A CHARACTERIZATION AND ANOTHER CONSTRUCTION OF JANKO'S GROUP J_3

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ABSTRACT. Graphs Γ with the following properties are classified: (i) Γ is (G, s) -transitive for some $s \geq 4$ and some group $G \leq \text{aut}(\Gamma)$ such that each vertex stabilizer in G is finite, (ii) $s \geq (g-1)/2$, where g is the girth of Γ , and (iii) Γ is connected. A new construction of J_3 is given.

1. Introduction. Let Γ be an undirected graph with vertex set V and let G be a subgroup of $\text{aut}(\Gamma)$. For each $x \in V$, we denote by $\Gamma(x)$ the set of vertices adjacent to x . An s -path is a sequence $(x_0, x_1, \dots, x_s)$ of $s+1$ vertices such that $x_i \in \Gamma(x_{i-1})$ for $1 \leq i \leq s$, and $x_i \neq x_{i-2}$ for $2 \leq i \leq s$. We say that Γ is (G, s) -transitive if G acts transitively on the set of s -paths but not on the set of $(s+1)$ -paths in Γ .

Suppose that Γ is connected and (G, s) -transitive for some s . Let $k = |\Gamma(x)|$ for $x \in V$ and suppose $k \geq 3$. By [12, (7.61)], $s \leq (g+2)/2$, where g denotes the girth of Γ . If we assume as well that G acts distance-transitively on Γ , then $s \geq (g-2)/2$. In [17] we classified the distance-transitive graphs which are also (G, s) -transitive for some $s \geq 4$ and some $G \leq \text{aut}(\Gamma)$ with $|G(x)| < \infty$ for each $x \in V$. For $s \geq g/2$, the classification did not require Γ to be distance-transitive. For $s = (g-1)/2$, it was shown, again without assuming distance-transitivity, that $k \leq 5$, $s = 4$, and $g = 9$, and that $G \cong L_2(17)$ and Γ is isomorphic to a certain graph Δ_{102} with 102 vertices when $k = 3$. (These results do rest on [14] and hence on the classification of 2-transitive permutation groups of degree k .) In this paper we extend these results as follows.

(1.1) **THEOREM.** *Suppose Γ is a connected, undirected graph and that G is a subgroup of $\text{aut}(\Gamma)$ such that $|G(x)| < \infty$ for each $x \in V$. Suppose that Γ is $(G, 4)$ -transitive, that the girth g of Γ is 9, and that the valency k of Γ is 4 or 5. Then $k = 5$ and either $|V| = 17442$ and $J_3 \leq G \leq \text{aut}(J_3)$, or $|V| = 52326$ and $J_3 \leq G \leq \text{aut}(J_3) \cdot Z_3$, where $\text{aut}(J_3)$ acts nontrivially on Z_3 . In both cases, Γ is uniquely determined.*

In the course of proving (1.1), we obtain a new construction of J_3 :

(1.2) **THEOREM.** *Let J be the group generated by elements a , r , and c and defined by the following relations:*

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$a^9 = 1, r^2 = 1, c^3 = 1, ac = ca, (cr)^3 = 1, (ara^{-1}r)^2 = 1, t^2 = 1$ where
 $t = a^{-1}ra^{-3}ra^3ra^{-3}, tat = a^{-1}, trt = s$ where $s = a^{-1}ra, tct = c^{-1}, d^3 = 1$
 where $d = a^{-2}(ra)^3ra^{-1}, (cd)^5 = 1, c^{-1}dcd^{-1}c^{-1} = ara^{-1}r, dc^irc^{-i}d^{-1} =$
 $c^irc^isc^i$ for $i = 0, 1$ and -1 , and $dc^isc^{-i}d^{-1} = c^{-i}rc^i$ for $i = 0, 1$ and -1 .

Then $|J| = 50\,232\,960$, J is simple and J contains a unique conjugacy class of involutions.

The proof of (1.2) depends on computer calculations which were kindly carried out by Charles Sims at Rutgers (see §6 below). The identification $J \cong J_3$ follows from [18]. The first step in the proof of (1.1) is to eliminate the case $k = 4$. For $k = 5$, we identify three elements of G which generate a normal subgroup of index dividing six and show that they must satisfy the relations of (1.2). Crucial to the proof is the construction of a trivalent subgraph Δ of Γ isomorphic to the graph Δ_{102} mentioned above. The methods used in the case $k = 4$ are very different from those used when $k = 5$; the reader more interested in the case $k = 5$ may skip over §3 below.

We recall that after its discovery by Janko [10] as part of the solution to a centralizer-of-an-involution problem, J_3 was first constructed by Higman and McKay [9] who used a computer to work with generators and relations based on a conjectured subgroup of index 6156. Subsequent constructions of J_3 were given in [7, 8] and [15, 16] (see also Conway's *Atlas of Finite Groups*). While both of these constructions have the advantage of being independent of computer calculations, they both have the defect of requiring detailed information about the group being constructed; neither approach could have led to the discovery of J_3 . The construction (1.2), on the other hand, is arrived at by completely natural means in the course of proving (1.1) and does not require even a suspicion that J_3 exists.

2. Preliminary facts. From now on, let Γ be a connected, undirected graph of girth g , let x be a fixed vertex of Γ , let $k = |\Gamma(x)| \geq 3$, and let G be a subgroup of $\text{aut}(\Gamma)$ such that $|G(x)| < \infty$ and such that Γ is $(G, 4)$ -transitive. For each $i > 0$ and each $u \in V$, let $\Gamma_i(u) = \{v | \partial(u, v) = i\}$, where ∂ is the distance function in Γ , and let $G_i(u)$ be the largest subgroup of $G(u)$ fixing $\{v | \partial(u, v) \leq i\}$ pointwise. For each $t \geq 0$ and each t -path $(x_0, \dots, x_t)$, let $G_i(x_0, \dots, x_t) = G_i(x_0) \cap \dots \cap G_i(x_t)$.

Let $q = k - 1$. By [14], we have $G(x)^{\Gamma(x)} \cong PGL_2(q)$, where $G(x)^{\Gamma(x)}$ denotes the permutation group induced by $G(x)$ on $\Gamma(x)$. By [13, (2.3) and (2.5)], we have:

(2.1) For each 3-path $(x_0, \dots, x_3)$, $G_1(x_0, x_1) \cap G(x_3) = 1$ and $|G_1(x_0, x_1)| = q$. $\square$

For the case $k = 5$, we will require a few facts about the case $k = 3$:

(2.2) If $k = 3$, then

(a) $G(x) \cong \Sigma_4$.

(b) Every involution in $G(x)$ lies in $G_1(u)$ for some $u \in \Gamma(x)$.

(c) If $g = 9$, then $\Gamma \cong \Delta_{102}$ and $G \cong L_2(17)$.

