathcal{F}(n, K)\}.$$

In his paper of 1957 A. Mori [M] (cf. [K, p. 81]) proved that

$$(1.3) \quad H_2(K) \leq e^{\pi K}.$$

Then in 1959 O. Lehto, K. I. Virtanen, and J. Väisälä [LVV] found the exact value of $H_2(K)$, namely

$$(1.4) \quad \begin{aligned} H_2(K) &= \lambda(K) = (\mu^{-1}(\pi/2K)/\mu^{-1}(\pi K/2))^2 \\ &= \frac{1}{16}e^{\pi K} - \frac{1}{2} + \delta(K), \quad 0 < \delta(K) < 2e^{-\pi K}, \end{aligned}$$

where μ is as defined in (2.4) below.

An n -dimensional version of (1.3) was established by F. W. Gehring, who proved in 1962 [G2, Lemma 8, pp. 371, 372] that

$$(1.5) \quad H_n(K) \leq d(n, K) = \exp\left(2(K\sigma_{n-1}/\gamma(\sqrt{2}))^{1/(n-1)}\right),$$

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where $d(2, K) = e^{\pi K}$. Here $\gamma(s)$ denotes the conformal capacity of the Grötzsch ring $R_{G,n}(s)$ in $\mathbf{R}^n$ (cf. (1.9), (2.1) below), and σ_{n-1} is the $(n-1)$ -dimensional surface area of the unit sphere S^{n-1} in $\mathbf{R}^n$.

Recently M. Vuorinen [Vu2] improved (1.5) by showing that

$$(1.6) \quad H_n(K) \leq c(n, K) < d(n, K),$$

where the *distortion coefficient* $c(n, K)$ is given by

$$(1.7) \quad c(n, K) = \left[\gamma^{-1} \left(\frac{1}{K} \gamma(\sqrt{2}) \right) \right]^2.$$

The definition of $c(n, K)$ is extended for $K \in (0, 1)$ in an obvious way. The proof of (1.6) was based on the study of a conformal invariant [Vu1] and on an inequality for the moduli of curve families due to F. W. Gehring [Vu1, Lemma 2.58]. Further, it was shown in [Vu2] that the constant $c(2, K)$ is asymptotically sharp, that is,

$$(1.8) \quad c(2, K)/H_2(K) \rightarrow 1 \quad \text{as } K \rightarrow \infty,$$

while $d(2, K)/H_2(K) \rightarrow 16$.

Continuing the work of [Vu1, Vu2] we shall develop in this paper a distortion theory for K -quasiconformal mappings in $\mathbf{R}^n$, $n \geq 2$, that differs essentially from previously known results in that it is independent of the dimension n . Nevertheless, this dimension-free theory preserves some important features of the n -dimensional theory such as Hölder-continuity, and some of our theorems yield better distortion results than the previously known ones also in the n -dimensional case.

The general plan of this paper is as follows. Applying some results of G. D. Anderson [An2] we prove a result of technical character which we call a “dimension-cancellation” lemma. This lemma enables us to find dimension-free lower and upper bounds for the special functions $t \mapsto \gamma^{-1}(A\gamma(t))$ and $t \mapsto \tau^{-1}(A\tau(t))$, $A > 0$, where

$$(1.9) \quad \begin{aligned} \gamma(s) &= \gamma_n(s) = \text{cap } R_{G,n}(s), & s > 1, \\ \tau(t) &= \tau_n(t) = \text{cap } R_{T,n}(t), & t > 0, \end{aligned}$$

are the conformal capacities of the Grötzsch and Teichmüller ring domains, respectively, in $\mathbf{R}^n$ (see §2). Next, applying these dimension-free bounds and F. W. Gehring’s modulus estimates in [G3, Corollary 1; Vu1, Lemma 2.58] we shall prove several dimension-free distortion theorems for quasiconformal mappings.

We now proceed to state some of our main theorems. The first three basic theorems show that the inequality (1.6) is indeed much better than (1.5).

1.10. THEOREM. $c(2, K) = 1 + \lambda(K)$ for all $K > 0$.

1.11. THEOREM. For all $n \geq 2$ and $K \geq 1$,

$$10c(n, K) < d(n, K) < \frac{1}{2}e^{2n}c(n, K)$$

and, moreover,

$$\lim_n c(n, K)^{1/n} = 1, \quad \lim_n d(n, K)^{1/n} = e^2.$$

1.12. THEOREM. For all $n \geq 2$ and $K > 0$,

$$\coth^2(\pi/2K) \leq c(n, K) \leq 1 + \lambda \left(\pi K / \left(2 \log \left(1 + \sqrt{2} \right) \right) \right).$$

As shown in [Vu2, Theorem 1.5], the constant $c(n, K)$ yields an upper bound for the linear dilatation $H(x, f)$ (see [V2, p. 78] for a definition) of a K -quasiconformal mapping $f: G \rightarrow fG$ of a subdomain G of $\mathbf{R}^n$, that is, $H(x, f) \leq c(n, K)$ for all $x \in G$. Thus Theorem 1.12 answers affirmatively a question arising from a problem of J. Väisälä: *The linear dilatation of a K -quasiconformal mapping $f: G \rightarrow fG$, $G \subset \mathbf{R}^n$, has an upper bound depending on K only for all $n \geq 2$.* The constant $d(n, K)$ in (1.5) also yields an upper bound for the linear dilatation (cf. [G2, Lemma 8]), but it follows from Theorem 1.11 that $\sup_n d(n, K) = \infty$.

We shall also prove a number of theorems about the behavior of *absolute* (cross) *ratios* under quasiconformal mappings of $\mathbf{R}^n$. As a special case, the following result for the class $\mathcal{F}(n, K)$ introduced in (1.2) is obtained.

1.13. THEOREM. *Let $f \in \mathcal{F}(n, K)$, $|x| \leq |y|$, and let $f(x) = \beta f(y)$ for some $\beta < 0$. Then*

$$|f(x)| \leq (c(n, K) - 1)|f(y)|.$$

This inequality is sharp for $n = 2$.

We next state a sharp lower bound for the expression $H_n(K)$ introduced in (1.2). Note that an upper bound for $H_n(K)$ was given in (1.6).

1.14. THEOREM. *$H_n(K) \geq \lambda(K^{1/(n-1)})$ for all $n \geq 2$ and $K \geq 1$. There is equality when $n = 2$.*

The Schwarz Lemma for plane quasiconformal mappings was proved in 1952 by J. Hersch and A. Pfluger [HP] and, with an improved explicit estimate, by Chuan-Fang Wang [W] (cf. [LV, (3.6), p. 65]) in 1960. In 1970 O. Martio, S. Rickman, and J. Väisälä [MRV, Theorem 3.1] proved an n -dimensional analogue of the Hersch-Pfluger result for the unit ball B^n in $\mathbf{R}^n$. Generalizing the proof of O. Lehto and K. I. Virtanen [LV, (3.6), p. 65] we shall give the following sharp dimension-free Schwarz Lemma.

1.15. THEOREM. *Let $f: B^n \rightarrow fB^n$ be K -quasiconformal with $f(0) = 0$ and $fB^n \subset B^n$. Then*

$$|f(x)| \leq \lambda_n^{1-\alpha} |x|^\alpha \leq 2^{1-1/K} K |x|^{1/K}$$

for $x \in B^n$, where $\alpha = K^{1/(1-n)}$ and λ_n is a constant depending only on n . For each $n \geq 2$ the inequality is sharp for $K = 1$.

Combining this result with our results about the absolute ratio, we shall show that a K -quasiconformal mapping of B^n into $\mathbf{R}^n$ is η -quasisymmetric (see §5 for the definition) in $\overline{B}^n(1/2) = \{x \in \mathbf{R}^n: |x| \leq 1/2\}$, with η depending only on K . This improves earlier results of P. Tukia and J. Väisälä [TV] and J. Väisälä [V2], where a dimension-dependent function η was found. Although the main point in our result is the fact that η is dimension-free it seems likely that the estimates established for the construction of η improve previously known functions of quasisymmetry also for each dimension $n \geq 2$. For $n = 2$, in particular, an improved form of a result of F. W. Gehring [G5, Theorem 2.6, p. 21] is obtained.

