SIMPLEXWISE LINEAR UNTANGLING

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ABSTRACT. In this paper we show how to canonically untangle simplexwise linear spanning arcs of a convex 2-cell. Specifically, we show that the space of such arcs is contractible. The main step in the contraction is a flow along the gradient field of an energy function. A 3-dimensional version of this result would imply the Smale Conjecture—Hatcher Theorem.

0. Introduction. We will show how to canonically untangle simplexwise linear arcs in the plane. Specifically, let K^2 be a triangulated convex 2-cell in $\mathbf{R}^2$. Let v and w be any two points that lie in the interior of distinct 1-simplexes in the boundary of K^2 . Let J_k be the (straight) segment joining v to w and subdivided into k subintervals. Let $\mathcal{T}(J_k, K^2)$ denote the space of all maps $f: J_k \rightarrow K^2$, which are simplexwise linear (i.e. linear on each 1-simplex of J_k) and are such that

$$f(v) = v, \quad f(w) = w, \quad \text{and} \quad f(J_k) \cap \partial K^2 = \{v\} \cup \{w\},$$

where ∂ denotes the boundary. Since $f \in \mathcal{T}(J_k, K^2)$ is determined by its values on the vertices $v_1, v_2, \dots, v_{k-1}$ of I_k , we can identify $\mathcal{T}(J_k, K^2)$ with an (open) subset of $(\mathbf{R}^2)^{k-1}$, where f is identified with the $(k-1)$ -tuple $(f(v_1), f(v_2), \dots, f(v_{k-1}))$. We shall prove

THEOREM. $\mathcal{T}(J_k, K^2)$ is contractible.

This is related to the main result of [BCH] which proves that the space of simplexwise linear homeomorphisms of K^2 , $\text{rel } \partial K^2$, is contractible. See [BCH] and [CHHS] for a discussion of history, results and questions concerning spaces of simplexwise maps. The main interest in the Theorem and its proof is that it gives a method for attempting a proof of the 3-dimensional version:

CONJECTURE. If A is a polyhedral spanning arc of I^3 and $\mathcal{T}(A, I^3)$ is the space of simplexwise linear unknotted embeddings of A into I^3 , $\text{rel } \partial I^3$, then $\mathcal{T}(A, I^3)$ is contractible.

See §6 for a discussion of this conjecture. According to a result in [Hen], this conjecture, if true, would imply the analogous result in the smooth category. This later result is known to be equivalent to the well-known Smale Conjecture, recently proved by A. Hatcher (see [Hat2]).

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1. Reduction of the Theorem. If P is a projective transformation of the extended plane which maps at most one point of K to infinity and if $f \in \mathcal{J}(J_k, K^2)$, then define $f_P: P(J_k) \rightarrow P(K)$ by setting $f_P(z) = P \circ f \circ P^{-1}(z)$ for z a vertex of J_k , and extending linearly. Note that $f_P \in \mathcal{T}(P(J_k), P(K^2))$ and $f_P(P(J_k)) = P \circ f \circ P^{-1}(P(J_k))$, but $f_P \neq P \circ f \circ P^{-1}$, since P is not linear on simplexes. Now pick a projective transformation P so that $P(J_k) = [0, k] \times \{0\}$ and so that the P -images of the 1-simplexes containing the endpoints of J_k are vertical and contain the segments $\{0\} \times [-1, 1]$ and $\{k\} \times [-1, 1]$ ($\subset \mathbf{R}^1 \times \mathbf{R}^1$) (see Figure 1). Note that if v and w belong to adjacent edges of ∂K , then P will send the vertex between v and w to ∞ .

FIGURE 1

(1.1) LEMMA. $\mathcal{T}(J_k, K^2)$ is homeomorphic to $\mathcal{T}(P(J_k), P(K^2))$ which contracts within itself into the subspace $\mathcal{T} \equiv \mathcal{T}(I_k, I^2)$, where $I_k \equiv P(J_k) = [0, k] \times \{0\}$ and $I^2 \equiv [0, k] \times [-1, 1]$.