PROOF. (a) holds by [13, (1.3)]. Thus $|G_1(x)| = 4$ and so $G_1(x)$ is the union of $G_1(x, u)$ for $u \in \Gamma(x)$. There are six subgroups of the form $G_1(u, v)$ for $u \in \Gamma(x)$ and $v \in \Gamma(u) - \{x\}$; they contain the other six involutions of $G(x)$ and (b) follows. By [17, (2.1)], Γ is distance-transitive if $g = 9$; (c) then follows by [2]. $\square$

From now on, suppose that $k > 3$. For each $u \in V$, we let $\overline{G}(u)$ denote the largest subgroup of $G(u)$ such that $\overline{G}(u)^{\Gamma(u)} \cong PGL_2(q)$ (so $\overline{G}(u) = G(u)$ always when $k = 4$). Following [13], we say that a path $(x_0, \dots, x_t)$ is good if $G(x_0, \dots, x_t) \cap \overline{G}(x_i)$ induces a $(q-1)$ -cycle on $\Gamma(x_i)$ for $0 < i < t$. By [13, (2.1)–(2.2)], we have:

(2.3) *Every t -path is good if $t \leq 4$. If $(x_0, \dots, x_t)$ is a good t -path with $t \geq 4$, then there is a unique vertex $u \in \Gamma(x_t)$ such that $(x_0, \dots, x_t, u)$ is a good $(t+1)$ -path.* $\square$

A subgraph A of Γ which is connected and regular of valency two will be called an apartment if every path of every length lying on A is good. (Notice that this corresponds to the usual notion of an apartment when Γ is the incidence graph of the desarguesian projective plane of order q , which we denote by $\Delta_{3,q}$ below.) By (2.3), every 4-path lies on a unique apartment.

(2.4) *If $(x_0, \dots, x_6)$ and $(x'_0, \dots, x'_6)$ are good 6-paths with $(x_0, \dots, x_3) = (x'_0, \dots, x'_3)$ but $x_4 \neq x'_4$, then $(x_6, x_5, x_4, x_3, x'_4, x'_6)$ is a good 6-path as well.*

PROOF. [13, (5.2)]. $\square$

(2.5) *Suppose that each apartment is a 9-circuit. Let $(x_0, \dots, x_3)$ be a 3-path and let $u_1, \dots, u_q$ be the points opposite $(x_0, \dots, x_3)$ on the q apartments passing through $(x_0, \dots, x_3)$. Then the $q+2$ vertices $u_1, \dots, u_q, x_0$, and x_3 are connected pairwise by 3-paths which are disjoint except for their endpoints.*

PROOF. This follows from (2.4). $\square$

(2.6) *If $k = 4$, then*

(a) *$G(x)$ is isomorphic to a maximal parabolic subgroup of $L_3(3)$; in particular, $G(x)$ contains no element of order 9.*

(b) *If $(x_0, \dots, x_4)$ is a 4-path, then $|G(x_0, \dots, x_4)| = 4$. If $H_i = G(x_0, \dots, x_4) \cap G_1(x_i)$ for $1 \leq i \leq 3$, then $|H_i| = 2$ and $G(x_0, \dots, x_4) = H_1 \cup H_2 \cup H_3$.*

(c) *Let $m = 0, 1$ or 2 . Suppose $(x_m, \dots, x_5)$ and $(x'_m, \dots, x'_5)$ are $(5-m)$ -paths such that $(x_m, \dots, x_3) = (x'_m, \dots, x'_3)$ but $x_4 \neq x'_4$. Suppose, too, that $(x_m, \dots, x_5)$ and $(x'_m, \dots, x'_5)$ are both not good if $m = 0$. Then G contains an involution exchanging $(x_m, \dots, x_5)$ and $(x'_m, \dots, x'_5)$.*

(d) *Let $t > 0$ be given. Suppose $G(x)$ acts transitively on $\Gamma_i(x)$ for all $i \leq t$. Then $|\Gamma(u) \cap \Gamma_{t+1}(x)| \geq 2$ for all $u \in \Gamma_t(x)$.*

PROOF. (a) holds by [13, (1.2)]. To check that (b) holds, it suffices, again by [13, (1.2)], to check that it holds when $\Gamma = \Delta_{3,3}$ (as defined above).

(c) If $(x_5, \dots, x_0)$ is a 5-path extending $(x_5, \dots, x_m)$ which is not good, then by (2.4), the 5-path $(x'_5, x'_4, x_3, \dots, x_0)$ is also not good. Thus, we may assume $m = 0$. Let u be the neighbor of x_3 distinct from x_2, x_4 , and x'_4 . By (b), $G(x_0, \dots, x_3, u)$ contains two involutions exchanging x_4 and x'_4 . Both involutions exchange the good 5-paths extending $(x_0, \dots, x_4)$ and $(x'_0, \dots, x'_4)$. By (2.1), their product does not lie in $G_1(x_4)$. Thus, one of them must map x_5 to x'_5 .

(d) By [11], Γ cannot be distance-transitive. The claim then follows by [17, (2.1)]. $\square$

Suppose that $k = 5$. For each $u \in V$, we denote by $\hat{G}(u)$ the subgroup of $G(u)$ generated by all the involutions of $\overline{G}(x)$.

(2.7) *If $k = 5$, then*

(a) *$\hat{G}(x)$ is isomorphic to a maximal parabolic subgroup of $L_3(4)$; in particular, $\hat{G}(x)$ contains no element of order greater than five.*

(b) $G(x)/\hat{G}(x) \lesssim \Sigma_3$.

(c) $\hat{G}(x) \cap G(y) \leq \hat{G}(y)$ for all $y \in \Gamma(x)$.

(d) $\overline{G}(x) \cap G(y) \leq \overline{G}(y)$ for $y \in \Gamma(x)$.

(e) $G_1(x) \cap \hat{G}(x)$ is an elementary abelian 2-group. In particular, $G_1(x, y)$ is elementary abelian for $y \in \Gamma(x)$. Every involution of $\overline{G}(x)$ lies in $G_1(u)$ for some $u \in \Gamma(x)$.

(f) Let $S = G(x_0, \dots, x_4)$ for some 4-path $(x_0, \dots, x_4)$. Then $|S| \mid 18$, $|S \cap \overline{G}(x_0)| \mid 9$, $|S \cap \hat{G}(x_0)| = 3$, and $S \cap \hat{G}(x_0)$ induces a 3-cycle on $\Gamma(x_0)$.

(g) Let A be an apartment through x . Then for each $v \in \Gamma(x)$ not on A , there exists an involution in $G_1(v)$ reflecting A .

(h) Suppose that each apartment is a 9-circuit. Then there are no elements in G of order 27 rotating A .

PROOF. Since (a)–(f) hold when $\Gamma = \Delta_{3,4}$, they hold in general by [13, (1.2)].

(g) Choose an arbitrary edge $\{u, v\}$ and an involution $a \in G_1(u)$ not in $G_1(v)$. There is a 4-path $(u_0, \dots, u_4)$ with $u_2 = v$ such that $a(u_i) = u_{4-i}$ for $0 \leq i \leq 4$. The element a reflects the unique apartment through $(u_0, \dots, u_4)$. Since G acts transitively on the set of 4-paths and $G(u_0, \dots, u_4)$ acts transitively on $\Gamma(u_2) - \{u_1, u_3\}$, the claim follows.

(h) By (2.5), any two vertices at distance 3 are contained in a unique set of six vertices which are pairwise at distance 3. We call such a set a sextet. Now let $(x_0, \dots, x_9)$ be a good 9-path and let $a \in G$ be a 3-element mapping $(x_0, \dots, x_8)$ to $(x_1, \dots, x_9)$. Then a^3 fixes the sets $\{x_0, x_3, x_6\}$ and $\{x_1, x_4, x_7\}$. Each of these sets lies on a unique sextet which must be fixed by a^3 , hence fixed pointwise by a^9 . It follows that $a^9 = G_1(x_0, x_1) \cap G(x_0, \dots, x_4) = 1$. $\square$

3. The case $k = 4$. Let $k = 4$. For each 4-path $(u_0, \dots, u_4)$, we denote by $A(u_0, \dots, u_4)$ the unique apartment through $(u_0, \dots, u_4)$. Let t be the number of vertices in an apartment. Let B be the function from the set of 4-paths to the set of $(t-6)$ -paths given by $B(u_0, \dots, u_4) = (u_5, \dots, u_{t-1})$ whenever $(u_0, \dots, u_{t-1})$ is a good $(t-1)$ -path. We will let $\Gamma_i = \Gamma_i(x)$ for all $i \geq 1$. For each $m \geq 0$ and each m -path $(u_0, \dots, u_m)$, we will denote by $[u_0, \dots, u_m]$ the $G(x)$ -orbit (in the set of all m -paths) containing $(u_0, \dots, u_m)$.

