A FINITENESS CONDITION ON REGULAR LOCAL OVERRINGS OF A LOCAL DOMAIN

BERNARD JOHNSTON

ABSTRACT. The local factorization theorem of Zariski and Abhyankar implies that between a given pair of 2-dimensional regular local rings, $S \supseteq R$, having the same quotient field, every chain of regular local rings must be finite. It is shown in this paper that this property extends to every such pair of regular local rings, regardless of dimension. An example is given to show that this does not hold if “regular” is replaced by “Cohen-Macaulay,” by “normal,” or by “rational singularity.” More generally, it is shown that the set $\mathcal{R}(R)$ of n -dimensional regular local rings birationally dominating a given n -dimensional local domain, R , and ordered by containment, satisfies the descending chain condition. An example is given to show that if R is regular the two examples of minimal elements of $\mathcal{R}(R)$ given by J. Sally do not exhaust the set of minimal elements of $\mathcal{R}(R)$.

1. Introduction. The local factorization theorem of Zariski and Abhyankar states that if R and S are 2-dimensional regular local rings with $R \subseteq S \subseteq$ quotient field of R , then S is obtainable from R by a unique finite sequence of quadratic transforms (see below for definitions) (cf. [10, Lemma, p. 538; 1, Theorem 3; and 9, §2]). Sally [8, Corollary 4.5] and Shannon [9, Example 3.2] showed that a straightforward generalization of this result does not extend to higher dimensions, even if monoidal transforms are considered in place of quadratic transforms. Shannon was able to show that no infinite ascending chain of regular local rings could exist between two higher-dimensional regular local rings, of the same dimension [9, Proposition 5.1]. This paper shows that no infinite descending chain of such rings can occur and so, using Shannon’s result, we are able to conclude, as Corollary 4.10 below, that a strong finiteness condition continues to hold between R and S , even if their mutual dimension is greater than 2; namely, every chain of regular local rings between them must be finite. In fact, we obtain, as Theorem 4.6, the somewhat more general statement that the n -dimensional regular local rings birationally containing a fixed local domain of the same dimension must satisfy the descending chain condition (d.c.c). As an application of these ideas, we note in Example

Received by the editors August 13, 1985 and, in revised form, February 25, 1986. The results in this paper were presented to the eight-hundred and twenty-third meeting of the American Mathematical Society at the University of Missouri, Columbia, on November 1, 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 13H05; Secondary 13E99, 14E40.

Key words and phrases. Regular local rings, birational, descending chain condition, domination, monoidal transform.

This paper forms part of the author’s Ph.D. thesis written at Purdue University under the supervision of Professor William Heinzer. The author wishes to thank Professor Heinzer and Professor Craig Huneke for many stimulating conversations and suggestions. Thanks are also due Professor David Lantz for a thorough reading of the manuscript and many helpful comments.

5.4 that the two types of minimal regular local overrings of a given n -dimensional regular local ring ($n > 2$) given by J. Sally cannot be the only such examples.

In Examples 4.9 and 5.2 we note that such strong finiteness conditions do not hold among some local domains which are more general than regular local rings.

The technical heart of the paper is contained in §3, in which the crucial result is Theorem 3.2. Preliminaries are given in §2, while §4 contains the main results, and §5 contains some extensions of these.

2. Preliminaries. If R is a quasilocal ring (i.e., R has a unique maximal ideal), we use the symbol $(R, \mathcal{M})$ to indicate explicitly that $\mathcal{M}$ is the unique maximal ideal of R . We use the word “local” to indicate a Noetherian quasilocal ring throughout. We use the symbol $\subset$ to indicate strict inclusion. By a regular local ring (RLR) we mean a local domain whose unique maximal ideal can be generated by n elements, where n is the Krull dimension of R , which we denote by $\dim R$. The height of a prime P is denoted $\text{ht } P$. If $(R, \mathcal{M})$ is a RLR, we use the phrase “regular system of parameters” for $\mathcal{M}$ to mean a minimal generating set for $\mathcal{M}$.

We denote the quotient field of any domain R by $\text{q.f.}(R)$. We say that S and R are “birational” if $\text{q.f.}(S) = \text{q.f.}(R)$. We say that “ $(S, \mathcal{N})$ dominates $(R, \mathcal{M})$ ” if $S \supseteq R$ and $\mathcal{N} \cap R = \mathcal{M}$. We use the symbol $S > R$ to indicate that S dominates R and S is birational to R . If $(S, \mathcal{N}) > (R, \mathcal{M})$ are quasilocal domains, then the field $S/\mathcal{N}$ contains the field $R/\mathcal{M}$ canonically. We denote the transcendence degree of $S/\mathcal{N}$ over $R/\mathcal{M}$ by $\text{restrdeg}_R S$, for “residual transcendence degree of S over R .”

NOTE 2.1. Let $(S, \mathcal{N})$ and $(R, \mathcal{M})$ be two local domains of the same dimension. Then $S > R$ if and only if $R \subseteq S \subseteq \text{q.f.}(R)$.

PROOF. ($\Rightarrow$) This is by definition of $S > R$.

($\Leftarrow$) We need only see that $\mathcal{N} \cap R = \mathcal{M}$. Certainly $\mathcal{N} \cap R \subseteq \mathcal{M}$. If $\mathcal{N} \cap R \subset \mathcal{M}$ then $\dim R_{\mathcal{N} \cap R} < \dim R = \dim S$, and $R_{\mathcal{N} \cap R} \subset S \subset \text{q.f.}(R) = \text{q.f.}(R_{\mathcal{N} \cap R})$. This contradicts the fact that any local domain between an n -dimensional local domain and its quotient field must itself have dimension $\leq n$ (see [9, Lemma 5.3]). $\square$

If $(R, \mathcal{M})$ is a RLR, several specific constructions of RLR's S such that $S > R$ are of particular interest. Namely, a “monoidal transform” of R is a ring of the form $S := R[x_2/x_1, x_3/x_1, \dots, x_i/x_1]_Q$, where $\{x_1, \dots, x_i\}$ is part of a regular system of parameters for $\mathcal{M}$, and Q is a maximal ideal of $R[x_2/x_1, \dots, x_i/x_1]$ whose height equals $\dim R$. A “simple monoidal transform” of R is a monoidal transform in which only one element is adjoined to R , e.g., $S = R[x_2/x_1]_Q$. A “quadratic transform” of R is a monoidal transform of R in which as many elements as possible have been adjoined to R , i.e., $S = R[x_2/x_1, \dots, x_n/x_1]_Q$, where $\{x_1, \dots, x_n\}$ constitutes all of a regular system of parameters for $\mathcal{M}$, i.e., $n = \dim R$.

