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SUFFICIENCY CONDITIONS FOR L^p MULTIPLIERS WITH GENERAL WEIGHTS

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ABSTRACT. Weighted norm inequalities in R^1 are proved for multiplier operators with the multiplier function satisfying Hörmander type conditions. The operators are initially defined on the space $\mathscr{S}_{0,0}$ of Schwartz functions whose Fourier transforms have compact support not including 0. This restriction on the domain of definition makes it possible to use a larger class of weight functions than usually considered; weight functions used here are of the form $|g(x)|^p V(x)$ where $g(x)$ is a polynomial of arbitrarily high degree and $V(x)$ is in A_p. For weight functions in A_p, the results hold for all Schwartz functions. The periodic case is also considered.

1. Introduction. This paper is concerned with proving norm inequalities of the form

$$\int_{-\infty}^{\infty} \left|(m\hat{f})^{\vee}(x)\right|^p W(x)\,dx \leqslant C\int_{-\infty}^{\infty} \left|f(x)\right|^p W(x)\,dx \tag{1.1}$$

for rather general classes of multipliers m and weight functions W. Initially, (1.1) will be proved for all f in $\mathscr{S}_{0,0}$, the Schwartz functions whose Fourier transforms have compact support not including 0. Restricting f to $\mathscr{S}_{0,0}$ allows a much greater variety of weight functions than is possible if (1.1) is required to hold for all Schwartz functions, and the additional weight functions are important for applications.

This paper is a continuation of **[18]**; there $W(x)$ was taken to be a power of $|x|$. As in **[18]** we consider the usual spaces of multiplier functions of Hörmander type, called $M(s,\lambda)$ here, which for λ a positive integer and s satisfying $1 \leqslant s \leqslant \infty$ consists of all m such that

$$B(m,s,\lambda) = \|m\|_\infty + \sup_{r>0} r^{\lambda-1/s}\left[\int_{r<|t|<2r} \left|m^{(\lambda)}(t)\right|^s dt\right]^{1/s} < \infty.$$

For the definition with λ fractional, see §2; except for $s=1$ and $s=\infty$, these are, as shown in §7 of **[18]**, two sided versions of the spaces $S(s,\lambda)$ used by Connett and Schwartz in **[6]** and the spaces $\mathrm{WBV}_{s,\lambda}$ used by Gasper and Trebels in **[9]**. For fixed s and $\lambda > 1/s$, we derive collections of weight functions W such that (1.1) holds for all m in $M(s,\lambda)$ and f in $\mathscr{S}_{0,0}$.

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The weight functions considered in (1.1) for a given p are of the form $|g(x)|^p V(x)$ where $g(x)$ is a polynomial and $V(x)$ satisfies the A_p condition

$$\left[\frac{1}{|I|}\int_I V(x)\,dx\right]\left[\frac{1}{|I|}\int_I V(x)^{-1/(p-1)}\,dx\right]^{p-1} \leqslant C,$$

where C is independent of the arbitrary interval I and $|I|$ denotes the length of I. Recent results of Ernst Adams **[1]** show that these are essentially the only weights for which (1.1) can hold. He showed that if (1.1) holds in the Hilbert transform case, $m(x) = \operatorname{sgn}(x)$, for all f in $\mathscr{S}_{0,0}$ and $\int_{-\infty}^{\infty} W(x)[1 + |x|]^{-N}\,dx < \infty$ for some $N > 0$, then W must have this form. He also showed that if (1.1) holds with $m(x) = \operatorname{sgn}(x)$ for all f in L^2 with a fixed number of moments equal to 0, then W must have this form. Since $\operatorname{sgn}(x)$ satisfies the condition $M(s, \lambda)$ for all $s \geqslant 1$ and $\lambda > 0$, this restriction on W seems natural.

Some of the main results proved here are the following.

THEOREM (1.2). *If* $1 < p < \infty$, $1 \leqslant s \leqslant \infty$, $l \geqslant 0$, $\lambda \geqslant l + 1$, $m \in M(s, \lambda)$, $V(x) \in A_p$, *$g(x)$ is a polynomial of degree l and* $W(x) = |g(x)|^p V(x)$, *then for every f in* $\mathscr{S}_{0,0}$

$$(1.3)\quad \left[\int_{-\infty}^{\infty} \left|(m\hat{f})^{\vee}(x)\right|^p W(x)\,dx\right]^{1/p} \leqslant CB(m, s, \lambda)\left[\int_{-\infty}^{\infty} |f(x)|^p W(x)\,dx\right]^{1/p},$$

where C is independent of m and f.

Theorem (1.5) illustrates the fact that the higher the degree of the polynomial, the larger λ must be. It is possible to have $\lambda < l + 1$ in many cases, but the conditions are more complicated; detailed statements of such theorems are given in Theorems (3.2), (6.1), (6.5) and (8.1).

The case $W(x) = (1 + |x|)^a \prod_{j=1}^{J} |x - b_j|^{a_j}$ is also considered separately. This is of interest since the periodic version is needed for the proofs in **[15]**, and the results are not immediate consequences of the theorems for general weight functions. The following result is proved; the notation $\operatorname{int}(x)$ is used for the greatest integer less than or equal to x.

THEOREM (1.4). *If* $1 < p < \infty$, $1 \leqslant s \leqslant \infty$, $\lambda > \max(\frac{1}{s}, |\frac{1}{p} - \frac{1}{2}|)$ *or* $\lambda = s = 1$, $m \in M(s, \lambda)$, $W(x) = (1 + |x|)^a \prod_{j=1}^{J} |x - b_j|^{a_j}$, *where the b_j's are real and distinct,* $a_0 = a + \sum_{j=1}^{J} a_j$,

$$\max\left(-1, -p\lambda, -1 + p\left(-\lambda + \tfrac{1}{2}\right)\right) < a_j < \min\left(p\lambda, -1 + p\left(\lambda + \tfrac{1}{2}\right), -1 + p\left(\lambda + 1 - \tfrac{1}{s}\right)\right)$$

and $(a_j + 1)/p$ is not an integer for $0 \leqslant j \leqslant J$, $\sum_{j=1}^{J} \operatorname{int}[(a_j + 1)/p] \leqslant \operatorname{int}[(a_0 + 1)/p]$ *and* $|a_j - a_k| < p\lambda$ *for* $1 \leqslant j, k \leqslant J$, *then for f in $\mathscr{S}_{0,0}$ (1.3) holds with C independent of m and f.*

The sufficiency theorems such as Theorems (1.2) and (1.4) can all be extended to functions f in more general classes than $\mathscr{S}_{0,0}$. As shown in §7, these theorems are valid if f is in L^2 and has its first l moments equal to 0, where l is the degree of the polynomial in Theorem (1.2) and $l = \operatorname{int}[(a_0 + 1)/p]$ in Theorem (1.4).

The last section, §9, is concerned with the periodic versions of these theorems. These are the results which, when combined with the transplantation theorems in [**15**], will produce multiplier theorems for Jacobi expansions. This application was one of the main reasons for developing the results of this paper.

The procedure used to obtain the main sufficiency theorems for $\lambda > \frac{1}{2}$ is given in §§2–6. Theorem (1.2) is a corollary of Theorem (6.1); Theorem (1.4) for $\lambda > \frac{1}{2}$ is proved as Theorem (6.7). Three sufficiency theorems for $\lambda > \frac{1}{2}$ for general weights are proved in §§2–6, Theorems (3.2), (6.1) and (6.5). The method consists of finding and using estimates of truncated kernels of the form $[m(x)\phi_N(x)]^\vee$ where ϕ_N is in C^∞, $\phi_N(x) = 0$ for $|x| > 2^{N+1}$ and $|x| < 2^{-N-1}$ while $\phi_N(x) = 1$ for $2^{-N+1} < |x| < 2^{N-1}$. This procedure has led us to the definition of the classes $M(s, \lambda)$ given in §2. Also stated in §2 are results from [**18**] concerning the kernels associated with multipliers in $M(s, \lambda)$. These estimates for integrals of the truncated kernels and their derivatives are the only way the $M(s, \lambda)$ assumption on m is used in later sections. The main theorems could, as a result, be stated with truncated kernel estimates as the hypothesis; this would, however, produce longer theorem statements.

In §3, results are obtained for (1.1) with f in the class $\mathscr{S}$ of Schwartz functions. This restricts W to being an A_p function but does give the basic result, Theorem (3.2), needed for later sections. For the more general theorems, various lemmas about A_p functions are needed; these are in §4. The main proofs are in §§5–6; §5 contains basic norm inequalities that are used repeatedly in §6.

If $s > 2$, there are values of λ that are greater than $1/s$ but less than or equal to $\frac{1}{2}$. This case is considered in §8; the method used is an adaptation of a proof by Calderón and Torchinsky in [**2**]. A sufficiency theorem for general weight functions for $\lambda \leqslant \frac{1}{2}$, Theorem (8.1), is proved in §8. Theorem (1.4) for $\lambda \leqslant \frac{1}{2}$ is proved in §8 as Theorem (8.7).

The following definitions and notations will be used throughout this paper except for a few changes in §9 noted at the beginning of §9. Given a nonnegative function W and $p \geqslant 1$, we define $\|f\|_{p,W} = [\int_{-\infty}^{\infty}|f(x)|^p W(x)\,dx]^{1/p}$ and $m_W(E) = \int_E W(x)\,dx$. In addition to the expression $\operatorname{int}(x)$ for the greatest integer less than or equal to x, the traditional $[x]$ will also be used when unambiguous. The spaces $\mathscr{S}$, $\mathscr{S}_{0,0}$ and A_p will be as defined above. The space A_∞ is the union of the spaces A_p for $p > 1$.

We will assume the following basic facts about the spaces A_p and A_∞; further information and proofs can be found in [**14** and **4**]. If $p > 1$ and $W \in A_p$, there is an $r < p$ such that $W \in A_r$. If W is in A_∞, there are positive constants C and δ such that for all intervals I and subsets E of I,

$$\frac{m_W(E)}{m_W(I)} \leqslant C\left[\frac{|E|}{|I|}\right]^\delta.$$

If W is in A_∞, W satisfies the doubling condition: there is a constant C such that for every interval I, $m_W(2I) \leqslant C m_W(I)$, where $2I$ is the interval with the same center as I and twice as long.

For integrable functions f, we define the Fourier transform by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-ixt}\, dt$$

and the inverse Fourier transform by

$$\check{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{ixt}\, dt.$$

For general locally integrable f, we define $\hat{f}$ to be the function that satisfies $\int_{-\infty}^{\infty} \hat{f}(x)\phi(x)\,dx = \int_{-\infty}^{\infty} f(x)\hat{\phi}(x)\,dx$ for every ϕ in C^∞ with compact support, provided such a function exists. The inverse Fourier transform $\check{f}$ for locally integrable functions is defined analogously. Similarly, the weak derivative of a function f on $(-\infty, \infty)$ is the function f' such that $\int_{-\infty}^{\infty} f(x)\phi'(x)\,dx = -\int_{-\infty}^{\infty} f'(x)\phi(x)\,dx$ for every ϕ in C^∞ with compact support, provided such a function exists.

Throughout this paper C will denote constants not necessarily the same at each occurrence. The letters i, j, k, l, m and n will be used for integers whether this is stated explicitly or not except for cases where i is obviously the square root of -1 or when they are names of functions. If g is an expression in x, $[g(x)]^\wedge$ will denote the Fourier transform of g at the point x. For a number p with $1 \leqslant p \leqslant \infty$, p' will denote $p/(p-1)$.

2. Definitions and basic results. Listed here are the definition and properties of the multiplier classes $M(s, \lambda)$. For further discussion and proofs, see [**18**].

We define the operator D^λ by $D^\lambda g(x) = [\check{g}(x)x^\lambda]^\wedge$, where x^λ is taken to be $|x|^\lambda e^{-i\pi\lambda}$ for $x < 0$, and the Fourier transforms are as defined in §1. To define the multiplier classes, choose a function $\psi(x)$ in C^∞ with support in $\frac{1}{2} < |x| < 2$ such that $\sum_{j=-\infty}^{\infty} \psi(2^{-j}x) = 1$ for $x \neq 0$. Given a function $m(x)$, define $m_j(x) = m(x)\psi(2^{-j}x)$, $k_j(x) = [m_j(x)]^\vee$ and $K_N(x) = \sum_{j=-N}^{N} k_j(x)$. For $1 \leqslant s \leqslant \infty$ and $\lambda \geqslant 0$, the multiplier class $M(s, \lambda)$ is the set of functions m such that $D^\lambda m_j$ is a locally integrable function for every j and

$$B(m, s, \lambda) = \|m\|_\infty + \sup_j 2^{j(\lambda - 1/s)} \left\| D^\lambda m_j(x) \right\|_s < \infty. \tag{2.1}$$

The class $M(s, \lambda)$ is independent of the choice of ψ; this is proved in Theorem (2.25) of [**18**]. Results from [**18**] that will be needed here are the following. They are respectively Theorem (2.12), Corollary (2.29), Lemma (3.1), Theorem (3.2), Theorem (3.4) and Theorem (3.11) of [**18**].

THEOREM (2.2). *If* $1 \leqslant s \leqslant \infty$, $1 \leqslant t \leqslant \infty$, $0 \leqslant \alpha \leqslant \lambda$, *$m$ is in* $M(s, \lambda)$ *and one of the following holds*:

(i) $\alpha - 1/t \leqslant \lambda - 1/s$, $s > 1$ *and* $t < \infty$,

(ii) $\alpha - 1/t \leqslant \lambda - 1/s$, $s = 1$ *and* $t = \infty$,

(iii) $\alpha - 1/t < \lambda - 1/s$,

then m is in $M(t, \alpha)$ *and* $B(m, t, \alpha) \leqslant CB(m, s, \lambda)$.

THEOREM (2.3). *If $\lambda > 0$, $1 \leqslant s \leqslant \infty$, $1 \leqslant p \leqslant \infty$, $W(x)$ is nonnegative and for all f in a subset S of L^2 and m in $M(s,\lambda)$ we have $\|(m\hat{f})^\vee\|_{p,W} \leqslant CB(m,s,\lambda)\|f\|_{p,W}$ with C independent of f and m, then for all $N \geqslant 0$ and f in S, $\|K_N * f\|_{p,W} \leqslant CB(m,s,\lambda)\|f\|_{p,W}$ with C independent of N, m and f.*

LEMMA (2.4). *If $1 \leqslant s \leqslant \infty$, $1 \leqslant p < \infty$, $t = \min(2, p', s)$, $l \geqslant 0$, $\lambda \geqslant 0$, $m(x)$ is in $M(s,\lambda)$ and $r > 0$, then*

$$\int_{r<|x|<2r} \left|k_j^{(l)}(x)\right|^p dx \leqslant CB(m,s,\lambda)^p (2^j r)^{p(l-\lambda+1/t)} r^{1-p(l+1)},$$

where C is independent of m, r and j.