Let $(x_0, \dots, x_9)$ be a 9-circuit with $x = x_0$. Then x_4 and $x_5 \in \Gamma_4$. By (2.6)(d), $|\Gamma_5 \cap \Gamma(x_4)| = 2$. It follows that $G(x_0, \dots, x_4) \leq G(x_0, \dots, x_5) \leq G(x_0, \dots, x_9)$ so $(x_0, \dots, x_9)$ is a good 9-path and $t = 9$. Moreover, $G(x_0, \dots, x_4)$ acts transitively on $\Gamma_5 \cap \Gamma(x_4)$ so $G(x)$ acts transitively on Γ_5 . Choose $x'_5 \in \Gamma_5 \cap \Gamma(x_4)$ and let $(x'_6, \dots, x'_9) = B(x_1, \dots, x_4, x'_5)$. By (2.6)(c), there is an involution in $G(x)$ reflecting $A(x_3, x_2, x_1, x'_9, x'_8)$. Hence $x'_6 \in \Gamma_5$. By (2.6)(b), $G(x) \cap G_1(x_4)$ induces a transposition on $\Gamma(x'_5)$. The subgroup $G(x) \cap G_1(x_4)$ also fixes $A(x_1, \dots, x_4, x'_5)$ and hence x'_6 as well, so $|\Gamma_6 \cap \Gamma(x'_5)| = 2$ and $G(x)$ acts transitively on Γ_6 . By (2.1), $G(x, u) = 1$ for $u \in \Gamma_6$.

Choose $x''_6 \in \Gamma_6 \cap \Gamma(x'_5)$ and let $(x''_7, \dots, x''_{10}) = B(x_2, x_3, x_4, x'_5, x''_6)$. By (2.6)(c), there is an involution in $G(x)$ reflecting $A(x_4, x_3, x_2, x''_{10}, x''_9)$. Hence $x''_7 \in \Gamma_6$. By (2.6)(d), we have $|\Gamma_7 \cap \Gamma(x''_7)| = 2$. Let $(v_5, \dots, v_8) = B(x_5, x_4, x'_5, x''_6, x''_7)$. Since $x_5 \notin A(x_4, x'_5, x''_6, x''_7, x''_8)$, it follows that $v_5 \neq x''_8$ and so $v_5 \in \Gamma_7$. Hence $v_8 \in \Gamma_5$. By (2.5) applied to $(x_4, x'_5, x''_6, x''_7)$, there is a 3-path $(w_1, \dots, w_4)$ with $w_1 = x''_{10}$, $w_4 = v_7$, and $w_3 \neq v_8$. Since $|\Gamma_5 \cap \Gamma(u)| = 1$ for $u \in \Gamma_6$, we conclude that $w_2, w_3 \in \Gamma_4$, $v_7 \in \Gamma_5$, and $v_6 \in \Gamma_6$.

Suppose that $G(x, v_5)$ acts intransitively on $\Gamma_6 \cap \Gamma(v_5)$. Since $G(x)$ acts transitively on Γ_6 and $|\Gamma_7 \cap \Gamma(u)| = 2$ for $u \in \Gamma_6$, it follows that $G(x)$ acts transitively on Γ_7 . By (2.6)(d), $|\Gamma_8 \cap \Gamma(v_5)| = 2$. If $w \in \Gamma_3 \cap \Gamma(w_3)$, then $A(w, w_3, v_7, v_6, v_5) \neq A(w_3, v_7, v_6, v_5, x''_7)$, so $A(w, w_3, v_7, v_6, v_5)$ must intersect Γ_8 . Since this is impossible, we conclude that $G(x, v_5)$ acts transitively on $\Gamma_6 \cap \Gamma(v_5)$. Since $G(x, u) = 1$ for $u \in \Gamma_6$, $|G(x, u)| = |\Gamma_6 \cap \Gamma(v_5)|$. Let g be the element of $G(x, v_5)$ mapping x''_7 to v_6 . We define a quintet to be a subset of five vertices pairwise at distance 3. By (2.5), each pair of vertices at distance 3 lies on a unique quintet. If Q is the quintet containing x''_9 and v_5 , then $w_3 = g(x''_9) \in Q$ and so $g(Q) = Q$. If $g(v_6) = x''_7$, then g must exchange x''_8 and v_7 as well and hence reflect $A(x''_8, x''_7, v_5, v_6, v_7)$. Since $w_1 \in \Gamma_3$ and $w_2 \in \Gamma_4$, this is impossible. Hence $g(v_6) \neq x''_7$. By (2.6)(b), there exists an involution $b \in G(x) \cap G_1(w_1)$. Then $b \in G(x''_9, w_3)$ and so $b(Q) = Q$. By (2.1), $b \notin G_1(x''_9)$ and so $b \notin G(v_5)$. If $v'_5 = b(v_5)$, then $v_5, v'_5, x''_9, g(x''_9)$, and $g^2(x''_9)$ are five vertices in Q . The quintet Q consists, therefore, of precisely these five vertices. In particular, $|g| = 3$ and $g \in G(v'_5)$. Moreover, there exist vertices v and v' such that (v_5, v, v', v'_5) is a 3-path and $g \in G_1(v, v')$. Since $G(x, u) = 1$ for $u \in \Gamma_6$, we have v and $v' \in \Gamma_8$. Since b exchanges v_5 and v'_5 , the $G(x)$ -orbits $[v]$ and $[v']$ coincide. Suppose $|\Gamma(v) \cap [v]| > 1$. Choose $v'' \in \Gamma(v) \cap [v]$ different from v' . Then g induces a 3-cycle on $\Gamma(v'')$ and so $|\Gamma_7 \cap \Gamma(v)| = 3$, which is impossible if $|\Gamma(v) \cap [v]| > 1$. Hence $|\Gamma(v) \cap [v]| = 1$. By (2.5), there is a vertex n such that $x, n, w_1, g(w_1)$, and $g^2(w_1)$ form a quintet. If (x, n', n'', n) is the 3-path from x to n , then $g \in G_1(n', n'')$. Since b fixes $x, w_1, g(w_1)$, and $g^2(w_1)$, it fixes n , too.