If $(R, \mathcal{M})$ is a local ring and $x \in R \setminus \{0\}$, then $x \notin \bigcap_{i=0}^{\infty} \mathcal{M}^i$ by Krull's Intersection Theorem [11, Corollary 2, p. 217]. Therefore there exists $r \geq 0$ such that $x \in \mathcal{M}^r \setminus \mathcal{M}^{r+1}$. We define a function on R , “ord,” by $\text{ord}(x) := r$ ($\text{ord } 0 := \infty$). We extend this to $\text{q.f.}(R)$ by $\text{ord}(a/b) := \text{ord}(a) - \text{ord}(b)$. If in addition R is regular then this order function is actually a valuation, i.e., we have $\text{ord}(ab) = \text{ord}(a) + \text{ord}(b)$ and $\text{ord}(a + b) \geq \min\{\text{ord}(a), \text{ord}(b)\}$ [12, Corollary 1, p. 302]. Among valuation rings, we reserve the symbol DVR for the valuation domains which are rank 1 and discrete, where “rank” means Krull dimension. We note that the DVR's contained in a given field are maximal elements among the set of quasilocal domains contained in that field and ordered by domination [3, Kriterium 2, p. 102].

By a “chain of RLR’s between R and S ” we mean a nonempty set $\{S_\lambda\}$ of distinct birational RLR’s linearly ordered by domination and such that $R < S_\lambda < S$ for all λ . Given an infinite descending sequence, $(S_0, M_0) \supseteq (S_1, M_1) \supseteq \cdots$ of quasilocal rings such that S_i dominates S_{i+1} for all i , we denote the intersection by “ (S_∞, M_∞) ”. With regard to this S_∞ we have the following lemma.

LEMMA 2.2. (S_∞, M_∞) is a quasilocal ring.

PROOF. If $\alpha \in M_\infty \subseteq M_0$ then α is not a unit in S_∞ , for if it were, we would have $\alpha^{-1} \in S_\infty \subseteq S_0$ so α would be a unit in S_0 , a contradiction. On the other hand, if $\alpha \in S_\infty \setminus M_\infty$ then $\alpha \in S_i \setminus M_i$ for some i and hence in $S_j \setminus M_j$ for all j ; so α^{-1} is in every S_j and hence in S_∞ . Thus the set of nonunits in S_∞ forms the ideal M_∞ . $\square$

3. Infinite descending chains of RLR’s. This section contains the main technical result of this paper, Theorem 3.2, which states that the quotient field of the intersection of a certain chain of regular local rings, all having the same quotient field, K , is strictly smaller than K . The following example is provided as motivation.

EXAMPLE 3.1. Let

$$(S_0, M_0) := (k[X, Y]_{(X, Y)}, (X, Y)k[X, Y]_{(X, Y)}),$$

where k is any field, and X and Y are indeterminates. Let

$$(S_i, M_i) := (k[X, X^i Y]_{(X, X^i Y)}, (X, X^i Y)k[X, X^i Y]_{(X, X^i Y)}).$$

Clearly $(S_0, M_0) \supset (S_1, M_1) \supset \cdots$, and since the matrix $\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$ has determinant 1, this is a strictly descending chain of RLR’s with the same quotient field, $K = k(X, Y)$. In this example there is no residue field extension, i.e., $S_i/M_i = k$ for all i . We have $S_\infty = k[X]_{(X)}$, so $\text{q.f.}(S_\infty) = k(X)$ is strictly smaller than $\text{q.f.}(S_i) = k(X, Y)$.

That the behavior exhibited in this simple example extends to the general case is the gist of the following theorem, and its proof.

THEOREM 3.2. Suppose that $(S_0, M_0) > (S_1, M_1) > \cdots$ is an infinite strictly descending sequence of regular local rings, all having the same quotient field, K , and the same dimension, n . Suppose, in addition, that the residue fields eventually stabilize, i.e., there exists $N \geq 0$ so that $S_i/M_i = S_{i+1}/M_{i+1}$ for all $i \geq N$. Then the quotient field of the intersection, (S_∞, M_∞) , is strictly contained in K .

Before proceeding with the proof we note a few remarks and lemmas.

REMARKS 3.3. (1) Deleting all but a cofinal set of rings will not change the intersection so, when convenient, we may ignore all but a cofinal set of rings in the chain. In particular, we may assume $S_0/M_0 = S_1/M_1 = \cdots = k$.

(2) The theorem is vacuously true if $n = 1$ since a 1-dimensional RLR is a DVR (see [5, Theorem 37] and [6, Theorem 12.4]) and any chain of DVR’s of the form $S_0 > S_1 > \cdots$ can occur only if $S_0 = S_1 = \cdots$, since DVR’s are maximal among the set of quasilocal domains within their quotient field (see §2). So we assume without loss of generality that $n \geq 2$ throughout this section.

LEMMA 3.4. *There is a cofinal subset C of the positive integers for which the following is true: for i in C the k -vector spaces $V_{i,j} := (\mathcal{M}_j + \mathcal{M}_i^2)/\mathcal{M}_i^2$ are equal for all j in C with $j > i$, and $\dim_k V_{i,j} = \dim_k V_{i',j} =: d$ whenever i, i', j are in C with $j > i$ and $j > i'$. Moreover, $d < n$ and the theorem is true if $d = 0$.*

PROOF. We will construct the set C by successively removing some of the rings in the original chain by the process described below and then renumbering. The rings in the chain which results, a subchain of the original chain, will be the rings indexed by C .

