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SUFFICIENCY CONDITIONS FOR L^p MULTIPLIERS WITH POWER WEIGHTS

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ABSTRACT. Weighted norm inequalities in R^1 are proved for multiplier operators with the multiplier function of Hörmander type. The operators are initially defined on the space $\mathscr{S}_{0,0}$ of Schwartz functions whose Fourier transforms have compact support not including 0. This restriction on the domain of definition makes it possible to use weight functions of the form $|x|^\alpha$ for α larger than usually considered. For these weight functions, if $(\alpha + 1)/p$ is not an integer, a strict inequality on α is shown to be sufficient for a norm inequality to hold. A sequel to this paper shows that the weak version of this inequality is necessary.

1. Introduction. This paper is concerned with proving norm inequalities of the form

$$\int_{-\infty}^{\infty} \left|(m\hat{f})^\vee(x)\right|^p |x|^\alpha \, dx \leqslant C\int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha \, dx \tag{1.1}$$

for multipliers m of Hörmander type. Initially, (1.1) will be proved for all f in $\mathscr{S}_{0,0}$, the Schwartz functions whose Fourier transforms have compact support not including 0. Restricting f to $\mathscr{S}_{0,0}$ allows much larger values of α than is possible if (1.1) is required to hold for all Schwartz functions, and the additional weight functions are important for applications.

This paper is a continuation of **[11]**; there p was taken to be 2, and we characterized for each $\alpha > -1$ all the multipliers for which (1.1) is valid for all f in $\mathscr{S}_{0,0}$. Here the approach is somewhat different; we consider the usual spaces of multiplier functions, called $M(s, \lambda)$ here, which for λ a positive integer and s satisfying $1 \leqslant s \leqslant \infty$ consists of all m such that

$$B(m, s, \lambda) = \|m\|_\infty + \sup_{r>0} r^{\lambda - 1/s}\left[\int_{r<|t|<2r} \left|m^{(\lambda)}(t)\right|^s dt\right]^{1/s} < \infty.$$

For the definition with λ fractional, see §2; except for $s = 1$ and $s = \infty$, these are two sided versions of the spaces $S(s, \lambda)$ used by Connett and Schwartz in **[4]** and the spaces $\mathrm{WBV}_{s,\lambda}$ used by Gasper and Trebels in **[5]**. The main result proved here is the following.

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THEOREM (1.2). *If* $1 < p < \infty$, $1 \leqslant s \leqslant \infty$, $\lambda > \max(\frac{1}{s}, |\frac{1}{p} - \frac{1}{2}|)$ *or* $\lambda = s = 1$, $m \in M(s, \lambda)$,

$$\max(-1, -p\lambda, -1 + p(-\lambda + \tfrac{1}{2})) < \alpha < \min(p\lambda, -1 + p(\lambda + \tfrac{1}{2}), -1 + p(\lambda + 1 - \tfrac{1}{s}))$$

and $(\alpha + 1)/p$ *is not an integer, then for* f *in* $\mathscr{S}_{0,0}$

$$\int_{-\infty}^{\infty} |(m\hat{f})^{\vee}(x)|^p |x|^\alpha \, dx \leqslant CB(m, s, \lambda)^p \int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha \, dx, \tag{1.3}$$

where C *is independent of* m *and* f.

If the hypothesis that f be in $\mathscr{S}_{0,0}$ in Theorem (1.2) is changed to require only that f be in the class $\mathscr{S}$ of Schwartz functions, then it can be shown that the theorem is false for α outside the interval $(-1, p - 1)$. This is done by observing that $i \operatorname{sgn} x$ is in $M(s, \lambda)$ for all $\lambda > 0$ and $1 \leqslant s \leqslant \infty$ and that $\mathscr{S}$ is dense in the space L^p_α of f with

$$\|f\|_{p,\alpha} = \left[\int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha \, dx\right]^{1/p} < \infty$$

for $1 < p < \infty$ and all real α. Therefore, the conclusion (1.3) asserts, in particular, that the Hilbert transform is a bounded operator on L^p_α. It is well known, see for example Theorem 9, p. 247 of [7], that the Hilbert transform is bounded on L^p_α if and only if $-1 < \alpha < p - 1$. Therefore, the interval for α for which Theorem (1.2) is true for all f in $\mathscr{S}$ cannot extend beyond $(-1, p - 1)$.

The conditions on α in Theorem (1.2) may seem peculiar, especially the fact that taking s larger than the minimum of p' and 2 does not increase the range of α. It turns out, however, that except possibly for the strictness of the inequalities, these conditions are essential as shown by the following result proved in [**10**].

THEOREM (1.4). *If* $1 < p < \infty$, $1 \leqslant s \leqslant \infty$, $\lambda \geqslant 1/s$ *and* (1.3) *holds for all* m *in* $M(s, \lambda)$ *and* f *in* $\mathscr{S}_{0,0}$, *then* $\alpha > -1$,

$$\max(-p\lambda, -1 + p(-\lambda + \tfrac{1}{2})) \leqslant \alpha \leqslant \min(p\lambda, -1 + p(\lambda + \tfrac{1}{2}), -1 + p(\lambda + 1 - \tfrac{1}{s}))$$

and $(\alpha + 1)/p$ *is not an integer.*

In at least some cases, the end values of the inequalities for α are included in the values for which (1.3) holds; a theorem of this type is given in §6.

Results for weight functions of the form $(1 + |x|)^a \prod_{j=1}^{J} |x - b_j|^{a_j}$ and for more general weight functions are given in [**12**]. Periodic analogues are also considered in [**12**].

Theorem (1.2) can be extended to functions in more general classes than $\mathscr{S}_{0,0}$. For example, if $Q_{-1} = L^2$ and for k a nonnegative integer Q_k is the set of f in $L^2 \cap L^1_k$ with $\int_{-\infty}^{\infty} f(x) x^j \, dx = 0$ for $0 \leqslant j \leqslant k$, then the following is true.

THEOREM (1.5). *If* $1 < p < \infty$, $\alpha > -1$, k *is an integer*, $k \geqslant -2 + (\alpha + 1)/p$, m *is bounded and*

$$\|(m\hat{f})^{\vee}\|_{p,\alpha} \leqslant C \|f\|_{p,\alpha} \tag{1.6}$$

for all f *in* $\mathscr{S}_{0,0}$, *then* (1.6) *is true for all* f *in* $Q_k \cap L^p_\alpha$ *with the same* C.

Theorem (1.5) is proved by fixing an f in $Q_{k-1} \cap L^p_\alpha$ and using Theorem (6.1) of [**11**] to produce a sequence $\{f_n\}$ of functions in $\mathscr{S}_{0,0}$ that converges to f in L^2 and L^p_α. By (1.6) and this density, the operator $(m\hat{f})^\vee$ on $\mathscr{S}_{0,0}$ has a unique extension to $Q_{k-1} \cap L^p_\alpha$. Call the image of a function g under this operator $T_m g$. Then $\|T_m f\|_{p,\alpha} \leqslant C\|f\|_{p,\alpha}$ and there is a subsequence f_{n_j} such that $T_m f_{n_j}$ converges to $T_m f$ almost everywhere. Since f_{n_j} converges to f in L^2 as $j \to \infty$, then $(m\hat{f}_{n_j})^\vee$ converges to $(m\hat{f})^\vee$ in L^2, and a subsequence converges to $(m\hat{f})^\vee$ almost everywhere. Therefore, $(m\hat{f})^\vee = T_m f$ almost everywhere, and (1.6) follows from the fact that $\|T_m f\|_{p,\alpha} \leqslant C\|f\|_{p,\alpha}$.

Theorem (1.2) for $\lambda > \frac{1}{2}$ is proved in §§2–4. The method consists of finding and using estimates of truncated kernels of the form $[m(x)\phi_N(x)]^\vee$ where ϕ_N is in C^∞, $\phi_N(x) = 0$ for $|x| > 2^{N+1}$ and $|x| < 2^{-N-1}$ while $\phi_N(x) = 1$ for $2^{-N+1} < |x| < 2^{N-1}$. This procedure has led us to the definition of the classes $M(s, \lambda)$ given in §2 and required most of the results there. Many of these are known because of the equivalence of our definition for $1 < s < \infty$ to the definitions in [**4** and **5**], but the approach is different because the definitions are different. In §3, estimates are obtained for integrals of the truncated kernels and their derivatives. This is the only way the $M(s, \lambda)$ assumption on m is used in later sections and the main theorems could, as a result, be stated with truncated kernel estimates as the hypothesis; this would, however, produce longer theorem statements.

In §4, a result is first obtained for (1.1) with f in the class $\mathscr{S}$ of Schwartz functions. This is then used to prove Theorem (1.2) for $\lambda > \frac{1}{2}$. The case $\lambda \leqslant \frac{1}{2}$ is considered in §5; the method used is an adaptation of a proof by Calderón and Torchinsky in [**1**]. Theorem (1.2) for $\lambda \leqslant \frac{1}{2}$ is proved in §5 as Theorem (5.1). As mentioned before, §6 contains a proof that in some cases Theorem (1.2) remains true for α equal to an endpoint of the interval in the hypothesis. In §7, it is shown that the multiplier classes $M(s, \lambda)$ are the two sided versions of the multiplier classes used by other authors.

The following definitions and notations will be used throughout this paper. In addition to the expression $\operatorname{int}(x)$ for the greatest integer less than or equal to x, the traditional $[x]$ will also be used when unambiguous. The spaces $\mathscr{S}$, $\mathscr{S}_{0,0}$ and L^p_α will be as defined above. For integrable functions f, we define the Fourier transform by $\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt}\,dt$ and the inverse Fourier transform by

$$\check{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{ixt}\,dt.$$

For general locally integrable f, we define $\hat{f}$ to be the function that satisfies $\int_{-\infty}^{\infty} \hat{f}(x)\phi(x)\,dx = \int_{-\infty}^{\infty} f(x)\hat{\phi}(x)\,dx$ for every ϕ in C^∞ with compact support, provided such a function exists. The inverse Fourier transform $\check{f}$ for locally integrable functions is defined analogously. Similarly, the weak derivative of a function f on $(-\infty, \infty)$ is the function f' such that $\int_{-\infty}^{\infty} f(x)\phi'(x)\,dx = -\int_{-\infty}^{\infty} f'(x)\phi(x)\,dx$ for every ϕ in C^∞ with compact support, provided such a function exists.

Throughout this paper C will denote constants not necessarily the same at each occurrence. The letters i, j, k, l, m and n will be used for integers whether this is stated explicitly or not except for cases where i is obviously the square root of -1 or when they are names of functions. If g is an expression in x, $[g(x)]^\wedge$ will denote the Fourier transform of g at the point x. For a number p with $1 \leqslant p \leqslant \infty$, p' will denote $p/(p-1)$.

2. The classes $M(s,\lambda)$. In this section we define the classes of multipliers $M(s,\lambda)$ that will be used in the rest of the paper and prove some basic results concerning them. For λ an integer, $M(s,\lambda)$ is the familiar space of multiplier functions m such that $\|m\|_\infty < \infty$ and for $r > 0$

$$\left[\int_{r\leqslant|x|\leqslant 2r} \left|m^{(\lambda)}(x)\right|^s dx\right]^{1/s} \leqslant Cr^{-\lambda+1/s}; \tag{2.1}$$

in particular, $M(1,1)$ is the Marcinkiewicz condition, $M(2,1)$ is the Hörmander condition and $M(\infty,1)$ is the Mihlin condition. For noninteger λ, it is shown in §7 that $M(s,\lambda)$ is the two sided version of the spaces $S(s,\lambda)$ in **[4]** and $\mathrm{WBV}_{s,\lambda}$ of **[5]** provided $1 < s < \infty$.

The main results of this section are as follows. Theorem (2.12) describes the inclusion relations between the spaces $M(s,\lambda)$ and relates their norms; it is needed frequently throughout this paper. Lemma (2.18) estimates the norm of the fractional derivative of a product and is needed for kernel estimation in §3. Theorem (2.30), which states that $xm'(x)$ is in $M(s,\lambda-1)$ if $m(x)$ is in $M(s,\lambda)$, is of independent interest and could have been used in several of our proofs.

We will use the following notations and definitions throughout the paper. The function $\psi(x)$ will be C^∞ with support in $\frac{1}{2} < |x| < 2$ and will satisfy $\sum_{j=-\infty}^{\infty}\psi(2^{-j}x) = 1$ for $x \neq 0$. If m is in L^∞, then $m_j(x) = \psi(2^{-j}x)m(x)$, $k_j(x) = [m_j(x)]^\vee$ and $K_N = \sum_{j=-N}^{N} k_j(x)$. D^λ will denote the operator defined by $D^\lambda g(x) = [\check{g}(x)x^\lambda]^\wedge$, where x^λ is taken to be $|x|^\lambda e^{-i\pi\lambda}$ for $x < 0$ and the Fourier transforms are as defined in §1. If $1 \leqslant s \leqslant \infty$ and $\lambda \geqslant 0$, then m is in $M(s,\lambda)$ if $D^\lambda m_j$ is a locally integrable function for every j and

$$B(m,s,\lambda) = \|m\|_\infty + \sup_j 2^{j(\lambda-1/s)}\|D^\lambda m_j(x)\|_s < \infty. \tag{2.2}$$

The classes M are independent of the choice of ψ, and the values of B are equivalent for different choices of ψ; this is an immediate consequence of Theorem (2.25). Note that if $1 < s < \infty$, the boundedness of the Hilbert transform on L^s shows that using $[|x|^\lambda \check{m}_j]^\wedge$ in place of $D^\lambda m_j$ will give an equivalent definition of $M(s,\lambda)$ and $B(m,s,\lambda)$. Note also that $\psi(2^{-j}x) + \psi(2^{-j-1}x) \equiv 1$ for $2^j < |x| < 2^{j+1}$. Consequently, if λ is a positive integer and m is in $M(s,\lambda)$, then $D^\lambda(m_j(x) + m_{j+1}(x))$ is the weak derivative of order λ of $m(x)$ on $2^j < |x| < 2^{j+1}$ and, therefore, (2.1) holds. Conversely, if $\lambda = n$ is an integer, $\|m\|_\infty < \infty$ and (2.1) holds, then $D^n m_j(x)$ can be written as a linear combination of terms of the form $[\psi(2^{-j}x)]^{(n-i)}[m(x)]^{(i)}$, $0 \leqslant i \leqslant n$. Now $|[\psi(2^{-j}x)]^{(n)}| \leqslant C2^{-nj}$ and it is known that if $\|m\|_\infty < \infty$ and (2.1) holds for $\lambda = n$, then (2.1) holds for $0 \leqslant \lambda \leqslant n$. These two facts imply that

$$\left\|[\psi(2^{-j}x)]^{(n-i)}[m(x)]^{(i)}\right\|_s \leqslant C2^{-j(n-1/s)}$$

and show that m is in $M(s,\lambda)$.