Let $a \in G(x)$ be an element mapping x_4 to x_5 . Since $|\Gamma_4 \cap \Gamma(u)| = 1$ for $u \in \Gamma_4$, we have $a(x_5) = x_4$. By (2.6)(b), we may assume that a exchanges x'_5 and v_8 . Then $a^2 \in G(x) \cap G_1(x_4, x_5) = 1$, so a must fix (as a set) one or all of the three apartments through (x'_5, x_4, x_5, v_8) . Since $v_7 \in \Gamma_5$ and $x''_6 \in \Gamma_6$, a does not fix $A(x'_5, x_4, x_5, v_8, v_7)$. It follows that there exist vertices $u_1, \dots, u_5$ such that $u_1 \in \Gamma_6 \cap \Gamma(x'_5)$, $(u_2, u_3, u_4, u_5) = B(v_8, x_5, x_4, x'_5, u_1)$, and a reflects $A(v_8, x_5, x_4, x'_5, u_1)$. Then $a \in G(u_3)$. Since $A(x''_7, x''_6, x'_5, x_4, x_5)$ cannot be reflected by an involution, $[u_2, u_1, x'_5, x_4, x_5] \neq [x''_7, x''_6, x'_5, x_4, x_5]$ and hence $[x''_7, x''_6] \neq [u_2, u_1]$. Thus u_2 and $u_4 = a(u_2) \in \Gamma_7$. Since $G(x, u) = 1$ for $u \in \Gamma_6$, we have $u_3 \in \Gamma_7 \cup \Gamma_8$; we leave the exact value of $\partial(x, u_3)$ undetermined for the moment. Since $A(v_5, v_6, v_7, w_3, w_2)$ intersects Γ_3 , we have $[v_5, v_6, v_7, w_3, w_2] \neq [u_2, u_1, x'_5, x_4, x_5]$ and hence $[v_5] \neq [u_2]$.

Let $v''_5 = a(v_5)$. The vertices x'_5, v_8, v_5, v''_5 , and u_3 form a quintet P . In particular, there are vertices q_1 and q_2 such that (v_5, q_1, q_2, v''_5) is a 3-path. Suppose that $q_1 = v$. Since $a(q_1) = q_2$ and $|\Gamma(v) \cap [v]| = 1$, we have $q_2 = v'$ and hence $g \in G(v''_5)$. It follows that g fixes P . Hence $g \in G(x'_5)$, which is impossible. We conclude that q_1 and $q_2 \in \Gamma_6$. Since P is a quintet, there is a vertex p in $\Gamma(v) \cap \Gamma(u_3)$. Since $a \in G(x, u_3)$ exchanges u_2 with u_4 and v_5 with v''_5 , a induces the product of two transpositions on $\Gamma(u_3)$.

We claim now that the orbits $[u_2], [v_5], [u_3], [v]$, and $[p]$ are all distinct. We know already that $[u_2] \neq [v_5]$ and that $v \in \Gamma_8$. Moreover, $[u_3] \neq [v_5]$ because $\Gamma(v_5) \cap [u_2] = \emptyset$. If $[u_2] = [u_3]$, then $|\Gamma_6 \cap \Gamma(u_3)| = 2$ since $a \in G(x, u_3)$ and $u_2, u_4 \in \Gamma_7 \cap \Gamma(u_3)$. Thus $p \in \Gamma_6 \cap \Gamma(v)$, which is impossible. Hence $[u_2] \neq [u_3]$. Suppose $[v] = [u_3]$. Then $|\Gamma(u_3) \cap [v_5]| \geq 1$. Since $|\Gamma(u_3) \cap [u_2]| \geq 2$ and $|\Gamma(v) \cap [v]| = 1$, we must have $|\Gamma(u_3) \cap [v_5]| = 1$. But this implies that $a \in G(x, v_5)$, which contradicts the fact that $|G(x, v_5)| = 3$. Hence $[v] \neq [u_3]$. Since $\Gamma(v_5) \cap [u_3] = \emptyset$,

$[p] \neq [v_5]$. If $[p] = [u_2]$, then u_2 must have neighbors in both $[u_3]$ and $[v]$, which implies that $|\Gamma_6 \cap \Gamma(u_2)| \leq 2$ and so $|G(x, u_2)| \leq 2$. This contradicts the fact that $g \in G(x, p)$. Hence $[p] \neq [u_2]$. Suppose $[v] = [p]$. Since $|\Gamma(v) \cap [v]| = 1$, $p = v'$. Thus, $\langle g, a \rangle \leq G(x, u_3)$ acts transitively on $\Gamma(u_3)$. This contradicts the fact that $[p] \neq [u_2]$. Hence $[v] \neq [p]$. Finally, if $[p] = [u_3]$, then $|\Gamma(u) \cap [u_2]| \geq 2$, $|\Gamma(u) \cap [u_3]| \geq 2$ (since $a(p) \neq p$), and $|\Gamma(u) \cap [v]| \geq 1$ for $u \in [u_3]$, which is impossible. This proves our claim.

We know that $|\Gamma(u_3) \cap [u_2]| = |\Gamma(u_3) \cap [p]| = 2$, $|\Gamma(p) \cap [u_3]| = 3$, and $|\Gamma(p) \cap [v]| = 1$. Let $\alpha = |\Gamma(v) \cap [v_5]|$, $\beta = |\Gamma(v) \cap [p]|$, $\gamma = |\Gamma_6 \cap \Gamma(u_2)|$, and $\delta = |\Gamma(u_2) \cap [u_3]|$. Since $|\Gamma(v) \cap [v]| = 1$, we have $\alpha + \beta \leq 3$. Since $|\Gamma(p) \cap [v]| = |\Gamma(v_5) \cap [v]| = 1$, $G(x, v)$ must act transitively on both $\Gamma(v) \cap [p]$ and $\Gamma(v) \cap [v_5]$. Thus $|G(x, v)| = 3 \cdot \alpha$ and so $\beta = 1$. Counting the vertices in $[p]$ in two different ways, we find that $\alpha \cdot \delta = \gamma$. Since $\alpha \leq 2$, we have $\gamma \leq 2$.

Let $(t_1, t_2, t_3, t_4) = B(u_2, u_1, x'_5, x_4, x_3)$. Then $t_1 \in \Gamma_4$. If $t_2 \in \Gamma_4$ as well, then $t_4 \in \Gamma_6$ so $\gamma = 2$ and $G(x, u_2)$ contains an involution reflecting $A(t_3, t_4, u_2, u_1, x'_5)$. Since $x_3 \in \Gamma_3$ and $t_1 \in \Gamma_4$, this is impossible. Hence $t_2 \in \Gamma_5$. By (2.6)(c), there is an involution in $G(x)$ reflecting $A(t_2, t_1, x_3, x_4, x'_5)$ and so $t_4 \in \Gamma_7$. Thus $\Gamma_7 \cap \Gamma(u_2) \neq \emptyset$. Hence $\delta = 1$ and $\alpha = \gamma$.

Recall that $G_1(n', n'') = \langle g \rangle = G_1(v, v')$ and that $b \in G(x, n)$ exchanges v and v' . Thus $\partial(n', v) = \partial(n', v') \leq 9$. It follows that there exists an edge $\{z, z'\}$ with $\partial(x, z) \leq 9$ and $[z] = [z']$ such that $G_1(x, y) = G_1(z, z')$ for some $y \in \Gamma(x)$. Choose a vertex $z'' \in \Gamma(z)$ nearest to x . Then $G(x, z'')$ induces a 3-cycle on $\Gamma(z'')$. If $z'' \in [v_5]$, then $[z, z'] = [v, v']$ and so $G_1(x, y) = G_1(z, z')$ is conjugate in $G(x)$ to $G_1(n', n'') = G_1(v, v')$; this is impossible since $G_1(x, n', n'') = 1$. We must therefore have $z'' \in \Gamma_8$, $z \in \Gamma_9$, $|\Gamma(z'') \cap [u_2]| = 3$, and $[z'']$ and $[z]$ are new orbits. In particular, $\gamma = 1$ and hence $\alpha = 1$.