PROOF. It is easy to check that $f \rightarrow f_P$ is a homeomorphism with inverse $g \rightarrow g_{P^{-1}}$. The contraction is accomplished by composing each function in $\mathcal{T}(P(J_k), P(K^2))$ with the homotopy of $\mathbf{R}^1 \times \mathbf{R}^1$, which takes

$$(x, y) \mapsto (x, (1 - t + td)y)$$

at time t for $d = \max\{\|y\| \mid y \text{ is the } y\text{-coordinate of some point in } P(K^2)\}^{-1}$. $\square$

In the remainder of the paper we will prove the Theorem by showing that $\mathcal{T} \equiv \mathcal{T}(I_k, I^2)$ is contractible.

2. The contraction of $\mathcal{T}$. In order to contract $\mathcal{T}$ it is sufficient to show that each compact subset $\mathcal{C} \subset \mathcal{T}$ can be contracted in $\mathcal{T}$. For $f \in \mathcal{C}$, we proceed by straightening out $f(I_k)$ one vertex at a time. That is, if

$$\mathcal{T}_j \equiv \{f \in \mathcal{T} \mid f(i) = i \text{ for } i = 1, 2, \dots, j\},$$

then we homotope $\mathcal{C}$ into $\mathcal{T}_1$ and then homotope (in $\mathcal{T}_1$) the (compact) image of $\mathcal{C}$ into $\mathcal{T}_2$, and so forth, until we reach $\mathcal{T}_{k-1}$, which consists only of the identity map. The homotopy of a compact subset of $\mathcal{T}_j$ (in $\mathcal{T}_j$) is accomplished in two steps:

(2.1) LEMMA. Any compact subset of $\mathcal{T}_j$ is homotopic (in $\mathcal{T}_j$) to a (compact) subset of

$$\mathcal{T}_j^* \equiv \{f \in \mathcal{T}_j \mid \pi_x f(j+1) = j+1\},$$

where π_x is the projection of $\mathbf{R}^1 \times \mathbf{R}^1$ onto $\mathbf{R}^1 \times \{0\}$.

(2.2) LEMMA. Any compact subset of $\mathcal{T}_j^*$ is homotopic (in $\mathcal{T}_j^*$) to a (compact) subset of $\mathcal{T}_{j+1}$.

PROOF OF LEMMA (2.1) [AND LEMMA (2.2)]. The construction of the homotopy is very similar in both steps: In the main text the homotopy of (2.1) will be detailed [the changes necessary for (2.2) will be indicated in square brackets]. The homotopy is defined to be horizontal [vertical]; i.e. $\pi_y f_t(i)$ [$\pi_x f_t(i)$] will be constant. With these restrictions the homotopy is defined as the flow along the gradient vector field of an energy function $E: \mathcal{T}_j \rightarrow [0, \infty)$ [$E: \mathcal{T}_j^* \rightarrow [0, \infty)$]. That is, the homotopy $f \mapsto f_t$ is defined by letting f_t be the solution of the system of equations

$$\frac{d}{dt} f_t(i) = \frac{-\partial}{\partial x(i)} E(f_t) \quad \left[= \frac{-\partial}{\partial y(i)} E(f_t) \right], \quad f_0 = f,$$

where $f_t(i) = (x(i), y(i))$. Part of E is a “spring energy”, S , acting as if there were a spring pulling $f(j+1)$ to where it is supposed to go. That is, for (2.1), define S on $\mathcal{T}_j$ by setting $S(f) = \|\{j+1\} - \pi_x f(j+1)\|^2$. [For (2.2), define S on $\mathcal{T}_j^*$ by setting $S(f) = \|\{j+1\} - f(j+1)\|^2$.] If this were all of E , then in most cases during the flow one of the vertices $f_t(i)$ would intersect the boundary, ∂I^2 , of I^2 or some $f_t([r, r+1])$ for $r \neq i-1, i$. To prevent this from happening, some “bumper energies”, B , are added, enabling points and segments to be kept apart. In particular, let $h: (0, \infty) \rightarrow [0, \infty)$ be defined by