The fact that the classes $M(s, \lambda)$ are two sided versions of the classes $S(s, \lambda)$ in [**4**] and of the classes $\mathrm{WBV}_{s,\lambda}$ in [**5**] for $1 < s < \infty$ will be shown in §7. These classes are also equivalent to classes defined similarly using Bessel potentials instead of Riesz potentials. This fact, which will be needed in §5, is proved in the following lemma.

LEMMA (2.3). *If* $1 < s < \infty$ *and* $\lambda \geqslant 0$, *then there is a constant* C, *independent of* m, *such that*

$$D(m, s, \lambda) = \|m\|_\infty + \sup_j 2^{-j/s} \left\| \left[\left(1 + 2^{2j}|x|^2\right)^{\lambda/2} k_j(x) \right]^\wedge \right\|_s$$

satisfies $D(m, s, \lambda) \leqslant CB(m, s, \lambda) \leqslant C^2 D(m, s, \lambda)$.

To prove the first inequality, assume that $B = B(m, s, \lambda) < \infty$. Then (2.2) and the boundedness of the Hilbert transform on L^s imply that

$$\left\| \left[2^{j\lambda}|x|^\lambda k_j(x) \right]^\wedge \right\|_s \leqslant CB2^{j/s},$$

while the fact that $|m| \leqslant B$ implies that

$$\left\| \left[k_j(x) \right]^\wedge \right\|_s \leqslant CB2^{j/s}.$$

These two inequalities imply

$$\left\| \left[\left(1 + 2^{j\lambda}|x|^\lambda\right) k_j(x) \right]^\wedge \right\|_s \leqslant CB2^{j/s}. \tag{2.4}$$

Now

$$\left(1 + 2^{2j}|x|^2\right)^{\lambda/2} / \left(1 + 2^{j\lambda}|x|^\lambda\right) \tag{2.5}$$

with $j = 0$ is the Fourier transform of a finite measure; see [**15**, pp. 133–134]. By a change of variables, (2.5) is, therefore, the Fourier transform of a finite measure with total variation independent of j. Therefore,

$$\left\| \left[\left(1 + 2^{2j}|x|^2\right)^{\lambda/2} k_j(x) \right]^\wedge \right\|_s \leqslant C \left\| \left[\left(1 + 2^{j\lambda}|x|^\lambda\right) k_j(x) \right]^\wedge \right\|_s,$$

and the fact that $D(m, s, \lambda) \leqslant CB(m, s, \lambda)$ follows from (2.4).

To prove the second inequality, assume that $D = D(m, s, \lambda) < \infty$. Since the statements above concerning (2.5) are also true of its reciprocal,

$$\left\| \left[\left(1 + 2^{j\lambda}|x|^\lambda\right) k_j(x) \right]^\wedge \right\|_s \leqslant C \left\| \left[\left(1 + 2^{2j}|x|^2\right)^{\lambda/2} k_j(x) \right]^\wedge \right\|_s.$$

Since $\|[k_j(x)]^\wedge\|_s \leqslant CD2^{j/s}$, it follows that $\|[|x|^\lambda k_j(x)]^\wedge\|_s \leqslant CD2^{j(1/s-\lambda)}$, and the boundedness of the Hilbert transform completes the proof.

LEMMA (2.6). *If* $f(x)$ *is integrable*, $f(x) = 0$ *for* x *not in a compact interval* I, $\lambda > -1$, *and* $D^\lambda f(x)$ *is a locally integrable function, then there is a* C, *depending only on* λ, *such that* $D^\lambda f(x) = 0$ *for almost every* x *to the right of* I *and*

$$D^\lambda f(x) = C \int_I \frac{f(t)}{(t-x)^{\lambda+1}}\, dt$$

for almost every x *to the left of* I.

To prove this, let ϕ be a C^∞ function with compact support disjoint from I. Then

$$(2.7)\qquad \int_{-\infty}^{\infty} D^\lambda f(x)\phi(x)\,dx = \int_{-\infty}^{\infty} \check{f}(x)x^\lambda\hat{\phi}(x)\,dx.$$

Since both f and $x^\lambda\hat{\phi}(x)$ are integrable, the right side of (2.7) equals

$$(2.8)\qquad \int_I f(x)\left[x^\lambda\hat{\phi}(x)\right]^\vee dx.$$

With x^λ defined as it is at the beginning of this section, it is easy to verify that $[x^\lambda]^\vee = 0$ for $x < 0$ and equals $Cx^{-\lambda-1}$ for $x > 0$, where $C = e^{-i\lambda\pi/2}/\Gamma(-\lambda)$. Note that if λ is a nonnegative integer, $C = 0$. Therefore for x in I, $[x^\lambda\hat{\phi}(x)]^\vee = C\int_{-\infty}^{x}\phi(t)(x-t)^{-\lambda-1}\,dt$. Substituting this into (2.8) shows that the integral in (2.7) equals

$$(2.9)\qquad C\int_I f(x)\left(\int_{-\infty}^{y} \frac{\phi(t)}{(x-t)^{\lambda+1}}\,dt\right)dx = C\int_{-\infty}^{y}\left(\int_I \frac{f(x)\,dx}{(x-t)^{\lambda+1}}\right)\phi(t)\,dt,$$

where y is the left end of I; the last equality follows from the fact that f and ϕ are integrable and $|x-t|^{-1-\lambda}$ is bounded on the set where $f(x)\phi(t) \neq 0$. The conclusion of the lemma follows since the right side of (2.9) equals the left side of (2.7) for all ϕ that are C^∞ with compact support disjoint from I.

COROLLARY (2.10). *If m is in L^∞, $\psi(x)$ is in C^∞ with support in $\frac{1}{2} \leqslant |x| \leqslant 2$, $m_j(x) = m(x)\psi(2^{-j}x)$, $\lambda > -1$, and $D^\lambda m_j(x)$ is a locally integrable function, then for almost every x satisfying $|x| > 2^{j+2}$ or $|x| < 2^{j-2}$,*

$$\left|D^\lambda m_j(x)\right| \leqslant C2^j\|m\|_\infty/\left(2^j + |x|\right)^{\lambda+1}.$$

LEMMA (2.11). *If $\lambda \geqslant 0$, $\lambda - 1 < \alpha < \lambda$, $f(x)$ is integrable, f has compact support if $\lambda > 0$, and $D^\lambda f(x)$ is a locally integrable function, then $D^\alpha f$ is a locally integrable function and for almost every x*

$$D^\alpha f(x) = C\int_x^\infty \frac{D^\lambda f(t)}{(t-x)^{1-\lambda+\alpha}}\,dt.$$

To prove this, let $\beta = \lambda - \alpha$ and let ϕ be a Schwartz function. If $\lambda > 0$, the hypothesis and Lemma (2.6) show that the function $D^\lambda f(x)$ is integrable on $(-\infty, \infty)$. If $\lambda = 0$, this is trivial. Therefore, since $\int_{-\infty}^{t}|\check{\phi}(x)|(t-x)^{\beta-1}\,dx$ is a bounded function of t,

$$\int_{-\infty}^{\infty}\left(\int_x^\infty \frac{D^\lambda f(t)}{(t-x)^{1-\beta}}\,dt\right)\check{\phi}(x)\,dx = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} \frac{\check{\phi}(x)}{(t-x)^{1-\beta}}\,dx\right)D^\lambda f(t)\,dt.$$

The right side equals

$$C\int_{-\infty}^{\infty}\left[t^{-\beta}\phi(t)\right]^\vee D^\lambda f(t)\,dt = C\int_{-\infty}^{\infty} t^\alpha\check{f}(t)\phi(t)\,dt = C\int_{-\infty}^{\infty}\left[D_\alpha f(t)\right]^\vee\phi(t)\,dt;$$

the first equality holds since $D^\lambda f$ and $|t|^{-\beta}\phi(t)$ are integrable. The equality in the conclusion of the lemma follows by comparing the end terms of this chain of equalities. The local integrability of $D^\alpha f(x)$ follows from the equality and the integrability of $D^\lambda f$.

By use of Theorem (7.3) we see that our next result is a strengthened version of part (a) of the Theorem on p. 37 of [**4**]. For s and t not equal to 1 or ∞, it is the same as Theorem 4 of [**5**].

THEOREM (2.12). *If* $1 \leqslant s \leqslant \infty$, $1 \leqslant t \leqslant \infty$, $0 \leqslant \alpha \leqslant \lambda$, *$m$ is in* $M(s, \lambda)$ *and one of the following holds*:
(i) $\alpha - 1/t \leqslant \lambda - 1/s$, $s > 1$ *and* $t < \infty$,
(ii) $\alpha - 1/t \leqslant \lambda - 1/s$, $s = 1$ *and* $t = \infty$,
(iii) $\alpha - 1/t < \lambda - 1/s$,
then m is in $M(t, \alpha)$ *and* $B(m, t, \alpha) \leqslant CB(m, s, \lambda)$.

The theorem is not true for $1/t - \alpha = 1/s - \lambda$ if $t = \infty$ and $1 < s < \infty$ or if $s = 1$ and $1 < t < \infty$; this will be shown at the end of the proof by giving examples.

The theorem will be proved by considering four cases. Case 1 is $1 \leqslant t \leqslant s \leqslant \infty$ and $\alpha = \lambda$. Case 2 is $1 < s < t < \infty$ and $\alpha - 1/t = \lambda - 1/s$. Case 3 is $s = 1$, $t = \infty$ and $\alpha = \lambda - 1$. Case 4 is $\alpha - 1/t < \lambda - 1/s$, $\alpha > \lambda - 1$ and either $s = 1$ or $t = \infty$. Successive use of these cases is clearly sufficient to obtain the theorem.

For each case we need to show that $D^\alpha m_j$ is a function and that

$$2^{j(\alpha-1/t)}\left[\int_{|x|\geqslant 2^{j+2}} |D^\alpha m_j(x)|^t\,dx\right]^{1/t} \tag{2.13}$$

and

$$2^{j(\alpha-1/t)}\left[\int_{|x|< 2^{j+2}} |D^\alpha m_j(x)|^t\,dx\right]^{1/t} \tag{2.14}$$

are bounded by $CB(m, s, \lambda)$. Lemma (2.11) implies that $D^\alpha f$ is a locally integrable function, and Corollary (2.10) shows that (2.13) has the bound $C\|m\|_\infty \leqslant CB(m, s, \lambda)$, unless $\alpha = 0$ and $t = 1$ for which the inequality is trivial. Therefore, we will complete the proof by estimating (2.14) in each of the four cases.

Case 1. Hölder's inequality shows that (2.14) is bounded by

$$2^{j(\lambda-1/t)}\left[\int_{|x|< 2^{j+2}} |D^\lambda m_j(x)|^s\,dx\right]^{1/s} 2^{j(1/t-1/s)},$$

which is bounded by $CB(m, s, \lambda)$.

Case 2. Lemma (2.11) and the usual fractional integral theorem, [**15**, p. 119], show that (2.14) is bounded by

$$2^{j(\alpha-1/t)}\left(\int_{-\infty}^{\infty} |D^\lambda m_j(x)|^s\,dx\right)^{1/s};$$

since $\alpha - 1/t = \lambda - 1/s$, this is bounded by $B(m, s, \lambda)$.

Case 3. By hypothesis, $D^\lambda m_j(x)$ is integrable, and by Corollary (2.10), $\lim_{x\to-\infty} D^\alpha m_j(x) = 0$. Therefore, $D^\alpha m_j(x) = \int_{-\infty}^{x} D^\lambda m_j(t)\,dt$, $\|D^\alpha m_j\|_\infty \leqslant \|D^\lambda m_j\|_1$ and the result follows immediately from the definition of $B(m, 1, \lambda)$.

Case 4. By Lemma (2.11), (2.14) is bounded by the sum of

$$C2^{j(\alpha-1/t)}\left[\int_{|x|<2^{j+2}}\left|\int_{|y|>2^{j+3}}\frac{|D^\lambda m_j(y)|}{|y-x|^{1-\lambda+\alpha}}\,dy\right|^t dx\right]^{1/t}$$

and

$$2^{j(\alpha-1/t)}\left[\int_{|x|<2^{j+2}}\left|\int_{|y|\leqslant 2^{j+3}}\frac{|D^\lambda m_j(y)|}{|y-x|^{1-\lambda+\alpha}}\,dy\right|^t dx\right]^{1/t}.$$

For the first, use Corollary (2.10) and the fact that $|y-x|>|y|/2$. For the second, if $s=1$, use Minkowski's integral inequality and the definition of $B(m,s,\lambda)$; if $t=\infty$, use Hölder's inequality with exponents s and s' on the inner integral. This completes the proof of Theorem (2.12).