We shall also study the so-called Θ -distortion theorem of F. W. Gehring [G2]. Although the original result [G2] again depends on the dimension in an essential way we shall show by applying our earlier results that this result also has a dimension-free counterpart.

Finally, we wish to point out a fundamental difference in the theory between the dimensions $n = 2$ and $n \geq 3$. In the first case, since $\gamma_2(s)$ has an explicit expression in terms of elliptic integrals (cf. [LV, p. 60]), one may approximate $\gamma_2(s)$ with arbitrary precision by successive application of Landen's Transformation for elliptic integrals (cf. [Bo, p. 72; BF, p. 39; H, p. 316]). A geometric development of the same iteration procedure is contained in [LV, pp. 60–62]. For $n \geq 3$, neither an explicit formula for $\gamma_n(s)$ nor inequalities with arbitrary precision for all $s > 1$ are known to us. It is therefore somewhat surprising that some of the n -dimensional results of this paper, when specialized to the case $n = 2$, yield nearly best possible results.

We shall adopt the relatively standard notation and terminology of [V1]. Unit vectors in the directions of the rectangular coordinate axes in $\mathbf{R}^n$ are denoted by $e_1, \dots, e_n$. For $x \in \mathbf{R}^n$ and $r > 0$ we let $B^n(x, r) = \{z \in \mathbf{R}^n : |x - z| < r\}$, $S^{n-1}(x, r) = \partial B^n(x, r)$, $B^n(r) = B^n(0, r)$, $S^{n-1}(r) = \partial B^n(r)$, $B^n = B^n(1)$, and $S^{n-1} = \partial B^n$. We follow J. Väisälä's definition of K -quasiconformality [V1], which is also equivalent to $K^{1/(n-1)}$ -quasiconformality in the definition given by F. W. Gehring [G2].

2. Distortion of the absolute ratio. A ring $R = R(C_0, C_1)$ is a domain (open connected set) in $\mathbf{R}^n$ whose complement in $\bar{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ consists of two components C_0 and C_1 , where C_0 is bounded. The (conformal) capacity $\text{cap } R$ of a ring R is defined by

$$(2.1) \quad \text{cap } R = \inf \int_R |\nabla u|^n dm,$$

where the infimum is taken over all continuous functions $u: \bar{\mathbf{R}}^n \rightarrow \mathbf{R}$ such that u is ACT (Absolutely Continuous in the sense of Tonelli [S]) in $\mathbf{R}^n$, $u = 0$ on C_0 and $u = 1$ on C_1 . The modulus $\text{mod } R$ is then

$$(2.2) \quad \text{mod } R = (\sigma_{n-1} / \text{cap } R)^{1/(n-1)}.$$

A ring is said to be *extremal* if it has the maximum modulus among all rings with a certain geometric property.

Two extremal rings are particularly important in our study. The first is the Grötzsch ring $R_{G,n}(s)$, $s > 1$, whose complementary components are the closed unit ball $\bar{B}^n$ and the ray $[s, \infty]$ along the x_1 -axis. The other is the Teichmüller ring $R_{T,n}(t)$, $t > 0$, whose complementary components are the segment $[-1, 0]$ and the ray $[t, \infty]$ along the x_1 -axis. It follows from the proof of Lemma 6 in [G1] (cf. [An4, Theorem 3]) that γ and τ are strictly decreasing on $(1, \infty)$ and $(0, \infty)$, respectively. For convenience we set $\gamma(s) = \infty$ for all $s \leq 1$ and $\tau(t) = \infty$ for all $t \leq 0$. The functions γ_n and τ_n are related by the basic formula

$$(2.3) \quad \gamma_n(s) = 2^{n-1} \tau_n(s^2 - 1)$$

(cf. [G1, §18; G4; Vu1, (2.10)]). For $n = 2$

$$(2.4) \quad \gamma_2(s) = 2\pi / \mu(1/s),$$

where $s > 1$ and μ satisfies the functional equations

$$(2.5) \quad \mu(r)\mu\left(\sqrt{1-r^2}\right) = \frac{\pi^2}{4}, \quad \mu(r)\mu\left(\frac{1-r}{1+r}\right) = \frac{\pi^2}{2}, \quad \mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1+r}\right)$$

(cf. [LV, (2.7), (2.9), (2.3), pp. 60, 61]) for $0 < r < 1$.

The extremal properties of these rings (cf. [G1, An1]) are indispensable in the study of distortion under quasiconformal mappings. In fact the principal tools in many of the proofs in this paper are the modulus estimates for the Grötzsch and Teichmüller rings that follow from Theorem 2 of [An2], and the following two inequalities due to F. W. Gehring that express extremal properties of the Teichmüller ring. We shall use q to denote the chordal distance in $\bar{\mathbf{R}}^n$ (see [G3, p. 226] or [V1, p. 37] for a definition).

2.6. LEMMA. Let $R = R(C_0, C_1)$ be a ring in $\mathbf{R}^n$ with $x, y \in C_0$ and $z, w \in C_1$. Then

$$\text{cap } R \geq \tau \left(\frac{q(x, z)q(y, w)}{q(x, y)q(z, w)} \right).$$

2.7. LEMMA. Let $R_1 = R([0, e_1], [t^2 e_1, \infty))$ and $R_2 = R([0, e], [t^2 e_1, \infty))$, where $e \in S^{n-1}$ and $t > 1$. Then

$$\text{cap } R_2 \leq \text{cap } R_1 = \tau(t^2 - 1).$$

Proofs of Lemmas 2.6, 2.7 may be found in [G3, Corollary 7; Vu1, Lemma 2.58], respectively.

Next, if x, y, z, w is an ordered quadruple of distinct points in $\bar{\mathbf{R}}^n$, the *absolute ratio* $|x, y, z, w|$ is defined by

$$(2.8) \quad |x, y, z, w| = \frac{q(x, y)q(z, w)}{q(x, w)q(y, z)} = |\phi(x)|,$$

where $\phi: \bar{\mathbf{R}}^n \rightarrow \bar{\mathbf{R}}^n$ is a Möbius transformation with $\phi(y) = 0$, $\phi(z) = e_1$, $\phi(w) = \infty$.

Our main results in this section include the following theorems. The first is a version of (1.1) that is invariant under Möbius transformations [Ah, Be].

2.9. THEOREM. Let $f: \bar{\mathbf{R}}^n \rightarrow \bar{\mathbf{R}}^n$ be a K -quasiconformal mapping. Let x, y, z, w be an ordered quadruple of distinct points in $\bar{\mathbf{R}}^n$. Then

$$(2.10) \quad \tau \left(\frac{1}{|x, y, z, w|} \right) \leq K \tau \left(\frac{1}{|f(x), f(y), f(z), f(w)|} - 1 \right).$$

PROOF. First, if the absolute ratio satisfies $|f(x), f(y), f(z), f(w)| \geq 1$, there is nothing to prove. Hence we may assume that $|f(x), f(y), f(z), f(w)| < 1$. Next, by invariance of the absolute ratio under Möbius transformations we may assume that $y = f(y) = 0, z = f(z) = e_1, w = f(w) = \infty$. Thus (2.10) reduces to the statement:

$$\tau \left(\frac{1}{|x|} \right) \leq K \tau \left(\frac{1}{|f(x)|} - 1 \right)$$

for $|f(x)| < 1$. Let R' denote the ring whose complementary components are $\{tf(x): 0 \leq t \leq 1\}$ and $\{te_1: 1 \leq t \leq \infty\}$, and let $R = f^{-1}(R')$. By Lemmas 2.6 and 2.7,

$$\tau \left(\frac{1}{|x|} \right) \leq \text{cap } R \leq K \text{cap } R' \leq K \tau \left(\frac{1}{|f(x)|} - 1 \right). \quad \square$$

We also prove the following dual of Theorem 2.9.