$$h(x) = \begin{cases} \frac{A}{x} \exp\left(\frac{-1}{d-x}\right) & \text{for } 0 < x < d, \\ 0 & \text{for } d \leq x, \end{cases}$$

which is a monotone decreasing C^∞ function depending on the two parameters A and d , which will be defined in §4. Then, if p is a point in the interior of I^2 and Q is ∂I^2 or a line segment disjoint from p , define $B(p, Q) \equiv h(D(p, Q))$, where $D(p, Q) \equiv \inf\{\|p - q\| \mid q \in Q\}$ is the distance from p to Q . Notice that $B(p, Q)$ is zero if p and Q are further apart than d and that, as p approaches Q , $B(p, Q) \rightarrow \infty$. We can now define

$$E(f) \equiv S(f) + \sum B(p, Q),$$

where the sum is over all $p = f(i)$ and $Q = \{\partial I^2 \text{ or } f([r, r+1]) \text{ for } r \neq i-1, i\}$. For $f \in \mathcal{T}_j$, the first j vertices of I_k are fixed and thus we can represent $f = (f(j+1), f(j+2), \dots, f(k+1)) \in (\mathbf{R}^2)^{k-1-j}$ and $\mathcal{T}_j$ is an open subset of $(\mathbf{R}^2)^{k-1-j}$ and $\mathcal{T}_j^*$ is an open subset of $(\{j+1\} \times \mathbf{R}) \times (\mathbf{R}^2)^{k-2-j}$. Define f_t to be the unique solution of the system of differential equations

$$\frac{d}{dt} f_t(i) = \frac{-\partial}{\partial x(i)} E(f_t), \quad f_0 = f \quad \left[\frac{d}{dt} f_t(i) = \frac{-\partial}{\partial y(i)} E(f_t), \quad f_0 = f \right]$$

for $j+1 \leq i \leq k-1$. The theory of ordinary differential equations applies because in §3 we will show that $\partial E / \partial x(i)$ [$\partial E / \partial y(i)$] are Lipschitz functions of $f \in \mathcal{T}_j$ [$\mathcal{T}_j^*$]. In order for f_t to move outside of $\mathcal{T}_j$ [$\mathcal{T}_j^*$] it is necessary that either f_t fails to be one-to-one or $f_t(i)$ intersects ∂I^2 for some $i = j+1, j+2, \dots, k-1$, but in either case some $B(f_t(i), Q)$ will be infinite, which is impossible since the energy $E(f_t)$ starts finite and, for $z = x$ [$z = y$],

$$\frac{d}{dt} E(f_t) = \nabla E(f_t) \cdot \frac{d}{dt} f_t = - \sum_{i=j+1}^{k-1} \left\| \frac{\partial}{\partial z(i)} E(f_t) \right\|^2 < 0.$$

Thus, if f is in the compact $\mathcal{C} \subset \mathcal{T}_j [\mathcal{T}_j^*]$, then f_t stays in the compact $E^{-1}E(\mathcal{C}) \subset \mathcal{T}_j [\mathcal{T}_j^*]$. It follows from the standard theory of differential equations (see, for example, [Spi, pp. 194–200] that $(f, t) \rightarrow f_t$ defines a continuous flow on $\mathcal{T}_j [\mathcal{T}_j^*]$ for $t \in [0, \infty)$. Let

$$\nabla_z E \equiv \left(\frac{\partial}{\partial z(j+1)} E, \dots, \frac{\partial}{\partial z(k-1)} E \right) \quad \text{for } z = x \text{ } [z = y].$$

In §5 we will show

(2.3) LEMMA. $S(f) \neq 0$ implies that $\nabla_z E(f) \neq 0$ for $z = x$ [$z = y$].