If v'' is the neighbor of v distinct from v_5, v' , and p , then $v'' \in \Gamma_9$ and g induces a 3-cycle on $\Gamma(v'')$. Since a 4-path beginning in Γ_5 and ending at v'' must lie on an apartment, we have $\Gamma(v'') \subseteq \Gamma_8 \cup \Gamma_9$. By a similar argument, there are now at most as many new $G(x)$ -orbits as there are neighbors of z outside of $[z'']$ and $[z]$. In every case, we find that the number of vertices of Γ is not divisible by 3. Since G contains elements rotating an apartment, G contains elements of order 9. Hence, $G(x)$ contains elements of order 9. This contradicts (2.6)(a). We conclude that $k = 4$ is impossible.

4. The subgraph Δ . Suppose from now on that $k = 5$. In this section, we show that Γ contains a trivalent subgraph Δ with the following properties:

- (I) $\Delta \cong \Delta_{102}$.
- (II) If D is the subgroup of G fixing Δ , then either $D \cong L_2(17)$ or $D \cong L_2(17) \times \mathbb{Z}_2$.
- (III) If $E = D'$, then E contains elements a, b , and t satisfying the relations

$$\begin{array}{ll}
 \text{(R1)} \ a^9 = 1, & \text{(R6)} \ b^t = b^{-1}, \\
 \text{(R2)} \ (ab^{-1})^2 = 1, & \text{(R7)} \ t = b^{-1}a^{-3}ba^2ba^{-4}, \\
 \text{(R3)} \ (a^2b^{-2})^2 = 1, & \text{(R8)} \ (a^{-2}b^4a^{-2})^3 = 1, \\
 \text{(R4)} \ t^2 = 1, & \text{(R9)} \ (a^{-1}b)^d = ab^{-1}, \text{ where } d = a^{-2}b^4a^{-2}, \text{ and} \\
 \text{(R5)} \ a^t = a^{-1}, & \text{(R10)} \ (ab^{-1})^d = (ab^{-1})(a^{-1}b).
 \end{array}$$

(We are using the conventions that $g^h = hgh^{-1}$ for elements g and $h \in G$ and that the elements of G act on V from the left.)

Let A be an apartment of Γ . By [17, (1.2) and (2.1)], A is a 9-circuit. We suppose first that there is an involution g fixing A pointwise. Let Δ_0 be the subgraph of Γ fixed pointwise by g , let Δ be the connected component of Δ_0 containing the vertices of A and let D be the subgroup of $C_G(g)$ fixing Δ . By (2.7)(f), $g \notin \overline{G}(x_0)$. Thus by (2.7)(d), Δ is trivalent. By (2.7)(f), $\langle g \rangle \in \text{Syl}_2(G(x_0, \dots, x_4))$ for each 4-path $(x_0, \dots, x_4)$ of Δ , so D acts transitively on the set of 4-paths in Δ . By (2.1) and (2.7)(f), $\langle g \rangle$ is the kernel of the action of D on Δ . By (2.2)(c), $\Delta \cong \Delta_{102}$ and $D/\langle g \rangle \cong L_2(17)$. Let $S \in \text{Syl}_2(D)$ and let h be an arbitrary element of S . Since the number of edges of Δ_{102} is odd, h fixes some edge $\{u, v\}$. Thus $h^2 \in G(u, v)$. By (2.7)(d), $h^2 g^i$ induces an even permutation on both $\Gamma(u)$ and $\Gamma(v)$ for $i = 0$ or 1. It follows that $h^4 = (h^2 g^i)^2 \in G_1(u, v)$ and hence $h^8 = 1$ by (2.7)(e). This implies that S is not generalized quaternion and so $D \not\cong SL_2(17)$. It follows that $D' \cong L_2(17)$. The group D' acts faithfully on Δ and Δ is $(D', 4)$ -transitive.

Next we produce Δ under the assumption that there is no involution fixing A pointwise. We define a star to be a subgraph σ of the form

which has the property that every 4-path $(v_0, \dots, v_4)$ in σ with $v_3 = u$ extends to a good 6-path in σ . The vertex u will be called the center of the star σ .

(4.1) Let τ be a subgraph of the form

such that every 4-path $(v_0, \dots, v_4)$ in τ with $v_3 = u$ extends to a good 5-path in τ . Then τ lies in a unique star with center u .

PROOF. Let σ be the subgraph spanned by τ and the collection of vertices z such that $(v_0, \dots, v_5, z)$ is a good 6-path for some good 5-path $(v_0, \dots, v_5)$ in τ with $v_3 = u$. By (2.3) and (2.4), σ is a star with center u . $\square$

(4.2) Every subgraph of the form

lies on a unique star with center u .

PROOF. We label vertices as follows:

By (2.3), there are vertices u_i , v_i , w_i , y_i , and z_i for $i = 1$ and 2 such that (x_i, w, u, y, z, u_i) , (u_i, z, y, u, v, v_i) , (v_i, v, u, w, x_1, w_i) , (w_i, x_1, w, u, y, y_i) , and (y_i, y, u, w, x_2, z_i) are all good 5-paths. By (2.4), $y_1 = z$. Any star with center u containing x_1 , x_2 , w , v , y , and z must contain all of these vertices. Let τ be the subgraph spanned by all of these vertices except u_1 and u_2 . Let $(p_0, \dots, p_4)$ be a 4-path in τ with $p_3 = u$. If $p_0 = w_1$ or w_2 , or if $p_0 = z_1$ or z_2 , and $p_4 = y$, then $(p_0, \dots, p_4)$ extends to a good 5-path in τ by construction. Suppose $p_0 = z_j$ for $j = 1$ or 2 and $p_4 = v$. By (2.1) and (2.7)(e), there is an element $g \in G_1(u, y)$ exchanging x_1 and x_2 . Since (w_i, x_1, w, u, y, y_i) and (z_i, x_2, w, u, y, y_i) are good paths and $g \in G(w, u, y, y_i)$, g must exchange z_i with w_i for $i = 1$ and 2 . Since (x_i, w, u, y, z, u_i) is good for $i = 1$ and 2 , g must exchange u_1 and u_2 . Since (v_i, v, u, y, z, u_i) is good for $i = 1$ and 2 , g must exchange v_1 and v_2 . Thus, $g(w_j, x_1, w, u, v, v_j) = (z_j, x_2, w, u, v, v_k)$ for $k \in \{1, 2\} - \{j\}$, so the 5-path $(p_0, \dots, p_4, v_k)$ is good. We conclude that τ is as in (4.1). Thus, there is a unique star σ with center u containing τ , u_1 , and u_2 . $\square$