THEOREM (2.5). *If $1 \leqslant s \leqslant \infty$, $1 \leqslant p < \infty$, $t = \min(2, p', s)$, $0 \leqslant l < \lambda - 1/t$, $m(x)$ is in $M(s,\lambda)$ and $r > 0$, then*

$$\int_{r<|x|<2r} \left|K_N^{(l)}(x)\right|^p dx \leqslant CB(m,s,\lambda)^p r^{1-p(l+1)},$$

where C is independent of r, m and N.

THEOREM (2.6). *If $1 \leqslant s \leqslant \infty$, $1 \leqslant p < \infty$, $t = \min(2, p', s)$, L is an integer, $0 \leqslant L < \lambda - 1/t < L + 1$, m is in $M(s,\lambda)$, $r > 0$ and $|y| < r/2$, then*

$$\int_{r<|x|<2r} \left|K_N(x-y) - \sum_{n=0}^{L} \frac{(-y)^n}{n!} K_N^{(n)}(x)\right|^p dx \leqslant CB(m,s,\lambda)^p \left[\frac{|y|}{r}\right]^{p\lambda - p/t} r^{1-p},$$

where C is independent of y, r, m and N.

THEOREM (2.7). *If $m(x)$ is in $M(s,\lambda)$, $1 \leqslant s \leqslant \infty$, $\lambda > \frac{1}{2}$, $\lambda > 1/s$ and $1 < p < \infty$, then there is a C, independent of m, N, y, f and r, such that*

$$\int_{|x|>2|y|} |K_N(x-y) - K_N(x)|\, dx \leqslant CB(m,s,\lambda), \tag{2.8}$$

$$\|f * K_N\|_p \leqslant CB(m,s,\lambda)\|f\|_p,$$

and, for $r > 0$,

$$\left|\{x: |(f * K_N)(x)| > r\}\right| \leqslant \frac{CB(m,s,\lambda)}{r}\|f\|_1.$$

3. Weight functions for multipliers initially defined on $\mathscr{S}$. In this section, we derive some theorems for multipliers defined as $(m\hat{f})^\vee$ for all f in $\mathscr{S}$. As mentioned in §1, defining a multiplier in this way, allowing m to be an arbitrary member of a class $M(s,\lambda)$ and requiring that the operator be bounded on L_W^p implies that W is in A_p. For some pairs (s,λ), the weight functions W must be in a proper subset of A_p.

The main results of this section are Theorems (3.2) and (3.3). For certain classes $M(s,\lambda)$ and p satisfying $1 < p < \infty$, Theorem (3.2) provides a large class of weight functions W for which the multiplier operator $(m\hat{f})^\vee$ is bounded on L_W^p. Theorem (3.3) does the same for W of the form $(1+|x|)^a \prod_{i=1}^J |x - b_i|^{a_i}$; it is proved using

Theorem (3.2) but is not a special case of Theorem (3.2). There are W's that satisfy the hypotheses of Theorem (3.3) but do not satisfy the hypotheses of Theorem (3.2).

Techniques similar to those used to prove Theorem (3.2) will prove a theorem in which less is assumed about W; this has a less restrictive A_p condition on W but requires that powers of W or its reciprocal or both satisfy B_p conditions as defined in formula (2.3) of [**10**]. The proof is technically complicated and does not produce a better version of Theorem (3.3). This version will not be pursued here.

The following facts based on §2 and known results are intended to put the theorems of this section in perspective. If T_m is initially defined for functions in $\mathscr{S}$ as $(m\hat{f})^\vee$, $1 \leqslant s \leqslant \infty$, $\lambda > \max(\frac{1}{2}, \frac{1}{s})$ and m is in $M(s, \lambda)$, then by Theorem (2.7), T_m is a bounded operator on (unweighted) L^p, $1 < p < \infty$. If $1 \leqslant s \leqslant \infty$, $\lambda \geqslant 1$ and m is in $M(s, \lambda)$, then T_m is a bounded operator on (unweighted) L^p, $1 < p < \infty$, by the Marcinkiewicz multiplier theorem [**19**, p. 108] and Theorem (2.2). More generally, we have the following.

THEOREM (3.1). *If $1 < p < \infty$, $W \in A_p$, $s \geqslant 1$, $\lambda \geqslant 1$, $m \in M(s, \lambda)$ and $f \in \mathscr{S}$, then $\|(m\hat{f})^\vee\|_{p,W} \leqslant CB(m, s, \lambda)\|f\|_{p,W}$, where C is independent of m and f.*

Theorem (3.1) was proved by Kurtz [**11**, p. 237], for $s = \lambda = 1$; the fact that the constant can be written as $CB(m, 1, 1)$ is clear from the proof. The form stated here is an immediate consequence of the case $s = \lambda = 1$ and Theorem (2.2). Except for the case $\lambda = s = 1$, Theorem (3.1) is also an immediate consequence of Theorems (3.2) and (2.2).

Since $W \in A_p$ is a necessary condition for the conclusion of Theorem (3.1), we cannot obtain more weight functions by placing more requirements on s and λ. We will consider, therefore, the case $\lambda < 1$. The basic results are as follows.

THEOREM (3.2). *If $1 < s \leqslant \infty$, $\max(\frac{1}{s}, \frac{1}{2}) < \lambda < 1$, $m \in M(s, \lambda)$, $1 < p < \infty$, $\max(1, 1/\lambda p) \leqslant u < \infty$, $u[1 - (1 - \lambda)p] \leqslant 1$, $W(x)^u \in A_{\lambda p u}$ and f is in $\mathscr{S}$, then*

$$\left\|(m\hat{f})^\vee\right\|_{p,W} \leqslant CB(m, s, \lambda)\|f\|_{p,W},$$

where C is independent of m and f.

THEOREM (3.3). *Let $1 < s \leqslant \infty$, $\max(\frac{1}{s}, \frac{1}{2}) < \lambda < 1$, $m \in M(s, \lambda)$, $1 < p < \infty$, $W(x) = (1 + |x|)^a \prod_{j=1}^J |x - b_j|^{a_j}$, where the b_j's are real and distinct, $a_0 = a + \sum_{j=1}^J a_j$, $\max(-1, -p\lambda) < a_j < \min(p - 1, p\lambda)$ for $0 \leqslant j \leqslant J$ and $|a_j - a_k| < p\lambda$ for $1 \leqslant j$, $k \leqslant J$. Then for all f in $\mathscr{S}$, we have $\|(m\hat{f})^\vee\|_{p,W} \leqslant CB(m, s, \lambda)\|f\|_{p,W}$, where C is independent of m and f.*

The proof of Theorem (3.2) is based on the following lemma.

LEMMA (3.4). *If $1 < s \leqslant \infty$, $\max(\frac{1}{s}, \frac{1}{2}) < \lambda < 1$, $m \in M(s, \lambda)$, $1/\lambda < p < \infty$, W is in $A_{p\lambda}$ and f is integrable, then $\|K_N * f\|_{p,W} \leqslant CB(m, s, \lambda)\|f\|_{p,W}$, where C is independent of f, m and N.*

We will prove Lemma (3.4) using the #-function of C. Fefferman and Stein [**8**]; this proof is essentially the proof of Theorem 1 of [**12**]. To do this, fix λ, m, p and W and choose q so that $1/\lambda < q < p$, $q \leqslant 2$, $q \leqslant s$ and W is in $A_{p/q}$. This is possible by Lemma 5, p. 214 of [**14**]. Let $f_q^*(x) = [(|f(x)|^q)^*]^{1/q}$, where f^* denotes the Hardy-Littlewood maximal function of f, and let $f^{\#}$ be the function

$$f^{\#}(x) = \sup_I \frac{1}{|I|}\int_I |f(y) - f_I|\, dy,$$

where $f_I = (1/|I|)\int_I f(t)\, dt$ and the sup is taken over all intervals I containing x. We will first show that for f in $\mathscr{S}$,

$$(K_N * f)^{\#}(x) \leqslant CB(m, s, \lambda) f_q^*(x), \tag{3.5}$$

where C is independent of f, m and N.

To prove (3.5), fix x and an interval I containing x and let $\delta = |I|$. It is sufficient to show that

$$\frac{1}{|I|}\int_I |(K_N * f)(y) - [K_N * f]_I|\, dy \leqslant CB(m, s, \lambda) f_q^*(x), \tag{3.6}$$

where C is independent of f, m, N, x and I. To do this, define

$$g_0(y) = \int_{|x-z| \leqslant 2\delta} K_N(y - z) f(z)\, dz$$

and

$$g_j(y) = \int_{2^j\delta \leqslant |x-z| \leqslant 2^{j+1}\delta} K_N(y - z) f(z)\, dz$$

for $j \geqslant 1$. The left side of (3.6) is bounded by the sum of

$$\frac{1}{|I|}\int_I |g_0(y) - [g_0]_I|\, dy \tag{3.7}$$

and

$$\sum_{j=1}^{\infty} \frac{1}{|I|}\int_I |g_j(y) - [g_j]_I|\, dy. \tag{3.8}$$

By Minkowski's inequality and Hölder's inequality, (3.7) is bounded by

$$2\left[\frac{1}{|I|}\int_I |g_0(y)|^q\, dy\right]^{1/q}.$$

By Theorem (2.7), the transformation $f \to K_N * f$ has L^q norm bounded by $CB(m, s, \lambda)$ with C independent of N. Therefore, (3.7) is bounded by

$$CB(m, s, \lambda)\left[\frac{1}{|I|}\int_{|x-y| \leqslant 2\delta} |f(y)|^q\, dy\right]^{1/q} \leqslant CB(m, s, \lambda) f_q^*(x)$$

as desired.

To estimate (3.8), let $c_j = g_j(x)$. Then (3.8) is bounded by

$$\sum_{j=1}^{\infty} \frac{1}{|I|}\int_I |g_j(y) - c_j - [g_j - c_j]_I|\, dy \leqslant 2\sum_{j=1}^{\infty} \sup_{y \in I} |g_j(y) - c_j|.$$

The right side equals

$$2\sum_{j=1}^{\infty} \sup_{y \in I}\left|\int_{2^j\delta \leqslant |x-z| \leqslant 2^{j+1}\delta} [K_N(y - z) - K_N(x - z)] f(z)\, dz\right|.$$

By Hölder's inequality, this is bounded by

$$2 \sum_{j=1}^{\infty} \sup_{y \in I} \left[\int_{2^j\delta \leqslant |x-z| \leqslant 2^{j+1}\delta} |K_N(y-z) - K_N(x-z)|^{q'} \, dz \right]^{1/q'} \times \left[\int_{|x-z| \leqslant 2^{j+1}\delta} |f(z)|^q \, dz \right]^{1/q}.$$

Since $y \in I$, $|x - y| \leqslant \delta$; note also that $1/q < \lambda < 1 + 1/q$. Hence, applying Theorem (2.6) with $p = q'$, $L = 0$ and $r = 2^j\delta$, we see that $t = q$ and that the last sum is bounded by

$$\sum_{j=1}^{\infty} \left[CB(m, s, \lambda)(2^{-j})^{\lambda - 1/q}(2^j\delta)^{-1/q} \right] \left[(2^j\delta)^{1/q} f_q^*(x) \right].$$

Since $\lambda > 1/q$, this also has the desired bound, and the proof of (3.6) and, consequently, of (3.5) is complete.

To complete the proof of Lemma (3.4), start with the fact from [7] that if $W \in A_p$, then $\|K_N * f\|_{p,W} \leqslant C\|(K_N * f)^{\#}\|_{p,W}$. By (3.5),

$$\|(K_N * f)^{\#}\|_{p,W} \leqslant CB(m, s, \lambda)\|f_q^*\|_{p,W}.$$

Since $W \in A_{p/q}$, the definition of f_q^* and Theorem 2 of [**14**] imply that $\|f_q^*\|_{p,W} \leqslant C\|f\|_{p,W}$. Combining these facts gives the conclusion of Lemma (3.4).

To prove Theorem (3.2), fix a W that satisfies the hypotheses. It is sufficient to prove that

$$\|K_N * f\|_{p,W} \leqslant CB(m, s, \lambda)\|f\|_{p,W}, \tag{3.9}$$

since this and Fatou's lemma imply the conclusion of Theorem (3.2).

If $u = \max(1, 1/\lambda p)$, there is an $r > u$ such that $W^r \in A_{\lambda pu}$ by Lemma 6 of [**14**]. In this case, $u[1 - (1 - \lambda)p] < 1$ and if $p < 1/(1 - \lambda)$, we can choose r so that it also satisfies $r < 1/[1 - (1 - \lambda)p]$. Since $A_{\lambda pu} \subset A_{\lambda pr}$, $W^r \in A_{\lambda pr}$. We may, therefore, by replacing u by r, assume that $u > \max(1, 1/\lambda p)$. Similarly, if $p < 1/(1 - \lambda)$ and $u = 1/[1 - (1 - \lambda)p]$, there is an r satisfying $\max(1, 1/\lambda p) < r < u$ such that $W^u \in A_{\lambda pr}$ by Lemma 5 of [**14**]. Since $0 < r < u$, $W^r \in A_{\lambda pr}$. We may, therefore, also assume that $u[1 - (1 - \lambda)p] < 1$.

Now choose p_0 and p_1 such that $1 < p_0 < 1/(1 - \lambda)$, $1/\lambda < p_1 < \infty$, $p_0 \leqslant p \leqslant p_1$ and

$$u = \frac{p_1 - p_0}{p_1 - p_0 - (1 - \lambda)p_0(p_1 - p)}; \tag{3.10}$$

to show this is possible, let $g(p_0, p_1)$ denote the expression on the right side of (3.10) and observe that

$$g(p_0, p_1) = \frac{1}{1 - (1 - \lambda)p_0((p_1 - p)/(p_1 - p_0))}$$
$$= \frac{1}{1 - (1 - \lambda)p_0(1 - (p - p_0)/(p_1 - p_0))}.$$

From these it follows that $g(p_0, p_1)$ is an increasing function in both variables for the indicated ranges of p_0 and p_1. The range of g is easily calculated and is seen to include u.