Then we conclude that, for all $S(f_t) > \varepsilon$, there is a δ such that

$$\frac{d}{dt} E(f_t) = \nabla_z E(f_t) \cdot \frac{d}{dt} f_t = -\|\nabla_z E(f_t)\|^2 < \delta < 0,$$

and thus $\lim_{t \rightarrow \infty} S(f_t) = 0$, since $E(f_t)$ is bounded below by 0. Now, we do not know that this flow can be extended to $t = \infty$; however, eventually $f_t(j+1)$ will be so close to its destination that we can stop the flow (for all $j+1 \leq i \leq k-1$ simultaneously) and continue the homotopy by moving only $f(j+1)$ along the straight line to its destination, $(j+1, \pi_y f(j+1)) [(j+1, 0)]$. This finishes the proof of (2.1) [(2.2)]. $\square$

3. $\partial E / \partial x(i)$ and $\partial E / \partial x(i)$ are Lipschitz functions. Since

$$\frac{\partial}{\partial z} E(f) = \frac{\partial}{\partial z} S(f) + \sum h'(D(p, Q)) \frac{\partial}{\partial z} D(p, Q)$$

for $z = x(i)$ [$y(i)$], it suffices to show that $(\partial / \partial z) D(p, Q)$ is Lipschitz for $p = f(s)$ and $Q = \{\partial I^2 \text{ or } f([r, r+1]), r \neq s-1, s\}$. We see that if $p = f(s)$ and $Q = f([r, r+1])$, then

$$\nabla_i D \equiv \left(\frac{\partial}{\partial x(i)} D(p, Q), \frac{\partial}{\partial x(i)} D(p, Q) \right)$$

is nonzero only if $i = s, r$ or $r+1$. We use the fact that the gradient, $\nabla_i D$, is in the direction that $f(i)$ must be moved to obtain the maximum rate of increase of D and that that rate is the magnitude of $\nabla_i D$. Let q be the closest point of Q to p . We have three cases: $q = f(r)$, $q = f(r+1)$, or $q \in f((r, r+1))$.

(i) $q = f(r)$. Then $D(p, Q) = \|f(s) - f(r)\|$ and

$$\nabla_s D = \frac{f(s) - f(r)}{\|f(s) - f(r)\|} = -\nabla_r D \quad \text{and} \quad \nabla_{r+1} D = 0.$$

(ii) $q = f(r+1)$. Then $D(p, Q) = \|f(s) - f(r+1)\|$ and

$$\nabla_s D = \frac{f(s) - f(r+1)}{\|f(s) - f(r+1)\|} = -\nabla_{r+1} D \quad \text{and} \quad \nabla_r D = 0.$$

(iii) $q \in f((r, r+1))$. Then, if μ is the unit vector perpendicular to Q and pointing towards p , we have

$$\nabla_s D = \mu, \quad \nabla_r D = \frac{-\|q - f(r+1)\|}{\|f(r) - f(r+1)\|} \cdot \mu, \quad \text{and}$$

$$\nabla_{r+1} D = \frac{-\|q - f(r)\|}{\|f(r) - f(r+1)\|} \cdot \mu.$$

We notice that these gradients are continuous, but not differentiable. But restricted to each of the three pieces each gradient is differentiable and bounded (in $\mathcal{T}$). Thus these gradients are Lipschitz.

4. The constants c , A , d , and the orderings $<$ and $\circ$. Throughout this section $\mathcal{C}$ is some compact subset of $\mathcal{T}_j$ [$\mathcal{T}_j^*$]. Notice that the homotopy of $\mathcal{T}_j$ [$\mathcal{T}_j^*$] into $\mathcal{T}_{j+1}$ [$\mathcal{T}_{j+1}^*$] will be contained in the set $\mathcal{C}^\sim$, $\mathcal{C} \subset \mathcal{C}^\sim \subset \mathcal{T}_j$ [$\mathcal{T}_j^*$], where we set $g \in \mathcal{C}^\sim$ iff there is an $f \in \mathcal{C}$ such that

- (a) $\pi_y f(t) = \pi_y g(t)$ [π_x] for all $t \in I_k$.
- (b) for all $t, t' \in I_k$, $\pi_y f(t) = \pi_y(t')$ [π_x] implies that $\{\pi_x f(t), \pi_x f(t')\}$ has the same order as $\{\pi_x g(t), \pi_x g(t')\}$ [π_y].
- (c) $E(g) \leq E(\mathcal{C}) \equiv \text{Max}\{E(f') \mid f' \in \mathcal{C}\}$.