2.11. THEOREM. *Let $f: \overline{\mathbf{R}}^n \rightarrow \overline{\mathbf{R}}^n$ be a K -quasiconformal mapping and let x, y, z, w be an ordered quadruple of distinct points in $\overline{\mathbf{R}}^n$ lying on a line (or circle). Then*

$$\tau\left(\frac{1}{|x, y, z, w|}\right) \geq \frac{1}{K} \tau\left(\frac{1}{|f(x), f(y), f(z), f(w)|}\right).$$

PROOF. By Möbius invariance we may assume that $x = -|x|e_1$, $y = f(y) = 0$, $z = f(z) = e_1$, and $w = f(w) = \infty$. Hence we need to prove that

$$\tau\left(\frac{1}{|x|}\right) \geq \frac{1}{K} \tau\left(\frac{1}{|f(x)|}\right).$$

Let R denote the ring whose complementary components are $\{te_1 : -|x| \leq t \leq 0\}$ and $\{te_1 : 1 \leq t \leq \infty\}$, and let $R' = f(R)$. Then by Lemmas 2.6 and 2.7 and K -quasiconformality,

$$\tau\left(\frac{1}{|x|}\right) = \text{cap } R \geq \frac{1}{K} \text{cap } R' \geq \frac{1}{K} \tau\left(\frac{1}{|f(x)|}\right). \quad \square$$

Theorem 2.9 will have several consequences, yielding estimates of the type (1.1). One of these we state here.

2.12. THEOREM. *Let $f: \overline{\mathbf{R}}^n \rightarrow \overline{\mathbf{R}}^n$ be a K -quasiconformal mapping with $f(0) = 0$ and $f(\infty) = \infty$. If $|x| \leq m_0|y|$, $m_0 > 0$, then*

$$(2.13) \quad |f(x)| \leq [1 + \tau^{-1}(\tau(m_0)/K)]|f(y)|.$$

PROOF. The result follows by application of Theorem 2.9 to the ordered quadruple $y, 0, x, \infty$ and use of the fact that τ is a decreasing function. $\square$

2.14. REMARK. By interchanging x and y in (2.10), we can get the slightly stronger form

$$\begin{aligned} & \max \left\{ \tau\left(\frac{1}{|x, y, z, w|}\right), \tau\left(\frac{1}{|y, x, z, w|}\right) \right\} \\ & \leq K \min \left\{ \tau\left(\frac{1}{|f(x), f(y), f(z), f(w)|}\right) - 1, \tau\left(\frac{1}{|f(y), f(x), f(z), f(w)|}\right) - 1 \right\}. \end{aligned}$$

2.15. REMARK. With $m_0 = 1$, the coefficient on the right side of (2.13) is precisely $c(n, K)$, but if $0 < m_0 < 1$ then it is less than $c(n, K)$ since τ is a strictly decreasing function. Theorem 2.12 improves earlier results of P. Tukia and J. Väisälä (see [TV, Theorem 3.10 and V2, Remark 2.8(1), p. 193]) as well as of F. W. Gehring [G2, Lemma 8; G5, Theorem 2.6]. This is true because a proof based on our Lemma 3.6 below will show that the constant $1 + \tau^{-1}(\tau(m_0)/K)$ has upper and lower bounds independent of n .

2.16. PROOF OF THEOREM 1.13. Since the points $f(y), 0, f(x), \infty$ are collinear, the result follows if we apply Theorem 2.11 to f^{-1} .

For $n = 2$ this theorem is sharp since, by Theorem 1.10 (see 3.2 below) and [LVV], $c(2, K) - 1 = \lambda(K)$ is the best possible constant. $\square$

2.17. COROLLARY. *Let $f: \overline{\mathbf{R}}^n \rightarrow \overline{\mathbf{R}}^n$, $n \geq 2$, be K -quasiconformal, and let $f(te_n) = te_n$ for all $t \in \mathbf{R}$, that is, f is the identity on the x_n -axis. Then*

$$|f(x)| \leq (c(n, K)^2 - 1)^{1/2}|x|.$$

PROOF. If x lies on the x_n -axis, there is nothing to prove. We shall consider only the case where the n th coordinate of $f(x)$ is nonnegative, since the proof is similar in the other case.

As in Remark 2.14,

$$\tau \left(\frac{1}{|-|x|e_n, 0, x, \infty|} \right) \leq K \tau \left(\frac{1}{|0, -|x|e_n, f(x), \infty|} - 1 \right),$$

so that

$$\begin{aligned} \tau(1) &\leq K \tau \left(\frac{|f(x) + |x|e_n|}{|x|} - 1 \right) \\ &\leq K \tau \left(\left(\left(\frac{|f(x)|}{|x|} \right)^2 + 1 \right)^{1/2} - 1 \right). \end{aligned}$$

Thus

$$\begin{aligned} |f(x)| &\leq [(1 + \tau^{-1}(\tau(1)/K))^2 - 1]^{1/2}|x| \\ &= [c(n, K)^2 - 1]^{1/2}|x|. \quad \square \end{aligned}$$

2.18. THEOREM. *Let $f: \overline{\mathbf{R}}^n \rightarrow \overline{\mathbf{R}}^n$ be a K -quasiconformal mapping with $f(0) = 0$, $f(\infty) = \infty$. Then*

$$|f(x)| \leq s(n, K)|f(-x)|,$$

where $s(n, K) = 1/\tau^{-1}(K\tau(1))$.

PROOF. The theorem follows by application of Theorem 2.11 to the quadruple $x, 0, -x, \infty$. $\square$

2.19. PROOF OF THEOREM 1.14. Let f be the K -quasiconformal self-mapping of $\overline{\mathbf{R}}^2$ with $f(-1) = -\lambda(K)$, $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$, as in [LVV, p. 11]. Then by the n -dimensional analogue of Theorem 1 in [AV1, p. 611], the self-mapping of $\overline{\mathbf{R}}^n$ obtained by rotating f about the x_1 -axis is K^{n-1} -quasiconformal. Hence $H_n(K^{n-1}) \geq \lambda(K)$. Now replace K by $K^{1/(n-1)}$. Finally, when $n = 2$, equality holds by [LVV, Theorem 1]. $\square$

3. Estimates for the distortion coefficient. We intend to show that $c(2, K)$ is closely related to a well-known distortion function in the theory of plane quasiconformal mappings introduced by Lehto, Virtanen, and Väisälä. First, however, let us observe that if $a, b \in (0, 1)$ then $\mu(a)\mu(b) = \pi^2/4$ if and only if $a^2 + b^2 = 1$. This follows immediately from the first equation in (2.5) and the fact that μ is strictly decreasing, hence one-to-one. From this observation it follows that if $u, v \in (0, \infty)$ then

$$(3.1) \quad (\mu^{-1}(u))^2 + (\mu^{-1}(v))^2 = 1 \quad \text{if and only if} \quad uv = \pi^2/4.$$

3.2. PROOF OF THEOREM 1.10. By [LVV, Theorem 1] and the above remark we have

$$(3.3) \quad \lambda(K) = (\mu^{-1}(\pi/2K)/\mu^{-1}(\pi K/2))^2.$$

Next, with $c(2, K) = c$, we have

$$\mu(c^{-1/2}) = \pi K/2,$$

so that $c^{-1/2} = \mu^{-1}(\pi K/2)$. The theorem then follows from the above remark, after a simple computation. $\square$

3.4. REMARK. From (3.3) it is clear that $\lambda(1/K) = 1/\lambda(K)$ for $K \in (0, \infty)$. Hence by Theorem 1.10 we obtain

$$c(2, 1/K) = c(2, K)/(c(2, K) - 1)$$

for all $K > 0$. By the definition of $s(2, K)$ and $c(2, K)$ (see 2.18 and (1.7)) we therefore have

$$s(2, K) = 1/\tau_2^{-1}(K\tau_2(1)) = 1/(c(2, 1/K) - 1) = 1/\lambda(1/K) = \lambda(K).$$