By a result of P. Jones [**5**], there exist A_1 functions V_0 and V_1 such that

$$W^u = V_0 V_1^{1-\lambda p u}.$$

Define

$$W_0 = V_0^{p_0\lambda - p_0 + 1} V_1^{1-p_0} \quad \text{and} \quad W_1 = V_0 V_1^{1-p_1\lambda}.$$

Now $p_0' > 1/\lambda$, and the function W_2 defined as $W_0^{-1/(p_0-1)}$ is in $A_{p_0'\lambda}$ as is easily seen by using Hölder's inequality. By Lemma (3.4),

$$\|K_N * f\|_{p_0', W_2} \leqslant CB(m, s, \lambda)\|f\|_{p_0', W_2}.$$

A standard duality argument then shows that

$$\|K_N * f\|_{p_0, W_0} \leqslant CB(m, s, \lambda)\|f\|_{p_0, W_0}.$$

Similarly, $p_1 > 1/\lambda$ and W_1 is in $A_{p_1\lambda}$. Therefore, by Lemma (3.4),

$$\|K_N * f\|_{p_1, W_1} \leqslant CB(m, s, \lambda)\|f\|_{p_1, W_1}.$$

The last two inequalities imply (3.9) by use of the following theorem about interpolation with change of measures; this theorem is a special case of Theorem (2.11), p. 164, of [**20**].

THEOREM (3.11). *If T is a linear operator such that $\|Tf\|_{p_i, W_i} \leqslant \|f\|_{p_i, W_i}$ for $i = 0$ and $i = 1$ and $1 \leqslant p_0 < p < p_1 < \infty$, then $\|Tf\|_{p,W} \leqslant \|f\|_{p,W}$, where $W(x) = [W_0(x)^{p_1-p} W_1(x)^{p-p_0}]^{1/(p_1-p_0)}$.*

To prove Theorem (3.3), let $R = \max_{1 \leqslant j \leqslant J}(|2b_j|, 2)$. Given f in $\mathscr{S}$, let f_1 and f_2 be functions in $\mathscr{S}$ such that $f_1(x) = 0$ for $|x| > 2R$, $f_2(x) = 0$ for $|x| < R$, and for all x we have $f(x) = f_1(x) + f_2(x)$, $|f_1(x)| \leqslant |f(x)|$ and $|f_2(x)| \leqslant |f(x)|$. To complete the proof of Theorem (3.3), it is sufficient to show that

$$(3.12) \quad \int_{|x| \leqslant 4R} \left|[m(x)\hat{f}_1(x)]^\vee\right|^p W(x)\,dx \leqslant CB^p \int_{|x| \leqslant 2R} |f(x)|^p W(x)\,dx,$$

$$(3.13) \quad \int_{|x| \geqslant 4R} \left|[m(x)\hat{f}_1(x)]^\vee\right|^p W(x)\,dx \leqslant CB^p \int_{|x| \leqslant 2R} |f(x)|^p W(x)\,dx$$

and

$$(3.14) \quad \int_{-\infty}^{\infty} \left|[m(x)\hat{f}_2(x)]^\vee\right|^p W(x)\,dx \leqslant CB^p \int_{|x| \geqslant R} |f(x)|^p W(x)\,dx,$$

where $B = B(m, s, \lambda)$ and C is independent of f and m.

We will use Theorem (3.2) to prove (3.12). To do this, we will need the existence of a u such that

$$(3.15) \qquad \max_{1 \leqslant j \leqslant J}\left(0, -a_j, 1 - (1-\lambda)p\right) < \frac{1}{u} < \min_{1 \leqslant j \leqslant J}\left(1, \lambda p, \lambda p - a_j\right).$$

The nine inequalities required to show that the left side is less than the right side are consequences of the hypotheses.

With u chosen to satisfy (3.15), it follows that $\max(1, 1/\lambda p) \leqslant u < \infty$ and $u[1-(1-\lambda)p] \leqslant 1$. Now define $V(x) = [1+|x|]^{-a_0}W(x)$. The function V^u is in $A_{\lambda p u}$ since $-1 < a_j u < \lambda p u - 1$ for $1 \leqslant j \leqslant J$ and V is bounded above and below by positive constants for $|x| \geqslant R$. Therefore, Theorem (3.2) implies

$$(3.16)\quad \int_{|x|\leqslant 4R} \left|[m(x)\hat{f}_1(x)]^{\vee}\right|^p V(x)\,dx \leqslant CB^p \int_{|x|\leqslant 2R} |f_1(x)|^p V(x)\,dx.$$

Since $W(x) \leqslant CV(x)$ and $V(x) = CW(x)$ for $|x| \leqslant 4R$, this implies (3.12).

To prove (3.13), let $a_j^* = \max(a_j, 0)$, $a^* = a_0^* - \sum_{j=1}^J a_j^*$ and

$$V(x) = (1+|x|)^{a^*} \prod_{j=1}^{J} |x-b_j|^{a_j^*}.$$

From the hypothesis it is easy to see that there is a u satisfying

$$(3.17)\qquad \max(0, 1-(1-\lambda)p) < \frac{1}{u} < \min_{0\leqslant j\leqslant J}\left(1, \lambda p - a_j^*\right).$$

Fix a u that satisfies (3.17). It follows that

$$\max(1, 1/\lambda p) < u < \infty \quad \text{and} \quad u[1-(1-\lambda)p] \leqslant 1.$$

The function $V(x)^u$ is in $A_{\lambda p u}$ since $0 \leqslant a_j^* u < \lambda p u - 1$ for $0 \leqslant j \leqslant J$. Therefore by Theorem (3.2)

$$(3.18)\quad \int_{|x|\geqslant 4R} \left|[m(x)\hat{f}_1(x)]^{\vee}\right|^p V(x)\,dx \leqslant CB^p \int_{|x|\leqslant 2r} |f_1(x)|^p V(x)\,dx.$$

Now for $|x| \geqslant 4R$ we have $V(x) \geqslant C(1+|x|)^{a_0^*} \geqslant C(1+|x|)^{a_0} \geqslant W(x)$ and for $|x| \leqslant 2R$ we have $V(x) \leqslant W(x)$. These facts and (3.18) prove (3.13).

To prove (3.14), let D be the set of j's for which $a_j < 0$ and $a_j < a_0$, let E be the set of j's for which $a_0 \leqslant a_j < 0$ and define

$$U(x) = |x|^{a_0} + |x+1|^{a_0} + \sum_{j\in D} |x-b_j|^{a_j} + \sum_{j\in E} |x-b_j|^{a_0}.$$

Since $W(x) \leqslant CU(x)$ for all x, we can replace W by U on the left side of (3.14). This produces a sum of integrals of the form $\int_{-\infty}^{\infty} |[m(x)\hat{f}_2(x)]^{\vee}|^p |x-b|^A\,dx$ with $\max(-1, -p\lambda) < A < \min(p-1, p\lambda)$. Theorem (3.2) can be applied to each of these if there is a u such that

$$\max(0, -A, 1-p+p\lambda) < 1/u < \min(1, \lambda p, \lambda p - A);$$

that there is such a u is easy to see by verifying that each term in the max is less than each term in the min. Alternatively, Theorem (1.2) of **[18]** can be used. It follows that the left side of (3.14) is bounded by $CB(m,s,\lambda)^p \int_{-\infty}^{\infty} |f_2(x)|^p U(x)\,dx$. The facts that $U(X) \leqslant CW(x)$ on $|x| > R$ and $|f_2(x)| \leqslant |f(x)|$ then complete the proof of Theorem (3.3).

4. Facts about A_p functions. To prove our multiplier results for general weight functions, we will need a number of results concerning A_p functions. The first seven are used in §5; the last is needed in §6.

LEMMA (4.1). *If* $1 < p < \infty$, $V(x) \in A_p$, *b is real and* $a \leqslant -p$, *then*

$$\int_{-\infty}^{\infty} \left| \int_{|y-b| \leqslant |x-b|} f(y)\, dy \right|^p |x-b|^a V(x)\, dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p |x-b|^{a+p} V(x)\, dx,$$

where C is independent of f.

By Theorem 1, p. 32, of [**13**], we need only verify that for $r > 0$

(4.2)

$$\left(\int_{|x-b|>r} |x-b|^a V(x)\, dx \right) \left(\int_{|x-b|<r} \left[|x-b|^{a+p} V(x) \right]^{-1/(p-1)} dx \right)^{p-1} \leqslant C$$

with C independent of r. Now since $V \in A_p$, $V \in A_{-a}$. Then the first factor in (4.2) is bounded by $Cr^a \int_{|x-b|<r} V(x)\, dx$ by Lemma 1, p. 232, of [**10**]. The second factor in (4.2) is bounded by

$$r^{-a-p} \left(\int_{|x-b|<r} V(x)^{-1/(p-1)}\, dx \right)^{p-1}$$

since $-a-p \geqslant 0$. Multiplying these estimates and using the definition of A_p then proves the lemma.

LEMMA (4.3). *If* $1 < p < \infty$, $V(x) \in A_p$, *b is real and* $a \geqslant 0$, *then*

$$\int_{-\infty}^{\infty} \left| \int_{|y-b| \geqslant |x-b|} f(y)\, dy \right|^p |x-b|^a V(x)\, dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p |x-b|^{a+p} V(x)\, dx,$$

where C is independent of f.

By Theorem 2, p. 32, of [**13**], we need only verify that

(4.4)

$$\left(\int_{|x-b|<r} |x-b|^a V(x)\, dx \right) \left(\int_{|x-b|>r} \left[|x-b|^{a+p} V(x) \right]^{-1/(p-1)} dx \right)^{p-1} \leqslant C$$

with C independent of r. The proof is essentially the same as for Lemma (4.1) using the fact that $V(x)^{-1/(p-1)} \in A_{p'}$ and $-(a+p)/(p-1) \leqslant -p'$.

LEMMA (4.5). *If* $1 \leqslant p < \infty$ *and* $V \in A_p$, *then there is a* $q > 1$ *such that for every interval I*,

$$\left[\int_I V(x)^q\, dx \right]^{1/q} \leqslant C |I|^{-1+1/q} \int_I V(x)\, dx,$$

where C is independent of I.

This is proved in [**14**, p. 214]; the conclusion is equivalent to the statement that V is in A_∞.

LEMMA (4.6). *If $1 \leqslant p < \infty$, $V \in A_p$ and g is a polynomial, then there is a $q > 1$ such that for every interval I*

$$\left[\int_I \left[|g(x)|^p V(x)\right]^q dx\right]^{1/q} \leqslant C|I|^{-1+1/q} \int_I |g(x)|^p V(x)\, dx,$$

where C is independent of I.

It should be noted that although Lemma (4.6) will be used in this form, the proof requires only that V satisfies A_∞.

To prove Lemma (4.6), let l denote the degree of g and fix an interval I. Then there is an open subinterval $J \subset I$ with $|J| \geqslant |I|/[3(l+1)]$ such that $3J$ contains no roots of g. For such a J and I, it is easy to verify by considering the individual factors of g that $\sup_I |g(x)| \leqslant C \inf_J |g(x)|$, where C is independent of I and J. Let $q > 1$ be a number for which Lemma (4.5) holds for this V. Then the left side of the conclusion of Lemma (4.6) is bounded by

$$C|I|^{-1+1/q}\left[\inf_J |g(x)|^p\right]\int_I V(x)\, dx.$$

Since V is in A_p, V satisfies the doubling condition and this is bounded by

$$C|I|^{-1+1/q}\left[\inf_J |g(x)|^p\right]\int_J V(x)\, dx.$$

Since this is bounded by the right side of the conclusion, the proof is complete.

LEMMA (4.7). *If $1 \leqslant p < \infty$, b is real, R and C are positive, $U(x)$ and $V(x)$ are in A_p, $U(x) \leqslant CV(x) \leqslant C^2U(x)$ for $R \leqslant |x-b| \leqslant 2R$, $W(x) = U(x)$ for $|x-b| < R$ and $W(x) = V(x)$ for $|x-b| \geqslant R$, then $W(x)$ is in A_p.*

Typical of the applications of the preceding lemma is the fact that if $R > 0$, $[|x-b|/(R+|x-b|)]^a U(x) \in A_p$ and $(R+|x-b|)^a U(x) \in A_p$, then $U(x) \in A_p$.

To prove Lemma (4.7), observe that W satisfies the definition of A_p trivially if $I \subset (-\infty, b-R]$, $I \subset [b-2R, b+2R]$ or $I \subset [b+R, \infty)$. If $[b+R, b+2R] \subset I$,

$$\int_{I\cap[b-R,b+R]} W(x)\, dx \leqslant \int_{b-R}^{b+R} U(x)\, dx \leqslant C\int_{b+R}^{b+2R} U(x)\, dx;$$

the last inequality follows from the fact that $U \in A_p$. Therefore,

$$\int_{I\cap[b-R,b+R]} W(x)\, dx \leqslant C\int_{b+R}^{b+2R} V(x)\, dx,$$

and we obtain

$$\int_I W(x)\, dx \leqslant C\int_I V(x)\, dx.$$

Similarly, from the fact that $U \in A_p$ we obtain

$$\left[\int_I W(x)^{-1/(p-1)}\, dx\right]^{p-1} \leqslant C\left[\int_I V(x)^{-1/(p-1)}\, dx\right]^{p-1},$$

and the defining inequality for W in A_p follows for $I \supset [b+R, b+2R]$ since V is in A_p. The case $I \supset [b-2R, b-R]$ is similar.

LEMMA (4.8). *If* $1 \leqslant p < \infty$, *b is real,* $0 < a < A$, $R > 0$, $h(x) = |x-b|$, $R + |x-b|$ *or* $|x-b|/(R+|x-b|)$, $V(x) \in A_p$ *and* $h(x)^A V(x) \in A_p$, *then* $h(x)^a V(x) \in A_p$.

To prove this, use Hölder's inequality to obtain

$$\int_I h(x)^a V(x)\,dx \leqslant \left[\int_I V(x)\,dx\right]^{(A-a)/A}\left[\int_I h(x)^A V(x)\,dx\right]^{a/A}.$$

Doing the same for the other integral, multiplying the estimates and using the hypotheses completes the proof.

LEMMA (4.9). *If* $1 < p < \infty$, *b is real,* $R \geqslant 0$, $V(x) \in A_p$ *and* $h(x) = |x-b|$, $R + |x-b|$ *or* $|x-b|/(R+|x-b|)$, *then there is an* $E > 0$ *such that* $0 \leqslant \varepsilon \leqslant E$ *implies* $h(x)^\varepsilon V(x) \in A_p$.