Now let Σ_0 be the graph whose vertices are the stars of Γ , where two stars are defined to be adjacent if their centers are adjacent in Γ and their intersection is a subgraph as in (4.1). It follows from (4.1) that Σ_0 is trivalent. The girth of Σ_0 is at least 9 since the centers of the stars forming a circuit in Σ_0 lie on a circuit of Γ . Choose a 9-path $(x_0, \dots, x_9)$ lying on A and an involution $g \in G(x_5)$ reflecting A . Let x'_0 be an arbitrary neighbor of x_0 not on A and let $x'_1 = g(x'_0)$. By (4.2), there exists a unique star σ with center x_0 containing the vertices x_7 , x_8 , x_0 , x_1 , x_2 , x'_0 and x'_1 , and the star σ is adjacent to $g(\sigma)$ in Σ_0 . Let Σ be the connected component of Σ_0 containing σ . By (2.7)(g), there is an involution $h \in G_1(x'_0)$ reflecting A . By (4.2), h fixes σ . The images of σ under $\langle gh \rangle$ form a circuit in Σ . By (2.7)(f) and (h) and the assumption that there are no involutions fixing A pointwise, we have $(gh)^9 = 1$. Thus, the girth of Σ is 9 and D acts transitively on the vertex set of Σ . By (2.1) and (2.7)(e), there exists an involution in $G_1(x_1, x_2)$ exchanging x_8 and x'_0 . This involution must fix the 3-path $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ of Σ with $\sigma_0 = \sigma$ and x_i the center of σ_i for $0 \leq i \leq 3$, and exchange the neighbors of σ having centers x_8 and x'_0 . It follows (see [12]) that D acts transitively on the set of 4-paths in Σ . By (2.1) and (2.7)(f), D acts faithfully on Σ . By (2.2)(c), $\Sigma \cong \Delta_{102}$ and $D \cong L_2(17)$. Since $L_2(17)$ acts primitively on the vertices of Δ_{102} , we may identify Σ with the subgraph Δ of Γ whose vertices are the centers of the stars of Σ . Thus, we have Δ fulfilling (I) and (II) above in this case, too. We let $E = D'$ (and withdraw the assumptions about the existence or nonexistence of an involution fixing A pointwise). We now turn our attention to the elements a , b , and t (compare [3, 4]).

Since E acts faithfully on Δ , (2.1) implies that $E(u_0, \dots, u_4) = 1$ for each 4-path $(u_0, \dots, u_4)$ lying in Δ . As above, we let $(x_0, \dots, x_9)$ be a 9-path on A and (extending the previous definition) we let x'_i be the unique neighbor of x_i in Δ

not lying on A for $0 \leq i \leq 8$. There are unique elements a and b in E such that $a(x_0, \dots, x_4) = (x_1, \dots, x_5)$ and $b(x_0, \dots, x_4) = (x_1, \dots, x_4, x'_4)$. The element a must fix A so $a^9 \in E(x_0, \dots, x_4) = 1$. Also $(ab^{-1})^2 \in E(x_0, \dots, x_4) = 1$ and $(a^2b^{-2})^2 \in E(x_1, \dots, x_5) = 1$. Thus, (R1)–(R3) hold.

Choose $x''_2 \in \Delta(x'_2) - \{x_2\}$. Then E contains an element t mapping $(x''_2, x'_2, x_2, x_3, x_4)$ to $(x''_2, x'_2, x_2, x_1, x_0)$. Since t^2 fixes x_1, x_2, x'_2, x''_2 and the other two neighbors of x''_2 in Δ as well, $t^2 = 1$. Hence, t reflects A and exchanges x'_4 and x'_0 . Since E contains a unique element mapping $(x_1, \dots, x_5)$ to $(x_0, \dots, x_4)$ and a unique element mapping $(x_1, \dots, x_4, x'_4)$ to $(x_0, \dots, x_4)$, we have $a^t = a^{-1}$ and $b^t = b^{-1}$. Thus, (R4)–(R6) hold.

Since $a^t = a^{-1}$, we have $(ta^4)^2 = 1$. Moreover, $ta^4 \in E(x'_0, x_0)$ but $ta^4 \notin E(x_1)$. By (2.2)(b), $ta^4 \in E_1(x'_0)$. Let $x''_0 = b^{-1}(x_0)$ and $x'_4 = b(x'_4)$. Since $b^t = b^{-1}$, t exchanges x''_0 and x'_4 . It follows that $a^4(x''_0) = x'_4$. We have $b(x_8) = x'_1 = a(x'_0)$ and so $a(x'_0)$ and $b(x_7) \in \Delta(x'_1) - \{x_1\}$. Since $b^{-1}a(x_7) = b^{-1}(x_8) \neq x''_0$, we have $b^{-1}a(x'_0) \neq x_7$ since $b^{-1}a \in E(x_0, \dots, x_3)$ is an involution. Thus $a(x'_0) \neq b(x_7)$ and so $x'_4 = a^4(x'_0) \neq a^3b(x_7)$. This implies that both ba^2b and a^3bta^4 map $(x'_0, x'_0, x_0, x_1, x_2)$ to $(x'_3, x_3, x_4, x'_4, u)$ where $u \in \Delta(x'_4) - \{x_4, x'_4\}$. Hence $ba^2b = a^3bta^4$ and (R7) follows.

Let $d = a^{-2}b^4a^{-2}$. Then $d \in G(x_2)$ and d induces the 3-cycle (x_1, x_3, x'_2) on $\Delta(x_2)$. By (2.2)(a), $d^3 = 1$. Since $E_1(x_2, x_3) = \langle ab^{-1} \rangle$ and $E_1(x_1, x_2) = \langle a^{-1}b \rangle$, we have $(a^{-1}b)^d = ab^{-1}$. By (2.2), $(ab^{-1})(a^{-1}b)$ is the unique involution in $E_1(x_2, x'_2)$. Thus, (R8)–(R10) hold.

Before going on to the next section, we point out the following fact:

(4.3) For each vertex u of Δ , $E(u) \leq \hat{G}(u)$.

PROOF. Let $v \in \Delta(u)$. By (2.2)(a) and (2.7)(b) and (c), $E_1(v) = E(v)'' \leq G(v)'' \cap G(u) \leq \hat{G}(v) \cap G(u) \leq \hat{G}(u)$. By (2.2), $E(u)$ is generated by the groups $E_1(v)$ for $v \in \Delta(u)$. $\square$

5. The remaining relations. By (2.7)(c) and (f), $\hat{G}(x_0) \cap G(x_0, \dots, x_4)$ is of order 3 and normalized by the stabilizer of A in G . Let c be a generator of $\hat{G}(x_0) \cap G(x_0, \dots, x_4)$. Then c induces a 3-cycle on $\Gamma(x_i)$ for $0 \leq i \leq 8$ and both a and t normalize $\langle c \rangle$. By (4.3), t induces an even permutation on $\Gamma(x_2)$. Also $a^9 = 1$. Thus:

$$(R11) \quad c^3 = 1, \quad (R12) \quad ac = ca, \quad (R13) \quad c^t = c^{-1}.$$

By (4.3), d and $ab^{-1} \in \hat{G}(x_2)$. The product cd induces a 5-cycle and the product cab^{-1} induces a 3-cycle on $\Gamma(x_2)$. By (2.7)(a), we have

$$(R14) \quad (cd)^5 = 1 \quad \text{and} \quad (R15) \quad (cab^{-1})^3 = 1.$$

Let $e = a^{-4}b^4$ (so $e = a^{-2}da^2$ and $e^3 = 1$) and $g = c^{-1}ece^{-1}c^{-1}$. Then $g \in \hat{G}(x_0) \cap G(x_1)$ and g induces a pair of transpositions on $\Gamma(x_0)$. By (2.7)(a), $g^4 = 1$. Since $a^4(x'_0) = x'_4$, we have $e(x_2) = x''_0$. Thus $e^2(x_2) = e(x'_0) = a^{-4}(x_2) = x_7$. Since $e^3 = 1$, $e^{-1}(x_2) = x_7$. It follows that $g \in G(x_2)$. By (2.7)(c), $g \in \hat{G}(x_1) \leq \bar{G}(x_1)$. Thus $g \in G_1(x_1)$ so $g^2 = 1$ by (2.7)(e). By [5, p. 67], $\langle e, c \rangle \cong A_5$. The involutions in $\langle e, c \rangle \cap G(x_1)$ are all in $G_1(x_1)$ by (2.7)(e). Since $c \in \langle e, c \rangle \cap G(x_2)$, we find that $\langle e, c \rangle \cap G(x_1) \leq G(x_2)$. Thus, the $\langle e, c \rangle$ -orbit containing x_2 is of length 5. Since $ec(x'_0) = c(x'_0)$, it follows that $ec(x'_0) = c(x'_0)$. This implies that $g(x_7) = x''_0$. Since $b^{-2}a^2(x_7) = x''_0$ and $b^{-2}a^2 = b^{-2}(a^2b^{-2})b^2$ is an involution fixing (x_0, x_1, x_2) and lying in $E(x_0) \leq \hat{G}(x_0)$, the product $gb^{-2}a^2$ fixes $(x_7, \dots, x_1, x_2)$ and x'_0 and