Now we will describe a function m such that m is in $M(s,\lambda)$ for given λ and s satisfying $\lambda>1/s$ and $1<s<\infty$ but m is not in $M(\infty,\lambda-1/s)$. To do this, define

$$h(x)=\frac{\chi_{[2,4]}(x)}{|x-3|^{1/s}|\log(|x-3|)|^a}+\phi(x)$$

where $1/s<a<1$ and ϕ is C^∞ with support in [2, 4]. By Lemma 2.6 of [2], ϕ can be chosen so that $\int_2^4 x^n h(x)\,dx=0$ for $0\leqslant n\leqslant[\lambda]+1$. Define $m(x)=[x^{-\lambda}\check{h}(x)]^\wedge$; note that m is bounded since $|\check{h}(x)|\leqslant C|x|^{[\lambda]+2}$ for $|x|\leqslant 1$ and $|\check{h}(x)|\leqslant C|x|^{-1+1/s}$ for $|x|>1$. If λ is an integer, it is easy to show that m is in $M(s,\lambda)$. If λ is not an integer, the proof is longer and uses Corollary (2.10), Lemma (2.11) and some facts about fractional integrals. These fractional integral facts include Lemmas (2.15) and (2.18) and the fact that if $0<\lambda<1$, then $\|D^\lambda f\|_s\leqslant C(\|f\|_s+\|f'\|_s)$. To show that m is not in $M(\infty,\lambda-1/s)$, use Lemma (2.11) to show that

$$D^{\lambda-1/s}m(x)=c\int_x^\infty\frac{h(t)}{(t-x)^{1-1/s}}\,dt.$$

The integral is easily estimated; it is bounded below for $5/2<x<3$ by a positive constant times $|\log(|x-3|)|^{1-a}$. By Lemma (2.6), it is easy to see that $\sum_{j<0,\,j>3}D^{\lambda-1/s}m_j(x)$ is bounded for $2<x<4$; therefore, $D^{\lambda-1/s}m_j(x)$ is unbounded on [2, 4] for at least one j satisfying $0\leqslant j\leqslant 3$.

The example of an m in $M(1,\lambda)$ but not $M(s,\lambda-1+1/s)$ for $\lambda>1$ and $1<s<\infty$ is similar. Define

$$h(x)=\frac{\chi_{[2,4]}(x)}{|x-3|\,|\log(|x-3|)|^a}+\phi(x),$$

where $1<a<1+1/s$ and ϕ is C^∞, has support in [2, 4] and is chosen so that $\int_2^4 x^n h(x)\,dx=0$ for $0\leqslant n\leqslant[\lambda]+1$. Define $m(x)=[x^{-\lambda}\check{h}(x)]^\wedge$. As before, it is easy to show that m is in $M(1,\lambda)$ if λ is an integer and more involved if λ is not an integer. The function $|[x^\alpha\check{m}(x)]^\wedge|$, where $\alpha=\lambda-1+1/s$, is bounded below for

$5/2 < x < 3$ by a positive constant times $|x-3|^{-1/s}|\log(|x-3|)|^{1-a}$ which is not in L^s. Therefore, $D^\alpha m$ is not in L^s on [2, 4]. As before, Lemma (2.6) implies that $\sum_{j<0, j>3} D^\alpha m_j(x)$ is in L^s on [2, 4] and completes the proof that m is not in $M(s, \alpha)$.

Similar examples can be given to show that Theorem (2.12) is false if $1/t - \alpha < 1/s - \lambda$; these are simpler since the power of $|\log(|x-3|)|$ is not needed in the definition of h.

To prove our last two theorems concerning the classes $M(s, \lambda)$, we will need two lemmas concerning fractional derivatives, Lemmas (2.15) and (2.18).

LEMMA (2.15). *If $0 < \alpha < 1$, f is in L^1 and either the function*

$$G^\alpha f(x) = \frac{d}{dx}\int_x^\infty \frac{f(t)}{(t-x)^\alpha}\,dt,$$

where the derivative is taken in the weak sense, or $D^\alpha f(x)$ is a locally integrable function, then the other is a locally integrable function and there is a nonzero constant C such that $D^\alpha f(x) = CG^\alpha f(x)$.

We will show that for all g in C^∞ with compact support

$$\int_{-\infty}^\infty \hat{g}(x)x^\alpha \check{f}(x)\,dx = C\int_{-\infty}^\infty g'(t)\left(\int_t^\infty \frac{f(x)}{(x-t)^\alpha}\,dx\right)dt. \tag{2.16}$$

This is sufficient, for if $D^\alpha f$ is a locally integrable function, the left side of (2.16) equals $C\int_{-\infty}^\infty g(x)D^\alpha f(x)\,dx$ and the equality $D^\alpha f(x) = CG^\alpha f(x)$ follows from the definition of the weak derivative. If $G^\alpha f$ is a locally integrable function, the right side of (2.16) equals $-C\int_{-\infty}^\infty g(x)G_\alpha f(x)\,dx$, and the asserted equality follows from the definition of the Fourier transform.

To prove (2.16), use the fact that f and $x^\alpha\hat{g}(x)$ are integrable and Plancherel's theorem to show that the left side of (2.16) equals

$$C\int_{-\infty}^\infty f(x)(\hat{g}(x)x^\alpha)^\vee\,dx = C\int_{-\infty}^\infty f(x)\left(\int_{-\infty}^x \frac{g'(t)}{(x-t)^\alpha}\,dt\right)dx. \tag{2.17}$$

Since f and g' are integrable and g' is bounded, $|x-t|^{-\alpha}|f(x)g'(t)|$ is integrable and Fubini's theorem shows that the right side of (2.17) equals the right side of (2.16). This completes the proof of Lemma (2.15).

LEMMA (2.18). *If $0 \leqslant \alpha < 1$, $1 \leqslant s \leqslant \infty$, f is integrable, $D^\alpha f$ is a locally integrable function, ϕ is differentiable with $\|\phi'\|_\infty < \infty$, and ϕ has support in a finite interval I, then $D^\alpha(\phi f)$ is a locally integrable function and*

$$\|D^\alpha(\phi f)\|_s \leqslant C\big(\|\phi\|_\infty + |I|\,\|\phi'\|_\infty\big)\big(\|\chi_I D^\alpha f\|_s + |I|^{-\alpha}\|f\|_s\big). \tag{2.19}$$

For $\alpha = 0$, this is immediate. Therefore, assume $\alpha > 0$. The expression

$$\frac{d}{dx}\left[\int_x^\infty \frac{\phi(x)f(t)}{(t-x)^\alpha}\,dt + \int_x^\infty \frac{[\phi(t)-\phi(x)]f(t)}{(t-x)^\alpha}\,dt\right], \tag{2.20}$$

where the derivative is taken in the weak sense, equals

$$C\phi(x)D_\alpha f(x) + C\int_x^\infty \frac{[\phi(t)-\phi(x)]f(t)}{(t-x)^{\alpha+1}}\,dt \tag{2.21}$$

by Lemma (2.15) since $D_\alpha f$ is a locally integrable function. From this, it follows that (2.20) is a locally integrable function, and by Lemma (2.15), (2.20) equals $CD^\alpha(\phi f)$. We can complete the proof, therefore, by showing that the L^s norm of (2.21) is bounded by the right side of (2.19).

The L^s norm of the first term in (2.21) is bounded by $\|\phi\|_\infty \|\chi_I D_\alpha f\|_s$ as desired. Using the fact that ϕ is 0 outside I, we see that the L^s norm of the second term in (2.21) is bounded by the sum of

$$C\left[\int_{2I}\left(\int_{2I}\frac{\|\phi'\|_\infty |f(t)|}{|t-x|^\alpha}dt\right)^s dx\right]^{1/s}, \tag{2.22}$$

$$C\left[\int_{(2I)^c}\left(\int_{I}\frac{\|\phi\|_\infty |f(t)|}{|t-x|^{\alpha+1}}dt\right)^s dx\right]^{1/s} \tag{2.23}$$

and

$$C\left[\int_{I}\left(\int_{(2I)^c}\frac{\|\phi\|_\infty |f(t)|}{|t-x|^{\alpha+1}}dt\right)^s dx\right]^{1/s}. \tag{2.24}$$

In (2.22), make the change of variables $u = t - x$ in the inner integral and use Minkowski's integral inequality to get the estimate $C\|f\|_s \|\phi'\|_\infty |I|^{1-\alpha}$. In (2.23), use Minkowski's integral inequality to get the bound $C\|\phi\|_\infty \|\chi_I f\|_1 |I|^{1/s-\alpha-1}$; then use Hölder's inequality to show that this is bounded by $C\|\phi\|_\infty \|f\|_s |I|^{-\alpha}$. For (2.24), use Hölder's inequality on the inner integral and the fact that

$$\int_{(2I)^c}|t-x|^{-s'(\alpha+1)}dt \leqslant C|I|^{1-s'(\alpha+1)} \quad \text{for } x \text{ in } I.$$

This shows that (2.24) is also bounded by $C\|\phi\|_\infty \|f\|_s |I|^{-\alpha}$ and completes the proof of Lemma (2.18).

The following theorem is needed to prove Corollary (2.28); it is also important since it implies immediately that the definition of $M(s, \lambda)$ is independent of the choice of the function ψ.

THEOREM (2.25). *If m is in $M(s,\lambda)$, $1 \leqslant s \leqslant \infty$, $\lambda > 0$, and ϕ has $[\lambda + 1]$ bounded derivatives and support in $\frac{1}{2} \leqslant |x| \leqslant 2$, then $D^\lambda[m(x)\phi(2^{-j}x)]$ is a locally integrable function and*

$$\left\|D^\lambda\left[m(x)\phi(2^{-j}x)\right]\right\|_s \leqslant CA(\phi)2^{j(-\lambda+1/s)}B(m,s,\lambda),$$

where C is independent of m and $A(\phi) = \sup_{0 \leqslant k \leqslant [\lambda+1]}\|\phi^{(k)}\|_\infty$.

Since $\phi(2^{-j}x)$ has support contained in $2^{j-1} \leqslant |x| \leqslant 2^{j+1}$,

$$m(x)\phi(2^{-j}x) = \sum_{k=j-1}^{j+1} m_k(x)\phi(2^{-j}x).$$

Therefore, since $D^\lambda = D^{\lambda-[\lambda]}D^{[\lambda]}$, it is sufficient to prove for $j - 1 \leqslant k \leqslant j+1$ and $0 \leqslant i \leqslant [\lambda]$ that

$$D^{\lambda-[\lambda]}\left(m_k^{([\lambda]-i)}(x)2^{-ij}\phi^{(i)}(2^{-j}x)\right) \tag{2.26}$$

is a locally integrable function and

$$(2.27) \quad 2^{-ij}\left\| D^{\lambda-[\lambda]}\left(m_k^{([\lambda]-i)}(x)\phi^{(i)}(2^{-j}x)\right)\right\|_s \leqslant CA(\phi)2^{j(-\lambda+1/s)}B(m,s,\lambda).$$

By Theorem (2.12) and Corollary (2.10), $m_k^{([\lambda]-i)}$ and $D^{\lambda-[\lambda]}m_k^{([\lambda]-i)}$ are integrable. Therefore, by Lemma (2.18), (2.26) is locally integrable and the left side of (2.27) has the bound

$$C2^{-ij}\left(\|\phi^{(i)}\|_\infty + \|\phi^{(i+1)}\|_\infty\right)\left(\|D^{\lambda-i}m_k\|_s + 2^{j([\lambda]-\lambda)}\|D^{[\lambda]-i}m_k\|_s\right).$$

This is bounded by

$$C2^{-ij}A(\phi)\left[2^{j(i-\lambda+1/s)}B(m,s,\lambda-i) + 2^{j(i-\lambda+1/s)}B(m,s,[\lambda]-i)\right].$$

Theorem (2.12) then completes the proof of (2.27), and thereby, of Theorem (2.25).

We will need the following simple consequences of Theorem (2.25).

COROLLARY (2.28). *If m is in $M(s,\lambda)$, $\lambda > 0$, $1 \leqslant s \leqslant \infty$ and $N \geqslant 0$, then $\hat{K}_N$ is in $M(s,\lambda)$ and $B(\hat{K}_N, s, \lambda) \leqslant CB(m,s,\lambda)$, where C is independent of m and N.*

Since $\hat{K}_N(x) = \sum_{j=-N}^{N} m(x)\psi(2^{-j}x)$, $\|\hat{K}_N\|_\infty \leqslant \|m\|_\infty$. If $\lambda = 0$, the result follows from this. For $\lambda > 0$, if j is not $N-1$, N, $N+1$, $-N$, $-N-1$ or $-N+1$, $\hat{K}_N(x)\psi(2^{-j}x)$ either equals $m(x)\psi(2^{-j}x)$ or 0 and the required estimate is immediate. For the six values of j listed, $\hat{K}_N(x)\psi(2^{-j}x) = m(x)\phi(2^{-j}x)$ where $\phi(x)$ is one of six infinitely differentiable functions with support in $1/2 \leqslant |x| \leqslant 2$. Theorem (2.25) then completes the proof.