Because of Lemmas (4.7) and (4.8), it is sufficient to prove that there is an $E > 0$ such that $|x-b|^E V(x) \in A_p$. To do this, use the fact that $V(x)^{-1/(p-1)} \in A_{p'}$ and Lemma (4.5) to show that there is a $q > 0$ such that

$$\left[\int_I V(x)^{-q/(p-1)}\,dx\right]^{1/q} \leqslant C|I|^{-1+1/q}\int_I V(x)^{-1/(p-1)}\,dx, \tag{4.10}$$

where C is independent of I. Choose E so that $E > 0$ and $Eq'/(p-1) < 1$. Hölder's inequality and (4.10) imply that

$$\int_I \left[V(x)|x-b|^E\right]^{-1/(p-1)} dx \leqslant C|I|^{-1+1/q}\left[\int_I V(x)^{-1/(p-1)}\,dx\right] \times\left[\int_I |x-b|^{-Eq'/(p-1)}\,dx\right]^{1/q'}.$$

With d equal to the distance from b to the more distant end of I, we get

$$\int_I \left[V(x)|x-b|^E\right]^{-1/(p-1)} dx \leqslant Cd^{-E/(p-1)}\int_I V(x)^{-1/(p-1)}\,dx.$$

It is immediate that

$$\int_I |x-b|^E V(x)\,dx \leqslant d^E \int_I V(x)\,dx.$$

These estimates and the fact that V is in A_p then show that $|x-b|^E V(x)$ is in A_p.

LEMMA (4.11). *Assume that* $\int_{-\infty}^{\infty} x^j f(x)\,dx = 0$ *for* $0 \leqslant j \leqslant l-1$, *$V(x)$ is in* A_p *and $g(x)$ is a polynomial of degree l whose real roots are* $b_1 < b_2 < \cdots < b_J$. *Let d be positive, and, if* $J \geqslant 2$, *assume that* $d < \min_{i \neq j}(|b_j - b_i|/4)$. *Then there exist functions* $f_i(x)$, $0 \leqslant i \leqslant J$, *such that* $\sum_{i=0}^J f_i(x) = f(x)$, $f_i(x) = f(x)$ *for* $1 \leqslant i \leqslant J$ *and* $|x-b_i| \leqslant d$, $f_i(x) = 0$ *for* $1 \leqslant i \leqslant J$ *and* $|x-b_i| \geqslant 2d$, $\int_{-\infty}^{\infty} x^j f_i(x)\,dx = 0$ *for*

$0 \leqslant j \leqslant l-1$ and $0 \leqslant i \leqslant J$, and for $0 \leqslant i \leqslant J$,

$$\int_{-\infty}^{\infty} |f_i(x)|^p |g(x)|^p V(x)\,dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p |g(x)|^p V(x)\,dx, \tag{4.12}$$

where C is independent of f. If f is in C^∞, then f_i can be chosen to be in C^∞.

To prove this, first choose h_i, $1 \leqslant i \leqslant J$, such that $h_i(x) = f(x)$ if $|x - b_i| < d$, $h_i(x) = 0$ if $|x - b_i| > 2d$, $|h_i(x)| \leqslant |f(x)|$ for all x and h_i is in C^∞ if f is in C^∞. For $1 \leqslant i \leqslant J$ and $0 \leqslant j \leqslant l-1$, choose C^∞ functions $\phi_{i,j}(x)$ with support in $(b_i + d, b_i + 2d)$ so that $\int_{-\infty}^{\infty} x^k \phi_{i,j}(x)\,dx = \delta_{j,k}$ for $0 \leqslant j,\ k \leqslant l-1$; this is possible by the proof of Lemma (2.6), p. 182, of [**3**]. For $1 \leqslant i \leqslant J$, define

$$f_i(x) = h_i(x) - \sum_{j=0}^{l-1} \phi_{i,j}(x) \int_{-\infty}^{\infty} t^j h_i(t)\,dt$$

and define $f_0(x) = f(x) - \sum_{i=1}^{J} f_i(x)$. The asserted properties other than (4.12) are then trivial. To prove (4.12), it is sufficient to show that $|\int_{-\infty}^{\infty} x^j h_i(x)\,dx|^p$ is bounded by the right side of (4.12) for $0 \leqslant j \leqslant l-1$ since $|h_i(x)| \leqslant |f(x)|$ and $\|\phi_{i,j}(x) g(x)\|_{p,V} \leqslant C$. Now

$$\left| \int_{d<|x-b_i|} x^j h_i(x)\,dx \right|^p \leqslant \left[\int_{d<|x-b_i|<2d} |x|^j |f(x)|\,dx \right]^p;$$

that this is bounded above by the right side of (4.12) is immediate by using Hölder's inequality, the fact that $|g(x)|$ has a positive lower bound on $d < |x - b_i| < 2d$ and the fact that $V^{-1/(p-1)}$ is locally integrable. It is, therefore, sufficient to prove that

$$\left| \int_{|x-b_i|<d} x^j f(x)\,dx \right|^p \leqslant C \int_{-\infty}^{\infty} |f(x)|^p |g(x)|^p V(x)\,dx \tag{4.13}$$

for $0 \leqslant j \leqslant l-1$.

To prove (4.13), let $\Sigma k_n(x)$ be the partial fraction decomposition of $x^j/g(x)$. Then the left side of (4.13) is bounded by a constant times a sum of terms of the form

$$\left| \int_{|x-b_i|<d} k_n(x) f(x) g(x)\,dx \right|^p. \tag{4.14}$$

Now if the denominator of k_n is not a power of $x - b_i$, k_n is bounded on $|x - b_i| < d$ and Hölder's inequality shows that (4.14) is bounded by the right side of (4.13). If $k_n(x) = C(x - b_i)^{-N}$ with $N \geqslant 1$, then since $k_n(x) g(x)$ is a polynomial of degree at most $l-1$, it follows that (4.14) equals

$$C \left| \int_{|x-b_i|>d} (x - b_i)^{-N} f(x) g(x)\,dx \right|^p.$$

By Hölder's inequality, this is bounded by the product of the right side of (4.13) and

$$\left(\int_{|x-b_i|>d} V(x)^{-1/(p-1)} |x - b_i|^{-Np'}\,dx \right)^{p-1}.$$

Now since $V(x)^{-1/(p-1)}$ is in $A_{p'}$, it is also in $A_{Np'}$ and Lemma 1, p. 232, of **[10]** shows that this last expression has the bound

$$d^{-Np}\left(\int_{|x-b_i|<d} V(x)^{-1/(p-1)}\,dx\right)^{p-1},$$

which is finite. This completes the proof of Lemma (4.11).

5. Four lemmas for general weight functions. The weighted L^p results proved here contain weight functions of the form $|g(x)|^pV(x)$ where $V(x)\in A_p$ and g is a polynomial. They are the basis for the main theorems in §6. Also stated here is Lemma (5.13) which is stated and proved as Lemma (4.3) of **[18]**. The proofs of Lemmas (5.1) and (5.5) use Lemma (5.13) and are essentially the same as the proof of Theorem (1.2) in **[18]**. Various technical problems occur since the weight functions are more general. The proofs are given after all of the lemmas are stated.

The results are obtained here for f in $\mathscr{S}$; this is convenient for these proofs but not essential. In §7 there are strengthened versions for more general classes of functions and more general weight functions.

LEMMA (5.1). *Assume that* $1\leqslant s\leqslant\infty$, $1<p<\infty$, $l\geqslant 1$, $V(x)\in A_p$, *b is real,* $R\geqslant 0$ *and Q is the least upper bound of all q such that*

$$\int_{r<|x-b|<2r} V(x)^q\,dx\leqslant Cr^{1-q}\left[\int_{r<|x-b|<2r} V(x)\,dx\right]^q \tag{5.2}$$

for all r satisfying $r>R$. Let $T=\min(2,\,(pQ')',s)$, $W(x)=|x-b|^{lp}V(x)$ *and assume that* $\lambda>l-1+1/T$ *and* $m\in M(s,\lambda)$. *For every h in $\mathscr{S}$, assume that* $\|K_N*h\|_{p,V_1}\leqslant HB(m,s,\lambda)\|h\|_{p,V_1}$, *where* $V_1(x)=|x-b|^{\alpha}V(x)$ *and α and H are constants independent of m, N and h. If* $\lambda<l+1/T$, *assume in addition that* $(R+|x-b|)^{p(l-\lambda+1/T)}V(x)\in A_p$. *Then for f in $\mathscr{S}$,*

$$\int_{|x-b|>R}\left|\int_{|y-b|>4R} K(x,y)f(y)\,dy\right|^p W(x)\,dx \tag{5.3}$$
$$\leqslant C(1+H^p)B(m,s,\lambda)^p\int_{|x-b|>4R}|f(x)|^pW(x)\,dx,$$

where

$$K(x,y)=K_N(x-y)-\sum_{n=0}^{l-1}\frac{(b-y)^n}{n!}K_N^{(n)}(x-b), \tag{5.4}$$

and C is independent of f, m and N.

LEMMA (5.5). *Assume that* $1\leqslant s\leqslant\infty$, $1<p<\infty$, $l\geqslant 1$, $V(x)\in A_p$, *b is real and* $R>0$. *Let Q be the least upper bound of all q such that* (5.2) *holds for all r satisfying* $0\leqslant r\leqslant 2R$, *let* $T=\min(2,(pQ')',s)$, $W(x)=|x-b|^{lp}V(x)$ *and assume that* $\lambda>l-1+1/T$ *and* $m\in M(s,\lambda)$. *For every h in $\mathscr{S}$, assume that* $\|K_N*h\|_{p,V_1}\leqslant HB(m,s,\lambda)\|h\|_{p,V_1}$, *where* $V_1(x)=|x-b|^{\alpha}V(x)$ *and α and H are constants independent of m, N and h. If* $\lambda<l+1/T$, *assume in addition that*

$$\left[\frac{|x-b|}{R+|x-b|}\right]^{p(l-\lambda+1/T)}V(x)\in A_p.$$

Then for f in $\mathscr{S}$,

$$(5.6)\qquad \int_{|x-b|<4R}\left|\int_{|y-b|<R} K(x,y)f(y)\,dy\right|^p W(x)\,dx \leqslant C(1+H^p)B(m,s,\lambda)^p\int_{|x-b|<R}|f(x)|^p W(x)\,dx,$$

where $K(x, y)$ is as defined in (5.4) *and C is independent of f, m and N.*

LEMMA (5.7). *Assume that $1 \leqslant s \leqslant \infty$, $1 < p < \infty$, $V(x) \in A_p$, l and a are integers with $l \geqslant 1$ and $l \geqslant a \geqslant 0$, $W(x) = [R + |x-b|]^{(l-a)p}|x-b|^{ap}V(x)$ and b is real, $R > 0$. Let Q be the least upper bound of all q for which there are constants C and R_1 such that* (5.2) *holds for all $r \geqslant R_1$. Let $T = \min(2,(pQ')', s)$ and assume that $\lambda > l - 1 + 1/T$ and $m \in M(s,\lambda)$. If $\lambda < l + 1/T$, assume in addition that $[R + |x-b|]^{p(l-\lambda+1/T)}V(x) \in A_p$; if $\lambda < a + 1/T$ (which implies $a = l$), assume in addition that $|x-b|^{p(l-\lambda+1/T)}V(x) \in A_p$. Then there exist positive R_0 and C such that for all f in $\mathscr{S}$,*

$$(5.8)\qquad \int_{|x-b|>R_0}\left|\int_{|y-b|<R} K(x,y)f(y)\,dy\right|^p W(x)\,dx \leqslant CB(m,s,\lambda)^p\int_{|x-b|<R}|f(x)|^p W(x)\,dx,$$

where $K(x, y)$ is as defined in (5.4) *and C and R_0 are independent of f, m and N.*

LEMMA (5.9). *Assume that $1 \leqslant s \leqslant \infty$, $1 < p < \infty$, $V(x) \in A_p$, $l \geqslant 1$, $g(x)$ is a polynomial, b is real, $R > 0$, $g(x) \neq 0$ for $|x-b| \leqslant 4R$, $R_0 > 4R$, and $W(x) = |g(x)(x-b)^l|^p V(x)$. Let Q be the least upper bound of all q such that*

$$(5.10)\qquad \int_{r<|x-b|<2r} W(x)^q\,dx \leqslant Cr^{1-q}\left[\int_{r<|x-b|<2r} W(x)\,dx\right]^q$$

for $2R < 2r < R_0$ and define $T = \min(2,(pQ')', s)$. Assume that $\lambda > l - 1 + 1/T$ and $m \in M(s,\lambda)$. If $\lambda < l + 1/T$, assume in addition that

$$(5.11)\qquad \left[\frac{|x-b|}{R+|x-b|}\right]^{p(l-\lambda+1/T)} V(x) \in A_p.$$

Then for all f in $\mathscr{S}$

$$(5.12)\qquad \int_{4R<|x-b|\leqslant R_0}\left|\int_{|y-b|<R} K(x,y)f(y)\,dy\right|^p W(x)\,dx \leqslant CB(m,s,\lambda)^p\int_{|x-b|<R}|f(x)|^p W(x)\,dx,$$

where $K(x, y)$ is as defined in (5.4) *and C is independent of f, m and N.*

LEMMA (5.13). *If $Tf(x) = \int_{-\infty}^{\infty} K(x,y)f(y)\,dy$, a and b are real, $U(x)$ and $W(x)$ are nonnegative and there is an A independent of h and r such that*

$$(5.14)\qquad \int_{r\leqslant|x-b|<2r}|Th(x)|^p|x-b|^a U(x)\,dx \leqslant A\int_{-\infty}^{\infty}|h(x)|^p|x-b|^a W(x)\,dx$$

for all h in C^∞ with support in $r/8 \leqslant |x-b| \leqslant 16r$, then for f in C^∞, $\|Tf\|_{p,U}^p$ is bounded by the sum of

$$(5.15)\quad C\int_0^\infty \left(\int_{|y-b|<r/4}\left[\int_{r/2<|x-b|<2r}|K(x,y)|^p U(x)\,dx\right]^{1/p}|f(y)|\,dy\right)^p \frac{dr}{r},$$

$$(5.16)\qquad CA\|f\|_{p,W}^p$$

and

$$(5.17)\quad C\int_0^\infty \left(\int_{|y-b|>4r}\left[\int_{r/2<|x-b|<2r}|K(x,y)|^p U(x)\,dx\right]^{1/p}|f(y)|\,dy\right)^p \frac{dr}{r},$$

where C is independent of f, K and W.

We will first prove Lemma (5.1) for f with support in $|x-b|>4R$ and $\lambda < l + 1/T$. For this we will prove the existence of a number q such that if $\tau = \min(2,(pq')', s)$ and $r \geqslant R$, then (5.2) holds, $q>1$,

$$(5.18)\qquad (R+|x-b|)^{p(l-\lambda+1/\tau)}V(x) \in A_p$$

and

$$(5.19)\qquad l-1+1/\tau < \lambda < l+1/\tau.$$

The existence of q is shown as follows. By Lemma (4.5), $Q>1$ and by Hölder's inequality (5.2) holds for $r \geqslant R$ for each q satisfying $1 \leqslant q < Q$. Since $\lim_{q\to Q}\tau = T$, the inequalities (5.19) will follow from the hypothesis $l-1+1/T<\lambda$ and the condition $\lambda < l+1/T$ if $1<q<Q$ and q is sufficiently close to Q. Furthermore, the hypothesis and Lemma (4.9) also imply (5.18) for q sufficiently close to Q. For the rest of this proof, q will denote a number satisfying these conditions.