lies in $\hat{G}(x_0)$. By (2.7)(f), $G(x_7, \dots, x_2) \cap \hat{G}(x_0) \cap G(x'_0) = 1$. Thus $gb^{-2}a^2 = 1$. Conjugating with a^2 and substituting a^2b^{-2} for b^2a^{-2} , we obtain the equation

$$(R16) \quad c^{-1}dcd^{-1}c^{-1} = a^2b^{-2}.$$

We now set $r = ab^{-1}$ (so $r = ba^{-1}$) and $s = a^{-1}b$. Let $i = 1$ or -1 . By (4.3), $E_1(u, v) \leq G_1(u, v)$ for each edge $\{u, v\}$ of Δ . Since $s \in E_1(x_1, x_2)$, we have $c^i sc^{-i} \in G_1(x_1, x_2)$. Since d maps (x_2, x_1) to (x_2, x_3) , we have $(c^i sc^{-i})^d \in G_1(x_2, x_3)$ and so $(c^i sc^{-i})^d = c^j rc^{-j}$ for $j = 0, 1$ or -1 . Since $c^i sc^{-i}(x_4)$ is not a vertex of Δ , we cannot have $j = 0$. The element $c^{-i}dc^i d^{-1}$ induces a 3-cycle on $\Gamma(x_2)$ and hence does not commute with r by (2.7)(a). By (R9), $s^d = r$. It follows that $i \neq j$. Thus,

$$(R17) \quad (c^i sc^{-i})^d = c^{-i}rc^i \quad \text{for } i = 1 \text{ and } -1.$$

The product $c^{-i}sc^i \cdot d$ induces a 3-cycle on $\Gamma(x_2)$. By (2.7)(a), $c^{-i}sc^i \cdot d$ has order 3. Thus $(c^{-i}sc^i)(c^{-i}sc^i)^d(c^{-i}sc^i)^{d^2} = 1$. By (R17), $(c^{-i}sc^i)^d = c^i rc^{-i}$ and $(c^{-i}sc^i)^{d^2} = (c^i rc^{-i})^d$. We conclude that

$$(R18) \quad (c^i rc^{-i})^d = (c^i rc^{-i})(c^{-i}sc^i) \quad \text{for } i = 1 \text{ and } -1.$$

If we now go back and make the substitutions $r = ab^{-1}$ and $s = a^{-1}b$ in (R1)–(R16), we obtain all the relations listed in (1.2).

6. The proof of (1.2). Let J be as defined in (1.2). For reference, we reproduce the relations (and definitions of t , d and s) here:

(i) $t = a^{-1}ra^{-3}ra^3ra^{-3}$,	(x) $t^2 = 1$,
(ii) $d = a^{-2}(ra)^3ra^{-1}$,	(xi) $tat = a^{-1}$,
(iii) $s = a^{-1}ra$,	(xii) $trt = s$,
(iv) $a^9 = 1$,	(xiii) $tct = c^{-1}$,
(v) $r^2 = 1$,	(xiv) $d^3 = 1$,
(vi) $c^3 = 1$,	(xv) $(cd)^5 = 1$,
(vii) $ac = ca$,	(xvi) $c^{-1}dcd^{-1}c^{-1} = ara^{-1}r$,
(viii) $(cr)^3 = 1$,	(xvii) $dc^i rc^{-i} d^{-1} = c^i rc^i sc^i$ for $i = 0, 1$ and -1 ,
(ix) $(ara^{-1}r)^2 = 1$,	(xviii) $dc^i sc^{-i} d^{-1} = c^{-i}rc^i$ for $i = 0, 1$ and -1 .

We set $H = \langle d, c, r \rangle$, $H_1 = \langle d, c \rangle$, $H_2 = \langle r, crc^{-1}, c^{-1}rc \rangle$, $H_3 = \langle s, csc^{-1}, c^{-1}sc \rangle$, and $H_4 = \langle H_2, H_3 \rangle$. By (xviii), $H_3 \leq H$. The relations (v), (vi), and (viii) imply that H_2 is a homomorphic image of $O_2(A_4)$. By (iii) and (vii), $H_3 = a^{-1}H_2a$. By (v), (vi), and (xvii), $((c^i rc^{-i})(c^{-i}sc^i))^2 = 1$ for $i = 0, 1$, and -1 . It follows that $[H_2, H_3] = 1$. Thus, H_4 is an elementary abelian 2-group of order at most 16. By (vi), (ix), (xiv)–(xvi), and [5, p. 67], H_1 is a homomorphic image of A_5 . By (xvii) and (xviii), H_1 acts transitively on the nontrivial elements of H_4 . In particular, $|H| \leq 960$.

Charles Sims, working with a VAX-780 at Rutgers, determined the following properties of J . The coset enumeration required about 20 minutes of CPU time. Approximately 1000 extra cosets were counted before enough were identified to arrive at the final number.

(6.1) (C. Sims) (a) $|X| = 52326$, where X is the set of left cosets of H in J .

(b) The stabilizer (in the permutation representation of J on X) of an element in X has exactly three fixed points in X . We will call such a triple of elements of X a triplet.

(c) Let z_0 be the triplet containing H . Let

$$f = a^2 da^4 d^{-1} ra^{-1} c^{-1} da^2 sa^2 dcdasacda^4.$$

Then f fixes z_0 and hence normalizes H .

(d) Let $K = \langle f, H \rangle$ and let Z denote the set of all triplets. Then K has 19 orbits in Z ; their lengths are 1, 5, 20, 80, 96, 120, 320, 480, 480, 960, 960, 960, 960, 960, 1440, 2880, 2880, and 2880 (compare [15]). $\square$

By (6.1)(c), $|K| \leq 3 \cdot |H| \leq 2880$. By (6.1)(d), $|K| \geq 2880$. Hence $|K| = 2880$, $|H| = 960$, H_4 is the only proper normal subgroup of H , and G acts faithfully on Z . Let $u = ara^{-1}r$. By (xvi)–(xviii), $[H_4, u] = H_2$. Thus, H has just two conjugacy classes of involutions with representatives u and rs . By (iii), $u = a(ra^{-1}ra)a^{-1} = a(rs)a^{-1}$, so these two classes are fused in J .