For $i = 0$, then for $i = 1$, etc., consider the descending sequence of k -vector spaces:

$$V_{i,i} := \mathcal{M}_i/\mathcal{M}_i^2 \supseteq V_{i,i+1} := (\mathcal{M}_{i+1} + \mathcal{M}_i^2)/\mathcal{M}_i^2 \supseteq V_{i,i+2} := (\mathcal{M}_{i+2} + \mathcal{M}_i^2)/\mathcal{M}_i^2 \supseteq \cdots$$

Since

$$n = \dim_k V_{i,i} \geq \dim_k V_{i,i+1} \geq \dim_k V_{i,i+2} \geq \cdots \geq 0,$$

there exists d_i , $n \geq d_i \geq 0$, and there exists $r_i > i$ so that

$$\begin{aligned} d_i &= \dim_k V_{i,r_i} = \dim_k (\mathcal{M}_{r_i} + \mathcal{M}_i^2)/\mathcal{M}_i^2 \\ &= \dim_k (\mathcal{M}_{r_i+1} + \mathcal{M}_i^2)/\mathcal{M}_i^2 = \dim_k V_{i,r_i+1} = \cdots \end{aligned}$$

Then removing S_j , $i < j < r_i$, and renumbering $r_i + l := i + l + 1$, $l \geq 0$, we have $d_i = \dim_k V_{i,j}$ for all $j > i$.

Notice that if $j > i \geq 1$ there exists a surjective k -vector space map

$$\begin{aligned} \Theta: V_{i,j} &= (\mathcal{M}_j + \mathcal{M}_i^2)/\mathcal{M}_i^2 \cong \mathcal{M}_j/\mathcal{M}_j \cap \mathcal{M}_i^2 \\ &\rightarrow \mathcal{M}_j/\mathcal{M}_j \cap \mathcal{M}_{i-1}^2 \cong (\mathcal{M}_i + \mathcal{M}_{i-1}^2)/\mathcal{M}_{i-1}^2 = V_{i-1,j}, \end{aligned}$$

so $d_i = \dim_k V_{i,j} \geq \dim_k V_{i-1,j} = d_{i-1}$. Thus we have $0 \leq d_0 \leq d_1 \leq \cdots \leq n$, since $d_i \leq \dim_k V_{i,i} = n$ for all i , so there exists $i' \geq 0$ such that $d_{i'} = d_{i'+1} = \cdots =: d$. If $i > 0$ we delete S_j , $0 \leq j \leq i' - 1$, and renumber $i' + l := l$, $l \geq 0$, to conclude (in any case) that $\dim V_{i,j} = d$ whenever $j > i \geq 0$ and moreover, $n \geq d \geq 0$.

Suppose $d = n$. Then in particular, $\dim_k (\mathcal{M}_1 + \mathcal{M}_0^2)/\mathcal{M}_0^2 = n$, so there exist $\eta_1, \dots, \eta_n \in \mathcal{M}_1$ such that $\bar{\eta}_1, \dots, \bar{\eta}_n$ (where “ $\bar{}$ ” indicates image mod $\mathcal{M}_0^2$) are linearly independent over k . It follows that $\bar{\eta}_1, \dots, \bar{\eta}_n$ generate $\mathcal{M}_0 \pmod{\mathcal{M}_0^2}$ over k , and hence (by Nakayama’s Lemma [5, 1.N]) that $\eta_1, \dots, \eta_n$ generate $\mathcal{M}_0$. But then $\mathcal{M}_1 S_0 = \mathcal{M}_0$, and so (by Zariski’s Main Theorem [2, Proposition 14]) $S_0 = S_1$, which contradicts our assumption that the chain is strictly decreasing. So $d \leq n - 1$.

On the other hand, suppose $d = 0$. Then $\dim_k (\mathcal{M}_i + \mathcal{M}_{i-1}^2)/\mathcal{M}_{i-1}^2 = 0$, giving $\mathcal{M}_i \subseteq \mathcal{M}_{i-1}^2 \subseteq \mathcal{M}_{i-2}^4 \subseteq \cdots \subseteq \mathcal{M}_0^{2^i}$ for $i \geq 1$, so $\mathcal{M}_\infty = \bigcap_{i=0}^\infty \mathcal{M}_i \subseteq \bigcap_{i=1}^\infty \mathcal{M}_0^{2^i} = 0$ (by Krull’s Intersection Theorem [11, Corollary 2, p. 217]). Hence $\text{q.f.}(S_\infty)$ is strictly smaller than $\text{q.f.}(S_0)$, and the theorem is true. $\square$

REMARK 3.5. The point of this lemma is this. In proving Theorem 3.2 we need to consider, by Remark 3.3(1), only a cofinal subset of the original chain of rings. Letting C be this cofinal subset, this lemma allows us to assume, in the proof of Theorem 3.2, that the k -vector spaces $V_{i,j}$ are equal whenever $j > i$, i fixed, that they have the same dimension, d , for all i , and that $1 \leq d \leq n - 1$. We make these assumptions throughout the remainder of this section.

LEMMA 3.6. *If $x_{i1}, \dots, x_{id} \in M_{i+1}$ are elements whose images in $V_{i,i+1}$ are a basis of $V_{i,i+1}$, then in fact their images in $V_{j,j+1}$ are a basis of $V_{j,j+1}$ for all j , $0 \leq j \leq i$. In particular, it follows that $\{x_{i1}, \dots, x_{id}\}$ is part of a regular system of parameters for M_j , $0 \leq j \leq i$.*

PROOF. By Remark 3.5 all the $V_{i,j}$ have the same dimension, d , so the surjection $\theta: V_{i,i+1} \twoheadrightarrow V_{i-1,i+1}$ given in the proof of Lemma 3.4 is actually a k -isomorphism which takes the images of $x_{i1}, \dots, x_{id}$ in $V_{i,i+1}$ to their images in $V_{i-1,i+1}$. Thus these images are a basis for $V_{i-1,i+1}$, which equals $V_{i-1,i}$. This proves the statement for $j = i - 1$ and it is clear we may iterate to achieve the result for all j . The last statement follows because, for all i , $V_{i,i+1} \subset V_{i,i}$. $\square$

The following proposition is the key reduction needed for computational purposes.