COROLLARY (2.29). *If $\lambda > 0$, $1 \leqslant s \leqslant \infty$, $1 \leqslant p \leqslant \infty$, $W(x)$ is nonnegative and for all f in a subset S of L^2 and m in $M(s,\lambda)$, we have $\|(m\hat{f})^\vee\|_{p,W} \leqslant CB(m,s,\lambda)\|f\|_{p,W}$ with C independent of f and m, then for all $N \geqslant 0$ and f in S, $\|K_N * f\|_{p,W} \leqslant CB(m,s,\lambda)\|f\|_{p,W}$ with C independent of N, m and f.*

Since $(K_N * f)(x) = (\hat{K}_N(x)\hat{f}(x))^\vee$ for almost every x, the conclusion follows from the hypothesis and Corollary (2.28).

Finally, we prove the following theorem which is of interest since it makes it possible to deduce properties of $M(s,\lambda)$ from properties of these classes with smaller values of λ. It could be used to prove some of our multiplier theorems by induction.

THEOREM (2.30). *If $1 \leqslant s \leqslant \infty$, $\lambda > 1 + 1/s$ and m is in $M(s,\lambda)$ then $xm'(x)$ is in $M(s,\lambda-1)$ and $B(xm'(x), s, \lambda-1) \leqslant CB(m,s,\lambda)$, where C is independent of m.*

By Theorem (2.12), m is in $M(\infty,1)$ and $B(m,\infty,1) \leqslant CB(m,s,\lambda)$. Therefore, $\|xm'(x)\|_\infty \leqslant CB(m,s,\lambda)$. Next we must show that

$$(2.31) \qquad \left\| D^{\lambda-1}\left(\psi(2^{-j}x)xm'(x)\right)\right\|_s \leqslant C2^{j(1-\lambda+1/s)}B(m,s,\lambda).$$

To do this, use the fact that $D^{\lambda-1} = D^{\lambda-[\lambda]}D^{[\lambda]-1}$, Leibniz' rule and Minkowski's inequality on the left side. This gives a sum of terms of the form

$$(2.32) \qquad C\left\| D^{\lambda-[\lambda]}\left[\left(x\psi(2^{-j}x)\right)^{(n)} m^{([\lambda]-n)}(x)\right]\right\|_s$$

for $0 \leqslant n \leqslant [\lambda] - 1$. We will use Lemma (2.18) with $\alpha = \lambda - [\lambda]$, I equal to $[2^{j-1}, 2^{j+1}]$ and $[-2^{j+1}, -2^{j-1}]$, $\phi(x) = (x\psi(2^{-j}x))^{(n)}$ and $f(x) = D^{[\lambda]-n}[\sum_{l=j-1}^{j+1} m_l(x)]$. It is easy to see that

$$\|\phi(x)\|_\infty + |I|\,\|\phi'(x)\|_\infty \leqslant C2^{j(1-n)}.$$

Furthermore,

$$\|D^{\lambda-[\lambda]}f\|_s + |I|^{[\lambda]-\lambda}\|f\|_s \leqslant C2^{j(n-\lambda+1/s)}\big(B(m,s,\lambda-n) + B(m,s,[\lambda]-n)\big)$$

by the definition of the function B. Now apply Lemma (2.18), these two estimates and Theorem (2.12) to show that (2.32) is bounded by the right side of (2.31). This completes the proof of Theorem (2.30).

3. Kernel estimates for functions in $M(s,\lambda)$. Recall the notation $k_j(x) = [\psi(2^{-j}x)m(x)]^\vee$ and $K_N(x) = [\sum_{j=-N}^{N} \psi(2^{-j}x)m(x)]^\vee$, where ψ is the function used in the definition of $M(s,\lambda)$. This section contains norm estimates for K_N that do not depend on N. The principal results are theorems (3.2) and (3.4); these are the basic facts used to prove the multiplier theorems in later sections.

LEMMA (3.1). *If* $1 \leqslant s \leqslant \infty$, $1 \leqslant p < \infty$, $t = \min(2, p', s)$, $l \geqslant 0$, $\lambda \geqslant 0$, $m(x)$ *is in* $M(s,\lambda)$ *and* $r > 0$, *then*

$$\int_{r<|x|<2r} \left|k_j^{(l)}(x)\right|^p dx \leqslant CB(m,s,\lambda)^p (2^j r)^{p(l-\lambda+1/t)} r^{1-p(l+1)},$$

where C is independent of m, r and j.

To prove this, start with the fact that the left side is bounded by

$$Cr^{-\lambda p}\int_{r<|x|<2r} \left|x^\lambda k_j^{(l)}(x)\right|^p dx.$$

Now $t \leqslant p'$ implies that $p \leqslant t'$, and Hölder's inequality gives the bound

$$Cr^{1-\lambda p-p/t'}\left[\int_{r<|x|<2r} \left|x^\lambda k_j^{(l)}(x)\right|^{t'} dx\right]^{p/t'}.$$

Since $t' \geqslant 2$, the Hausdorff-Young inequality implies that this is bounded by $Cr^{1-\lambda p-p/t'}\|D^\lambda(x^l m_j(x))\|_t^p$. Using the fact that $D^\lambda = D^{\lambda-[\lambda]}D^{[\lambda]}$ shows that it is sufficient to estimate terms of the form

$$Cr^{1-\lambda p-p/t'}\left\|D^{\lambda-[\lambda]}\left(x^{l-n}m_j^{([\lambda]-n)}(x)\right)\right\|_t^p,$$

where $0 \leqslant n \leqslant [\lambda]$. To estimate these, use Lemma (2.18) with

$$\phi(x) = x^{l-n}\left[\psi(2^{1-j}x) + \psi(2^{-j}x) + \psi(2^{-1-j}x)\right]$$

and $f(x) = m_j^{([\lambda]-n)}(x)$; this gives

$$Cr^{1-\lambda p-p/t'}(2^{j(l-n)p})\left(\left\|D^{\lambda-n}m_j\right\|_t^p + 2^{-jp(\lambda-[\lambda])}\left\|D^{[\lambda]-n}m_j\right\|_t^p\right).$$

Then by Theorem (2.12), $B(m,t,\lambda-n)$ and $B(m,t,[\lambda]-n)$ are both bounded by $CB(m,s,\lambda)$ since $t \leqslant s$. These inequalities and the definition of B complete the proof of Lemma (3.1).

THEOREM (3.2). *If* $1 \leqslant s \leqslant \infty$, $1 \leqslant p < \infty$, $t = \min(2, p', s)$, $0 \leqslant l < \lambda - 1/t$, $m(x)$ *is in* $M(s, \lambda)$ *and* $r > 0$, *then*

$$\int_{r<|x|<2r} |K_N^{(l)}(x)|^p \, dx \leqslant CB(m, s, \lambda)^p r^{1-p(l+1)},$$

where C *is independent of* r, m *and* N.

To prove this, we will show that

$$\int_{r<|x|<2r} |k_j^{(l)}(x)|^p \, dx \leqslant CB^p(2^j r)^{p(l+1)} r^{1-p(l+1)}, \tag{3.3}$$

where $B = B(m, s, \lambda)$. This is sufficient since

$$\left[\int_{r<|x|<2r} |K_N^{(l)}(x)|^p \, dx\right]^{1/p} \leqslant \sum_{j=-N}^{N} \left[\int_{r<|x|<2r} |k_j^{(l)}(x)|^p \, dx\right]^{1/p},$$

and we can estimate each term of the sum using (3.3) if $2^j r \leqslant 1$ and the conclusion of Lemma (3.1) if $2^j r > 1$.

To prove (3.3), observe that since

$$|m(x)| \leqslant B \quad \text{and} \quad |k_j^{(l)}(x)| = C|(x^l m(x)\psi(2^{-j}x))^\vee|,$$

we have

$$|k_j^{(l)}(x)| \leqslant C\|x^l m(x)\psi(2^{-j}x)\|_1 \leqslant C2^{j(l+1)}B.$$

Using this inequality in the left side of (3.3) proves (3.3) and completes the proof of Theorem (3.2).

THEOREM (3.4). *If* $1 \leqslant s \leqslant \infty$, $1 \leqslant p < \infty$, $t = \min(2, p', s)$, $0 \leqslant L < \lambda - 1/t < L + 1$, m *is in* $M(s, \lambda)$, $r > 0$ *and* $|y| < r/2$, *then*

$$\int_{r<|x|<2r} \left| K_N(x-y) - \sum_{n=0}^{L} \frac{(-y)^n}{n!} K_N^{(n)}(x) \right|^p dx \leqslant CB(m, s, \lambda)^p \left(\frac{|y|}{r}\right)^{p\lambda - p/t} r^{1-p},$$

where C *is independent of* y, r, m *and* N.

We will show that the hypotheses imply

$$\int_{r<|x|<2r} \left| k_j(x-y) - \sum_{n=0}^{L} \frac{(-y)^n}{n!} k_j^{(n)}(x) \right|^p dx \leqslant CB^p(2^j|y|)^{p(L-\lambda+1/t)} \left(\frac{|y|}{r}\right)^{p(\lambda-1/t)} r^{1-p} \tag{3.5}$$

if $2^j|y| \geqslant 1$ and

$$\int_{r<|x|<2r} \left| k_j(x-y) - \sum_{n=0}^{L} \frac{(-y)^n}{n!} k_j^{(n)}(x) \right|^p dx \leqslant CB^p(2^j|y|)^{p(L+1-\lambda+1/t)} \left(\frac{|y|}{r}\right)^{p(\lambda-1/t)} r^{1-p} \tag{3.6}$$

if $2^j|y| < 1$, where $B = B(m, s, \lambda)$ and C is independent of y, r, j, and m. As in the proof of Theorem (3.2), this is sufficient by using Minkowski's inequality and either (3.5) or (3.6) according to the value of j.

To prove (3.5) for $2^j|y| \geqslant 1$, we will estimate

$$\int_{r<|x|<2r} |k_j(x-y)|^p\,dx \tag{3.7}$$

and

$$\int_{r<|x|<2r} |y^n k_j^{(n)}(x)|^p\,dx \tag{3.8}$$

for $0 \leqslant n \leqslant L$. For (3.7) use the fact that $|y| < r/2$ to get the bound

$$\int_{r/2<|u|<3r} |k_j(u)|^p\,du. \tag{3.9}$$

We will now use Lemma (3.1) to bound all these terms. For (3.8) we get

$$CB^p(2^j|y|)^{p(1/t+n-\lambda)}(|y|/r)^{p(\lambda-1/t)}r^{1-p};$$

for (3.9) we get the same with $n = 0$. Since $n \leqslant L$ and $2^j|y| \geqslant 1$, these are all bounded by the right side of (3.5).

To prove (3.6) for $2^j|y| \leqslant 1$, start with the fact that the left side is bounded by

$$Cr^{-\lambda p}\int_{r<|x|<2r}\left|x^\lambda\left(k_j(x-y) - \sum_{n=0}^{L}\frac{(-y)^n}{n!}k_j^{(n)}(x)\right)\right|^p dx.$$

Since $p \leqslant t'$, we can use Hölder's inequality to get the bound

$$Cr^{-\lambda p+1-p/t'}\left(\int_{r<|x|<2r}\left|x^\lambda\left(k_j(x-y) - \sum_{n=0}^{L}\frac{(-y)^n}{n!}k_j^{(n)}(x)\right)\right|^{t'} dx\right)^{p/t'}.$$

Since $t' \geqslant 2$, the Hausdorff-Young theorem gives the estimate

$$Cr^{-\lambda p+1-p/t'}\left\|D^\lambda\left[\left(e^{-ixy} - \sum_{n=0}^{L}\frac{(ixy)^n}{n!}\right)m_j(x)\right]\right\|_t^p.$$

Since $D^\lambda = D^{\lambda-[\lambda]}D^{[\lambda]}$, it is sufficient to estimate

$$Cr^{-\lambda p+1-p/t'}\left\|D^{\lambda-[\lambda]}\left[\left(e^{-ixy} - \sum_{n=0}^{L}\frac{(ixy)^n}{n!}\right)^{([\lambda]-l)} m_j^{(l)}(x)\right]\right\|_t^p \tag{3.10}$$

for $0 \leqslant l \leqslant [\lambda]$. This will be done using Lemma (2.18) with $\alpha = \lambda - [\lambda]$, $I = [2^{j-1}, 2^{j+1}]$ or $[-2^{j+1}, -2^{j-1}]$,

$$\phi(x) = \left(e^{-ixy} - \sum_{n=0}^{L}\frac{(ixy)^n}{n!}\right)^{([\lambda]-l)} \chi_I(x)$$

and $f(x) = m_j^{(l)}(x)$. It is easy to verify by Taylor's theorem that

$$\|\phi\|_\infty + |I|\,\|\phi'\|_\infty \leqslant C|y|^{L+1}(2^j)^{L+1+l-[\lambda]}.$$

From the definition of $B(m, s, \lambda)$,

$$\|D^{\lambda-[\lambda]}f\|_t + |I|^{[\lambda]-\lambda}\|f\|_t \leqslant C2^{j([\lambda]-\lambda-l+1/t)}\big(B(m,t,\lambda-[\lambda]+l) + B(m,t,l)\big).$$

Since $t \leqslant s$ and $0 \leqslant l \leqslant [\lambda]$, Theorem (2.12) shows that

$$B(m,t,\lambda-[\lambda]+l) + B(m,t,l) \leqslant CB(m,s,\lambda).$$

Now with these inequalities and Lemma (2.18), we get the bound

$$Cr^{-\lambda p+1-p/t'}\big[B(m,s,\lambda)|y|^{L+1}2^{j(L+1-\lambda+1/t)}\big]^p$$

for (3.10). This equals the right side of (3.6), and the proof of Theorem (3.4) is complete.