We will now use Lemma (5.13) with $a = \alpha - lp$, $U(x) = W(x)\chi_{\{|x-b|>R\}}(x)$ and $K(x,y)$ as defined in (5.4). To prove inequality (5.14), fix an h in C^∞ with support in $r/8 \leqslant |x-b| \leqslant 16r$. By hypothesis, $\|K_N * h\|_{p,V_1} \leqslant BH\|h\|_{p,V_1}$; here and throughout the proof B will denote $B(m,s,\lambda)$. Since $V_1(x) = |x-b|^a W(x)$, this implies (5.14) with $A = C(BH)^p$ and Th replaced by $K_N * h$. To complete the proof of (5.14), we must estimate

$$\int_{r\leqslant|x-b|\leqslant 2r}|Th(x)-(K_N*h)(x)|^p|x-b|^a U(x)\,dx.$$

From the definition of $K(x,y)$, we see that this is bounded by a sum of terms of the form

$$(5.20)\qquad C\left|\int_{r/8\leqslant|y-b|\leqslant 16r}(b-y)^n h(y)\,dy\right|^p \times\left[\int_{r\leqslant|x-b|\leqslant 2r}|K_N^{(n)}(x-b)|^p|x-b|^a U(x)\,dx\right]$$

with $0 \leqslant n \leqslant l-1$. We will show that these terms are bounded by $Cr^\alpha B^p\|h\|_{p,V}^p$. This is sufficient since $r^\alpha V(x) \leqslant C|x-b|^a W(x)$ on the support of h.

Hölder's inequality shows that the first term in (5.20) has the bound

$$(5.21)\quad Cr^{pn}\left[\int_{-\infty}^{\infty}|h(y)|^{p}V(y)\,dy\right]\left[\int_{r/8\leqslant|y-b|\leqslant 16r}V(y)^{-1/(p-1)}\,dy\right]^{p-1}.$$

For the second term in (5.20), use Hölder's inequality with exponents q' and q. If $r \geqslant R$, use (5.2) to get the bound

$$(5.22)\quad Cr^{\alpha-1/q'}\left[\int_{r\leqslant|x-b|<2r}V(x)\,dx\right]\left[\int_{r\leqslant|x-b|\leqslant 2r}\left|K_N^{(n)}(x-b)\right|^{pq'}dx\right]^{1/q'}.$$

This bound also holds trivially if $r \leqslant R/2$ since the second term in (5.20) is 0 in this case. If $R/2 < r < R$, use the facts that

$$(5.23)\quad \int_{r\leqslant|x-b|\leqslant 2r}U(x)^{q}\,dx \leqslant CR^{lpq}\int_{R\leqslant|x-b|\leqslant 2R}V(x)^{q}\,dx,$$

(5.2) and then

$$(5.24)\quad \int_{R\leqslant|x-b|\leqslant 2R}V(x)\,dx \leqslant C\int_{r/2<|x-b|<2r}V(x)\,dx,$$

which follows from the hypothesis $V \in A_p$. These show that (5.22) is a bound for the second term in (5.20) for this case also and that, therefore, this bound is valid for all $r > 0$.

Since $n \leqslant l-1$, we have from (5.19) that $n < \lambda - 1/\tau$. Theorem (2.5) then shows that (5.22) is bounded by

$$(5.25)\quad CB^{p}r^{\alpha-p(n+1)}\int_{r\leqslant|x-b|\leqslant 2r}V(x)\,dx.$$

Multiplying (5.21) and (5.25) and using the definition of A_p then shows that (5.20) is bounded by $Cr^{\alpha}B^{p}\|h\|_{p,V}^{p}$ as asserted. This completes the proof that (5.14) holds with $A = C(1+H^{p})B^{p}$ and C independent of m, N and h.

To complete the proof of Lemma (5.1) for f with support in $|x-b| > 4R$ and $\lambda < l + 1/T$, we must show that (5.15) and (5.17) are bounded by the right side of (5.3). Because of the support restriction, it is sufficient to show that (5.15) and (5.17) are bounded by

$$(5.26)\quad C(1+H^{p})B^{p}\int_{-\infty}^{\infty}|f(x)|^{p}W(x)\,dx$$

for these f's.

To show this for (5.15), use Hölder's inequality with exponents q' and q and then use (5.2) to show that the inner integral in (5.15) has the bound

$$(5.27)\quad Cr^{pl-1+1/q}\left[\int_{r/2<|x-b|<2r}|K(x,y)|^{pq'}dx\right]^{1/q'}\int_{r/2<|x-b|<2r}V(x)\,dx$$

provided $r > 2R$. Since (5.19) holds and $|y-b| < r/4$ in (5.15), we can use Theorem (2.6) with $L = l-1$ to estimate (5.27); the result is

$$CB^{p}\left[\frac{|y-b|}{r}\right]^{p\lambda-p/\tau}r^{p(l-1)}\int_{r/2<|x-b|<2r}V(x)\,dx.$$

Now replace the inner integral in (5.15) with this estimate for $r > 2R$. Since $f(y)$ is 0 for $|y - b| < 4R$, we see that (5.15) is bounded by

$$CB^p \int_{16R}^{\infty} \left[\int_{r/2<|x-b|<2r} V(x)\,dx \right] g(r)^p r^{-1+p(l-1-\lambda+1/\tau)}\,dr,$$

where

$$g(r) = \int_{|y-b|<r/4} |y - b|^{\lambda - 1/\tau} |f(y)|\,dy.$$

Now interchange the order of the first two integrals to get

$$CB^p \int_{|x-b|>8R} \left[\int_{|x-b|/2<r<2|x-b|} r^{-1+p(l-1-\lambda+1/\tau)} g(r)^p\,dr \right] V(x)\,dx.$$

Next, replace $g(r)$ by $g(2|x - b|)$, which is larger, and perform the integration in r to obtain the bound

$$CB^p \int_{|x-b|>8R} g(2|x - b|)^p |x - b|^{p(l-1-\lambda+1/\tau)} V(x)\,dx.$$

Since $|x - b| > 8R$, we have

$$|x - b|^{p(l-\lambda+1/\tau)} V(x) \leqslant C(R + |x - b|)^{p(l-\lambda+1/\tau)} V(x),$$

and this is in A_p by (5.18). Lemma (4.1) then gives the bound

$$CB^p \int_{-\infty}^{\infty} |f(x)|^p |x - b|^{p(\lambda-1/\tau)} (R + |x - b|)^{p(l-\lambda+1/\tau)} V(x)\,dx.$$

Since f is supported on $|x - b| \geqslant 4R$, this is bounded by (5.26) as asserted.

To estimate (5.17), observe that the inner integral is the same as the inner integral in (5.15) and, therefore, has the bound (5.27) for $r > 2R$. For $R/2 < r < 2R$, the bound (5.27) is also valid; the proof uses the obvious modification of (5.23) and (5.24). Note also that for $r < R/2$ this integral is 0 by the definition of U. We will estimate the first integral in (5.27) for $|y - b| > 4r$ by estimating the integral of each term in the definition of $K(x, y)$. First

$$\int_{r/2<|x-b|<2r} |K_N(x - y)|^{pq'}\,dx \leqslant \int_{|y-b|/2<|u|<2|y-b|} |K_N(u)|^{pq'}\,du$$

since $|y - b| > 4r$. Then since (5.19) holds and $l \geqslant 1$, we can use Theorem (2.5) to get the estimate

$$\left[\int_{r/2<|x-b|<2r} |K_N(x - y)|^{pq'}\,dx \right]^{1/q'} \leqslant CB^p |y - b|^{-p+1/q'}.$$

The other terms are

$$|y - b|^{np} \left[\int_{r/2<|x-b|<2r} |K_N^{(n)}(x - b)|^{pq'}\,dx \right]^{1/q'} \leqslant CB^q \left(\frac{|y - b|}{r} \right)^{np} r^{-p+1/q'}$$

for $0 \leqslant n \leqslant l - 1$; the inequality follows from Theorem (2.5) by using (5.19). Since $|y - b| > 4r$, we can combine these to obtain

$$\left(\int_{r/2<|x-b|<2r} |K(x, y)|^{pq'}\,dx \right)^{1/q'} \leqslant CB^p \left(\frac{|y - b|}{r} \right)^{(l-1)p} r^{-p+1/q'}.$$

Using this in (5.27) and replacing the inner integral in (5.17) by the resulting estimate shows that (5.17) has the bound

$$CB^p\int_{R/2}^{\infty}\left[\int_{r/2<|x-b|<2r}V(x)\,dx\right]\left[\int_{|y-b|>4r}|y-b|^{l-1}|f(y)|\,dy\right]^p\frac{dr}{r}$$

since the integrand in (5.17) is 0 for $r \leqslant R/2$. Now, as in the estimation of (5.15), interchange the order of the first two integrals, enlarge the integration set in the y integral to $|y-b|>|x-b|$ and perform the integration in r. This gives the estimate

$$CB^p\int_{|x-b|>R/4}\left[\int_{|y-b|>|x-b|}|y-b|^{l-1}|f(y)|\,dy\right]^p V(x)\,dx.$$

Now since $V(x) \in A_p$, we can use Lemma (4.3) to show that (5.17) also has the bound (5.26). This completes the proof of Lemma (5.1) for f with support in $|x-b|>4R$ and $\lambda < l + 1/T$.

We will now consider the case with f's support in $|x-b|>4R$ and $\lambda \geqslant l+1/T$. For this, choose μ such that $(R+|x-b|)^{p(l-\mu+1/T)}V(x) \in A_p$ and $l-1+1/T < \mu < l+1/T$; this is possible by Lemma (4.9). By the case just proved with $\lambda = \mu$, the left side of (5.3) is bounded by $C(1+H^p)B(m,s,\mu)^p\|f\|_{p,W}^p$. Theorem (2.2) then completes the proof of this case.

Finally, we consider general f in $\mathscr{S}$. Given such an f, choose a sequence of functions f_n in $\mathscr{S}$ such that $f_n(x) = f(x)$ for $|x-b|>4R+1/n$, $f_n(x)=0$ for $|x-b|\leqslant 4R$ and $|f_n(x)| \leqslant |f(x)|$ for all x. Then f_n converges to $f\chi_{|x-b|>4R}$ in L_W^p and $\int_{-\infty}^{\infty}K(x,y)f_n(y)\,dy$ converges pointwise to $\int_{|y-b|>4R}K(x,y)f(y)\,dy$. Applying the case already proved to the f_n's and then using Fatou's lemma completes the proof of Lemma (5.1).

The proof of Lemma (5.5) is similar to the proof of Lemma (5.1), and only the differences will be described. For the case $\lambda < l+1/T$ and f with support in $|x-b|\leqslant R$, we choose $q>1$ such that if $\tau = \min(2,(pq')',s)$ and $0 \leqslant r \leqslant 2R$, then (5.2) holds,

$$\left(\frac{|x-b|}{R+|x-b|}\right)^{p(l-\lambda+1/\tau)}V(x) \in A_p \tag{5.28}$$

and (5.19) is true. Lemma (5.13) is applied with $a = \alpha - lp$ and $U(x) = W(x)\chi_{\{|x-b|<4R\}}(x)$ and the same $K(x,y)$. The proof of inequality (5.14) is the same.

To estimate (5.15), note first that the inner integral has the bound (5.27) for $r<8R$; for $r<2R$ this is done as before while for $2R<r<8R$ use is made of the facts

$$\int_{r/2<|x-b|<2r}U(x)^q\,dx \leqslant CR^{lpq}\int_{R\leqslant|x-b|\leqslant 4R}V(x)^q\,dx$$

and

$$\int_{R\leqslant|x-b|\leqslant 4R}V(x)\,dx \leqslant C\int_{r/2\leqslant|x-b|<2r}V(x)\,dx.$$

As in the last proof, (5.27) is estimated by using Theorem (2.6); this shows that (5.15) is bounded by

$$CB^p\int_0^{8R}\left(\int_{r/2<|x-b|<2r}V(x)\,dx\right)\times\left(\int_{|y-b|<r/4}|y-b|^{\lambda-1/\tau}|f(y)|\,dy\right)^p r^\beta\,dr,$$

where $\beta=-1+p(l-1-\lambda+1/\tau)$. The same procedure as before leads to the bound

$$CB^p\int_{|x-b|<16R}\left[\int_{|y-b|<|x-b|}|y-b|^{\lambda-1/\tau}|f(y)|\,dy\right]^p|x-b|^{\beta+1}V(x)\,dx.$$

Since $|x-b|<16R$,

$$|x-b|^{p(l-\lambda+1/\tau)}V(x)\leqslant C\left[\frac{|x-b|}{R+|x-b|}\right]^{p(l-\lambda+1/\tau)}V(x)$$

and this is in A_p by (5.28). Lemma (4.1) then gives the bound

$$CB^p\int_{-\infty}^{\infty}|f(x)|^p|x-b|^{p(\lambda-1/\tau)}\left[\frac{|x-b|}{R+|x-b|}\right]^{p(l-\lambda+1/\tau)}V(x)\,dx.$$

Since f is supported on $|x-b|\leqslant R$, this is bounded by $CB^p\|f\|_{p,W}^p$ as asserted.

For the estimation of (5.17), we obtain the bound (5.27) for the inner integral if $r<2R$ and, as before, show that (5.27) is bounded by

$$CB^p|y-b|^{(l-1)p}\int_{r/2<|x-b|<2r}V(x)\,dx$$

and (5.17) has the bound

$$CB^p\int_0^{R/4}\left[\int_{r/2<|x-b|<2r}V(x)\,dx\right]\left[\int_{|y-b|>4r}|y-b|^{l-1}|f(y)|\,dy\right]^p\frac{dr}{r}.$$

With the same procedure as before, we get the bound

$$CB^p\int_{|x-b|<R}\left[\int_{|y-b|>|x-b|}|y-b|^{l-1}|f(y)|\,dy\right]^pV(x)\,dx.$$

Since $V\in A_p$, Lemma (4.3) gives the needed bound. This completes the proof of Lemma (5.5) if $\lambda<l+1/T$ and f has support in $|x-b|\leqslant R$. The extension to $\lambda\geqslant l+1/T$ and general f in $\mathscr{S}$ is done as it was in the proof of Lemma (5.1).