PROPOSITION 3.7. *Let d be as in Remark 3.5. For all $i \geq 0$ there exists $\{x_{i1}, \dots, x_{id}\}$, part of a regular system of parameters for M_i , such that for each j , $0 \leq j \leq i$, there exists $y_{j1}, \dots, y_{j,n-d} \in M_j$, with $\{x_{i1}, \dots, x_{id}, y_{j1}, \dots, y_{j,n-d}\}$ a regular system of parameters for M_j , and such that $y_{jl} \in M_0^{j+1}$, $1 \leq l \leq n - d$.*

In particular, this gives, for each $i \geq 0$, a regular system of parameters $\{x_{i1}, \dots, x_{id}, y_{i1}, \dots, y_{i,n-d}\}$ for M_i such that $\bar{x}_{i1}, \dots, \bar{x}_{id}$ are linearly independent (mod M_0^2) over k , and $y_{i1}, \dots, y_{i,n-d} \in M_0^{i+1}$.

PROOF. Fix i and choose fixed $x_{i1}, \dots, x_{id} \in M_{i+1}$ whose images in $V_{i,i+1}$ are a basis. By Lemma 3.6, $\{x_{i1}, \dots, x_{id}\}$ is part of a regular system of parameters for M_j , $0 \leq j \leq i$. We will show by induction on j that for all such j there exists $\{y_{j1}, \dots, y_{j,n-d}\}$ as in the statement.

Case $j = 0$. Clear.

Case $j = 1$. Let $z_{11}, \dots, z_{1,n-d}$ be any elements of M_0 , so that $\{x_{i1}, \dots, x_{id}, z_{11}, \dots, z_{1,n-d}\}$ forms a regular system of parameters for M_1 . Then for each l , $1 \leq l \leq n - d$, we may write, in $V_{0,1}$, $\bar{z}_{1l} = \bar{\alpha}_{1l}\bar{x}_{i1} + \dots + \bar{\alpha}_{dl}\bar{x}_{id}$, where $\bar{\alpha}_{sl} \in k$, $1 \leq s \leq d$. Picking preimages of $\bar{\alpha}_{sl}$ in S_1 we have $z_{1l} = \alpha_{1l}x_{i1} + \dots + \alpha_{dl}x_{id} + y_{1l}$, where $y_{1l} \in M_0^2 \cap S_1$. Then

$$(x_{i1}, \dots, x_{id}, y_{11}, \dots, y_{1,n-d})S_1 = (x_{i1}, \dots, x_{id}, z_{11}, \dots, z_{1,n-d})S_1 = M_1,$$

so $\{x_{i1}, \dots, x_{id}, y_{11}, \dots, y_{1,n-d}\}$ gives the desired regular system of parameters for M_1 .

Case $1 < j \leq i$. We apply the induction hypothesis to assume that $\{x_{i1}, \dots, x_{id}, y_{j-1,1}, \dots, y_{j-1,n-d}\}$ is a regular system of parameters for M_{j-1} so that $y_{j-1,1}, \dots, y_{j-1,n-d} \in M_0^j$.

We now apply the case $j = 1$ to the truncated descending chain starting at S_{j-1} . That case implies that we can find $z_{j1}, \dots, z_{j,n-d} \in M_j$ such that $x_{i1}, \dots, x_{id}, z_{j1}, \dots, z_{j,n-d}$ form a regular system of parameters for M_j and such that each z_{jl} is in M_{j-1}^2 , $1 \leq l \leq n - d$. Thus each z_{jl} can be written as a homogeneous polynomial of degree two in $x_{i1}, \dots, x_{id}, y_{j-1,1}, \dots, y_{j-1,n-d}$ with coefficients in S_{j-1} . Since we have equality of residue fields, any monomial in this polynomial whose coefficient is a unit in S_{j-1} can be written as the sum of two monomials, one with coefficient a unit in S_j , and the other with a coefficient in M_{j-1} . After performing this process for each monomial in z_{jl} , we collect all monomials whose

coefficient is a unit in S_j , and call their sum p_{1l} , while the sum of the monomials whose coefficient is in M_{j-1} we call q_{1l} . Of course, $z_{jl} = p_{1l} + q_{1l}$ and $q_{1l} \in M_{j-1}^3$.

Hence q_{1l} can be written as a homogeneous polynomial of degree three in $x_{i1}, \dots, x_{id}, y_{j-1,1}, \dots, y_{j-1,n-d}$ and so we may repeat the above process to write $q_{1l} = p_{2l} + q_{2l}$, where p_{2l} is a sum of monomials (of degree three) whose coefficients are units in S_{j1} and $q_{2l} \in M_{j-1}^4$. Now $z_{jl} = p_{1l} + p_{2l} + q_{2l}$. We iterate until reaching $z_{kl} = p_{1l} + p_{2l} + \dots + p_{j-1,l} + q_{j-1,l}$. Note that $q_{j-1,l} \in M_{j-1}^{j+1} \subseteq M_0^{j+1}$.

Let $p_l := p_{1l} + \dots + p_{j-1,l}$. Then p_l is a polynomial in $x_{i1}, \dots, x_{id}, y_{j-1,1}, \dots, y_{j-1,n-d}$ in which every monomial has degree at least two and every coefficient is a unit in S_j . But now p_l can be rewritten as $p_l = p'_l + p''_l$ where p'_l is the sum of those terms of p_l which involve only the x_i 's, and p''_l is the sum of those terms which involve at least one $y_{j-1,r}$. We now have $z_{jl} = p'_l + p''_l + q_{j-1,l}$. Set $y_{jl} := z_{jl} - p'_l$. Since every term in p'_l involves at least two x_{ir} 's and has a coefficient which is a unit in S_j , we see that $p'_l \in M_j^2$. Therefore, since $\{x_{i1}, \dots, x_{id}, z_{j1}, \dots, z_{j,n-d}\}$ is a regular system of parameters for M_j , so is $\{x_{i1}, \dots, x_{id}, y_{j1}, \dots, y_{j,n-d}\}$.