(4.1) LEMMA. *There is a constant $c \equiv c(\mathcal{C}) < \frac{1}{4}$ such that, for all $1 \leq r, s \leq k-1$ and $g \in \mathcal{C}^\sim$, whenever the horizontal [vertical] line through $g(s)$ intersects $g([r-1, r])$ and $g([r, r+1])$ on opposite sides of $g(s)$, then $\|\pi_y g(s) - \pi_y g(r)\| \geq c$ [$\|\pi_x g(s) - \pi_x g(r)\| \geq c$].*

PROOF. We shall prove (4.1) only for the case of $\mathcal{T}_j$ and leave to the reader the modifications necessary for $\mathcal{T}_j^*$. Suppose that $\{g_i\} \subset \mathcal{C}^\sim$ such that the horizontal line through $g_i(s)$ intersects $g_i([r-1, r])$ at p_i and intersects $g_i([r, r+1])$ at q_i on opposite sides of $g(s)$ and suppose that

$$\|\pi_y g_i(s) - \pi_y g_i(r)\| \rightarrow 0.$$

Then, for the corresponding $f_i \in \mathcal{C}$, some subsequence of $\{f_i\}$ will converge to $f \in \mathcal{C}$, since $\mathcal{C}$ is compact. Thus

$$\lim \pi_y(p_i) = \lim \pi_y(q_i) = \pi_y f(s) = \pi_y f(r)$$

and $\pi_x f(s)$ is between $\lim \pi_x(p_i)$ and $\lim \pi_x(q_i)$. Therefore, $f(s) \in f([r-1, r+1])$, which is impossible since $f \in \mathcal{T}$ is one-to-one. $\square$

(4.2) DEFINITIONS. (i) Set $b \equiv b(\mathcal{C}) \equiv 2(k/c)^2 > 2k/c > 8k$.

(ii) Choose $d = d(\mathcal{C}) > 0$ so small that $b^{2k}d \leq c/2$.

(iv) Pick $A = A(\mathcal{C})$ so large that $h(x) \leq E(\mathcal{C})$ implies $\frac{3}{4}d \leq x$.

For what follows, assume that $f \in \mathcal{C}^\sim$. We now use the above defined constants to construct several orderings on the segments: $P_r = f([r, r+1])$, $0 \leq r \leq k-1$.

(4.3) DEFINITION. (a) $P_i <_{\mathbf{a}} P_r$ iff there is a horizontal [vertical] line in I^2 which first (traveling from left to right [down to up]) intersects P_i and then intersects P_r .

(b) $P_i <_{\mathbf{b}} P_r$ iff $P_i <_{\mathbf{a}} P_{r_1} <_{\mathbf{a}} P_{r_2} <_{\mathbf{a}} \cdots <_{\mathbf{a}} P_{r_n} <_{\mathbf{a}} P_r$, for some n .

(c) $P_i <_{\mathbf{c}} P_r$ iff P_r not- $<_{\mathbf{b}} P_i$ and the projections $\pi_y P_i$ [$\pi_x P_i$] and $\pi_y P_r$ [$\pi_x P_r$] are disjoint and, for all those endpoints, $p \in P_i$ and $q \in P_r$, whose π_y [π_x]-images are closest, it is true that these images are within d of each other (see (4.2)), and $\pi_x p < \pi_x q$ [π_y] or $\{\pi_y(\pi_y|P_i)^{-1}\pi_y p \cap \pi_x(\pi_y|P_r)^{-1}\pi_y q\}$ is nonempty and $\pi_y p < \pi_y q$ [interchange π_x and π_y].