THEOREM (3.11). *If $m(x)$ is in $M(s,\lambda)$, $1 \leqslant s \leqslant \infty$, $\lambda > \frac{1}{2}$, $\lambda > 1/s$ and $1 < p < \infty$, then there is a C, independent of m, N, y, f and r, such that*

$$\int_{|x|>2|y|} |K_N(x-y) - K_N(x)|\,dx \leqslant CB(m,s,\lambda), \tag{3.12}$$

$$\|f * K_N\|_p \leqslant CB(m,s,\lambda)\|f\|_p \tag{3.13}$$

and, for $r > 0$,

$$\big|\{x : |(f * K_N)(x)| > r\}\big| \leqslant \frac{CB(m,s,\lambda)}{r}\|f\|_1. \tag{3.14}$$

By Theorem (2.12), we may assume that $1/t < \lambda < 1 + 1/t$ where $t = \min(2,s)$. By Theorem (3.4) with $p = 1$, $L = 0$, $n \geqslant 1$ and $r = 2^n|y|$,

$$\int_{2^n|y|<|x|<2^{n+1}|y|} |K_N(x-y) - K_N(x)|\,dx \leqslant CB(m,s,\lambda)2^{n(-\lambda+1/t)}.$$

Adding these for $n \geqslant 1$ proves (3.12). For $1 < p < \infty$, (3.13) follows from (3.12) and the boundedness of m; see, e.g., the corollary on p. 34 of **[15]**. The weak type inequality (3.14) is obtained in the proof of the corollary in **[15]**.

4. Proof of Theorem (1.2) for $\lambda > \frac{1}{2}$. To do this, we first derive Theorem (4.2) for multipliers defined as $(m\hat{f})^\vee$ for all f in the set $\mathscr{S}$ of Schwartz functions. As mentioned in §1, defining a multiplier in this way, allowing m to be an arbitrary member of a class $M(s,\lambda)$ and requiring that the operator be bounded on L_α^p implies that $-1 < \alpha < p-1$. For some pairs (s,λ), α must be in a proper subset of $(-1, p-1)$.

It should be noted that for each set of values λ, p, s, Theorem (4.2) asserts the boundedness of the multipliers in $M(s,\lambda)$ for weight $|x|^\alpha$ if α is in a certain open interval I, while Theorem (1.4) and the fact that α must be in $(-1, p-1)$ show that the conclusion of Theorem (4.2) is false for α not in the closure of I. Therefore, Theorem (4.2) cannot be greatly improved. Theorem (1.4) does show that α cannot equal -1 or $p-1$; in most other cases we state no result for α an endpoint of I. However, since $\mathscr{S}_{0,0}$ is dense in $\mathscr{S}$ in L_α^p metric for $-1 < \alpha < p-1$ and $1 < p < \infty$

by Theorem 6.1 of **[11]**, the results of §6 do provide some examples in which the multipliers in $M(s,\lambda)$ are bounded on $\mathscr{S}$ for α equal to an endpoint of I.

Theorem (4.2) is proved using a sequence of three lemmas. The first, Lemma (4.3), is a statement of what is commonly known as the "three parts proof". It is stated in more generality than needed here since it will also be used in **[12]** and is no harder to prove in this general form. The next two lemmas, Lemma (4.9) and Lemma (4.10), are proved using Lemma (4.3) and the results in §3. They are like Theorem (4.2) but with overly restrictive hypotheses. Theorem (4.2) is then proved from Lemmas (4.9) and (4.10) using an interpolation argument. Finally, Theorem (1.2) for $\lambda > \frac{1}{2}$ is proved using Theorem (4.2) and Lemma (4.3) except for the case $\lambda = s = 1$ for which we quote a known result.

The following facts based on §§2 and 3 and known results are intended to put Theorem (4.2) in perspective. If T_m is initially defined for functions in $\mathscr{S}$ as $(m\hat{f})^\vee$, $1 \leqslant s \leqslant \infty$, $\lambda > \max(\frac{1}{2}, \frac{1}{s})$ and m is in $M(s,\lambda)$, then by Theorem (3.11), T_m is a bounded operator on (unweighted) L^p, $1 < p < \infty$. If $1 \leqslant s \leqslant \infty$, $\lambda \geqslant 1$ and m is in $M(s,\lambda)$, then T_m is a bounded operator on (unweighted) L^p, $1 < p < \infty$, by the Marcinkiewicz multiplier theorem **[15**, p. 108] and Theorem (2.12). More generally, we have the following.

THEOREM (4.1). *If $1 < p < \infty$, $-1 < \alpha < p-1$, $s \geqslant 1$, $\lambda \geqslant 1$, $m \in M(s,\lambda)$ and $f \in \mathscr{S}$, then $\|(m\hat{f})^\vee\|_{p,\alpha} < CB(m,s,\lambda)\|f\|_{p,\alpha}$, where C is independent of m and f.*

Theorem (4.1) was proved for $\lambda = s = 1$ by Hirschman, Theorem 6.1, p. 60 of **[6]** in the periodic case and by Kurtz, Theorem 2, p. 237 of **[8]** on the line with more general weight functions. The fact that the constant in these results can be written as $CB(m,1,1)$ is clear from the proofs. The form stated here is an immediate consequence of the case $s = \lambda = 1$ and Theorem (2.12). Except for the case $\lambda = s = 1$, Theorem (4.1) is also an immediate consequence of Theorem (4.2).

Since $\alpha \in (-1, p-1)$ is a necessary condition for the conclusion of Theorem (1.1), we cannot obtain more weight functions by placing more requirements on s and λ. What is needed here is the following result for $\lambda < 1$.

THEOREM (4.2). *If $1 < s \leqslant \infty$, $\max(\frac{1}{s}, \frac{1}{2}) < \lambda < 1$, $m \in M(s,\lambda)$, $1 < p < \infty$, $\max(-1, -p\lambda) < \alpha < \min(p-1, p\lambda)$ and f is in $\mathscr{S}$, then*

$$\left\|(m\hat{f})^\vee\right\|_{p,\alpha} \leqslant CB(m,s,\lambda)\|f\|_{p,\alpha},$$

where C is independent of m and f.

The lemmas to be used to prove this are the following. The first is a statement of what is commonly known as the "three parts proof."

LEMMA (4.3). *If $Tf(x) = \int_{-\infty}^{\infty} K(x,y)f(y)\,dy$, a and b are real, $r > 0$, $U(x)$ and $W(x)$ are nonnegative and there is an A independent of h and r such that*

$$(4.4)\quad \int_{r \leqslant |x-b| < 2r} |Th(x)|^p |x-b|^a U(x)\,dx \leqslant A\int_{-\infty}^{\infty} |h(x)|^p |x-b|^a W(x)\,dx$$

for all h in C^∞ with support in $r/8 \leqslant |x-b| \leqslant 16r$, then for f in C^∞, $\|Tf\|_{p,U}^p$ is bounded by the sum of

$$(4.5)\quad C\int_0^\infty \left(\int_{|y-b|<r/4}\left[\int_{r/2<|x-b|<2r}|K(x,y)|^p U(x)\,dx\right]^{1/p}|f(y)|dy\right)^p \frac{dr}{r},$$

$$(4.6)\quad CA\int_{-\infty}^{\infty}|f(x)|^p W(x)\,dx$$

and

$$(4.7)\quad C\int_0^\infty \left(\int_{|y-b|>4r}\left[\int_{r/2<|x-b|<2r}|K(x,y)|^p U(x)\,dx\right]^{1/p}|f(y)|dy\right)^p \frac{dr}{r},$$

where C is independent of f, K and W.

This is proved by starting with the fact that

$$\|Tf\|_{p,U}^p = \sum_{n=-\infty}^{\infty}\int_{2^n<|x-b|<2^{n+1}}\left|\int_{-\infty}^{\infty}K(x,y)f(y)\,dy\right|^p U(x)\,dx.$$

Now write $f(y) = \sum_{j=1}^3 f_j(y)$ where $f_j(y)$ is in C^∞, $|f_j(y)| \leqslant |f(y)|$ for $1 \leqslant j \leqslant 3$, $f_1(y) = 0$ for $|y-b| > 2^{n-2}$, $f_2(y) = 0$ for $|y-b| < 2^{n-3}$ and $|y-b| > 2^{n+4}$ and $f_3(y) = 0$ for $|y-b| < 2^{n+3}$. Then $\|Tf\|_{p,U}^p$ is bounded by 3^p times the sum of

$$(4.8)\quad \sum_{n=-\infty}^{\infty}\int_{2^n<|x-b|<2^{n+1}}\left|\int_{-\infty}^{\infty}K(x,y)f_j(y)\,dy\right|^p U(x)\,dx$$

for $1 \leqslant j \leqslant 3$.

To estimate (4.8) with $j = 1$, use Minkowski's integral inequality to get the bound

$$\sum_{n=-\infty}^{\infty}\left(\int_{|y-b|<2^{n-2}}\left[\int_{2^n<|x-b|<2^{n+1}}|K(x,y)|^p U(x)\,dx\right]^{1/p}|f_1(y)|\,dy\right)^p.$$

This is bounded by

$$2\sum_{n=-\infty}^{\infty}\int_{2^n}^{2^{n+1}}\left(\int_{|y-b|<r/4}\left[\int_{r/2<|x-b|<2r}|K(x,y)|^p U(x)\,dx\right]^{1/p}|f(y)|\,dy\right)^p\frac{dr}{r},$$

which is bounded by (4.5).

For (4.8) with $j = 2$, we have the bound

$$\sum_{n=-\infty}^{\infty} C2^{-na}\int_{2^n\leqslant|x-b|\leqslant 2^{n+1}}\left|\int_{2^{n-3}<|y-b|<2^{n+4}}K(x,y)f_2(y)\,dy\right|^p|x-b|^a U(x)\,dx.$$

By (4.4), this is bounded by

$$\sum_{n=-\infty}^{\infty} CA2^{-na}\int_{2^{n-3}<|x-b|<2^{n+4}}|f_2(x)|^p|x-b|^a W(x)\,dx,$$

and this is bounded by (4.6).

The fact that (4.7) dominates (4.8) with $j = 3$ is proved in the same way that (4.8) with $j = 1$ was estimated. This completes the proof of Lemma (4.3).

LEMMA (4.9). *If* $1 < s \leqslant \infty$, $\max(\frac{1}{s}, \frac{1}{2}) < \lambda < 1$, $m \in M(s, \lambda)$, $1 < p < 1/(1-\lambda)$, $p - p\lambda - 1 < \alpha < p - 1$ *and* f *is integrable, then* $\|K_N * f\|_{p,\alpha} \leqslant CB(m, s, \lambda)\|f\|_{p,\alpha}$, *where* C *is independent of* f, m *and* N.

To prove this, we will apply Lemma (4.3) with $K(x, y) = K_N(x - y)$, $a = -\alpha$, $b = 0$ and $U(x) = W(x) = |x|^\alpha$. To complete the proof, we will show that (4.5) and (4.7) have the bound $CB(m, s, \lambda)^p\|f\|_{p,\alpha}^p$; that (4.6) has this bound is immediate.

To estimate (4.5), replace $|x|^\alpha$ by cr^α and enlarge the integration set in the inner integral to get the estimate

$$C\int_0^\infty \left(\int_{|y|<r/4} \left[\int_{r/4<|x-y|<4r} |K_N(x-y)|^p r^\alpha \, dx \right]^{1/p} |f(y)| \, dy \right)^p \frac{dr}{r}.$$

By Theorem (3.2), this is bounded by

$$CB(m, s, \lambda)^p \int_0^\infty \left[\int_{|y|<r/4} |f(y)| \, dy \right]^p r^{\alpha - p} \, dr.$$

Now since $\alpha < p - 1$, we have $\alpha - p < -1$ and Hardy's inequality, Lemma 3.14, p. 196 of [**16**], shows this is bounded by $CB(m, s, \lambda)^p\|f\|_{p,\alpha}^p$ as desired.

To estimate (4.7), replace $|x|^\alpha$ by Cr^α to get the estimate

$$C\int_0^\infty \left(\int_{|y|>4r} \left[\frac{1}{r} \int_{r/2<|x|<2r} |K_N(x-y)|^p \, dx \right]^{1/p} |f(y)| \, dy \right)^p r^\alpha dr.$$

To estimate this, note that the hypothesis $\alpha > p - p\lambda - 1$ implies that $p/(\alpha + 1) < 1/(1 - \lambda)$. We can, therefore, choose a q satisfying $\max(p, p/(\alpha + 1)) < q < 1/(1 - \lambda)$ and use Hölder's inequality to obtain the bound

$$C\int_0^\infty \left(\int_{|y|>4r} \left[\frac{1}{r} \int_{r/2<|x|<2r} |K_N(x-y)|^q \, dx \right]^{1/q} |f(y)| \, dy \right)^p r^\alpha \, dr.$$

Now enlarge the integration set of the inner integral to $|y|/2 < |x - y| < 2|y|$ and use Theorem (3.2). This gives the bound

$$CB(m, s, \lambda)^p \int_0^\infty \left[\int_{|y|>4r} |y|^{-1+1/q} |f(y)| \, dy \right]^p r^{\alpha - p/q} \, dr.$$

Now since $p/(\alpha + 1) < q$, we have $\alpha - p/q > -1$. Hardy's inequality, Lemma 3.14, p. 196 of [**16**], then shows that this part also has the asserted bound. This completes the proof of Lemma (4.9).