To prove Lemma (5.7) for $\lambda<l+1/T$, define for $R_0>4R$ the quantity Q_0 as the least upper bound of all q for which (5.2) holds for $r>R_0$ and let $T_0=\min(2,(pQ_0')',s)$. Since $T_0\leqslant T$ and $T_0\to T$ as $R_0\to\infty$, choosing R_0 sufficiently large will make

(5.29) $$l-1+1/T_0<\lambda<l+1/T_0.$$

Because of Lemma (4.9), we will also have

(5.30) $$(R+|x-b|)^{p(l-\lambda+1/T_0)}V(x)\in A_p$$

for R_0 large. Similarly, if $\lambda < a + 1/T$, then

$$|x - b|^{p(l-\lambda+1/T_0)}V(x) \in A_p \tag{5.31}$$

for large R_0. Choose R_0 so that (5.29) and (5.30) hold and so that (5.31) holds if $\lambda < a + 1/T$. Note that R_0 is independent of f, m and N.

To estimate the left side of (5.8) for this R_0, choose q so that $1 < q < Q_0$ and (5.18) and (5.19) hold, where $\tau = \min(2, (pq')', s)$; this is possible by (5.29), (5.30) and Lemma (4.9) since $\tau \leqslant T_0$ and $\tau \to T_0$ as $q \to Q_0$. Furthermore, if $\lambda < a + 1/T$, choose q so that

$$|x - b|^{p(l-\lambda+1/\tau)}V(x) \in A_p; \tag{5.32}$$

this is possible by (5.31) and Lemma (4.9).

By Minkowski's integral inequality, the left side of (5.8) is bounded by

$$\left[\int_{|y-b|<R}\left[\int_{|x-b|>R_0} |K(x, y)|^p W(x)\,dx\right]^{1/p} |f(y)|\,dy\right]^p. \tag{5.33}$$

The inner integral is bounded by

$$C\sum_{n=0}^{\infty}\int_{2^nR_0<|x-b|<2^{n+1}R_0} |K(x, y)|^p [2^nR_0]^{lp}V(x)\,dx.$$

Now apply Hölder's inequality with exponents q' and q and then use (5.2) to show that this is bounded by

$$C\sum_{n=0}^{\infty} 2^{n(lp-1+1/q)}\left[\int_{2^nR_0\leqslant|x-b|<2^{n+1}R_0} |K(x, y)|^{pq'}\,dx\right]^{1/q'}$$
$$\times\int_{2^nR_0\leqslant|x-b|<2^{n+1}R_0} V(x)\,dx.$$

Since $4|y - b| < R_0$ and (5.19) holds, Theorem (2.6) gives the bound

$$C\sum_{n=0}^{\infty} B^p\left[\frac{|y-b|}{2^n}\right]^{p\lambda-p/\tau} 2^{np(l-1)}\int_{2^nR_0\leqslant|x-b|<2^{n+1}R_0} V(x)\,dx.$$

This is bounded by

$$CB^p|y - b|^{p\lambda-p/\tau}\int_{|x-b|\geqslant R_0}\frac{[R + |x - b|]^{p(l-\lambda+1/\tau)}}{|x - b|^p}V(x)\,dx.$$

Lemma 1 of **[10]** and (5.18) then show that this has the bound

$$CB^p|y - b|^{p\lambda-p/\tau}\int_{R_0\leqslant|x-b|<2R_0} V(x)\,dx. \tag{5.34}$$

Now replace the inner integral in (5.33) with (5.34). This shows that (5.33) is bounded by a constant times the product of

$$\left[\int_{|y-b|<R} |y - b|^{\lambda-1/\tau}|f(y)|\,dy\right]^p \tag{5.35}$$

and

$$B^p \int_{R_0 \leqslant |x-b| < 2R_0} V(x)\,dx. \tag{5.36}$$

If $\lambda \geqslant a + 1/T$, then since $\lambda < l + 1/T$ we conclude that $a \leqslant l - 1$. By (5.19) we have $\lambda - 1/\tau > a$ and, therefore, $|y-b|^{\lambda - 1/\tau} \leqslant C[W(y)/V(y)]^{1/p}$ on $|y-b| < R$. Using this and Hölder's inequality, we see (5.35) is bounded by

$$C\left[\int_{|y-b|<R} V(y)^{-1/(p-1)}\,dy\right]^{p-1} \int_{|y-b|<R} |f(y)|^p W(y)\,dy.$$

Now multiply by (5.36), enlarge the interval of integration in the two integrals containing V to $(b - 2R_0, b + 2R_0)$ and use the definition of A_p. This shows that the left side of (5.8) is bounded by the right side of (5.8) if $\lambda \geqslant a + 1/T$ and $\lambda < l + 1/T$. If $\lambda < a + 1/T$, then since $\lambda > l - 1 + 1/T$ and $a \leqslant l$, we conclude that $a = l$. Hölder's inequality then shows that (5.35) is bounded by

$$C\left[\int_{|y-b|<R} \left[|y-b|^{p(l-\lambda+1/\tau)} V(y)\right]^{-1/(p-1)} dy\right]^{p-1} \int_{|y-b|<R} |f(y)|^p W(y)\,dy.$$

Now (5.36) is bounded by

$$CB^p \int_{|x-b|<2R_0} |x-b|^{p(l-\lambda+1/\tau)} V(x)\,dx;$$

multiplying these two expressions and using (5.32) then completes the proof of Lemma (5.7) for $\lambda < l + 1/T$. The extension to $\lambda \geqslant l + 1/T$ is done as it was for Lemma (5.1).

To prove Lemma (5.9) for $\lambda < l + 1/T$, use Lemma (4.9) to choose a q such that $1 < q < Q$, (5.19) holds and

$$\left[\frac{|x-b|}{R + |x-b|}\right]^{p(l-\lambda+1/\tau)} V(x) \in A_p, \tag{5.37}$$

where $\tau = \min(2, (pq')', s)$. The left side of (5.12) is bounded by

$$\left[\int_{|y-b|<R} \left(\int_{4R<|x-b|\leqslant R_0} |K(x,y)|^p W(x)\,dx\right)^{1/p} |f(y)|\,dy\right]^p. \tag{5.38}$$

Following the procedure used to estimate (5.33), using (5.10) in place of (5.2), shows that the inner integral in (5.38) is bounded by

$$CB^p |y-b|^{p\lambda - p/\tau} \int_{4R<|x-b|\leqslant R_0} W(x)\,dx. \tag{5.39}$$

Since the integral in (5.39) is finite, we need only show that

$$CB^p \left[\int_{|y-b|<R} |y-b|^{\lambda - 1/\tau} |f(y)|\,dy\right]^p \tag{5.40}$$

is bounded by the right side of (5.12).

Since $|y-b|^{lp}V(x) \leqslant CW(y)$ on $|y-b| < R$, Hölder's inequality applied to (5.40) gives the bound

$$CB^p\left[\int_{|y-b|<R}\left[|y-b|^{p(l-\lambda+1/T)}V(y)\right]^{-1/(p-1)}dy\right]^{p-1}\int_{|y-b|<R}|f(y)|^p W(y)\,dy.$$

The first integral is finite because of (5.37). This completes the proof of Lemma (5.9) if $\lambda < l + 1/T$; the extension to $\lambda \geqslant l + 1/T$ is done as it was for Lemma (5.1).

6. Results for general weights. Here we prove results of the form $\|(m\hat{f})^\vee\|_{p,W} \leqslant C\|f\|_{p,W}$ for general W. The W's will all have the form $|g(x)|^p V(x)$, where $g(x)$ is a polynomial and $V(x)$ is in A_p. As mentioned in §1, this form is necessary if such inequalities are to hold for Schwartz functions f with a fixed number of zero moments and $m(x) = \operatorname{sgn}(x)$. This form is also a necessary condition for the inequality with $m(x) = \operatorname{sgn}(x)$ for all f in $\mathscr{S}_{0,0}$ if the weight satisfies a doubling condition.

The general results proved here are Theorem (6.1) for $\lambda \geqslant 1$ and Theorem (6.5) for $\frac{1}{2} < \lambda < 1$. Theorem (6.7) concerns W of the form $(1+|x|)^a \prod_{j=1}^J |x-b_j|^{a_j}$; it is the case $\lambda > \frac{1}{2}$ of Theorem (1.4). Theorems (6.1) and (6.7) are stated for f in $\mathscr{S}_{0,0}$ and Theorem (6.5) for f in $\mathscr{S}$. As shown in §7, however, the conditions on f can be weakened.

THEOREM (6.1). *Assume that* $1 \leqslant s \leqslant \infty$, $1 < p < \infty$, $\lambda \geqslant 1$, $l_0 \geqslant 0$, $m \in M(s,\lambda)$, $V(x) \in A_p$, $g(x)$ *is a polynomial of degree* l_0 *and* $W(x) = |g(x)|^p V(x)$. *Let* Q_0 *be the least upper bound of all* q *for which there are constants* C *and* R *such that with* $b = 0$

$$\int_{r\leqslant|x-b|\leqslant 2r} W(x)^q\,dx \leqslant Cr^{1-q}\left[\int_{r\leqslant|x-b|\leqslant 2r} W(x)\,dx\right]^q \tag{6.2}$$

for all $r > R$. *Let* Q_i *be the least upper bound of all* q *for which* (6.2) *holds for all* $r > 0$ *with* $b = b_i$, *where* $\{b_i\}_{i=1}^J$ *is the set of distinct real roots of* g. *For* $0 \leqslant i \leqslant J$ *let* $T_i = \min(2,(pQ_i')', s)$, *for* $1 \leqslant i \leqslant J$ *let* l_i *be the multiplicity of the root* b_i *in* g, *and for* $0 \leqslant i \leqslant J$ *assume that* $\lambda > l_i - 1 + 1/T_i$. *If* $\lambda < l_0 + 1/T_0$, *assume in addition that*

$$(1+|x|)^{p(l_0-\lambda+1/T_0)}V(x) \in A_p; \tag{6.3}$$

if $\lambda < l_i + 1/T_i$ *for* $1 \leqslant i \leqslant J$, *assume in addition that*

$$\left[\frac{|x-b_i|}{1+|x-b_i|}\right]^{p(l_i-\lambda+1/T_i)} V(x) \in A_p. \tag{6.4}$$

Then for every f *in* $\mathscr{S}_{0,0}$,

$$\|(m\hat{f})^\vee\|_{p,W} \leqslant CB(m,s,\lambda)\|f\|_{p,W},$$

where C *is independent of* m *and* f.

Note that Theorem (1.2) is an immediate corollary of this for $\lambda \geqslant \frac{1}{2}$.

For $\lambda < 1$, Theorem (3.2) provides norm inequalities with W in A_p. If $\lambda < 1$ and the weight function has the form $|x|^\alpha$, Theorem (1.4) of **[16]** shows that $\alpha < -1 + 3p/2$. This suggests that if $\lambda < 1$, V is in A_p, g is a polynomial and $|g(x)|^p V(x)$ is a weight function for $(m\hat{f})^\vee$ for all m in $M(s,\lambda)$, then g has degree at most one. Note also that if $1 < \lambda \leqslant \frac{3}{2}$, Theorem (6.1) requires that g's degree is at most one. Norm inequalities for weights of this type are provided by using the following theorem with Theorem (3.2).

THEOREM (6.5). *Assume that* $1 \leqslant s \leqslant \infty$, $1 < p < \infty$, $\lambda < 1$, $m \in M(s,\lambda)$ *and* $V(x) \in A_p$. *Let* $W(x) = |g(x)|^p V(x)$, *where* $g(x)$ *is a polynomial of degree* 1, *and assume for every* $m \in M(s,\lambda)$ *and* $h \in \mathscr{S}$ *that* $\|(m\hat{h})^\vee\|_{p,V_1} \leqslant HB(m,s,\lambda)\|h\|_{p,V_1}$, *where* $V_1(x) = |g(x)|^\beta V(x)$ *and* β *and* H *are constants independent of* f *and* m *with* $\beta \leqslant p$. *Let* Q_0 *be the least upper bound of all* q *for which there are constants* C *and* R *such that* (6.2) *holds with* $b = 0$ *and* $r > R$. *Let* $T_0 = \min(2,(pQ_0')', s)$ *and assume that* $\lambda > 1/T_0$ *and* $(1+|x|)^{p(1-\lambda+1/T_0)}V(x) \in A_p$. *If the root* b *of* $g(x)$ *is real, let* Q *be the least upper bound of all* q *for which* (6.2) *holds for* $r > 0$; *let* $T = \min(2,(pQ')', s)$ *and assume that* $\lambda > 1/T$ *and*

$$\left[\frac{|x-b|}{1+|x-b|}\right]^{p(1-\lambda+1/T)} V(x) \in A_p. \tag{6.6}$$

Then for every f *in* $\mathscr{S}$ *with* $\hat{f}(0) = 0$ *and every* m *in* $M(s,\lambda)$, *we have*

$$\|(m\hat{f})^\vee\|_{p,W} \leqslant C(H+1)B(m,s,\lambda)\|f\|_{p,W},$$

where C *is independent of* m *and* f.

An important consequence of Theorems (6.1), (6.5) and (3.3) to be proved in this section is the following. This is Theorem (1.4) for $\lambda > \frac{1}{2}$.

THEOREM (6.7). *Assume that* $1 \leqslant s \leqslant \infty$, $1 < p < \infty$, $\lambda > \max(\frac{1}{2}, \frac{1}{s})$, *or* $\lambda = s = 1$, $m \in M(s,\lambda)$, $W(x) = (1+|x|)^a \prod_{j=1}^J |x-b_j|^{a_j}$, *where the* b_j*'s are real and distinct. Define* $t = \min(2, p', s)$, $a_0 = a + \sum_{j=1}^J a_j$ *and* $l_j = \operatorname{int}[(a_j+1)/p]$. *Assume that* $\sum_{j=1}^J l_j \leqslant l_0$ *and for* $0 \leqslant j \leqslant J$ *that* $a_j > -1$, $a_j > -\lambda p$, $(a_j+1)/p$ *is not an integer, and* $a_j < (\lambda + 1 - 1/t)p - 1$. *For* $1 \leqslant i, j \leqslant J$, *assume that* $|a_i - a_j| < p\lambda$. *Then for* f *in* $\mathscr{S}_{0,0}$,

$$\|(m\hat{f})^\vee\|_{p,W} \leqslant CB(m,s,\lambda)\|f\|_{p,W},$$

where C *is independent of* m *and* f.

Theorem (6.1) will be proved by showing that for f in $\mathscr{S}_{0,0}$,

$$\|K_N * f\|_{p,W}^p \leqslant CB(m,s,\lambda)^p \|f\|_{p,W}^p, \tag{6.8}$$

where C is independent of f, m and N. This is sufficient since for f in $\mathscr{S}_{0,0}$, $K_N * f = \sum_{j=-N}^N [m_j(x)\hat{f}(x)]^\vee$ equals $(m\hat{f})^\vee$ for sufficiently large N. We may assume $l_0 \geqslant 1$ since for $l_0 = 0$ Theorem (6.1) is an immediate consequence of Theorem (3.1).