It remains to see that each $y_{jl} \in M_0^{j+1}$. But we have $y_{jl} = z_{jl} - p'_l = p''_l + q_{j-1,l}$, $q_{j-1,l} \in M_0^{j+1}$, so we need only see that $p''_l \in M_0^{j+1}$. This is clear since each $y_{j-1,r} \in M_0^j$ and each $x_{ir} \in M_0$, so each term in p''_l is in M_0^{j+1} . $\square$

The technique employed in the latter part of the proof of Proposition 3.7 was essentially the expansion of elements in M_i as power series with respect to a fixed regular system of parameters for M_i . Had S_i contained a coefficient field, for all i , then we would have been able to actually expand these elements as power series by passing to the completion and using Cohen's Structure Theorem [6, Theorem (31.1)]. The far clumsier technique of the text seems necessary in order to avoid the assumption of the existence of a coefficient field. A similar technique is also used in the proof of Lemma 3.8.

LEMMA 3.8. *For each positive integer i , and for each j , $1 \leq j \leq i$, let $\{x_{i1}, \dots, x_{id}, y_{j1}, \dots, y_{j,n-d}\}$ be a regular system of parameters for M_j as in Proposition 3.7. If $z \in M_0^j \cap M_j$, then z can be written $z = f + g$, where f is a homogeneous polynomial of degree j in $x_{i1}, \dots, x_{id}$ whose nonzero coefficients are units in S_0 , and where $g \in M_0^{j+1}$.*

PROOF. Suppose $z \in M_j^s$, $s \geq 1$. Then z may be written as a homogeneous polynomial of degree s in the given basis elements for M_j , coefficients in S_j . By grouping the monomials in this expression accordingly, we can write $z = a + b + c$, where a is a homogeneous polynomial of degree s in $x_{i1}, \dots, x_{id}$ whose nonzero coefficients are units in S_j ; b is a homogeneous polynomial of degree s , each of whose monomials contains at least one y_{jl} , and whose nonzero coefficients are units in S_j ; and c is a homogeneous polynomial of degree s whose coefficients are in M_j . Thus $c \in M_j^{s+1}$.

Concerning this expression for z we observe, first, that $b \in M_0^{j+1}$ —which is clear, since each of its terms contains an element, y_{jl} , $1 \leq l \leq n-d$, which is in M_0^{j+1} . Secondly, $a = 0$ if $s < j$. For, noting that units in S_j are also units in S_0 , $a \neq 0$ would yield $a \in M_0^s \setminus M_0^{s+1}$; on the other hand $b \in M_0^{j+1} \subseteq M_0^{s+1}$, and $c \in M_j^{s+1} \subseteq M_0^{s+1}$, so we would get $z = a + b + c \in M_0^s \setminus M_0^{s+1}$, a contradiction to

$z \in M_0^j \subseteq M_0^{s+1}$. Finally we see, if $a = 0$, that $z \in M_0^j$ together with $b \in M_0^{j+1}$ imply $c (= z - b) \in M_0^j$; and that, if $a \neq 0$, then $s = j$, so $c \in M_j^{s+1} \subseteq M_0^{s+1} = M_0^{j+1}$.

Now if $s = j$ we set $f := a$ and $g := b + c$ to obtain the desired result. In the contrary case we use the observations above, that $c \in M_0^j \cap M_j^{s+1}$, to repeat the above procedure, with c in place of z , so that we have $c = a_1 + b_1 + c_1$, where a_1 is a homogeneous polynomial of degree $s + 1$ in the x_{im} 's, $1 \leq m \leq d$, with unit coefficients; b_1 is a homogeneous polynomial of degree $s + 1$, with unit coefficients in the regular parameters for M_j , each term of which includes at least one y_{jl} ; and c_1 has coefficients in M_j . Now we have $b_1 \in M_0^{j+1}$, $c_1 \in M_j^{s+2}$, $z = b + a_1 + b_1 + c_1$, and, again, if $s + 1 < j$ then $a_1 = 0$.

Upon the $(j - s + 1)$ st iteration of this expansion we reach

$$z = a_{j-s} + (b + b_1 + b_2 + \cdots + b_{j-s}) + c_{j-s}.$$

Then setting $f := a_{j-s}$ and $g := (b + b_1 + \cdots + b_{j-s}) + c_{j-s}$ we are done, as a_{j-s} is a homogeneous polynomial of the prescribed sort, as $b_t \in M_0^{j+1}$ for all t , and as $c_{j-s} \in M_j^{s(j-s+1)} = M_j^{j+1} \subseteq M_0^{j+1}$. $\square$

We are finally prepared for the

PROOF OF THEOREM 3.2. For each $i \geq 1$, let $\{x_{i1}, \dots, x_{id}, y_{i1}, \dots, y_{i, n-d}\}$ be as in Proposition 3.7. As Lemma 3.6 shows that the images of $x_{i1}, \dots, x_{id}$ form a basis of $V_{0,1}$, and as $d < n$, we have $V_{0,1} \subset V_{0,0}$, so we can find a $y \in M_0$ such that, for all i , $\{x_{i1}, \dots, x_{id}, y\}$ is part of a regular system of parameters for M_0 . We will show that $y \notin \text{q.f.}(S_\infty)$ by showing that we cannot write $y = z_1/z_2$ with $z_1, z_2 \in M_\infty$.

Suppose, by way of contradiction, that $y = z_1/z_2$, $z_1, z_2 \in M_\infty$. Since $y \neq 0$, $z_1 \neq 0$, so there exists a positive integer r such that $z_1 \in M_0^r \setminus M_0^{r+1}$. Since $z_1 \in M_\infty \subseteq M_r$, we may apply Lemma 3.8, with $i := r$, to write $z_1 = f_1 + g_1$ with $g_1 \in M_0^{r+1}$, and with f_1 a nonzero homogeneous polynomial of degree r in $x_{r1}, \dots, x_{rd}$ whose nonzero coefficients are units in S_0 . On the other hand we have $z_1 = yz_2$, so, since the ord_{M_0} function is a valuation, with $\text{ord}_{M_0} z_1 = r$ and $\text{ord}_{M_0} y = 1$, we must have $\text{ord}_{M_0} z_2 = r - 1$, i.e., $z_2 \in M_0^{r-1} \setminus M_0^r$.