3.5. COROLLARY. $c(2, K) = e^{\pi K}/16 + 1/2 + \delta(K)$, where for each $K \geq 1$, $\delta(K) \in (0, 2e^{-\pi K})$.

PROOF. This follows from Theorem 1.10 above and Theorem 3 of [LVV]. $\square$

3.6. THEOREM. For $n \geq 2$ and $1 \leq K < \infty$,

$$c(n, K) \geq c(2, K^{1/(n-1)}).$$

PROOF. First,

$$\text{mod } R_{G,2}(c(2, K)^{1/2}) = K \text{ mod } R_{G,2}(\sqrt{2})$$

by the definition of $c(2, K)$, while

$$(3.7) \quad \text{mod } R_{G,n}(c(2, K)^{1/2}) \leq K \text{ mod } R_{G,n}(\sqrt{2})$$

by the n -dimensional analogue of [AV1, §2] (see [AV2, §§3, 4]). But

$$(3.8) \quad \text{mod } R_{G,n}(c(n, K^{n-1})^{1/2}) = K \text{ mod } R_{G,n}(\sqrt{2})$$

by the definition of $c(n, K)$. Since $\text{mod } R_{G,n}(s)$ is a monotone increasing function of s for $s > 1$, it follows from (3.7) and (3.8) that $c(n, K^{n-1}) \geq c(2, K)$. Now replace K by $K^{1/(n-1)}$. $\square$

By applying [An2, Theorem 2] we next prove a “dimension-cancellation” lemma that will allow us to prove that $c(n, K)$ has lower and upper bounds independent of n . In an implicit form this result was used in the dimension-free distortion estimates [Vu1, 3.3(2), 5.17(2)]. The lemma will also be useful in §5 in our treatment of quasisymmetry and in §6 in our new version of the Θ -distortion theorem.

3.9. LEMMA. For any integer $n \geq 2$ and for $0 < K < \infty$, the capacity functions γ and τ satisfy the following dimension-free inequalities:

$$(3.10) \quad \coth\left(\frac{\pi^2}{4K\mu(1/s)}\right) \leq \gamma^{-1}\left(\frac{1}{K}\gamma(s)\right) \leq \frac{1}{\nu(\pi^2 K/2 \log((s+1)/(s-1)))},$$

where $\nu = \mu^{-1}$, and

$$(3.11) \quad \text{csch}^2\left(\frac{\pi^2}{4K\mu(r)}\right) \leq \tau^{-1}\left(\frac{1}{K}\tau(t)\right) \leq \lambda\left(\frac{\pi K}{\log((1+r)/(1-r))}\right),$$

for $1 < s < \infty$ and $0 < t < \infty$, where $r = (1+t)^{-1/2}$.

PROOF. It follows from the second equation in (2.5) and [An2, Theorem 2] that for $n \geq 2$ and $1 < s < \infty$,

$$(3.12) \quad A_n\left(\frac{4}{\pi^2}\mu\left(\frac{1}{s}\right)\right)^{1/(n-1)} \leq \text{mod } R_{G,n}(s) < A_n\left(\frac{1}{2}\log\frac{s+1}{s-1}\right)^{1/(1-n)}.$$

Hence by (2.2)

$$(3.13) \quad u \equiv \frac{\sigma_{n-1}}{2A_n^{n-1}} \log \frac{s+1}{s-1} < \gamma(s) \leq \frac{\sigma_{n-1}\pi^2}{4A_n^{n-1}\mu(1/s)} \equiv v.$$

If we solve for s in terms of u on the left in (3.13) and in terms of v on the right, and then use the fact that γ is a decreasing function of its argument, we see that

$$(3.14) \quad \coth \left(A_n^{n-1} \frac{\omega}{\sigma_{n-1}} \right) \leq \gamma^{-1}(\omega) \leq \frac{1}{\nu(\pi^2 \sigma_{n-1} / (4A_n^{n-1} \omega))}.$$

Now setting $\omega = (1/K)\gamma(s)$ and using (3.13) we obtain (3.10).

For (3.11) we first compare (2.3) and (3.13) to obtain

$$(3.15) \quad \begin{aligned} u &\equiv \frac{\sigma_{n-1}}{2(2A_n)^{n-1}} \log \left(\frac{\sqrt{t+1}+1}{\sqrt{t+1}-1} \right) < \tau(t) \\ &\leq \frac{\sigma_{n-1}\pi^2}{4(2A_n)^{n-1}\mu((t+1)^{-1/2})} \equiv v. \end{aligned}$$

Then solving for t in terms of u on the left in (3.15) and for t in terms of v on the right, and using the fact that τ is a decreasing function, we derive the double inequality

$$(3.16) \quad \operatorname{csch}^2 \left(\frac{(2A_n)^{n-1}\omega}{\sigma_{n-1}} \right) < \tau^{-1}(\omega) \leq \lambda \left(\frac{\sigma_{n-1}\pi}{2(2A_n)^{n-1}\omega} \right).$$

The final step is to combine the bounds for τ^{-1} in (3.16) with those for τ in (3.15). Then (3.11) follows. $\square$

The next theorem includes Theorem 1.12.

3.17. THEOREM. *For any integer $n \geq 2$ and for $0 < K < \infty$,*

$$a_0(K) \leq a(n, K) \leq c(n, K) \leq b(n, K) \leq b_0(K),$$

where

$$\begin{aligned} a_0(K) &= \coth^2 \left(\frac{\pi}{2K} \right), \quad a(n, K) = \coth^2 \left(\frac{T}{K} \right), \quad b(n, K) = \left[\nu \left(\frac{\pi^2 K}{4T} \right) \right]^{-2}, \\ b_0(K) &= \left[\nu \left(\frac{\pi^2 K}{4 \log(\sqrt{2}+1)} \right) \right]^{-2} = 1 + \lambda \left(\frac{\pi K}{2 \log(\sqrt{2}+1)} \right), \end{aligned}$$

and where

$$\begin{aligned} T &= \left(\frac{A_n}{\operatorname{mod} R_{G,n}(\sqrt{2})} \right)^{n-1}, \\ A_n &= \left(\int_0^{\pi/2} \sin^{n-2} t \, dt \right)^{1/(n-1)} \int_0^{\pi/2} (\sin t)^{(2-n)/(n-1)} \, dt \end{aligned}$$

(see [An1, Theorem 2]), and $\nu = \mu^{-1}$ as in (2.4).

PROOF. Setting $\omega = \gamma(\sqrt{2})/K$ in (3.14) and recalling the definition of $c(n, K)$, we have

$$\coth(T/K) \leq \sqrt{c(n, K)} \leq 1/\nu(\pi^2 K/4T),$$

so that $a(n, K) \leq c(n, K) \leq b(n, K)$ as claimed.

Next, since $\mu(1/\sqrt{2}) = \pi/2$ (cf. (2.5)) we obtain

$$\log(1 + \sqrt{2}) \leq T \leq \pi/2$$

from (3.12), so that the bounds $a_0(K) \leq a(n, K)$ and $b(n, K) \leq b_0(K)$ follow. $\square$

3.18. REMARK. Note that

$$b(2, K) = [\nu(\pi K/2)]^{-2} = 1 + \lambda(K) = c(2, K)$$

and that

$$a(2, K) = \coth^2(\pi/2K) = a_0(K),$$

so that the estimates $a(n, K) \leq c(n, K) \leq b(n, K)$ are in some sense sharp.

3.19. COROLLARY. For $n \geq 2$ and $1 \leq K < \infty$,

$$1 + \lambda(K^{1/(n-1)}) \leq c(n, K) \leq 1 + \lambda \left[\frac{\pi K}{2 \log(\sqrt{2} + 1)} \right].$$

PROOF. These inequalities are clear from Theorems 1.10, 3.6, and 3.17. $\square$

3.20. REMARK. Since $H_n(K) \leq c(n, K)$ by (1.6), where $H_n(K)$ is defined in (1.2), Theorem 3.17 and Corollary 3.19 yield dimension-free upper bounds for $H_n(K)$.