LEMMA (4.10). *If* $1 < s \leqslant \infty$, $\max(\frac{1}{s}, \frac{1}{2}) < \lambda < 1$, $m \in M(s, \lambda)$, $1/\lambda < p < \infty$, $-1 < \alpha < p\lambda - 1$ *and* f *is integrable, then* $\|K_N * f\|_{p,\alpha} \leqslant CB(m, s, \lambda)\|f\|_{p,\alpha}$, *where* C *is independent of* f, m *and* N.

To prove this, observe that the dual of L_α^p is $L_{-\alpha/(p-1)}^{p'}$. A standard duality argument then derives Lemma (4.10) from Lemma (4.9).

To prove Theorem (4.2), fix p and α satisfying the hypotheses. It is possible to choose p_0 such that $1 < p_0 < \min(p, 1/(1 - \lambda))$ and $p_0 - p_0\lambda - 1 < \alpha p_0/p < p_0 - 1$ because each of these inequalities is equivalent to requiring an upper bound or a lower bound on p_0, and it is easy to show that the lower bounds are strictly less than the upper bounds. By Lemma (4.9),

$$(4.11)\quad \int_{-\infty}^{\infty} \left| |x|^{\alpha/p}(K_N * f)(x) \right|^{p_i} dx \leqslant CB(m,s,\lambda) \int_{-\infty}^{\infty} \left| |x|^{\alpha/p} f(x) \right|^{p_i} dx$$

for $i = 0$. Similarly, there is also a p_1 satisfying $\max(1/\lambda, p) < p_1 < \infty$ and $-1 < \alpha p_1/p < p_1\lambda - 1$; by Lemma (4.10) this gives (4.11) for $i = 1$. Interpolation of operators, Theorem (1.3), p. 179 of **[16]**, then shows that $\|K_N * f\|_{p,\alpha} \leqslant CB(m,s,\lambda)\|f\|_{p,\alpha}$. Since $(m\hat{f})^{\vee} = \lim_{N\to\infty} K_N * f$ almost everywhere, Fatou's lemma completes the proof of Theorem (4.2).

To prove Theorem (1.2) for $\lambda > \frac{1}{2}$, observe for $\lambda > \frac{1}{2}$, $s > 1$ and $\alpha < p - 1$ that Theorem (1.2) is a corollary of Theorem (4.2). For $\lambda > \frac{1}{2}$, $s = 1$ and $\alpha < p - 1$ Theorem (1.2) is a consequence of Theorem (4.1). Therefore, to complete the proof of Theorem (1.2) for $\lambda > \frac{1}{2}$, we need only consider the case $\alpha > p - 1$. Now fix p, s, $\lambda > \frac{1}{2}$ and $\alpha > p - 1$ that satisfy the hypotheses of Theorem (1.2), let $t = \min(2, p', s)$ and let l be the positive integer for which $lp - 1 < \alpha < (l+1)p - 1$. Then since $\alpha < -1 + p(1 + \lambda - 1/t)$, we have $l - 1 + 1/t < \lambda$. Since $\alpha < -1 + p(1 + (l + 1/t) - 1/t)$, there is a λ_1 satisfying $l - 1 + 1/t < \lambda_1 < l + 1/t$ and $\alpha < -1 + p(1 + \lambda_1 - 1/t)$. It $\lambda \geqslant l + 1/t$, then $\lambda_1 < \lambda$, m is in $M(s, \lambda_1)$ and $B(m, s, \lambda_1) \leqslant CB(m, s, \lambda)$ by Theorem (2.12). Therefore, proving the case $\alpha > p - 1$ for $l - 1 + 1/t < \lambda < l + 1/t$ is sufficient to establish it in general.

Now apply Lemma (4.3) with $a = -\alpha$, $b = 0$, $U(x) = W(x) = |x|^\alpha$ and

$$K(x,y) = K_N(x-y) - \sum_{n=0}^{l-1} \frac{(-y)^n}{n!} K_N^{(n)}(x).$$

Since f is in $\mathscr{S}_{0,0}$, $\int K(x,y)f(y)\,dy = \int K_N(x-y)f(y)\,dy$, and inequality (4.4) holds with $A = C[B(m,s,\lambda)]^p$ by Theorem (3.11). To complete the proof, we will show that (4.5) and (4.7) have the bound $CB(m,s,\lambda)^p\|f\|_{p,\alpha}^p$; that (4.6) has this bound is immediate.

By Theorem (3.4), (4.5) has the bound

$$\int_0^{\infty} \left(\int_{|y|<r/4} r^{\alpha/p} \left[CB^p \left(\frac{|y|}{r} \right)^{p\lambda - p/t} r^{1-p} \right]^{1/p} |f(y)|\,dy \right)^p \frac{dr}{r},$$

where $B = B(m,s,\lambda)$. This is bounded by

$$CB^p \int_0^{\infty} \left[\int_{|y|<r} |y|^{\lambda - 1/t} |f(y)|\,dy \right]^p r^{\alpha - p - p\lambda + p/t}\,dr.$$

Now since $\alpha < -1 + p(1 + \lambda - 1/t)$, we have $\alpha - p - p\lambda + p/t < -1$. Therefore, Hardy's inequality, Lemma 3.14, p. 196 of **[16]**, shows this is bounded by $CB^p\|f\|_{p,\alpha}^p$ as desired.

For (4.7), we use the fact that since $|y| > 4r$, the inner integral is bounded by a constant times the sum of

$$\int_{|y|/2<|t|<2|y|} |K_N(t)|^p r^\alpha\,dt$$

and

$$\int_{r/2<|x|<2r} |y|^{np}\left|K_N^{(n)}(x)\right|^p r^\alpha\,dx$$

for $0 \leqslant n \leqslant l-1$. Theorem (3.2) gives the respective bounds $CB^p r^{1-p+\alpha}(|y|/r)^{1-p}$ and $CB^p r^{1-p+\alpha}(|y|/r)^{np}$. Since $|y|/r \geqslant 1$, these are all bounded by $CB^p r^{1-p+\alpha}(|y|/r)^{p(l-1)}$. Using this estimate in (4.7) gives the bound

$$CB^p \int_0^\infty \left(\int_{|y|>4r} |y|^{l-1}|f(y)|\,dy\right)^p r^{\alpha-lp}\,dr.$$

Since $\alpha - lp > -1$, Hardy's inequality gives the desired estimate; this completes the proof that $\|K_N * f\|_{p,\alpha} \leqslant CB\|f\|_{p,\alpha}$ for m, p, α and f satisfying the conditions of the theorem. The fact that $\|(m\hat{f})^\vee\|_{p,\alpha} \leqslant CB\|f\|_{p,\alpha}$ follows from Fatou's lemma.

5. The case $\lambda \leqslant \frac{1}{2}$. This section contains the proof of Theorem (5.1), which is Theorem (1.2) for $\lambda \leqslant \frac{1}{2}$. This proof is based on a similar one in §4 of [**1**].

THEOREM (5.1). *If* $1 < p < \infty$, $2 < s \leqslant \infty$, $\max(\frac{1}{s}, |\frac{1}{p} - \frac{1}{2}) < \lambda \leqslant \frac{1}{2}$, $m \in M(s,\lambda)$, $\max(-p\lambda, -1 + p(\frac{1}{2} - \lambda)) < \alpha < \min(-1 + p(\lambda + \frac{1}{2}), p\lambda)$ *and f is in* $\mathscr{S}$, *then*

$$\left\|(m\hat{f})^\vee\right\|_{p,\alpha} \leqslant CB(m,s,\lambda)\|f\|_{p,\alpha}$$

with C independent of m and f.

We will use the following notation. Define $\theta(x) = \psi(x/2) + \psi(x) + \psi(2x)$, ψ as in §2, and define the operator D_l^λ by

$$D_l^\lambda f(x) = \left[\left[1 + 2^{2l}|x|^2\right]^{\lambda/2}\hat{f}(x)\right]^\vee.$$

Given a bounded function $m(x)$, complex z and λ, positive ε and σ, and an integer N, define

$$m(z,x) = \sum_{l=-N}^{N} \theta(2^{-l}x) D_l^{(z-1)/2-\varepsilon}\left[\left|D_l^\lambda m_l\right|^{\sigma(1-z)/2} \operatorname{sgn} D_l^\lambda m_l\right], \tag{5.2}$$

where $\operatorname{sgn} z = z/|z|$ for $z \neq 0$ and $\operatorname{sgn} 0 = 0$.

LEMMA (5.3). *If* $\sigma > 2$, $\frac{1}{\sigma} < \lambda \leqslant \frac{1}{2}$, $\varepsilon = \lambda - \frac{1}{\sigma}$, *$m$ is in $M(\sigma,\lambda)$ and v is real, then*

$$m\left(1 - \frac{2}{\sigma}, x\right) = \sum_{l=-N}^{N} m_l(x), \tag{5.4}$$

$$\|m(1+iv,x)\|_\infty \leqslant C(1+v^2) \tag{5.4}$$

and

$$B\left(m(iv,x), 2, \varepsilon + \tfrac{1}{2}\right) \leqslant C(1+v)^2[B(m,\sigma,\lambda)]^{\sigma/2}, \tag{5.6}$$

where C is independent of v and m.

Equality (5.4) is immediate since the hypothesis and Lemma (2.3) imply that $D_l^\lambda m_l$ is a function and $\theta(2^{-l}x) = 1$ on the support of m_l.

To prove (5.5), we use the fact that

$$D_l^{iv/2-\varepsilon}f(x) = \int_{-\infty}^{\infty} f(x-t)\left[\left[1+2^{2l}|t|^2\right]^{iv/4-\varepsilon/2}\right]^{\vee} dt \tag{5.7}$$

provided this integral exists. Now define for $0 < \alpha < \frac{1}{2}$ and β real

$$h(\alpha,\beta,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}[1+t^2]^{i\beta-\alpha}\,dt. \tag{5.8}$$

We will estimate $|h(\alpha,\beta,x)|$. To do this for $|x|<1$, divide the defining integral for h into integrals over $|t| < 1/|x|$ and $|t| > 1/|x|$. In the first, replace e^{ixt} by 1, and in the second, integrate by parts and replace e^{ixt} by 1 to get

$$|h(\alpha,\beta,x)| \leqslant C(1+|\beta|)|x|^{2\alpha-1}, \qquad |x|<1. \tag{5.9}$$

For $|x| > 1$, integrate by parts twice and replace e^{ixt} by 1 to get

$$|h(\alpha,\beta,x)| \leqslant C(1+\beta^2)x^{-2}, \qquad |x|>1. \tag{5.10}$$

Now with $f(x) = |D_l^{\lambda}m_l|^{-i\sigma v/2}\operatorname{sgn} D^{\lambda}m_l$,

$$\left|D_l^{iv/2-\varepsilon}f(x)\right| = \left|\int_{-\infty}^{\infty} f(x-t)2^{-l}h\left(\frac{\varepsilon}{2},\frac{v}{4},2^{-l}t\right)dt\right|.$$

Since $|f(x)| \leqslant 1$, this is bounded by $\|h(\varepsilon/2, v/4, t)\|_1 \leqslant C(1+v^2)$. Use this fact in the definition of $m(1+iv,x)$ and the fact that at most five terms in the sum defining $m(z,x)$ can be nonzero for any x to complete the proof of (5.5).

To prove (5.6), it is sufficient by Lemma (2.3) to show that

$$\|m(iv,x)\|_{\infty} \leqslant C(1+v^2)[D(m,\sigma,\lambda)]^{\sigma/2} \tag{5.11}$$

and

$$\left\|D_j^{\varepsilon+1/2}m(iv,x)_j\right\|_2 \leqslant C2^{j/2}[D(m,\sigma,\lambda)]^{\sigma/2}. \tag{5.12}$$

To prove (5.11), use (5.7) and (5.8) to show that

$$\|m(iv,x)\|_{\infty} \leqslant \left\|\sum_{l=-N}^{N}\theta(2^{-l}x)\int_{-\infty}^{\infty}2^{-l}\left|h\left(\frac{2\varepsilon+1}{4},\frac{v}{4},2^{-l}(x-t)\right)\right|\left|D_l^{\lambda}m_l(t)\right|^{\sigma/2}dt\right\|_{\infty}.$$

Apply Schwarz' inequality to show that the integral is bounded by

$$\left[\int_{-\infty}^{\infty}2^{-2l}\left|h\left(\frac{2\varepsilon+1}{4},\frac{v}{4},2^{-l}x\right)\right|^2dx\right]^{1/2}\left[\int_{-\infty}^{\infty}\left|D_l^{\lambda}m_l(t)\right|^{\sigma}dt\right]^{1/2}.$$

In the first integral, change the variable and use the estimates (5.9) and (5.10) to obtain the bound $C2^{-l/2}(1+v^2)$. For the second integral, use Theorem (2.3) to obtain the bound $[2^lB(m,\sigma,\lambda)^{\sigma}]^{1/2}$. Since for a given x, $\theta(2^{-l}x) > 0$ for at most five values of l, this completes the proof of (5.11).