Now let $y_{r-1,1}, \dots, y_{r-1, n-d} \in M_{r-1} \cap M_0^r$ such that $\{x_{r1}, \dots, x_{rd}, y_{r-1,1}, \dots, y_{r-1, n-d}\}$ is a regular system of parameters for M_{r-1} , as in Proposition 3.7 (here, $j := r - 1$). We apply Lemma 3.8 to z_2 , since $z_2 \in M_0^{r-1} \cap M_{r-1}$, to write $z_2 = f_2 + g_2$, where f_2 is a nonzero homogeneous polynomial of degree $r - 1$ in $x_{i1}, \dots, x_{id}$ whose nonzero coefficients are units in S_0 , and where $g_2 \in M_0^r$. Thus $z_1 = yz_2$ yields

$$(*) \quad f_1 + g_1 = yf_2 + yg_2$$

Consider for a moment the function $\varphi: S_0 \setminus \{0\} \rightarrow \text{gr}^{M_0}(S_0)$ (the associated graded ring of S_0 with respect to M_0) given by $\varphi(h) = \bar{h} \in M_0^t/M_0^{t+1}$, where $h \in M_0^t \setminus M_0^{t+1}$. Then $\varphi(h_1 h_2) = \varphi(h_1) \varphi(h_2)$ (where the multiplication on the right-hand side is that in $\text{gr}^{M_0}(S_0)$) because ord_{M_0} is a valuation, and $\varphi(h_1 + h_2) = \varphi(h_1) + \varphi(h_2)$ if $\varphi(h_1) + \varphi(h_2) \neq 0$ and $h_1, h_2 \in M_0^t \setminus M_0^{t+1}$. Applying φ to $(*)$ gives us

$$\varphi(f_1) = \varphi(f_1 + g_1) = \varphi(yf_2 + yg_2) = \varphi(yf_2) = \varphi(y)\varphi(f_2).$$

But, letting

$$\sum_{u=1}^{e_1} m_u := f_1, \quad \sum_{v=1}^{e_2} m_v := f_2,$$

be the expressions for f_1 and f_2 as sums of monomials in $x_{i1}, \dots, x_{id}$ whose nonzero coefficients are units in S_0 , we get

$$\sum_{u=1}^e \varphi(m_u) = \sum_{v=1}^{e_2} \varphi(y) \varphi(m_v),$$

a contradiction, since $\varphi(x_{i1}), \dots, \varphi(x_{id}), \varphi(y)$ are algebraically independent over k in the polynomial ring $\text{gr}^{(\mathcal{M}_0)}(S_0)$. $\square$

The author is grateful to the referee for providing greatly improved versions of the original proofs in this section.

4. Finite chain conditions. We begin by recalling a few results from valuation theory (cf. [12, Appendix 2]).

DEFINITION 4.1. Given a quasilocal domain $(R, \mathcal{M})$, a *prime divisor* of R is a DVR, $(V, \mathcal{M}_v)$, such that $V > R$ and $\text{restrdeg}_R V = \dim R - 1$.

PROPOSITION 4.2. *Any positive-dimensional local domain $(R, \mathcal{M})$ has a prime divisor.*

PROOF. [7, PROPOSITION 1]. $\square$

PROPOSITION 4.3. *If $(V, \mathcal{M}_v)$ is any valuation ring birationally dominating the local domain $(R, \mathcal{M})$ then $\text{restrdeg}_R V \leq \dim R - \text{rank } V$.*

PROOF. [12, PROPOSITION 1, p. 330]. $\square$

PROPOSITION 4.4. *If $(V, \mathcal{M}_v)$ is a prime divisor of $(R, \mathcal{M})$ then $V/\mathcal{M}_v$ is a finitely generated field extension of $R/\mathcal{M}$.*

PROOF. [12, COROLLARY 2, p. 339]. $\square$

LEMMA 4.5. *If $(S, \mathcal{N}) > (R, \mathcal{M})$ are two n -dimensional local domains ($n \leq 1$) then $S/\mathcal{N}$ is a finite algebraic field extension of $R/\mathcal{M}$.*

PROOF. Using Proposition 4.2, choose a prime divisor $(V, \mathcal{M}_v)$ for $(S, \mathcal{N})$. Then $V > R$ as well, and we have, using Proposition 4.3, that

$$\begin{aligned} n - 1 = \dim R - \text{rank } V &\geq \text{restrdeg}_R V \\ &= \text{restrdeg}_S V + \text{restrdeg}_R S = n - 1 + \text{restrdeg}_R S. \end{aligned}$$

This implies, first, that $\text{restrdeg}_R S = 0$, i.e., $S/\mathcal{N}$ is algebraic over $R/\mathcal{M}$, and second, that V is a prime divisor of R . Then, by Proposition 4.4, $V/\mathcal{M}_v$ is finitely generated over $R/\mathcal{M}$, which implies that $S/\mathcal{N}$ is also finitely generated over $R/\mathcal{M}$, since $V/\mathcal{M}_v \supseteq S/\mathcal{N} \supseteq R/\mathcal{M}$. $\square$

Now we are ready to prove the main result of this paper. We denote by $\mathcal{R}(R)$ the set of RLR's properly containing a local domain R , having the same dimension as R , and contained in the quotient field of R . We order this set by inclusion.

THEOREM 4.6. *For any local domain $(R, \mathcal{M})$ of positive dimension, $\mathcal{R}(R)$ satisfies d.c.c.*

PROOF. Let $(S_0, \mathcal{M}_0) \supseteq (S_1, \mathcal{M}_1) \supseteq \cdots \supseteq R$ be a descending chain of elements of $\mathcal{R}(R)$. By Note 2.1, we have $S_i > R$ for all i , so $S_0/\mathcal{M}_0 \supseteq S_1/\mathcal{M}_1 \supseteq \cdots$ is a descending chain of finite algebraic field extensions of $R/\mathcal{M}$ by Lemma 4.5. Therefore this chain of residue fields must stabilize. If the chain of RLR's fails to stabilize then it has infinitely many different rings, all dominating R , and with stabilizing residue field, the situation to which Theorem 3.2 applies. Then Theorem 3.2 tells us that $\text{q.f.}(R) \subseteq \text{q.f.}(S_\infty) \subset \text{q.f.}(S_0) = \text{q.f.}(R)$, a contradiction. $\square$

DEFINITION 4.7. By a *minimal RLR over R* we mean a minimal element of $\mathcal{R}(R)$.