Now fix d with $d > 0$ and $4d < \min_{i \neq j}|b_i - b_j|$ if $J \geqslant 2$. Let $f_i(x)$, $0 \leqslant i \leqslant J$, be the functions whose existence is asserted by Lemma (4.11); if g has no real roots, take $f_0 = f$ and $J = 0$. Since by Lemma (4.11) we have $\int_{-\infty}^{\infty} x^j f_i(x)\,dx = 0$ for $0 \leqslant j \leqslant l_0 - 1$,

$$(6.9) \quad (K_N * f_i)(x) = \int_{-\infty}^{\infty} \left[K_N(x-y) - \sum_{n=0}^{L} \frac{(b-y)^n}{n!} K_N^{(n)}(x-b) \right] f_i(y)\,dy$$

for any $L \leqslant l_0 - 1$ and any b.

By Minkowski's inequality, the left side of (6.8) is bounded by a constant times the sum of

$$(6.10) \qquad \int_{-\infty}^{\infty} |(K_N * f_0)(x)|^p W(x)\,dx,$$

$$(6.11) \qquad \sum_{i=1}^{J} \int_{|x-b_i|>8d} |(K_N * f_i)(x)|^p W(x)\,dx$$

and

$$(6.12) \qquad \sum_{i=1}^{J} \int_{|x-b_i|\leqslant 8d} |(K_N * f_i)(x)|^p W(x)\,dx.$$

The proof of Theorem (6.1) will be completed by showing that (6.10)–(6.12) are bounded by the right side of (6.8).

To estimate (6.10), we must first choose a large number R_m as follows. Given $R_m > 0$, let Q_m be the least upper bound of all q for which (6.2) holds with $b = 0$ and $r > R_m$ and let $T_m = \min(2, (pQ_m')', s)$. Since $T_m \to T_0$ as $R_m \to \infty$, if R_m is sufficiently large we will have

$$(6.13) \qquad \lambda > l_0 - 1 + 1/T_m.$$

We also want to choose R_m so that if $\lambda < l_0 + 1/T_m$, then

$$(6.14) \qquad (1 + |x|)^{p(l_0 - \lambda + 1/T_m)} V(x) \in A_p.$$

To do this if $\lambda > l_0 + 1/T_0$ we will choose R_m so that $\lambda > l_0 + 1/T_m$ and (6.14) is not needed. If $\lambda \leqslant l_0 + 1/T_0$, then since $(1 + |x|)^{p(l_0 - \lambda + 1/T_0)} V(x) \in A_p$, we will have (6.14) for sufficiently large R_m by Lemma (4.9). Choose R_m so large that (6.13) holds, (6.14) holds if $\lambda < l_0 + 1/T_m$, and $|g(x)|$ has a positive lower bound on $|x| \geqslant R_m$.

To estimate (6.10), we will first use Lemma (5.7) to show the existence of $R_0 > 16R_m$ and C independent of f, m and N such that

$$(6.15) \quad \int_{|x|>R_0} \left| \int_{|y|<4R_m} K(x,y) f_0(y)\,dy \right|^p W(x)\,dx \leqslant CB^p \int_{|y|<4R_m} |f_0(y)|^p W(y)\,dy,$$

where $K(x,y)$ is as in (5.4) with $l = l_0$. To do this, take l, a, b, R and V in Lemma (5.7) to be respectively l_0, 0, 0, R_m and V. Then Q and T of Lemma (5.7) are Q_0 and T_0 respectively, and the hypotheses of Lemma (5.7) are restatements of those of

Theorem (6.1). The conclusion of Lemma (5.7) gives (6.15) since $[R + |x - b|]^{lp} V(x) \leqslant C|g(x)|^p V(x)$ on the support of f_0.

Using (6.9) with $L = l_0 - 1$, then (6.15) and (4.12), we see that the proof that (6.10) is bounded by the right side of (6.8) can be completed by estimating

$$\int_{|x| \leqslant R_0} |(K_N * f_0)(x)|^p W(x)\, dx \tag{6.16}$$

and

$$\int_{|x| \geqslant R_0} \left| \int_{|y| \geqslant 4R_m} K(x, y) f_0(y)\, dy \right|^p W(x)\, dx \tag{6.17}$$

where $K(x, y)$ is as in (5.4) with $l = l_0$. For (6.16), use the fact that g is bounded on $|x| \leqslant R_0$ to obtain the bound

$$\int_{|x| \leqslant R_0} |(K_N * f_0)(x)|^p V(x)\, dx. \tag{6.18}$$

By Theorem (3.1) and Theorem (2.3), this has the bound $CB^p \|f_0\|_{p,V}^p$ since $V \in A_p$. Since $V(x) \leqslant CW(x)$ on the support of f_0 and (4.12) holds, it follows that (6.16) is bounded by the right side of (6.8).

We will apply Lemma (5.1) to (6.17) with $R = R_m$, $l = l_0$, $b = 0$, $Q = Q_m$, $T = T_m$, $\alpha = 0$ and the same V. Note that the W of Theorem (6.1) is not the W of Lemma (5.1). The inequality $\|K_N * h\|_{p,V} \leqslant CB(m, s, \lambda)\|h\|_{p,V}$ follows from Theorem (3.1) and Theorem (2.3). The other hypotheses of Lemma (5.1) are properties of R_m established before. Lemma (5.1) shows that (6.17) is bounded by $CB^p \|f_0\|_{p,W}^p$ since $V(x) \leqslant CW(x)$ on $|x| \geqslant 4R_m$. Inequality (4.12) shows this is bounded by the right side of (6.8); this completes the estimation of (6.10).

To estimate the terms in (6.11), fix an i and first apply Lemma (5.7) with $l = l_0$, $a = l_i$, $b = b_i$, $R = 2d$, $Q = Q_0$ and $T = T_0$. That $\lambda > l_0 - 1 + 1/T_0$ and that (6.3) holds if $\lambda < l_0 + 1/T_0$ are hypothesized in Theorem (6.1). If $\lambda < l_i + 1/T_0$, then $l_i = l_0$. Since $T_i \leqslant T_0$, the hypothesis asserts that (6.4) holds. Then since $0 \leqslant p(l_0 - \lambda + 1/T_0) \leqslant p(l_i - \lambda + 1/T_i)$, Lemma (4.8) shows that

$$\left[\frac{|x - b_i|}{1 + |x - b_i|} \right]^{p(l_0 - \lambda + 1/T_0)} V(x) \in A_p.$$

Combining this with (6.3) and using Lemma (4.7) then shows that

$$|x - b_i|^{p(l_i - \lambda + 1/T_0)} V(x) \in A_p.$$

Thus, the hypotheses of Lemma (5.7) are verified. Using (6.9) with $L = l_0 - 1$, and the facts that $W(x) \leqslant C|x - b_i|^{pl_i}[R + |x - b_i|]^{p(l_0 - l_i)} V(x)$ for all x and the reverse inequality holds on the support of f_i, we obtain an R_i, independent of N, m and f, such that

$$\int_{|x - b_i| > R_i} |(K_N * f_i)(x)|^p W(x)\, dx \leqslant CB^p \int_{-\infty}^{\infty} |f_i(x)|^p W(x)\, dx.$$

Inequality (4.12) is then used to show that this is bounded by the right side of (6.8).

To complete the consideration of (6.11), we must show that

$$(6.19)\quad \int_{8d<|x-b_i|<R_i} |(K_N * f_i)(x)|^p W(x)\,dx \leqslant CB^p \int_{-\infty}^{\infty} |f(x)|^p W(x)\,dx.$$

For this we will apply Lemma (5.9) with $R = 2d$, $R_0 = R_i$, $l = l_i$, $b = b_i$ and $g(x)$ the $g(x)$ of Theorem (6.1) divided by $(x - b_i)^{l_i}$. Take Q to be the least upper bound of all q for which (6.2) holds with $b = b_i$ and $4d < 2r < R_i$; then $Q \geqslant Q_i$ and $T = \min(2, (pQ')', s) \geqslant T_i$. The hypotheses of Lemma (5.9) are then immediate consequences of the hypotheses of Theorem (6.1). Using (6.9) with $L = l_i - 1$ on the left side of (6.19) and then applying Lemma (5.9) and (4.12) completes the proof of (6.19).

For the estimate of (6.12) we will apply Lemma (5.5) to each term with $R = 2d$, $l = l_i$, $b = b_i$, Q the least upper bound of all q for which (6.2) holds with $b = b_i$ and $0 \leqslant r \leqslant 4d$, $T = \min(2, (pQ')', s)$ and $\alpha = 0$. Then $Q \geqslant Q_i$ and $T \geqslant T_i$; from this and the hypothesis $\lambda > l_i - 1 + 1/T_i$ we conclude that $\lambda > l_i - 1 + 1/T$. The inequality $\|K_N * h\|_{p,V} \leqslant CB(m, s, \lambda)\|h\|_{p,V}$ follows from Theorem (3.1) and Theorem (2.3). If $\lambda < l_i + 1/T_i$, then (6.4) holds by hypothesis. Therefore, using (6.9) with $L = l_i - 1$, the fact that $|g(x)| \leqslant |x - b_i|^{l_i}$ for $|x - b_i| \leqslant 8d$ and Lemma (5.5) shows that the ith term in (6.12) is bounded by

$$CB^p \int_{|x-b_i|<2d} |f_i(x)|^p |x - b_i|^{pl_i} V(x)\,dx.$$

Since b_i is the only root of $g(x)$ in $|x - b_i| < 4d$, we can use (4.12) to show that this is bounded by the right side of (6.8). This completes the proof of Theorem (6.1).

The proof of Theorem (6.5) is like the proof of Theorem (6.1). As in Theorem (6.1), it is sufficient to prove (6.8). The same decomposition of f is used; note that there will be at most two functions in this decomposition and that d can be taken to be 1. The proof is completed by estimating (6.10)–(6.12).

The estimation of (6.10) is the same as in Theorem (6.1) until we reach the estimation of (6.16). For this, instead of (6.18), we use the bound

$$\int_{|x|\leqslant R_0} |(K_N * f_0)(x)|^p V_1(x)\,dx;$$

this is valid because the boundedness of g on $|x| \leqslant R_0$ and the fact that $\beta \leqslant p$ imply $W(x) \leqslant CV_1(x)$ on $|x| \leqslant R_0$. By the hypothesis and Theorem (2.3), this has the bound $CB^pH^p\|f_0\|_{p,V_1}^p$. Since $\beta \leqslant p$, we have $V_1(x) \leqslant CW(x)$ on the support of f_0, and it follows that (6.16) is bounded by the right side of (6.8).

For (6.17), Lemma (5.1) is applied with $R = R_m$, $l = 1$, $b = 0$, $Q = Q_m$, $T = T_m$ and $\alpha = \beta$. As before, this gives the bound $CB^p\|f_0\|_{p,W}^p$; this completes the estimation of (6.10).

There can be at most one term in (6.11); if there is one, then b_1 is real and $l_1 = 1$. The proof that (6.11) is bounded by the right side of (6.8) is done as it was in the proof of Theorem (6.1). In (6.12) there is also at most one term, and if there is one, then b_1 is real and $l_1 = 1$. We apply Lemma (5.5) with $R = 2$, $l = 1$, $b = b_1$, Q the least upper bound of all q for which (6.2) holds with $b = b_1$ and $0 \leqslant r \leqslant 4$,

$T = \min(2, (pQ')', s)$ and $\alpha = \beta$. The inequality $\|K_N * h\|_{p,V_1} \leqslant CB(m, s, \lambda)\|h\|_{p,V_1}$ for $h \in \mathcal{S}$ follows from the hypothesis $\|(m\hat{h})^\vee\|_{p,V_1} \leqslant CB(m, s, \lambda)\|h\|_{p,V_1}$ and Theorem (2.3). This part is completed in the same way as it was in the proof of Theorem (6.1). This completes the proof of Theorem (6.5).

To prove Theorem (6.7), we will assume that $W(x)$ has the equivalent form $|x + i|^a \prod_{j=1}^J |x - b_j|^{a_j}$, where $i = \sqrt{-1}$. Define $L = l_0 - \sum_{j=1}^J l_j$, $g(x) = (x + i)^L \prod_{j=1}^J (x - b_j)^{l_j}$ and $V(x) = W(x)|g(x)|^{-p}$. The hypothesis insures that $L \geqslant 0$ and that, therefore, $g(x)$ is a polynomial. The degree of g is l_0. It is easy to see by Lemma (4.7) that a function of the form $|x + i|^u \prod_{j=1}^J |x - b_j|^{u_j}$, where the b_j are real and distinct, is in A_p if and only if $-1 < u_j < p - 1$ for $1 \leqslant j \leqslant J$ and $-1 < u + \sum_{j=1}^J u_j < p - 1$. When written in this form, V has $u_j = a_j - pl_j$ for $1 \leqslant j \leqslant J$, and since $u = a - pL$, we have

$$u + \sum_{j=1}^J u_j = a_0 - \sum_{j=1}^J a_j - p\left(l_0 - \sum_{j=1}^J l_j\right) + \sum_{j=1}^J (a_j - pl_j) = a_0 - pl_0.$$

To show that V is in A_p, therefore, we need to show that $-1 < a_j - pl_j < p - 1$ for $0 \leqslant j \leqslant J$. This follows from the definition of l_j and the fact that $(1 + a_j)/p$ is not an integer; therefore V is in A_p.

For the case $\lambda \geqslant 1$, we will apply Theorem (6.1) with this $V(x)$ and $g(x)$. Since (6.2) holds for all q if $r > 1 + 2\max(|b_i|)$, we have $Q_0 = \infty$ and $T_0 = t$. For $1 \leqslant i \leqslant J$, (6.2) will hold for $r > 0$ with $b = b_i$ provided $qa_j > -1$ for $1 \leqslant j \leqslant J$ and $j \neq i$. Therefore, $Q_i = 1/\max_{1 \leqslant j \neq i}(0, -a_j)$ and

$$T_i = \min\left(t, \frac{p}{p - 1 + \max_{1 \leqslant j \neq i}(0, -a_j)}\right)$$

for $1 \leqslant i \leqslant J$. To show that $\lambda > l_i - 1 + 1/T_i$ for $0 \leqslant i \leqslant J$, observe that since $l_i < (a_i + 1)/p$, it is sufficient to prove that

$$p\lambda > a_i + 1 - p + p/T_i \tag{6.20}$$

for $0 \leqslant i \leqslant J$. If $T_i = t$, this is hypothesized; otherwise, $i \geqslant 1$, $\max_{1 \leqslant j \neq i}(0, -a_j) > 0$ and (6.20) reduces to $p\lambda > a_i + \max_{1 \leqslant j \neq i}(-a_j)$. This holds since $|a_i - a_j| < p\lambda$ for $1 \leqslant i, j \leqslant J$.