3.21. PROOF OF THEOREM 1.11. By [LV, p. 62] we obtain, for $s > 1$,

$$\mu(1/s) \geq \log s(1 + (1 - s^{-2})^{1/4})^2 \geq \log(s + 3(s^2 - 1)^{1/2}).$$

Further, by [An2, Theorem 1] and the above inequality,

$$(3.22) \quad \gamma(s) \leq \sigma_{n-1} \mu(1/s)^{1-n} \leq \sigma_{n-1} (\log(s + 3(s^2 - 1)^{1/2}))^{1-n}.$$

It follows from (3.22) that

$$\gamma^{-1}(u) \leq (3(\beta^2 + 8)^{1/2} - \beta)/8, \quad \beta = \exp \left[\left(\frac{\sigma_{n-1}}{u} \right)^{1/(n-1)} \right].$$

Next, since $\mu(1/\sqrt{2}) = \pi/2$ by (2.5), it follows from (3.22) that

$$\exp \left(\left(\frac{K \sigma_{n-1}}{\gamma(\sqrt{2})} \right)^{1/(n-1)} \right) \geq \exp \frac{\pi}{2}$$

for all $K \geq 1$. Therefore

$$c(n, K) = \left(\gamma^{-1} \left(\frac{\gamma(\sqrt{2})}{K} \right) \right)^2 \leq \left(\frac{\delta}{8} \right)^2 \exp \left[2 \left(\frac{K \sigma_{n-1}}{\gamma(\sqrt{2})} \right)^{1/(n-1)} \right] < \frac{1}{10} d(n, K),$$

where $\delta = -1 + 3(1 + 8e^{-\pi})^{1/2} = 2.480 \dots$. This completes the proof of the lower bound.

The upper bound follows from the n -dimensional version of [G1, Lemma 8, p. 518] and the upper estimate in [AV3, Theorem, p. 134] (cf. [An3]).

Finally, it follows from Theorem 1.10 that $\lim_n c(n, K)^{1/n} = 1$, while it follows from [An2, Theorem 5, p. 18] that $\lim_n d(n, K)^{1/n} = e^2$. $\square$

3.23. REMARK. Let f be a K -quasiconformal self-mapping of $\mathbf{R}^n$, and let x, y, z, w be an ordered quadruple of distinct points in $\mathbf{R}^n$. By applying Theorem 2.9 to f and to f^{-1} we obtain

$$1/(1 + \tau^{-1}(K^{-1}\tau(1/t))) \leq t' \leq 1/\tau^{-1}(K\tau(1/t - 1))$$

for $t = |x, y, z, w|$ and $t' = |f(x), f(y), f(z), f(w)|$. By (3.11) it is then clear that $|f(x), f(y), f(z), f(w)|$ has upper and lower bounds that depend only upon K and $|x, y, z, w|$ but not upon n .

4. A dimension-free quasiconformal Schwarz Lemma. Given an integer $n \geq 2$ and $K > 0$, let $\phi_K = \phi_{K,n}: (0, 1) \rightarrow (0, 1)$ denote the strictly increasing function defined by

$$(4.1) \quad \gamma_n \left(\frac{1}{\phi_K(r)} \right) = K\gamma_n \left(\frac{1}{r} \right).$$

For our convenience in theorems 5.23 and 6.2 we now extend the definition of ϕ_K by setting $\phi_K(0) = 0$ and $\phi_K(1) = 1$. Since γ_n is a decreasing function, it follows that $\phi_K(r) \geq r$ for $K \geq 1$ and $\phi_K(r) \leq r$ for $0 < K \leq 1$, with equality in each case if and only if $K = 1$. Clearly $\phi_{K,n}^{-1} = \phi_{1/K,n}$.

We have the following distortion theorem, a quasiconformal counterpart of the Schwarz Lemma (see also [MRV, 3.1 and Vu1, 3.3]).

4.2. THEOREM. Let f be a K -quasiconformal mapping of the unit ball B^n onto a subdomain G of B^n with $f(0) = 0$. Then $|f(x)| \leq \phi_K(|x|)$ for all $x \in B^n$. Moreover, if $G = B^n$, then $\phi_{1/K}(|x|) \leq |f(x)| \leq \phi_K(|x|)$ for all $x \in B^n$.

PROOF. This theorem is the n -dimensional version of [LV, (3.2), p. 63 and (3.4), p. 64], and it has an analogous proof. $\square$

4.3. REMARK. Note that for $K > 0$, $c(n, K)^{-1/2} = \phi_{1/K,n}(1/\sqrt{2})$, that is,

$$\phi_{K,n}(c(n, K)^{-1/2}) = 2^{-1/2}$$

by (1.7). Hence we have lower and upper bounds for $\phi_{K,n}^{-1}(1/\sqrt{2}) = \phi_{1/K,n}(1/\sqrt{2})$ for $K > 0$, as a consequence of the bounds for $c(n, K)$ in Theorem 3.17.

We now obtain bounds independent of n for $\phi_{K,n}(r)$ for each $r \in (0, 1)$ and $K > 0$.

4.4. THEOREM. Let $n \geq 2$ and $r \in (0, 1)$. If $K \geq 1$ then

$$(4.5) \quad \tanh(K \operatorname{artanh} r) \leq \phi_{K,n}(r) \leq \tanh((K/2)\mu((1-r)/(1+r))),$$

and if $0 < K \leq 1$ then

$$(4.6) \quad \mu^{-1}(\pi^2/(4K \operatorname{artanh} r)) \leq \phi_{K,n}(r) \leq \tanh(K \operatorname{artanh} r).$$

The lower estimate in (4.5) and the upper estimate in (4.6) are sharp for $K = 1$.

PROOF. The first inequality in (4.6) follows directly from (3.10), and the second inequality in (4.5) follows from (3.10) and [LV, (2.9), p. 61]. (In fact, both of these inequalities hold for all $K > 0$.)

Next, for $K \geq 1$, let $s = \phi_{K,n}(r)$, hence $M_n(s) = \alpha M_n(r)$. Since $s > r$ and since $(\log(1+r)/(1-r))^{1/(n-1)} M_n(r)$ is monotone increasing for $0 < r < 1$ [An1, Theorem 3 and Corollary 1] it then follows that

$$(4.7) \quad \alpha(\log((1+s)/(1-s)))^{1/(n-1)} \geq (\log((1+r)/(1-r)))^{1/(n-1)}.$$

By solving (4.7) for s we obtain the first inequality in (4.6). The second inequality in (4.6) follows by a similar proof. $\square$

4.8. REMARK. It is easily checked that each of the bounds in (4.5) and (4.6) is monotone increasing from 0 to 1 for $r \in (0, 1)$. From l'Hôpital's Rule and [BF, #112.01] it follows that the lower and upper bounds in (4.5) have the asymptotic orders Kr and $\pi^2 K / (4 \log(1/r))$, respectively, as $r \rightarrow 0$. Next, it follows from [LV, (2.11), p. 62] that $\mu^{-1}(s) \sim 4e^{-s}$ as $s \rightarrow \infty$. Since artanh is an odd function, it is thus easy to see that the lower bound in (4.6) has the asymptotic order $4 \exp(-\pi^2 / (4Kr))$ as $r \rightarrow 0$.

Next, we compare the values of the distortion function $\phi_{K^{n-1},n}$ with those of $\phi_{K,2}$.