(d) $P_i < P_r$ iff $P_i <_{\mathbf{e}_0} P_{r_1} <_{\mathbf{e}_1} P_{r_2} <_{\mathbf{e}_2} \cdots <_{\mathbf{e}_{n-1}} P_{r_n} <_{\mathbf{e}_n} P_r$ for some n , where for each i , $\mathbf{e}_i = \mathbf{a}$ or $\mathbf{e}_i = \mathbf{c}$.

Notice that $<_c$ is discontinuous with respect to f in the sense that, for some f , $P_i <_c P_r$ but, for some g arbitrarily near f , $P_r <_c P_i$. This is the main source of the g_f constructed in §5 being discontinuous if f . We now prove

(4.4) LEMMA. *It is not true that $P_r < P_r$. Thus $<$ is a partial ordering on the $\{P_i\}$ for each $f \in \mathcal{C}^\sim$.*

PROOF. Suppose $P_{r_1} < P_{r_2} < \dots < P_{r_n}$ are adjacent elements in the ordering, i.e., for each i , there is no P_s such that $P_{r_i} \leq P_s \leq P_{r_{i+1}}$. It follows that, for each $1 \leq i \leq n-1$, $P_{r_i} \leq_{e_i} P_{r_{i+1}}$, where $e_i = \mathbf{a}$ or $e_i = \mathbf{c}$. Let $p_i \in P_{r_i}$ and $q_{i+1} \in P_{r_{i+1}}$ be such that $\pi_y(p_i)\pi_y(q_{i+1})[\pi_y]$ if $e_i = \mathbf{a}$, or $\|\pi_y(p_i - q_{i+1})\| < a[\pi_x]$ if $e_i = \mathbf{c}$. Look at the piecewise-linear path

$$\Lambda \equiv p_1 - q_2 - p_2 - q_3 - p_3 - \dots - q_{n-1} - p_{n-1} - q_n.$$

At each vertex of Λ whose y [x]-coordinate is a local extremum the path turns towards the right [top] and thus the first (in traversing Λ from p_1 to q_n) point $p \in \Lambda$, if any, which intersects a point on the path before p must be as in Figure 4.

FIGURE 4

If there is such a p then there are four possibilities: For some $r < s$, (i) $p = [q_r, p_r] \cap [q_s, p_s]$, (ii) $p = [q_r, p_r] \cap [p_s, q_{s+1}]$, (iii) $p = [p_r, q_{r+1}] \cap [q_s, p_s]$, or (iv) $p = [p_r, q_{r+1}] \cap [p_s, q_{s+1}]$. Since $[q_i, p_i] \subset P_{r_i}$, if (i) were true then we would have $P_{r_r} = P_{r_s} \equiv P$ and from Figure 4 we see that p_r, q_r must be in the same order on P as q_s, p_s and thus either $p_r \in [q_s, p_s]$ or $q_s \in [p_r, q_r]$. However, both of these are impossible since q_{r+1} is left of [below] p_r and p_{s-1} is right of [above] q_s and p is the first point of intersection; thus we conclude that (i) is impossible. If $e_s = \mathbf{a}$ then (ii) is impossible since p_s and p_{s+1} are assumed to be adjacent in the ordering. If $e_s = \mathbf{c}$ then (ii) would imply that P_{r_r} is less than $\frac{3}{4}d$ from either P_{r_s} or $P_{r_{s+1}}$, which is not possible by (4.2). Similarly, (iii) is impossible. From Figure 4 we see that at least one of the intervals of Λ containing p must point towards the left [bottom], thus (iv) is impossible. $\square$

In the construction of g_f in §5, we shall use the above partial ordering of the P_i and a related total ordering induced by a bijection:

(4.5) DEFINITION. Choose $o: \{0, 1, \dots, k-1\} \rightarrow \{0, 1, \dots, k-1\}$ to be a one-to-one function so that:

- (a) If $o(r) < o(s)$, then either $P_r < P_s$, or P_r and P_s are incomparable.

(b) If no segment $P_i \in f(I_k)$ is between (in the $<$ -partial ordering) two adjacent segments $\{P_{r-1}, P_r\}$, then $o(r-1) = o(r) + 1$, or $o(r) = o(r+1) + 1$.