To prove (5.12), use the fact that $\theta(2^{-l}x)\psi(2^{-j}x) \equiv 0$ for $|j-l| > 2$ to show that the left side is bounded by

(5.13)

$$\sum_{l=j-2}^{j+2}\left\|D_j^{\varepsilon+1/2}\left[D_l^{(iv-1)/2-\varepsilon}\left\{\left|D_l^{\lambda}m_l\right|^{\sigma(1-iv)/2}\operatorname{sgn} D_l^{\lambda}m_l\right\}\theta(2^{-l}x)\psi(2^{-j}x)\right]\right\|_2.$$

To estimate this we need the fact that for $\alpha > 0$

$$\| D_l^\alpha(fg)\|_2 \leqslant 2^{\alpha/2}\| D_l^\alpha f\|_2 \|(D_l^\alpha g)^\wedge\|_1. \tag{5.14}$$

To show (5.14), start with the fact that

$$1 + 2^{2l}x^2 \leqslant 2\left[1 + 2^{2l}(x-t)^2\right]\left[1 + 2^{2l}t^2\right]$$

to show that the quantity

$$\left|[D_l^\alpha(fg)]^\wedge\right| = [1 + 2^{2l}x^2]^{\alpha/2}\left|\int_{-\infty}^{\infty} \hat{f}(x-t)\hat{g}(t)\,dt\right|$$

is bounded by

$$2^{\alpha/2}\int_{-\infty}^{\infty}\left[1 + (x-t)^2 2^{2l}\right]^{\alpha/2}|\hat{f}(x-t)|(1 + t^2 2^{2l})^{\alpha/2}|\hat{g}(t)|\,dt.$$

Therefore,

$$\left|[D_l^\alpha(fg)]^\wedge\right| \leqslant 2^{\alpha/2}\left|[D_l^\alpha f]^\wedge\right| * \left|[D_l^\alpha g]^\wedge\right|$$

and (5.14) follows by using Plancherel's identity and Young's inequality.

Applying (5.14) to (5.13) shows that the terms in (5.13) are bounded by a constant times the product of

$$\left\| D_j^{\varepsilon+1/2}D_l^{-\varepsilon-1/2}D_l^{iv/2}\left[|D_l^\lambda m_l|^{\sigma(1-iv)/2}\operatorname{sgn} D_l^\lambda m_l\right]\right\|_2 \tag{5.15}$$

and

$$\left\|[1 + 2^{2j}|x|^2]^{(2\varepsilon+1)/4}[\theta(2^{-l}x)\psi(2^{-j}x)]^\wedge\right\|_1. \tag{5.16}$$

Since $(1 + 2^{2j}|x|^2)^{\varepsilon+1/2}(1 + 2^{2l}|x|^2)^{-\varepsilon-1/2} \leqslant C$ for $|j - l| \leqslant 2$, (5.15) is bounded for $|j - l| \leqslant 2$ by

$$C\left\||D_l^\lambda m_l|^{\sigma/2}\right\|_2 = C\left[\|D_l^\lambda m_l\|_\sigma\right]^{\sigma/2} \leqslant C2^{j/2}[D(m,\sigma,\lambda)]^{\sigma/2}.$$

A change of variables in (5.16) gives

$$\left\|[1 + |x|^2]^{(2\varepsilon+1)/4}[\theta(2^{j-l}x)\psi(x)]^\wedge\right\|_1,$$

which is finite, having one of five possible values. Therefore, (5.13) is bounded by $C2^{j/2}[D(m,\sigma,\lambda)]^{\sigma/2}$; this completes the proof of (5.6).

To complete the proof of Theorem (5.1), fix $m \in M(s,\lambda)$ with $B(m,s,\lambda) = 1$. Let σ satisfy $1/\lambda < \sigma \leqslant s$ and note that $\sigma > 2$. Additional requirements on σ of the form $\sigma - 1/\lambda < C$ will be given below. Let $\varepsilon = \lambda - 1/\sigma$, $r = 4p/(2p + 2\sigma - p\sigma)$ and $\beta = \alpha\sigma r/2p$, and define $m(z,x)$ by (5.2). By taking $\sigma - 1/\lambda$ sufficiently small we will have $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{\sigma}$, and it is easy to see that this implies $1 < r < \infty$ and $2p/\sigma r > 0$. From the fact that

$$\lim_{\sigma\to 1/\lambda}\frac{2p(r-1)}{\sigma r} = -1 + p\lambda + \frac{p}{2} > \alpha,$$

we see that by taking σ sufficiently close to $1/\lambda$ we have $\alpha < 2p(r-1)/\sigma r$, and, consequently, $\beta < r - 1$. Similarly, since

$$\lim_{\sigma\to 1/\lambda}\left(\frac{2p}{\sigma r}\right)\max\left[-1, -r\left(\varepsilon + \frac{1}{2}\right)\right] = \max\left[-p\lambda, -1 + p\left(\frac{1}{2} - \lambda\right)\right] < \alpha$$

and

$$\lim_{\sigma\to 1/\lambda}\left(\frac{2p}{\sigma r}\right)\left[-1+r\left(1+\varepsilon+\frac{1}{2}-\frac{1}{\min(2,r')}\right)\right]$$
$$=\min\left[p\lambda,-1+p\left(\lambda+\frac{1}{2}\right)\right]>\alpha,$$

we have

$$\max\left(-1,-r\left(\varepsilon+\frac{1}{2}\right)\right)<\beta<-1+r\left[1+\varepsilon+\frac{1}{2}-\frac{1}{\min(2,r')}\right]$$

if σ is close enough to $1/\lambda$. Now assume that σ is close enough to $1/\lambda$ that all these inequalities are satisfied. Theorem (1.2) with its λ, s, p, and α taken as $\varepsilon+\frac{1}{2}$, 2, r and β respectively and Theorem (1.5) then imply that for $f\in L^r_\beta\cap L^2$,

$$\int_{-\infty}^{\infty}\left|[m(iv,x)\hat{f}(x)]^{\vee}\right|^r|x|^\beta\,dx\leqslant C\left[B\left(m(iv,x),2,\varepsilon+\frac{1}{2}\right)\right]^r\int_{-\infty}^{\infty}|f(x)|^r|x|^\beta\,dx.$$

Now apply Theorem (2.12), the conclusion (5.6) of Lemma (5.3) and the fact that $B(m,s,\lambda)=1$ to show that for $f\in L^r\cap L^2_{-2\beta/r}$ we have

$$\int_{-\infty}^{\infty}|T(iv)f(x)|^r\,dx\leqslant C(1+v^2)^r\int_{-\infty}^{\infty}|f(x)|^r\,dx,$$

where

$$T(z)f(x)=|x|^{(1-z)\beta/r}\left[m(z,x)\left(f(x)|x|^{(z-1)\beta/r}\right)^{\wedge}\right]^{\vee}.$$

Conclusion (5.5) of Lemma (5.3) implies for $f\in L^2$ that

$$\int_{-\infty}^{\infty}|T(1+iv)f(x)|^2\,dx\leqslant C(1+v^2)^2\int_{-\infty}^{\infty}|f(x)|^2\,dx.$$

Complex interpolation, Theorem 4.1, p. 205 of **[16]**, then implies for f in $L^p\cap L^2_{-4\beta/\sigma r}$

$$\int_{-\infty}^{\infty}\left|T\left(1-\frac{2}{\sigma}\right)f(x)\right|^p dx\leqslant C\int_{-\infty}^{\infty}|f(x)|^p\,dx,$$

since $1/p=[1-(1-2/\sigma)]/r+\frac{1}{2}(1-2/\sigma)$. Conclusion (5.4) of Lemma (5.3), the fact that $2\beta/r\sigma=\alpha/p$ and letting $N\to\infty$ then completes the proof of Theorem (5.1) for m with $B(m,s,\lambda)=1$. The general result follows from the fact that $B(\gamma m,s,\lambda)=|\gamma|B(m,s,\lambda)$.

6. Endpoint results. If $1\leqslant s\leqslant\infty$, $\lambda>\max(|\frac{1}{p}-\frac{1}{2}|,\frac{1}{s})$ and $1<p<\infty$, Theorem (1.2) asserts that (1.3) holds for all m in $M(s,\lambda)$ and f in $\mathscr{S}_{0,0}$ if $\alpha<-1+p(\lambda+1-1/t)$, where $t=\min(2,p',s)$, provided $(\alpha+1)/p$ is not an integer and $\alpha>\max(-1,-p\lambda,-1+p(-\lambda+\frac{1}{2}))$. Theorem (1.4) asserts that (1.3) does not hold for all m in $M(s,\lambda)$ and f in $\mathscr{S}_{0,0}$ if $\alpha>-1+p(\lambda+1-1/t)$ or if $(\alpha+1)/p$ is an integer. Neither theorem makes an assertion about $\alpha=-1+p(\lambda+1-1/t)$ if $\lambda-1/t$ is not an integer. In this section, we show that in some cases (1.3) does hold for this value of α. The proof is interesting both because the result partially fills the gap between Theorems (1.2) and (1.4) and also because the technique is quite different from that of previous sections. The proof uses Pitt's theorem and the $M(s, \lambda)$ condition directly rather than using the results of §3. The results to be proved are the following.

THEOREM (6.1). *If $1 < s \leqslant p \leqslant 2$, $\lambda \geqslant 1$, $\lambda - [\lambda] = 1/s - 1/p$, m is in $M(s, \lambda)$ and $\alpha = -1 + p(\lambda + 1 - 1/s)$, then for all f in $\mathscr{S}_{0,0}$*

$$(6.2) \quad \int_{-\infty}^{\infty} \left|[m(x)\hat{f}(x)]^{\vee}\right|^p |x|^{\alpha}\,dx \leqslant C[B(m, s, \lambda)]^p \int_{-\infty}^{\infty} |f(x)|^p |x|^{\alpha}\,dx,$$

where C is independent of f and m.

THEOREM (6.3). *If $2 \leqslant p \leqslant s \leqslant \infty$, λ is a positive integer, m is in $M(s, \lambda)$ and $\alpha = p\lambda$, then for all f in $\mathscr{S}_{0,0}$ (6.2) holds with C independent of f and m.*

The proof of Theorems (6.1) and (6.3) is based on the following special case of Pitt's theorem. For a proof of the periodic case of Pitt's theorem, see [**14**, p. 489].

THEOREM (6.4). *If $1 < p < \infty$, $-1 < \beta \leqslant \min(p - 2, 0)$ and f is in $L^2 \cap L^p_{p-2-\beta}$, then*

$$\int_{-\infty}^{\infty} |\hat{f}(x)|^p |x|^{\beta}\,dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p |x|^{p-2-\beta}\,dx,$$

where C is independent of f.

Theorems (6.1) and (6.3) will be proved simultaneously; where differences occur, they will be distinguished by considering the cases $p \leqslant 2$ for Theorem (6.1) and $p \geqslant 2$ for Theorem (6.3). Since the left side of (6.2) is equal to

$$\int_{-\infty}^{\infty} \left|\left[\frac{d^k}{dx^k}[m(x)\hat{f}(x)]\right]^{\vee}\right|^p |x|^{\alpha - kp}\,dx,$$

where k denotes $[\lambda]$, it is sufficient to show that for $0 \leqslant j \leqslant k$ we have

$$(6.5) \quad \int_{-\infty}^{\infty} \left|\left[m^{(j)}(x)[x^{k-j}f(x)]^{\wedge}\right]^{\vee}\right|^p |x|^{\alpha-kp}\,dx \leqslant CB^p \int_{-\infty}^{\infty} |f(x)|^p |x|^{\alpha}\,dx,$$

where $B = B(m, s, \lambda)$.

We will first prove (6.5) for $j = 0$. To do this observe that if $p \leqslant 2$, then $\alpha - kp = p - 2$, and if $p \geqslant 2$, then $\alpha - kp = 0$. Therefore, since $k \geqslant 1$, we have $-p\lambda < -1 < \alpha - kp < \alpha = -1 + p(\lambda + 1 - 1/t)$. Therefore, we can use Theorem (1.2) with α replaced by $\alpha - kp$ to show that the left side of (6.5) with $j = 0$ is bounded by $CB^p \int_{-\infty}^{\infty} |x^k f(x)|^p |x|^{\alpha - kp}\,dx$ as asserted.

For $1 \leqslant j \leqslant k$, we use the fact that $\alpha - kp = \min(p - 2, 0)$ to apply Theorem (6.4) to the left side of (6.5). This gives the bound

$$(6.6) \quad \int_{-\infty}^{\infty} \left|m^{(j)}(x)[x^{k-j}f(x)]^{\wedge}\right|^p |x|^{kp+p-\alpha-2}\,dx.$$

To estimate (6.6), use the fact that f is in $\mathscr{S}_{0,0}$ to write it as

$$\int_{-\infty}^{\infty} \left|m^{(j)}(x) \int_{-\infty}^{\infty} \left[e^{-ixt} - \sum_{m=0}^{j-1} \frac{(-ixt)^m}{m!}\right] t^{k-j} f(t)\,dt\right|^p |x|^{kp+p-\alpha-2}\,dx.$$

This is bounded above by a constant times the sum of

$$(6.7)\qquad \int_{-\infty}^{\infty}\left[|m^{(j)}(x)|\int_{|t|\geqslant 1/|x|}|tx|^{j-1}|t|^{k-j}|f(t)|\,dt\right]^p|x|^{kp+p-\alpha-2}\,dx$$

and

$$(6.8)\qquad \int_{-\infty}^{\infty}\left[|m^{(j)}(x)|\int_{|t|\leqslant 1/|x|}|tx|^{j}|t|^{k-j}|f(t)|\,dt\right]^p|x|^{kp+p-\alpha-2}\,dx.$$

These will be estimated separately.