4.9. THEOREM. For $n \geq 2$, $K \geq 1$, $0 \leq r \leq 1$,

$$\phi_{K^{n-1},n}(r) \geq \phi_{K,2}(r) \geq 4^{1-1/K} r^{1/K} (1 - 12r^{2/K}).$$

PROOF. Let f be the plane extremal K -quasiconformal self-mapping of the unit disk B^2 with $se_1 = f(re_1)$, where $s = \phi_{K,2}(r)$. Let F be the extension of f to B^n by rotation about the x_1 -axis. Then F is a K^{n-1} -quasiconformal self-mapping of B^n with $F(0) = 0$, $F(re_1) = se_1$ (see [AV1, §2; AV2, §4]). Hence $\phi_{K^{n-1},n}(r) \geq s = \phi_{K,2}(r)$. The second inequality is [LV, (3.8), p. 65]. $\square$

4.10. THEOREM. For $n \geq 2$ and $0 \leq r \leq 1$,

$$(4.11) \quad r^\alpha \leq \phi_{K,n}(r) \leq \lambda_n^{1-\alpha} r^\alpha \quad \text{for } K \geq 1$$

and

$$(4.12) \quad \lambda_n^{1-\alpha} r^\alpha \leq \phi_{K,n}(r) \leq r^\alpha \quad \text{for } 0 < K \leq 1,$$

where $\alpha = K^{1/(1-n)}$ and λ_n is the constant defined by

$$(4.13) \quad \log \lambda_n = \lim_{s \rightarrow \infty} (\operatorname{mod} R_{G,n}(s) - \log s)$$

(cf. [G2]).

PROOF. Since $\phi_{K,n}(0) = 0$ and $\phi_{K,n}(1) = 1$, (4.11) and (4.12) are obvious for $r = 0$ or 1 . Let $0 < r < 1$. For the upper estimate in (4.11) fix $n \geq 2$, $K \geq 1$, and let $M(r) = \operatorname{mod} R_{G,n}(1/r)$. Setting $r' = \phi_{K,n}(r)$ as in [LV, pp. 64, 65] we have $M(r') = \alpha M(r)$ and $0 < r \leq r' < 1$.

By the proof of [G1, Lemma 6, p. 514] we know that $M(r) + \log r$ is a decreasing function on $(0, 1)$ so that

$$M(r) + \log r \geq M(r') + \log r'.$$

Adding $-\log \lambda_n$ to both sides and reversing the sign, we have

$$0 \leq \log \frac{\lambda_n}{r} - M(r) \leq \log \frac{\lambda_n}{r'} - M(r').$$

Now since $0 < \alpha \leq 1$ we get

$$\alpha \log \frac{\lambda_n}{r} - \alpha M(r) \leq \log \frac{\lambda_n}{r'} - M(r').$$

Adding $\alpha M(r) = M(r')$ and exponentiating, we obtain

$$\phi_{K,n}(r) = r' \leq \lambda_n^{1-\alpha} r^\alpha.$$

Now define $f(x) = |x|^{\alpha-1}x$ for $0 < |x| < 1$ and $f(0) = 0$. Since $f(re_1) = r^\alpha e_1$ and f is K -quasiconformal by [V1, p. 49] the left inequality in (4.11) follows from Theorem 4.2.

Next, solving $\phi_{K,n}(r) = 1/\gamma_n^{-1}(K\gamma_n(1/r)) \geq r^\alpha$ for r in terms of r^α when $K \geq 1$ and then replacing r by $r^{1/\alpha}$ gives $\phi_{K,n}(r) \leq r^{1/\alpha}$, which is equivalent to the second inequality in (4.12). The first inequality in (4.12) follows in an analogous way from the second inequality in (4.11). $\square$

4.14. COROLLARY. For $K \geq 1$, $n \geq 2$, and $r \in [0, 1]$,

$$\phi_{K,n}(r) \leq 2^{1-1/K} K r^\alpha, \quad \alpha = K^{1/(1-n)}.$$

There is equality when $K = 1$.

PROOF. By [AV3, (5) or An3, (4)], we know that $\log \lambda_n \leq n - 1 + \log 2$ for $n \geq 2$, while it is easy to see that $1 - \alpha \leq 1 - 1/K$ and $(1 - \alpha)(n - 1) \leq \log K$. Hence

$$(1 - \alpha) \log \lambda_n \leq (1 - \alpha)(n - 1) + (1 - \alpha) \log 2 \leq \log K + (1 - 1/K) \log 2,$$

so that $\lambda_n^{1-\alpha} \leq 2^{1-1/K} K$ for $n \geq 2$. $\square$

4.15. COROLLARY. For $K \geq 1$, $n \geq 2$, and $r \in [0, 1]$, $\phi_{K,n}(r)$ has the dimension-free upper bound $\phi_{K,n}(r) \leq 2^{1-1/K} K r^{1/K}$.

PROOF. This follows from Corollary 4.14 since $\alpha \geq 1/K$ for $n \geq 2$. $\square$

4.16. PROOF OF THEOREM 1.15. This result follows directly from Theorems 4.2, 4.10, and the proof of Corollary 4.14. $\square$

4.17. REMARK. In earlier results similar to Corollary 4.15 (cf. [G3, pp. 233–236; MRV, 3.1; Vul, 3.3(1)]) the right side is unbounded as n tends to ∞ with K fixed. For $n = 2$ the upper bound in (4.11) is due to Chuan-Fang Wang [W]; however, our proof for this upper bound is similar to the proof of [LV, (3.6), p. 65].

4.18. REMARK. From Theorems 4.11 and 4.12 it follows that $r^{-\alpha} \phi_{K,n}(r)$ has lower and upper limits at $r = 0$ between $4^{1-\alpha}$ and $\lambda_n^{1-\alpha}$. When $n = 2$ the corresponding lower and upper limits coincide and are equal to $4^{1-1/K}$ [LV, p. 65].

5. An application to quasisymmetric mappings. Let G be a domain in $\mathbf{R}^n$. A one-to-one continuous mapping $f: G \rightarrow \mathbf{R}^n$ is called η -quasisymmetric if there exists a homeomorphism $\eta: [0, \infty) \rightarrow [0, \infty)$ such that

$$(5.1) \quad \frac{|f(x) - f(a)|}{|f(x) - f(b)|} \leq \eta \left(\frac{|x - a|}{|x - b|} \right)$$

for all $x, a, b \in G$ with $x \neq b$. The function f is called *quasisymmetric* if it is η -quasisymmetric for some η . (In this section we do not specify the image domain, and fG may be any subdomain of $\mathbf{R}^n$.)

Clearly this definition is invariant under similarities; that is, if f is quasisymmetric and g, h are similarities, then $g \circ f \circ h$ is also quasisymmetric with the same η .

P. Tukia and J. Väisälä [TV, V2] have proved that a K -quasiconformal mapping $f: B^n \rightarrow fB^n \subset \mathbf{R}^n$ is η -quasisymmetric in $\overline{B^n}(s)$, $s \in (0, 1)$, with η depending only on n, K, s . However, their function η is unbounded as n tends to ∞ .

In this section we modify the function η and obtain bounds for it that are independent of n . Such dimension-free estimates should help lay the groundwork for a study of quasiconformal mappings in infinite-dimensional spaces.

In order to carry out these estimates we find it convenient to present the following background material on Möbius transformations [Ah, Be].

5.2. *Möbius transformations in $\overline{\mathbf{R}}^n$.* To begin, let a be a prescribed point of B^n . As shown in [Ah, p. 24] (see also [Be, pp. 36–42]), a sense-preserving Möbius transformation $T_a: \overline{\mathbf{R}}^n \rightarrow \overline{\mathbf{R}}^n$ with $T_a B^n = B^n$ and $T_a(a) = 0$ can be constructed in the following way. If $a = 0$, set $T_a = I$, the identity mapping. Otherwise, define $T_a x = (p_a \circ \sigma_a)(x)$, where σ_a is an inversion in the sphere $S^{n-1}(a^*, r)$, $r = (|a|^{-2} - 1)^{1/2}$, orthogonal to S^{n-1} , that is,

$$(5.3) \quad \sigma_a(x) = a^* + r^2(x - a^*)^*, \quad u^* \equiv u/|u|^2,$$

and p_a is a reflection in the $(n - 1)$ -dimensional linear subspace orthogonal to a . Conversely, a prescribed Möbius transformation $g: \overline{\mathbf{R}}^n \rightarrow \overline{\mathbf{R}}^n$ with $gB^n = B^n$ has a canonical representation [Ah, p. 24; Be, pp. 36–42]

$$(5.4) \quad g = k \circ T_a, \quad k \in O(n), \quad a \in B^n,$$

where $O(n)$ denotes the set of all orthogonal mappings of $\overline{\mathbf{R}}^n$.