(c) If $S \equiv \{P_0, P_1, \dots, P_s\}$ is maximal with respect to satisfying (b) for each pair $\{P_r, P_{r+1}\}$, $0 \leq r < s$, then $o(i) < o(0)$ iff $P_i < P_r$ for some $r \leq s$. (Note that $j \leq s$.)

Define $m(i) \equiv \min\{o(i), o(i-1)\}$ and $M(i) \equiv \max\{o(i), o(i-1)\}$. Note that $f(i)$ belongs to P_{i-1} and P_i , and so $m(i)$ gives the first place and $M(i)$ the last place that $f(i)$ is represented in the ordering o .

(4.6) LEMMA. *Let $f \in \mathcal{C}^-$. If $\pi_y f(i)$ is within d of $\pi_y(P_{r-1} \cup P_r) [\pi_x]$ and $\pi_y f(i) [\pi_x]$ is within $c = c(\mathcal{C})$ of $\pi_y f(r) [\pi_x]$, then either $m(i) \geq M(r)$ or $M(i) \leq m(r)$, with equality only if $i = r \pm 1$.*

PROOF. By (4.1) it is impossible that $f(i)$ be between P_r and P_{r-1} . Therefore, either (a) $\pi_y f(i)$ intersects both $\pi_y P_r$ and $\pi_y P_{r-1} [\pi_x]$, (b) $\pi_y P_r \cap \pi_y P_{r-1} = \pi_y f(r) [\pi_x]$, or (c) π_{P_r} or $\pi_{P_{r-1}}$ is contained in $[\pi_y f(i), \pi_y f(r)] [\pi_x]$. Thus, by (4.1) and (4.5)(b), we see in each case that neither P_{i-1} nor P_i can come between P_r and P_{r-1} in the $>$ -ordering. Therefore the desired inequalities hold with equality only if $P_r = P_{i-1}$ or $P_{r-1} = P_i$. $\square$

5. Proof of Lemma (2.3). Preparatory to proving (2.3) at the end of this section, we construct a certain function $g_f \in \mathcal{C}^-$ for each $f \in \mathcal{C}^-$.

(5.1) LEMMA. *If $f \in \mathcal{C}^-$ and $S(f) \neq 0$, then there is a $g_f \in \mathcal{C}^-$ such that:*

- (a) $\pi_y g_f(i) = \pi_y f(i) [\pi_x]$, $0 \leq i \leq k$,
- (b) for all $0 \leq s \leq 1$, $sf + (1-s)g_f \in \mathcal{T}_j [\mathcal{T}_j^*]$,
- (c) $E(g_f) = S(g_f) = 0$,
- (d) $(f(j+1) - g_f(j+1)) \cdot \nabla_{j+1} S(f) > 0$,
- (e) for all $0 \leq s \leq 1$, $B(sf + (1-s)g_f) < B(f)$, where $B(\cdot)$ ranges over all $B = B(p, Q)$ that appear in the definition of $E(\cdot)$.

We will prove (5.1) after giving the inductive construction of g_f in (5.2). The g_f which we construct will not vary continuously with f ; in fact, note that if we could find the g_f that varied continuously with f , then (5.1) would produce the desired homotopy of $\mathcal{T}_j [\mathcal{T}_j^*]$.

In order to construct g_f we make use of the constants d , b , c , and the orderings $<$, o of §4. Recall $m(i) \equiv \min\{o(i), o(i-1)\}$ and $M(i) \equiv \max\{o(i), o(i-1)\}$, and let $r \equiv o^{-1}: \{0, 1, \dots, k-1\} \rightarrow \{0, 1, \dots, k-1\}$. We define g_f inductively, one segment at a time. In the case that $\pi_x f(j+1) > j+1$ [$\pi_y f(j+1) > 0$], we define g_f first on $[r(0), r(0)+1]$ and then on $[r(1), r(1)+1]$ and so on in order until $[r(k-1), r(k-1)+1]$. In the case that $\pi_x f(j+1) < j+1$ [$\pi_y f(j+1) < 0$], we define g_f on the segments in the reverse order. We shall describe in detail only the first case; the changes necessary for the second case should be clear to the reader.