Next, we observe

$$(6.2) \quad H_4 \leq H \cap H^a \text{ and } [H: H \cap H^a] = 5.$$

PROOF. By (vii), $c \in H^a$. By (iii), $r = s^a \in H^a$. By (ii), (v), and (xvi), $a^{-2}rara = d \cdot c^{-1}dcd^{-1}c^{-1} \in H$ and hence $a^{-1}rar = sr \in H^a$. Thus $H_4 \leq H^a$. Also $u = a(rs)a^{-1} \in H^a \cap H_1$ so $H^a \cap H_1$ contains $\langle c, u \rangle$. By (xvi), $\langle c, u \rangle = \langle c, c^d \rangle$. The group $\langle c, c^d \rangle$ is of index 5 in $\langle c, d \rangle \cong A_5$. Since $J = \langle H, a \rangle$ and H acts faithfully on X , we cannot have $H = H^a$. Thus $[H: H \cap H^a] = 5$. $\square$

$$(6.3) \quad [K: K \cap K^a] = 5.$$

PROOF. By (6.2), $1 \neq [K: K \cap K^a] \mid [K: H \cap H^a] = 15$. By (6.1.d), $[K: K \cap K^a] = 5$. $\square$

Suppose that B is a block of imprimitivity of J in its action on Z and suppose that B properly contains z_0 . By (6.3), a carries z_0 to the unique K -orbit of length 5. Since $J = \langle H, a \rangle$, $B = Z$ follows if B contains this K -orbit and, since $a^9 = 1$, $|B| \neq 17442/2$. Inspection of the K -orbit lengths now yields the conclusion $B = Z$. Thus:

$$(6.4) \quad J \text{ acts primitively on } Z. \quad \square$$

The group H contains no subgroup of index 2. Thus, the subgroup of J consisting of an even permutations (in the action on Z) contains K . By (6.4), this subgroup acts transitively on K and hence equals J . Since $|Z|/2$ is odd, every involution of J has fixed points. Since the two conjugacy classes of involutions in H are fused in J , J has a unique conjugacy class of involutions.

Let N be a normal subgroup of J with $N \neq 1$. By (6.4), N acts transitively on Z . In particular, $|N|$ is even. Since J has a unique conjugacy class of involutions and H is generated by its involutions, $H \leq N$ and, by (x), $t \in N$. By (iv) and (xi), $a \in N$. Hence $J = \langle H, a \rangle \leq N$. It follows that J is simple. This concludes the proof of (1.2).

Let Ω_1 denote the graph with vertex set X , where two cosets g_1H and g_2H are defined to be adjacent whenever $g_1^{-1}g_2 \in HaH$. The following observation implies that adjacency does not depend on the order of the two cosets:

$$(6.5) \quad HaH = Ha^{-1}H.$$

PROOF. Since K is maximal in J , we have $K = N_J(H)$. By (ii), (iii), (v), (xi), and (xii), $d^t = d^{-1}$ and $r^t \in H$. Thus by (xiii), $t \in N_J(H)$. By (x), $t \in H$ follows. By (xi), $a^{-1} = tat \in HaH$. $\square$

Thus Ω_1 is an undirected graph; by (6.2), its valency is 5 and H acts as A_5 on the neighbors of H . We identify J with its image in $\text{aut}(\Omega_1)$.

(6.6) Ω_1 satisfies the hypotheses of (1.1).

PROOF. By (6.4), Ω_1 has either one or three connected components. Since J is simple, Ω_1 must be connected. There exists an integer n such that Ω_1 is $(J, n+1)$ -transitive. By [13], 4^n is the highest power of 2 dividing $|H|$. Thus $n = 3$. Since $a^9 = 1$, the girth of Ω_1 must be 9. $\square$

Now let Y be the set of left cosets of K (which we may identify with the elements of Z). Let Ω_2 be the graph with vertex set Y , where two cosets g_1K and g_2K are defined to be adjacent whenever $g_1^{-1}g_2 \in KaK$. By (6.5), Ω_2 is an undirected graph. We identify J with its image in $\text{aut}(\Omega_2)$.

(6.7) Ω_2 satisfies the hypotheses of (1.1).

PROOF. By (6.3), the valency of Ω_2 is 5. Thus, K acts as A_5 on the set of neighbors of K , and the remaining properties follow as in the proof of (6.6). $\square$

(6.8) $\text{aut}(\Omega_1)$ contains an element of order three centralizing J .

PROOF. Let f be a 3-element of K not in H and let p be the permutation of X given by $p(gH) = gfH$ for all $g \in J$. Since $H \trianglelefteq K$, p is well defined. Because $|K/H| = 3$, $|p| = 3$. Clearly $[p, J] = 1$. To show that $p \in \text{aut}(\Omega_1)$, we need to show that f normalizes HaH . We consider a and f in their action on Ω_2 and refer to (2.7). Let A be the apartment through K rotated by a . Suppose first that f fixes A . If $h \in H$ is a nontrivial element fixing A pointwise, then $\langle a, f, h \rangle$ is the stabilizer of A in J . Both $\langle h \rangle$ and $\langle h, f \rangle$ are normal in $\langle a, f, h \rangle$ and $|\langle f, h \rangle / \langle h \rangle| = 3$. Thus $[a^{-1}, f] \in H$ and so $a^f \in aH$. The group H acts transitively on the set of 4-paths beginning at K and f , being a 3-element, fixes some of these 4-paths. It follows that if f does not fix A , then gfg^{-1} does for some $g \in H$. Then $gfg^{-1} \cdot a \cdot g^{-1} \in aH$ and hence $a^f \in HaH$. $\square$

7. Conclusion. Let M be the subgroup of G generated by the subgroups $G_1(u, v)$ for all edges $\{u, v\}$ of Γ and let $N = \langle a, b, c \rangle$, where a , b , and c are the elements introduced in §§4 and 5 above. Then $G_1(x_2, x_3) \leq \langle ab^{-1}, c \rangle \leq N$. The subgroup $\langle a^2b^{-2}, a^{-2}b^2, c \rangle \leq N(x_2)$ acts transitively on $\Gamma(x_2)$ and $a \in N$ maps x_2 to x_3 . Since Γ is connected, it follows that N acts transitively on the edge set of Γ . Thus $M \leq N$. We have seen that N is a homomorphic image of J and that J is simple. Thus, $N \cong J$. Since $M \trianglelefteq G$ and N is simple, $M = N$. Let $L = N \cap G(x)$. Then $L \geq \hat{G}(x) \cong H$. Since K is the only proper subgroup of J properly containing H , we have $L \cong H$ or K . Since N acts transitively on V , we can identify the vertices of Γ with the left cosets of L in N . Two cosets g_1L and g_2L correspond to adjacent vertices whenever $g_1^{-1}g_2 \in LaL$. Thus $\Gamma \cong \Omega_1$ or Ω_2 .

It remains only to determine the full automorphism groups of Ω_1 and Ω_2 . By [6], J contains a unique conjugacy class of subgroups isomorphic to K . Since J has a unique suborbit of length 5 on Z , $\text{aut}(J) \leq \tilde{\text{aut}}(\Omega_2)$. By (2.7)(a) and (b), $|\text{aut}(\Omega_2)| \leq 2 \cdot |J|$. By [9] or [16] (see also [1]), $|\text{aut}(J)| \geq 2 \cdot |J|$. Thus $\text{aut}(\Omega_2) \cong \text{aut}(J)$. In particular, J has an outer automorphism g with $H^g = H$ and $a^g = a$ (recall the construction of Δ), so $\text{aut}(J) \leq \text{aut}(\Omega_1)$. By (2.7)(b),

$\text{aut}(\Omega_1)/J \cong \Sigma_3$. By (6.8), $3 \mid |\text{aut}(\Omega_1)/J|$. Thus $\text{aut}(\Omega_1)/J \cong \Sigma_3$. This concludes the proof of (1.1).

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