A fundamental formula specialized to T_a [Be, p. 26, (3.1.5)] states that

$$(5.5) \quad |T_a x - T_a y| = |\sigma_a x - \sigma_a y| = \frac{r^2 |x - y|}{|x - a^*| |y - a^*|},$$

where $r^2 = |a|^{-2} - 1$.

We shall need the following lemma in our estimation of the function η .

5.6. LEMMA. *Let $s \in (0, 1)$. Then*

$$(5.7) \quad \frac{1 - s^2}{(1 + s^2)^2} |x - y| \leq |T_z x - T_z y| \leq \frac{|x - y|}{1 - s^2}$$

for all $x, y, z \in \overline{B}^n(s)$,

$$(5.8) \quad \frac{1 - s^2}{1 + s^2} \frac{|x - y|}{|x - z|} \leq \frac{|T_w x - T_w y|}{|T_w x - T_w z|} \leq \frac{1 + s^2}{1 - s^2} \frac{|x - y|}{|x - z|},$$

for $x, y, z, w \in \overline{B}^n(s)$, with $x \neq z$, and

$$(5.9) \quad |T_a x| \leq \left| T_a \left(-\frac{|x|}{|a|} a \right) \right| \leq \frac{2}{s + 1/s}$$

for all $x, a \in \overline{B}^n(s)$, $a \neq 0$.

PROOF. For (5.7) we note that $|x - z^*| \in [|z|^{-1} - s, |z|^{-1} + s]$. The proof then follows from (5.5).

Next, if $w = 0$, (5.8) is trivial since T_0 is the identity. Otherwise, by (5.5) we have

$$\frac{|T_w x - T_w y|}{|T_w x - T_w z|} = \frac{|z - w^*|}{|y - w^*|} \frac{|x - y|}{|x - z|}.$$

Since $|z - w^*|$ and $|y - w^*|$ lie in the interval $[|w|^{-1} - s, |w|^{-1} + s]$ and since $(1 + st)/(1 - st)$ is increasing as a function of t on $0 \leq t \leq s$, (5.8) holds.

Finally, by (5.5) and direct calculation,

$$\begin{aligned} \left| T_a \left(-\frac{|x|}{|a|} a \right) \right|^2 - |T_a x|^2 &= \left(\frac{|x| + |a|}{1 + |a||x|} \right)^2 - \left(\frac{|x - a|}{|a||x - a^*|} \right)^2 \\ &= \frac{2(1 - |a|^2)(1 - |x|^2)(|a||x| + a \cdot x)}{(1 + |a||x|)^2 |a|^2 |x - a^*|^2} \\ &\geq 0, \end{aligned}$$

with equality if and only if $x = -|x|a/|a|$, since $|a| < 1$ and $|x| < 1$. It is easy to see that

$$\frac{|x| + |a|}{1 + |a||x|} \leq \frac{2}{s + 1/s},$$

since $(|a| + t)/(1 + |a|t)$ is increasing as a function of t for $0 \leq t \leq s$, and (5.9) follows. $\square$

The following technical lemma provides the framework for deriving dimension-free bounds for the function η of quasisymmetry.

5.10. LEMMA. *Let $s \in (0, 1)$, f a K -quasiconformal mapping of B^n into $\mathbf{R}^n$, $y, z \in \overline{B}^n(s), z \neq 0$. Then*

$$(5.11) \quad \frac{|f(y) - f(0)|}{|f(z) - f(0)|} \leq 1 + \tau^{-1} \left(\frac{1}{2K} \tau \left(\frac{1 + s|y|}{1 - s|z|} \right) \right)$$

and

$$(5.12) \quad fB^n(|z|) \subset B^n(f(0), |f(0) - f(z)|a(s, K)),$$

where

$$a(s, K) = 1 + \tau^{-1} \left(\frac{1}{2K} \tau \left(\frac{1 + s}{1 - s} \right) \right).$$

PROOF. We may obviously assume that $|f(y) - f(0)| > |f(z) - f(0)|$, since otherwise we have nothing to prove.

Let F' be the segment $\{tf(0) + (1 - t)f(z): 0 \leq t \leq 1\}$ and let E' be the ray $\{f(0) + t(f(y) - f(0)): t \geq 1\}$. Let $E = f^{-1}(E' \cap f(B^n))$ and $F = f^{-1}(F' \cap f(B^n))$. Let $\Gamma' = \Delta(E', F'; fB^n)$, the family of arcs in fB^n that join E' and F' , and let $\Gamma = f^{-1}\Gamma'$. Then by Lemma 2.7,

$$(5.13) \quad M(\Gamma') \leq \tau \left(\frac{|f(y) - f(0)|}{|f(z) - f(0)|} - 1 \right),$$

while by a symmetry property of the modulus [G4]

$$(5.14) \quad M(\Gamma) \geq M(\Delta(E, F; R^n))/2.$$

If we perform a spherical symmetrization of E, F with center 0 in the negative and positive x_1 -axis, respectively, we obtain sets E^*, F^* with

$$E^* \supset (-e_1, -|y|e_1], \quad F^* \supset [0, |z|e_1].$$

By (5.8) and the fact that spherical symmetrization decreases the modulus of a curve family, we obtain

$$(5.15) \quad M(\Delta(E, F; \mathbf{R}^n)) \geq M(\Delta(E^*, F^*; \mathbf{R}^n)) \geq \tau \left(\frac{|y|}{|z|} \frac{1 + s}{1 - s} \right).$$

Since $M(\Gamma) \leq KM(\Gamma')$ by K -quasiconformality, inequality (5.11) now follows from (5.13), (5.14), and (5.15).

Finally, (5.12) follows from (5.11) because $|y|/|z| \leq 1$ for $y \in B^n(|z|)$. $\square$

5.16. COROLLARY. *Let $s \in (0, 1)$, f a K -quasiconformal mapping of B^n into $\mathbf{R}^n$, and x, y, z distinct points of $B^n(s)$. Then*

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \leq 1 + \tau^{-1} \left(\frac{1}{2K} \tau \left(a(s) \frac{|x - y|}{|x - z|} \right) \right),$$

where

$$a(s) = \frac{(1 + s)(1 + s^2)}{(1 - s)^3}.$$

PROOF. It follows from (5.9) that $|T_x w| \leq 2/(s + 1/s)$ for all $w \in \overline{B}^n(s)$. Now if we set $f \circ T_x^{-1} = g$, Lemma 5.10 yields

$$\begin{aligned} \frac{|f(x) - f(y)|}{|f(x) - f(z)|} &= \frac{|g(0) - g(T_x y)|}{|g(0) - g(T_x z)|} \\ &\leq 1 + \tau^{-1} \left(\frac{1}{2K} \tau \left(\alpha \frac{|T_x y|}{|T_x z|} \right) \right), \end{aligned}$$

where

$$\alpha = \frac{1 + 2/(s + 1/s)}{1 - 2/(s + 1/s)} = \frac{(1 + s)^2}{(1 - s)^2}.$$

Further, by (5.8),

$$\frac{|T_x y|}{|T_x z|} \leq \frac{1 + s^2}{1 - s^2} \frac{|x - y|}{|x - z|}.$$

The corollary follows since τ is strictly decreasing. $\square$

5.17. THEOREM. *Let $s \in (0, 1)$, f a K -quasiconformal mapping of B^n into $\mathbf{R}^n$, and x, y, z distinct points of $\overline{B}^n(s)$. If $|x - y|/|x - z| < (1 - s^2)/(1 + s^2)$, then*

$$(5.18) \quad |T_x y| < |T_x z|$$

and

$$(5.19) \quad \frac{|f(x) - f(y)|}{|f(x) - f(z)|} \leq b(s, K) \phi_K \left(\frac{1 + s^2}{1 - s^2} \frac{|x - y|}{|x - z|} \right),$$

where ϕ_K is the distortion function defined in (4.1) and where

$$b(s, K) = 1 + \tau^{-1} \left(\frac{1}{2K} \tau \left(\left(\frac{1 + s}{1 - s} \right)^2 \right) \right).$$

PROOF. First, (5.18) follows from (5.8).