COROLLARY 4.8. *If $S > R$ is a pair of n -dimensional RLR's, then any chain of RLR's $\{S_\lambda\}$, between R and S must contain a minimal element, $S', S_\lambda > S' > R$ for all λ .*

PROOF. Otherwise we would be able to find an infinite strictly descending chain in $\mathcal{R}(R)$, contrary to Theorem 4.6 $\square$

This corollary shows that birational RLR's have a nice structure. That the structure is not so nice in more general classes of local rings is shown by the following example.

EXAMPLE 4.9. Let k be any field. Let

$$R := k[X, Y]_{(X, Y)}, \quad S := k[X, Y, Y/X]_{(X, Y/X)},$$

$$S_i := k[X, Y, Y^{i+1}/X]_{(X, Y, Y^{i+1}/X)}, \quad i \geq 1.$$

Then $S > S_1 > S_2 > \cdots > R$. Since a local domain caught between two local domains of dimension n must also have dimension n (see [9, Lemma 5.3]), $\dim S_i = 2$ for all i . Since

$$S_i \cong k[X, Y, Z]_{(X, Y, Z)} / (XZ - Y^{i+1}) k[X, Y, Z]_{(X, Y, Z)},$$

it is easily verified that S_i satisfies Serre's conditions $\mathcal{S}_2$ and $\mathcal{R}_1$, so that S_i is normal (cf. [5, Theorem 39]) and hence Cohen-Macaulay, since $\dim S_i = 2$ [12, Example (3), p. 397].

Moreover, since any 2-dimensional normal local ring birationally dominating a 2-dimensional RLR has a rational singularity [4, Proposition (1.2)], each S_i must have a rational singularity.

Again restricting our attention to RLR's, however, and using a related result of Shannon's, we are able to obtain the following corollary. This corollary answers a question raised by Shannon [9, Remark 3.8].

COROLLARY 4.10. *If $S > R$ is a pair of n -dimensional RLR's then any chain between them is finite in length.*

PROOF. Let $\{S_\lambda\}$ be a chain of RLR's between R and S . If S is minimal over R , then $S > R$ is the only such chain, and is clearly finite. Otherwise, use Corollary 4.8 to pick $S_1 \in \{S_\lambda\}$ minimal over R , $S > S_\lambda > S_1 > R$ for all λ . If S is minimal over S_1 , we are done; otherwise pick S_2 minimal over S_1 , $S > S_\lambda > S_2 > S_1 > R$ for all λ . Either there eventually exists an N such that S is minimal over S_N , so that $S > S_N > S_{N-1} > \cdots > S_1 > R$ forms a saturated chain, or else we

obtain an infinite ascending chain of RLR's between R and S . That this latter situation cannot hold was shown by Shannon [9, Proposition 5.1]. Thus the chain must become saturated after finitely many steps. $\square$

Note, however, that two saturated chains need not have the same length, as is illustrated in the following 3-dimensional example, due to Shannon [9, Remark 3.8].

EXAMPLE 4.11. Let k be any field, and set

$$\begin{aligned} R_0 &:= k[X, Y, Z]_{(X, Y, Z)}, \\ R_1 &:= k[X, Z, Y/X]_{(X, Z, Y/X)} = R_0[Y/X]_{(\mathcal{M}_0 R[Y/X], Y/X)}, \\ R_2 &:= k[X, Y/X, Z/X]_{(X, Y/X, Z/X)} = R_1[Z/X]_{(\mathcal{M}_1 R_1[Z/X], Z/X)}, \\ R_3 &:= k[X, Y/X, Z/Y]_{(X, Y/X, Z/Y)} = R_2[Z/Y]_{(\mathcal{M}_2 R_2[Z/Y], Z/Y)}. \end{aligned}$$

Then $R_0 < R_1 < R_2 < R_3$, each R_i is a RLR of dimension 3, and each R_i is obtained from R_{i-1} by a simple monoidal transformation, so this chain is saturated (because simple monoidal transforms of R are minimal over R —see [8, Theorem 5.1]), and has length 3. On the other hand, letting

$$\begin{aligned} S_1 &:= k[X, Y, Z/Y]_{(X, Y, Z/Y)} = R_0[Z/Y]_{(\mathcal{M}_0 R_0[Z/Y], Z/Y)}, \\ S_2 &:= k[X, Y/X, Z/Y]_{(X, Y/X, Z/Y)} = S_1[Y/X]_{(\mathcal{M}_{S_1} S_1[Y/X], Y/X)} = R_3 \end{aligned}$$

then $R_0 < S_1 < S_2 = R_3$ is again a saturated chain of RLR's, each a simple monoidal transform of the previous one. Though these two chains both lie between R_0 and $R_3 = S_2$, they have different lengths.

QUESTION 4.12. Given two RLR's of dimension n , $S > R$, does there exist a uniform upper bound, N , depending only on S and R , such that every saturated chain of RLR's between S and R has length $\leq N$?

In relation to this question, we should note that Hironaka has apparently shown that a similar question for projective models has an affirmative answer. Namely, if X and Y are two n -dimensional, birational normal projective models such that Y dominates X , then any birational sequence of normal models between X and Y has finite length, bounded uniformly by a fixed number depending only on X and Y (see [9, Remark 3.8]).

5. Some related questions. In this section we discuss some applications and directions for further research. One immediate question is the following.

QUESTION 5.1. Are RLR's the broadest class of local domains having the property that, given two rings, $S > R$, in this class, any chain of rings in this class between S and R must be finite in length?

Corollary 4.10 says that RLR's have this property in any dimension, while Example 4.9 shows that the class of Cohen-Macaulay local domains, the class of normal local rings, and the class of rational singularities do not have this property, even in dimension 2.

Another generalization of RLR's is the class of normal local rings whose order function is a valuation. The following example of Shannon's [9, Example 3.9] shows that this class also fails to have this property of RLR's.