To find a bound for (6.7), make the change of variables $t = 1/u$ to get

$$(6.9)\qquad \int_{-\infty}^{\infty}\left[\int_{|u|\leqslant|x|}|u|^{-k-1}\left|f\left(\frac{1}{u}\right)\right|du\right]^p|x|^{jp+kp-\alpha-2}|m^{(j)}(x)|^p\,dx.$$

We will show that this is bounded by

$$(6.10)\qquad C[B(m,s,\lambda)]^p\int_{-\infty}^{\infty}\left|x^{-k-1}f\left(\frac{1}{x}\right)\right|^p|x|^{kp+p-\alpha-2}\,dx,$$

which equals the right side of (6.2). By Theorem 1 of [**9**], (6.9) is bounded by (6.10) provided that for $r > 0$

(6.11)

$$\left[\int_{|x|\geqslant r}|x|^{jp+kp-\alpha-2}|m^{(j)}(x)|^p\,dx\right]\left[\int_{|x|\leqslant r}|x|^{(\alpha+2-kp-p)/(p-1)}\,dx\right]^{p-1}\leqslant CB^p,$$

where $B = B(m,s,\lambda)$. Since $\alpha - kp = \min(p-2,0)$, we have

$$(\alpha + 2 - kp - p)/(p-1) > -1;$$

therefore, the left side of (6.11) is bounded by

$$(6.12)\qquad Cr^{\alpha+1-kp}\sum_{n=0}^{\infty}\int_{2^n r\leqslant|x|\leqslant 2^{n+1}r}[2^n r]^{jp+kp-\alpha-2}|m^{(j)}(x)|^p\,dx.$$

Since $j \leqslant k \leqslant \lambda$ and $j - 1/p \leqslant k - 1/p \leqslant \lambda - 1/s$, Theorem (2.12) shows that m is in $M(p,j)$ and $B(m,p,j) \leqslant CB(m,s,\lambda)$. Therefore, (6.12) is bounded by

$$Cr^{jp-1}\sum_{n=0}^{\infty}2^{n(jp+kp-\alpha-2)}[2^n r]^{1-jp}B^p.$$

Since $kp - \alpha - 1 < 0$, this is bounded by CB^p. This completes the proof that (6.7) is bounded by the right side of (6.2).

To estimate (6.8), make the change of variables $t = 1/u$ to get

$$(6.13)\qquad \int_{-\infty}^{\infty}\left[\int_{|u|\geqslant|x|}|u|^{-k-2}\left|f\left(\frac{1}{u}\right)\right|du\right]^p|x|^{jp+kp+p-\alpha-2}|m^{(j)}(x)|^p\,dx.$$

To show this is bounded by (6.10) we need, by Theorem 2 of [**9**], the inequality

$$\left[\int_{|x|\leqslant r}|x|^{jp+kp+p-\alpha-2}|m^{(j)}(x)|^p\,dx\right]\left[\int_{|x|\geqslant r}|x|^{(\alpha+2-kp-2p)/(p-1)}\,dx\right]^{p-1}\leqslant CB^p.$$

Since $\alpha - kp = \min(p-2, 0)$, we have $(\alpha + 2 - kp - 2p)/(p-1) < -1$, and the left side is bounded by

$$Cr^{\alpha+1-kp-p} \sum_{n=-\infty}^{0} \int_{2^{n-1}r \leqslant |x| \leqslant 2^n r} [2^n r]^{jp+kp+p-\alpha-2} |m^{(j)}(x)|^p \, dx.$$

As before, we use the fact that $B(m, p, j) \leqslant CB(m, s, \lambda)$ to show this is bounded by

$$Cr^{jp-1} \sum_{n=-\infty}^{0} 2^{n(jp+kp+p-\alpha-2)} [2^n r]^{1-pj} B^p.$$

Since $kp + p - \alpha - 1 > 0$, this is bounded by CB^p as desired. This completes the proof of Theorems (6.1) and (6.3).

7. Equivalence of various multiplier conditions. It was shown in Lemma (2.3) that if $1 < s < \infty$ and $\lambda > 0$, then the classes $M(s, \lambda)$ are equivalent to similar classes defined using Bessel potentials. In this section, we will relate $M(s, \lambda)$ to the classes $\mathrm{WBV}_{s,\lambda}$ of **[5]** and $S(s, \lambda)$ of **[4]**. Since functions in $M(s, \lambda)$ are defined on $(-\infty, \infty)$ while functions in $\mathrm{WBV}_{s,\lambda}$ and $S(s, \lambda)$ are defined on $(0, \infty)$, the classes are not identical. The appropriate comparison is to show that a function m is in $M(s, \lambda)$ if and only if $m(x)$ and $m(-x)$ restricted to $(0, \infty)$ are in $\mathrm{WBV}_{s,\lambda}$ and $S(s, \lambda)$. This will be done by showing that m is in $M(s, \lambda)$ if and only if these restrictions are in the space $\mathrm{RL}(s, \lambda)$ of **[3]** and using the facts, proved in **[3** and **5]**, that $\mathrm{RL}(s, \lambda)$ is equivalent to $\mathrm{WBV}_{s,\lambda}$ and $S(s, \lambda)$. We shall prove the following.

THEOREM (7.3). *If $1 < s < \infty$ and $\lambda > 1/s$, then m is in $M(s, \lambda)$ if and only if $m_1(x) = m(x)$ and $m_2(x) = m(-x)$ restricted to $(0, \infty)$ are in $WBV_{s,\lambda}$. Furthermore, there is a constant C, independent of m, such that*

$$B(m, s, \lambda) \leqslant C\left(\|m_1\|_{s,\lambda;W} + \|m_2\|_{s,\lambda;W}\right) \leqslant C^2 B(m, s, \lambda). \tag{7.4}$$

It should be noted that Theorem 3, p. 246 of **[5]** asserts that $\mathrm{WBV}_{s,\lambda}$ is the same as $S(s, \lambda)$ of **[4]** and the norms are equivalent for $1 < s < \infty$ and $\lambda > 1/s$. Therefore, Theorem (7.3) remains true if $\mathrm{WBV}_{s,\lambda}$ is replaced by $S(s, \lambda)$.

To prove Theorem (7.3), we will need two lemmas: Lemma (7.5) and Lemma (7.10).

LEMMA (7.5). *If $1 \leqslant s \leqslant \infty$ and $\lambda \geqslant 0$, then m is in $M(s, \lambda)$ if and only if $m\chi_{[0,\infty)}$ and $m\chi_{(-\infty,0]}$ are in $M(s, \lambda)$. Furthermore,*

$$B(m, s, \lambda) \leqslant \left[B(m\chi_{[0,\infty)}, s, \lambda) + B(m\chi_{(-\infty,0]}, s, \lambda)\right] \leqslant CB(m, s, \lambda) \tag{7.6}$$

with C independent of m.

To prove Lemma (7.5) observe first that if both $m\chi_{[0,\infty)}$ and $m\chi_{(-\infty,0]}$ are in $M(s,\lambda)$, then the fact that m is in $M(s, \lambda)$ and the first inequality in (7.6) follow immediately from the definition of $M(s, \lambda)$. For the converse, assume that m is in $M(s, \lambda)$. Then since

$$m(x)\chi_{[0,\infty)}(x)\psi(2^{-j}x) = \left[\chi_{[0,\infty)}(x) \sum_{k=j-1}^{j+1} \psi(2^{-k}x)\right]\left[m(x)\psi(2^{-j}x)\right],$$

Lemma (2.18) shows that $D^\lambda(m(x)\chi_{[0,\infty)}(x)\psi(2^{-j}x))$ is a locally integrable function. From this and the fact that $D^\lambda m_j$ is a locally integrable function, it follows that $D^\lambda(m\chi_{(-\infty,0]})_j$ is a locally integrable function. Then Minkowski's inequality shows that $\|D^\lambda(m\chi_{[0,\infty)})_j\|_s$ is bounded by the sum of

$$\left[\int_0^\infty \left|D^\lambda(m_j)\right|^s dx\right]^{1/s}, \tag{7.7}$$

$$\left[\int_0^\infty \left|D^\lambda(m\chi_{(-\infty,0]})_j\right|^s dx\right]^{1/s} \tag{7.8}$$

and

$$\left[\int_{-\infty}^0 \left|D^\lambda(m\chi_{(-\infty,0]})_j\right|^s dx\right]^{1/s}. \tag{7.9}$$

By the definition of $B(m,s,\lambda)$ we have (7.7) bounded by $2^{j(-\lambda+1/s)}B(m,s,\lambda)$. By Lemma (2.6), (7.8) is 0 and (7.9) is bounded by

$$C\left[\int_{-\infty}^0\left[\int_{2^{j-1}}^{2^{j+1}}\frac{|m(t)|\,dt}{(2^j-x)^{\lambda+1}}\right]^s dx\right]^{1/s} \leqslant C\frac{2^j\|m\|_\infty 2^{j/s}}{2^{j+j\lambda}}.$$

From these facts and the definition of $B(m,s,\lambda)$, we have

$$2^{j(\lambda-1/s)}\left\|D^\lambda(m\chi_{[0,\infty)})_j\right\|_s \leqslant CB(m,s,\lambda).$$

This shows that $m\chi_{[0,\infty)}$ is in $M(s,\lambda)$ and $B(m\chi_{[0,\infty)},s,\lambda)\leqslant CB(m,s,\lambda)$. Minkowski's inequality then shows that $B(m\chi_{(-\infty,0]},s,\lambda)\leqslant CB(m,s,\lambda)$. This completes the proof of Lemma (7.5).

The other lemma needed to prove Theorem (7.3), Lemma (7.10), relates the space $M(s,\lambda)$ to the space $\mathrm{RL}(s,\lambda)$ of **[3]**. The space $\mathrm{RL}(s,\lambda)$ is defined as follows for $1\leqslant s\leqslant\infty$ and $\lambda>0$. A locally integrable function m on $(0,\infty)$ is in $\mathrm{RL}(s,\lambda)$ if, given a nonnegative C^∞ function ϕ with support in [1, 2], the quantity

$$\|m\|_{\mathrm{RL}(s,\lambda)} = \sup_{t>0}\left\|D^\lambda(\phi(x)m(tx))\right\|_s$$

is finite. We shall prove the following.

LEMMA (7.10). *If $m(x)=0$ for $x\leqslant 0$, $m_1(x)$ is the restriction of m to $(0,\infty)$, $1\leqslant s\leqslant\infty$ and $\lambda>0$, then $m(x)$ is in $M(s,\lambda)$ if and only if $m_1(x)$ is in $\mathrm{RL}(s,\lambda)$. Furthermore, there is a C, independent of m, such that $\|m_1\|_{\mathrm{RL}(s,\lambda)}\leqslant CB(m,s,\lambda)$ and $B(m,s,\lambda)\leqslant c\|m_1\|_{\mathrm{RL}(s,\lambda)}$.*

To prove that if m is in $M(s,\lambda)$ then m_1 is in $\mathrm{RL}(s,\lambda)$, fix a nonnegative C^∞ function ϕ with support in [1, 2]. By a change of variables we have for $t>0$,

$$\left\|D^\lambda(m_1(tx)\phi(x))\right\|_s = t^{\lambda-1/s}\left\|D^\lambda(m(x)\phi(x/t))\right\|_s. \tag{7.11}$$

Now let j be the least integer with $2^j\geqslant t$ and let $\phi_1(x)=\phi(2^jx/t)$. The right side of (7.11) is bounded by $C2^{j(\lambda-1/s)}\|D^\lambda m(x)\phi_1(2^{-j}x)\|_s$. Since ϕ_1 has support in $[\frac{1}{2},2]$, we can apply Theorem (2.25) to get the bound $CB(m,s,\lambda)$. This proves that m_1 is in $\mathrm{RL}(s,\lambda)$ and the first inequality of the lemma.

Conversely, if m_1 is in $\mathrm{RL}(s, \lambda)$, let ψ be a nonnegative function in C^∞ with support in $\frac{1}{2} \leqslant |x| \leqslant 2$ such that $\sum \psi(2^{-j}x) = 1$. A change of variables shows that

$$2^{j(\lambda-1/s)}\left\| D^\lambda\left(m(x)\psi(2^{-j}x)\right)\right\|_s = \left\| D^\lambda\left(m_1(2^j x)\psi(x)\right)\right\|_s.$$

By Lemma 1 of [3] the right side is bounded by $C\|m_1\|_{\mathrm{RL}(s,\lambda)}$. This completes the proof of Lemma (7.10).

To complete the proof of Theorem (7.3), it is sufficient by Theorem 2 of [3] to prove a version of Theorem (7.3) with $\mathrm{WBV}_{s,\lambda}$ replaced by $\mathrm{RL}(s, \lambda)$ and $\| \;\|_{s,\lambda;W}$ replaced by $\| \;\|_{\mathrm{RL}(s,\lambda)}$. To prove this version, assume first that m is in $M(s, \lambda)$. Since the Hilbert transform is bounded on L^s for $1 < s < \infty$, we have

$$B\left(m(-x)\psi_{[0,\infty)}(x), s, \lambda\right) \leqslant CB\left(m(x)\psi_{(-\infty,0]}(x), s, \lambda\right).$$

Combining this with the second inequality in (7.6) and Lemma (7.10) shows that

$$\|m_1\|_{\mathrm{RL}(s,\lambda)} + \|m_2\|_{\mathrm{RL}(s,\lambda)} \leqslant CB(m, s, \lambda).$$

Conversely, if m_1 and m_2 are in $\mathrm{RL}(s, \lambda)$, the boundedness of the Hilbert transform gives

$$B\left(m(x)\psi_{(-\infty,0]}(x), s, \lambda\right) \leqslant CB\left(m(-x)\psi_{[0,\infty)}(x), s, \lambda\right).$$

Combining this with the first inequality in (7.6) and Lemma (7.10) shows that

$$B(m, s, \lambda) \leqslant C\left(\|m_1\|_{\mathrm{RL}(s,\lambda)} + \|m_2\|_{\mathrm{RL}(s,\lambda)}\right).$$

This completes the proof of Theorem (7.3).

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