(5.2)_s LEMMA. In the case that $\pi_x f(j+1) > j+1$ [$\pi_y(j+1) > 0$], the function g_f can be defined on

$$Cs \equiv \bigcup_{i=0}^{s-1} [r(i), r(i) + 1]$$

so that the following hold:

(a)_s g_f is linear on each interval $[i, i+1] \in Cs$, $\pi_y g_f(p) = \pi_y f(p)$ [$\pi_x g_f(p) = \pi_x f(p)$], and, for $0 \leq i \leq j$, $g_f(i) = (i, 0)$ and $g_f(k+1) = (k+1, 0)$, whenever defined.

(b)_s There is a finite subset

$$Ts \equiv \{g_f(i) | M(i) \geq s\} \cup \{p_1, p_2\} \subset I^2,$$

where $p_1 = (b^{2k}d, \text{sign}(\pi_y f(j+1))1)$ [$= (0, -1 + b^{2k}d)$] and $p_2 = (b^{2k}d, 0)$ [$= (k, -1 + b^{2k}d)$] if $s \leq o(0)$, and $p_1 = (j+1 + b^{2k}d, -)$ [$= (0, b^{2k}d)$] and $p_2 = (j+1 + b^{2k}d, 1)$ [$= (k, b^{2k}d)$] if $s > o(0)$. For $p \in Ts$, define $M(p) = M(i)$ if $p = g_f(i)$, and $M(p_1) = M(p_2) = o(0)$ if $s \leq o(0)$, and $M(p_1) = M(p_2) = k$ if $s > o(0)$. We order Ts by the order on $\pi_y Ts$ [$\pi_x Ts$]. Except for p_1, p_2 , Ts is just those vertices of $g_f(Cs)$ which are endpoints of only one simplex in $g_f(Cs)$.

(c)_s Let q, q' be in Ts . If, for every r between q and q' in Ts , $M(r) < M(q)$ and $M(r) < M(q')$ and if $[q, q']$ is not contained in Cs , then, for all $0 < \tau < 1$, either (i) $p \equiv \tau \cdot q + (1 - \tau) \cdot q'$ is at least as far horizontally [vertically] as

$$\begin{aligned} D(q, q'; \tau; s) &\equiv (\tau b^{2M(q)} + (1 - \tau)b^{2M(q')})d(b^{-1} - sb^{-2}) \\ &= \tau(b^{2M(q)-1} - sb^{2M(q)-2})d + (1 - \tau)(b^{2M(q)-1} - sb^{2M(q)-2})d \end{aligned}$$

from any point within d of Cs or (ii) $\pi_y p$ [$\pi_x p$] is within c of $\pi_y q$ (or $\pi_y q'$) [π_x] and p is closer horizontally [vertically] than $D(q, q'; \tau; s)$ to the g_f -image of some interval containing q (or q'), where $M(p) \equiv \tau \cdot M(q) + (1 - \tau) \cdot M(q)$ (see Figure 5.1).

(d)_s Let $q \in Ts$. If $M(q) < o(0)$ and $p \in Jq$ with $\pi_y p = \pi_y f(j+1)$ [π_x], then $(j+1) - \pi_x p$ [$0 - \pi_y p$] $\geq b^{2k}d$. If $M(q) > o(0)$ and $q \in Ts$, then (c) holds for $q' \equiv (j+1 + b^{2k}d, \pm 1)$ [$\equiv (0, b^{2k}d)$ or $(k, b^{2k}d)$] with $M(q') \equiv k$.

(e)_s If $m(i) \leq o(0) \leq M(i)$, then $j+1 \leq \pi_x g_f(i) \leq j+1 + b^{2k}d$ [$0 \leq \pi_y g_f(i) \leq b^{2k}d$].

FIGURE 5.1