EXAMPLE 5.2. Let k be a field of characteristic 0,

$$(R, \mathcal{M}) := (k[X, Y]_{(X, Y)}, (X, Y)R)$$

and for $i \geq 1$,

$$R_i := k[X, Y, Y^2/X, \dots, Y^{i+1}/X^i]_{(X, Y, Y^2/X, \dots, Y^{i+1}/X^i)},$$

$$S := k[X, Y/X]_{(X, Y/X)}.$$

Then we have $R < R_1 < R_2 < \dots < S$, R and S are RLR's of dimension 2, S is a quadratic transform of R , R_i has dimension 2 for all i (since $R < R_i < S$ —see [9, Lemma 5.3]), R_i is normal for all i (since $k[X, Y, Y^2/X, \dots, Y^{i+1}/X^i]$ is the integral closure of $k[X, Y^{i+1}/X^i]$ in $k(X, Y)$ —cf. [2, Theorem 7]), and this is a strictly ascending chain. Furthermore, $\text{ord}_{M_i|R_i}$ (where $M_i := (X, Y, Y^2/X, \dots, Y^{i+1}/X^i)R_i$) $= \text{ord}_{M|R_i}$, since if $(V, M_v) =$ the valuation ring of the valuation ord_M , $M_v^n \cap R_i = M_i^n$ for all n , which implies that $\text{ord}_M = \text{ord}_{M_i}$ and so ord_{M_i} is a valuation. This gives infinitely many normal local rings, each of whose ord function is a valuation, between two RLR's, so Corollary 4.10 does not extend to this class of rings.

Corollary 4.8 implies that every n -dimensional RLR, $(S, \mathcal{N})$, birationally dominating a given n -dimensional RLR, $(R, \mathcal{M})$, must contain a minimal RLR over R . It was proven by J. Sally [8, Theorem 5.1] and by D. Shannon [9, Proposition 3.7] that if $n > 1$, simple monoidal extensions of R are minimal elements of $\mathcal{R}(R)$. If $n > 2$, another type of minimal extension was given by J. Sally [8, Theorem 5.1], namely an extension of the form $R[a/b]_{\mathcal{N}}$, where $a, b \in R$ are relatively prime, $a \neq 0 \neq b$, $a \in \mathcal{M} \setminus \mathcal{M}^2$, $b \in \mathcal{M}^2$, $(a, b)R$ is a prime ideal, and $\mathcal{N}$ is any height n maximal ideal of $R[a/b]$. We denote this type of extension as a “Sally” extension. Example 5.4 below, due to C. Huneke, shows that simple monoidal extensions and Sally extensions are not the only minimal elements of $\mathcal{R}(R)$, if R is an arbitrary RLR. We need a preliminary lemma.

LEMMA 5.3. *If $(S, \mathcal{N})$ and $(R, \mathcal{M})$ are n -dimensional RLR's, $n > 2$, and S is either a simple monoidal extension or a Sally extension of R , then MS requires at most $n - 1$ generators.*

PROOF. Suppose $\mathcal{M} = (x_1, \dots, x_n)R$, where $S = R[x_2/x_1]_Q$ (if S is a simple monoidal extension) or $S = R[x_1/b]_Q$ (if S is a Sally extension). Clearly, in the first case, $MS = (x_1, x_3, x_4, \dots, x_n)S$ is $(n - 1)$ -generated. In the second case, $MS = (x_2, x_3, \dots, x_n, b)S$, but since $b \in \mathcal{M}^2 \subseteq \mathcal{N}^2$, we have $MS = (x_2, x_3, \dots, x_n)S$ by Nakayama's lemma. $\square$

EXAMPLE 5.4. Let $S := k[X, Y, Z]_{(X, Y, Z)}$, where k is any field,

$$R := k[XYZ, Y^2Z, X^3Z]_{(XYZ, Y^2Z, X^3Z)}.$$

Then R and S are 3-dimensional RLR's and since

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix} = -1,$$

we have $S > R$. If there existed a simple monoidal or Sally extension, T , of R , $S > T > R$, then by Lemma 5.3, MT would be at most 2-generated, so $MS = (MT)S$ would be at most 2-generated. But $MS = (XYZ, Y^2Z, X^3Z)k[X, Y, Z]_{(X, Y, Z)}$ is 3-generated for, if it were generated by less than three elements, then, since we are in a local ring, one such smaller generating set would have to be a subset of $\{XYZ, Y^2Z, X^3Z\}$, but none of these three elements lies in the ideal generated

by the other two in S . On the other hand, Corollary 4.8 above states that there must exist some minimal element of $\mathcal{R}(R)$ between S and R (it might be S). Thus there must exist at least one new type of minimal regular local overring. Specific calculations show that in fact S is minimal over R .

REFERENCES

1. S. Abhyankar, *On the valuations centered in a local domain*, Amer. J. Math. **78** (1956), 321–348.
2. —, *Uniformization of Jungian local domains*, Math. Ann. **159** (1965), 1–43.
3. W. Krull, *Idealtheorie*, Springer-Verlag, Berlin and New York, 1968.
4. J. Lipman, *Rational singularities, with applications to algebraic surfaces and unique factorization*, Inst. Hautes Etudes Sci. Publ. Math. **36** (1969), 195–279.
5. H. Matsumura, *Commutative algebra*, (2nd ed.), Benjamin/Cummings, Reading, Mass., 1980.
6. M. Nagata, *Local rings*, Interscience, New York, 1962.
7. M. Sakuma, *Existence theorems of valuations centered in a local domain with preassigned dimension and rank*, J. Sci. Hiroshima Univ. **21** (1957), 61–67.
8. J. Sally, *Regular overrings of regular local rings*, Trans. Amer. Math. Soc. **171** (1972), 291–300.
9. D. Shannon, *Monoidal transforms of regular local rings*, Amer. J. Math. **95** (1973), 294–320.
10. O. Zariski, *Reductions of the singularities of algebraic three dimensional varieties*, Ann. of Math. (2) **45** (1944), 472–542.
11. O. Zariski and P. Samuel, *Commutative algebra*, vol. I, Van Nostrand, Princeton, N. J., 1958.
12. —, *Commutative algebra*, vol. II, Van Nostrand, Princeton, N. J., 1960.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907

Current address: Department of Mathematics, University of Utah, Salt Lake City, Utah 84112