Next, let $g = f \circ T_x^{-1}$, and let $u \in B^n(|T_x z|)$. Then by Lemmas 5.6 and 5.10 we get

$$\begin{aligned} \frac{|g(u) - g(0)|}{|g(T_x z) - g(0)|} &\leq 1 + \tau^{-1} \left(\frac{1}{2K} \tau \left(\frac{1 + |T_x z|}{1 - |T_x z|} \cdot \frac{|u|}{|T_x z|} \right) \right) \\ &\leq 1 + \tau^{-1} \left(\frac{1}{2K} \tau \left(\frac{1 + 2/(s + 1/s)}{1 - 2/(s + 1/s)} \right) \right) \\ &= 1 + \tau^{-1} \left(\frac{1}{2K} \tau \left(\left(\frac{1 + s}{1 - s} \right)^2 \right) \right) = b(s, K). \end{aligned}$$

This implies that

$$g: B^n(|T_x z|) \rightarrow B^n(f(x), b(s, K)|f(x) - f(z)|)$$

and hence that

$$|f(x) - f(y)|/|f(x) - f(z)| \leq b(s, K)$$

for $x, y, z \in \overline{B}^n(s)$ with $|x - y|/|x - z| < (1 - s^2)/(1 + s^2)$, hence by Theorem 4.2 that

$$(5.20) \quad \frac{|g(w) - f(x)|}{b(s, K)|f(x) - f(z)|} \leq \phi_K \left(\frac{|w|}{|T_x z|} \right)$$

for $|w| < |T_x z|$.

Now apply (5.20) with $w = T_x y$ and with $|x - y|/|x - z| < (1 - s^2)/(1 + s^2)$ (whence $|T_x y| < |T_x z|$ by (5.18)) to obtain

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \leq b(s, K)\phi_K \left(\frac{|T_x y|}{|T_x z|} \right).$$

This inequality and (5.8) imply (5.19). $\square$

5.21. REMARK. For $s \in (0, 1)$ and $K \in [1, \infty)$, let $a(s)$ and $b(s, K)$ be as in Corollary 5.16 and Theorem 5.17, respectively. Next, let

$$\eta_1(t) = b(s, K)\phi_K \left(\frac{1 + s^2}{1 - s^2} t \right) \quad \text{for } t \in \left[0, \frac{1 - s^2}{1 + s^2} \right]$$

and

$$\eta_2(t) = 1 + \tau^{-1} \left(\frac{1}{2K} \tau(a(s)t) \right) \quad \text{for } t \in (0, \infty).$$

Define

$$(5.22) \quad \eta(t) = \begin{cases} \eta_1(t) & \text{for } t \in [0, (1 - s^2)/(1 + s^2)], \\ \eta_2(t) & \text{for } t \in [(1 - s^2)/(1 + s^2), \infty). \end{cases}$$

Because η_1 and η_2 are continuous, strictly increasing functions agreeing at $t = (1 - s^2)/(1 + s^2)$, it is clear that $\eta: [0, \infty) \rightarrow [0, \infty)$ is a homeomorphism. Moreover, by Theorem 4.4 and Lemma 3.9, η has an upper bound independent of n . Thus we may formulate a dimension-free version of the Tukia-Väisälä theorem, which was mentioned at the beginning of this section, as follows.

5.23. THEOREM. *Let G be a domain in $\mathbf{R}^n$, $n \geq 2$, and f a K -quasiconformal mapping of G into $\mathbf{R}^n$. Suppose $B^n(x, r) \subset G$ and that $s \in (0, 1)$. Then $f|B^n(x, sr)$ is η -quasisymmetric, where η depends only on K and s , but not on n .*

PROOF. By Lemma 3.9 it is clear how the function η in (5.22) should be modified so as to make it independent of n . Alternatively, one may obtain a dimension-free function η that satisfies a Hölder condition at 0 if one uses Corollary 4.15. $\square$

6. A dimension-free Θ -distortion theorem.

6.1. *The Θ -distortion theorem.* We shall prove the following dimension-free variant of a distortion theorem due to F. W. Gehring [G2, Theorem 11].

6.2. THEOREM. For each $K \geq 1$ there exists a homeomorphism $\Theta^*: [0, 1) \rightarrow [0, \infty)$ with the following property. If $n \geq 2$ and $f: G \rightarrow fG$ is a K -quasiconformal mapping with $G, fG \subset \mathbf{R}^n$, and if $x, y \in G$ with $|y - x| < d(x, \partial G)$, then

$$(6.3) \quad \frac{|f(x) - f(y)|}{d(f(x), \partial fG)} \leq \Theta^* \left(\frac{|x - y|}{d(x, \partial G)} \right).$$

PROOF. Define $\Theta: [0, 1) \rightarrow [0, \infty)$ by

$$\Theta(t) = \begin{cases} a\left(\frac{1}{2}, K\right) \phi_K(2t), & \text{for } 0 \leq t \leq \frac{1}{2}, \\ a(t, K), & \text{for } \frac{1}{2} \leq t < 1, \end{cases}$$

where $a(s, K)$ is as in (5.12) and ϕ_K is as in (4.1). Then Θ is clearly a homeomorphism, and from Corollary 4.15 and Lemma 3.9 it follows that Θ has a majorant Θ^* that is independent of n . One such function Θ^* is

$$(6.4) \quad \Theta^*(t) = \begin{cases} 2K \left[1 + \lambda \left(\frac{2\pi K}{\log 3} \right) \right] t^{1/K}, & 0 \leq t \leq \frac{1}{2}, \\ 2^{1-1/K} K \left[1 + \lambda \left(\frac{2\pi K}{\log((1+r)/(1-r))} \right) \right], & \frac{1}{2} \leq t < 1, \end{cases}$$

where $r = ((1 - t)/2)^{1/2}$. Moreover, Θ^* is even a homeomorphism since it is continuous on $[0, 1)$ and strictly increasing on each subinterval $[0, 1/2]$ and $[1/2, 1)$. The function Θ^* is also Hölderian at 0.

To show that Θ satisfies inequality (6.3) we consider two cases. First, suppose $|x - y| < d(x, \partial G)/2$. We may assume $x \neq y$. After performing preliminary Möbius transformations we may employ Lemma 5.10. Fix $z \in S^{n-1}(x, d(x, \partial G)/2)$ with $|f(z) - f(x)| < d(f(x), \partial fG)$. Then by (5.12),

$$fB^n(x, |x - z|) \subset B^n(f(x), d(f(x), \partial fG)a\left(\frac{1}{2}, K\right)).$$

Applying Theorem 4.2 to the function $f|_{B^n(x, d(x, \partial G)/2)}$, we obtain

$$\frac{|f(x) - f(y)|}{a\left(\frac{1}{2}, K\right)d(f(x), \partial fG)} \leq \phi_K \left(\frac{2|x - y|}{d(x, \partial G)} \right),$$

from which it follows that

$$\frac{|f(x) - f(y)|}{d(f(x), \partial fG)} \leq \Theta \left(\frac{|x - y|}{d(x, \partial G)} \right)$$

for $|x - y| < d(x, \partial G)/2$.

Finally, if $\frac{1}{2}d(x, \partial G) \leq |x - y| < d(x, \partial G)$ we let $t = |x - y|/d(x, \partial G) < 1$ and choose $z \in S^{n-1}(x, |x - y|)$ with $|f(z) - f(x)| < d(f(x), \partial fG)$. Then by (5.11) we have

$$\frac{|f(y) - f(x)|}{d(f(x), \partial fG)} < \frac{|f(x) - f(y)|}{|f(x) - f(z)|} \leq a(t, K) = \Theta(t). \quad \square$$

6.5. REMARK. The distortion function in [G2, Theorem 11] is $1/\tau^{-1}(K\gamma(1/t))$, which has a lower bound of order $\frac{1}{16}((1+t)/(1-t))^{2^{n-3}}K$ as n tends to ∞ with t and K fixed.

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