If $0 \leqslant i \leqslant J$ and $\lambda < l_i + 1/T_i$, we must verify that

$$-1 < p(l_i - \lambda + 1/T_i) + a_i - pl_i < p - 1$$

to prove (6.3) if $i = 0$ or (6.4) if $i \neq 0$. The left inequality follows from the facts that $\lambda < l_i + 1/T_i$ and $a_i + 1 \geqslant pl_i$. If $T_i = t$, the right inequality follows from the assumption $a_i < -1 + p(\lambda + 1 - 1/t)$. If $T_i \neq t$, then $i \geqslant 1$ and the right inequality reduces to $-p\lambda + \max_{1 \leqslant j \neq i}(-a_j) + a_i < 0$ which follows from the assumption $|a_i - a_j| < p\lambda$ for $1 \leqslant i,\ j \leqslant J$. This completes the proof that W satisfies the conditions of Theorem (6.1) for the case $\lambda \geqslant 1$; Theorem (6.1) then implies the conclusion of Theorem (6.7) in this case.

For the case $\max(\frac{1}{2}, \frac{1}{s}) < \lambda < 1$, we will use Theorems (3.3) and (6.5). Note first that since $t \leqslant p'$, the condition $a_j < -1 + p(\lambda + 1 - 1/t)$ implies $a_j < \lambda p$. If $l_j = 0$ for $0 \leqslant j \leqslant J$, we also have $a_j < p - 1$ for $0 \leqslant j \leqslant J$ and $W(x)$ satisfies the hypotheses of Theorem (3.3). The conclusion of Theorem (6.7) then follows from Theorem (3.3) in this case.

There remains the case $\lambda < 1$ and $l_j > 0$ for some j satisfying $0 \leqslant j \leqslant J$. We will show for this case that $W(x)$ satisfies the hypotheses of Theorem (6.5). Since $\lambda < 1$, we have $a_0 < -1 + p(\lambda + 1 - 1/t) < 2p - 1$, and, therefore, $l_0 \leqslant 1$. Since $l_0 \geqslant \sum_{j=1}^{J} l_j$ and $l_j \geqslant 0$ for $0 \leqslant j \leqslant J$, we must have $l_0 = 1$ and at most one other l_j can equal 1 while the rest are 0. Therefore, $g(x)$ has degree 1. It was shown previously that $V(x)$ is in A_p.

If $l_k = 1$ for some $k > 0$, define $V_1(x) = |x - b_k|^{-p\lambda} W(x) = |x - b_k|^{p-p\lambda} V(x)$; otherwise, let $V_1(x) = |x + i|^{-p\lambda} W(x) = |x + i|^{p-p\lambda} V(x)$. We will prove that $\|(m\hat{h})^\vee\|_{p,V_1} \leqslant CB\|h\|_{p,V_1}$ for h in $\mathscr{S}$ by showing that V_1 satisfies the hypotheses of Theorem (3.3). Since $t \leqslant p'$, we have $a_j < -1 + p(\lambda + 1 - 1/t) \leqslant p\lambda$ and the exponents that are the same in V_1 and W clearly satisfy the hypotheses of Theorem (3.3). Since $p - 1 < a_0 < p\lambda$, we also obtain $\max(-1, -p\lambda) < a_0 - p\lambda < \min(p - 1, \lambda p)$. The same holds with a_0 replaced by a_k if $l_k = 1$. If $k \geqslant 1$, $l_k = 1$, $j \geqslant 1$ and $j \neq k$, we must also show that $-p\lambda < a_j - a_k + p\lambda < p\lambda$; this follows from the hypothesis $|a_j - a_k| < p\lambda$ and the fact that since $l_j = 0$, we have $a_j < a_k$. Also note that the β of Theorem (6.5) equals $p - p\lambda \leqslant p$.

The Q_0 of Theorem (6.5) is ∞ and $T_0 = t$. Since $l_0 = 1$ and $t \leqslant p'$, we have $p - 1 < a_0 < -1 + p(\lambda + 1 - 1/p')$, which implies $\lambda > 1/p'$ and $\lambda > 1/t = 1/T_0$. The requirement $(1 + |x|)^{p(1-\lambda+1/T_0)} V(x) \in A_p$ will be satisfied provided $-1 < p(1 - \lambda + 1/t) + a_0 - p < p - 1$. The left inequality follows because $\lambda < 1$ and $a_0 > p - 1$; the right side is a consequence of $a_0 < -1 + p(1 + \lambda - 1/t)$.

If the root of $g(x)$ is real, it equals b_k for the k with $l_k = 1$, and $a_k > p - 1 > 0$. In this case, $Q = 1/\max_{j \geqslant 1}(0, -a_j)$ and $T = \min(t, p/(p - 1 - \min_{j \geqslant 1}(0, a_j)))$. To show $\lambda > 1/T$, it is sufficient to prove $\lambda > 1/t$ and $\lambda p > p - 1 - \min_{j \geqslant 1}(0, a_j)$. That $\lambda > 1/t$ was shown in the preceding paragraph; $\lambda p > p - 1$ is a consequence of $\lambda > 1/t \geqslant 1/p'$. The inequality $\lambda p > p - 1 - a_j$ for $j \geqslant 1$ follows from the facts that $a_k - a_j < p\lambda$ for $j \geqslant 1$ and $a_k > p - 1$. To prove (6.6), we need to show that

$$-1 < p(1 - \lambda + 1/T) + a_k - p < p - 1.$$

The left inequality follows from $\lambda < 1$ and $a_k > p - 1$. If $T = t$, the right inequality follows from $a_k < -1 + p(1 + \lambda - 1/t)$. If $T < t$, the right inequality becomes $-\lambda p - \min_{j \geqslant 1} a_j + a_k < 0$, which follows from $|a_k - a_j| < \lambda p$ for $1 \leqslant j, k \leqslant J$.

This completes the verification that W satisfies the hypotheses of Theorem (6.5) in this case. Since an f in $\mathscr{S}_{0,0}$ has $\hat{f}(0) = 0$, the conclusion of Theorem (6.5) implies the conclusion of Theorem (6.7) for this case. This completes the proof of Theorem (6.7).

7. Extensions. As mentioned in §§5 and 6, the basic norm inequalities can be extended to more general classes of functions than $\mathscr{S}$ or $\mathscr{S}_{0,0}$ by using density theorems. This section contains a few such results and comments on the possibility of further extensions.

To state our results we need the following definition. Let Q_k be the set of functions in $L^2 \cap L^2_{|x|^k}$ such that $\int_{-\infty}^{\infty} f(x)x^j\,dx = 0$ for $0 \leqslant j \leqslant k$ and define Q_{-1} to be L^2. The main density result is the following.

THEOREM (7.1). *If $1 < p < \infty$, $l \geqslant 0$, $g(x)$ is a polynomial of degree l, $V(x) \in A_p$, $W(x) = |g(x)|^p V(x)$ and f is in $L^p_W \cap Q_{l-1}$, then there is a sequence of functions in $\mathscr{S}_{0,0}$ that converges to f in L^2 and L^p_W.*

To prove this, observe first that the definition of A_p implies that $\int_{-n}^{n} V(x)\,dx \leqslant Cn^p$ for $n \geqslant 1$. Since any V in A_p is also in A_r for some $r < p$, we also have $\int_{-n}^{n} V(x)\,dx = o(n^p)$ as $n \to \infty$. From this, $\int_{-n}^{n} W(x)\,dx = o(n^{lp+p})$, and Theorem (6.13) of **[17]** gives the asserted sequence.

The following is the principal extension theorem; it can be applied to Theorems (1.2), (1.4), (3.2), (3.3), (6.1), (6.5) and (6.7).

THEOREM (7.2). *If $1 < p < \infty$, $l \geqslant 0$, $m(x)$ is bounded, $V(x) \in A_p$, $g(x)$ is a polynomial of degree l, $W(x) = |g(x)|^p V(x)$ and*

$$\|(m\hat{f})^\vee\|_{p,W} \leqslant C\|f\|_{p,W} \tag{7.3}$$

for all f in $\mathscr{S}_{0,0}$, then (7.3) is true for all f in $Q_{l-1} \cap L^p_W$ with the same C.

To prove this, fix an f in Q_{l-1} and let $\{f_n\}$ be a sequence of functions in $\mathscr{S}_{0,0}$ that converges to f in L^2 and L^p_W by using Theorem (7.1). Theorem (7.1) and (7.3) imply that the operator $(m\hat{f})^\vee$ on $\mathscr{S}_{0,0}$ has a unique extension to $Q_{l-1} \cap L^p_W$. Call the image of f under this operator $T_m f$. Then $\|T_m f\|_{p,W} \leqslant C\|f\|_{p,W}$, and there is a subsequence f_{n_j} such that $T_m f_{n_j}$ converges to $T_m f$ almost everywhere. Since f_{n_j} converges to f in L^2 as $j \to \infty$, then $(m\hat{f}_{n_j})^\vee$ converges to $(m\hat{f})^\vee$ in L^2 and a subsequence converges to $(m\hat{f})^\vee$ almost everywhere. Therefore, $(m\hat{f})^\vee = T_m f$ and the result follows.

If $W(x) = |x|^{lp}V(x)$, where $V \in A_p$, Theorem (6.19) of **[17]** shows that $\mathscr{S}_{0,0}$ is dense in L^p_W and the operator $(m\hat{f})^\vee$, defined initially on $\mathscr{S}_{0,0}$, has a unique extension. More generally, if $W(x) = |g(x)|^p V(x)$, where g is a polynomial with no complex roots, then $\mathscr{S}_{0,0}$ is dense in L^p_W as shown in Corollary (8.11) of **[22]**. This density can also be obtained by modifying the proof of Theorem (6.19) of **[17]**. Therefore, in this case the operator also has a unique extension.

If the polynomial g has complex roots, it is not hard to show that $\mathscr{S}_{0,0}$ is not dense in L^p_W and the extension is not unique. If $g(x) = g_1(x)g_2(x)$ where g_1 has complex roots, g_2 has real roots and g_1 has degree d, then by Theorem (8.13) of **[22]**, $\mathscr{S}_{0,0}$ is dense in the subspace of L^p_W with $\int fg_2x^k\,dx = 0$ for $0 \leqslant k \leqslant d-1$.

Even if the extension is unique, the extended operator will not, in general, equal $(m\hat{f})^\vee$ even for f in $L^2 \cap L^p_W$. Various ways of expressing $T_m f$ for the case $p = 2$, $g(x) = x^l$, $V(x) = |x|^a$ with $-1 < a < 1$ are given in §7 of **[17]**, similar expressions can be obtained for the general case.

8. The case $\lambda \leqslant \frac{1}{2}$. This section contains the proof of two multiplier theorems for $\lambda \leqslant \frac{1}{2}$. Theorem (8.1) is a result for general weight functions and resembles Theorem (3.2). Theorem (8.7) is the case of Theorem (1.4) for $\lambda \leqslant \frac{1}{2}$. The proof of Theorem

(8.1) is based on a similar one in §4 of [**2**] and is like the proof of Theorem (5.1) of [**18**]. Theorem (8.7) is proved from (8.1) in the same way that Theorem (3.3) was obtained from Theorem (3.2). The first theorem to be proved is as follows.

THEOREM (8.1). *If* $1 < p < \infty$, $2 < s \leqslant \infty$, $\max(\frac{1}{s}, |\frac{1}{p} - \frac{1}{2}|) < \lambda \leqslant \frac{1}{2}$, $m \in M(s, \lambda)$, $u \geqslant \max(2/(2p\lambda + 2 - p), 1/\lambda p)$, $u(1 - p/2) \leqslant 1$, $W(x)^u \in A_{\lambda p u}$ *and* f *is in* $\mathcal{S}$, *then*

$$\|(m\hat{f})^{\vee}\|_{p,W} \leqslant CB(m, s, \lambda)\|f\|_{p,W}$$

with C *independent of* m *and* f.

We will use the following notation. Define $\theta(x) = \psi(x/2) + \psi(x) + \psi(2x)$, ψ as in §2, and define the operator D_l^λ by

$$D_l^\lambda f(x) = \left[\left(1 + 2^{2l}|x|^2\right)^{\lambda/2}\hat{f}(x)\right]^{\vee}.$$

Given a bounded function $m(x)$, complex z and λ, positive ε and σ, and an integer N, define

$$m(z, x) = \sum_{l=-N}^{N} \theta(2^{-l}x) D_l^{(z-1)/2-\varepsilon}\left[\left|D_l^\lambda m_l\right|^{\sigma(1-z)/2} \operatorname{sgn} D_l^\lambda m_l\right], \tag{8.2}$$

where $\operatorname{sgn} z = z/|z|$ for $z \neq 0$ and $\operatorname{sgn} 0 = 0$. We will need the following fact proved as Lemma (5.3) of [**18**].

LEMMA (8.3). *If* $\sigma > 2$, $\frac{1}{\sigma} < \lambda \leqslant \frac{1}{2}$, $\varepsilon = \lambda - \frac{1}{\sigma}$, m *is in* $M(\sigma, \lambda)$ *and* v *is real, then*

$$m\left(1 - \frac{2}{\sigma}, x\right) = \sum_{l=-N}^{N} m_l(x), \tag{8.4}$$

$$\|m(1 + iv, x)\|_\infty \leqslant C(1 + v^2) \tag{8.5}$$

and

$$B\left(m(iv, x), 2, \varepsilon + \tfrac{1}{2}\right) \leqslant C(1 + v)^2[B(m, \sigma, \lambda)]^{\sigma/2}, \tag{8.6}$$

where C *is independent of* v *and* m.

To prove Theorem (8.1) for the case $B(m, s, \lambda) = 1$, fix m, let σ satisfy $1/\lambda < \sigma \leqslant s$, let $\varepsilon = \lambda - 1/\sigma$ and define $m(z, x)$ by (8.2). Let $r = 4p/(2p + 2\sigma - p\sigma)$. By taking $\sigma - 1/\lambda$ sufficiently small we will have $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{\sigma}$, and this implies that $1 < r < \infty$. Next define $t = 2pu/r\sigma$ and $U(x) = W(x)^{u/t}$. As in the proof of Theorem (3.2), we may assume, by changing u if necessary, that the hypothesized inequalities for u are strict. It is then easy to verify that

$$\lim_{\sigma \to 1/\lambda} \frac{1}{t}\max\left(1, \frac{1}{r(\varepsilon + \frac{1}{2})}\right) = \max\left(\frac{2}{u(2\lambda p + 2 - p)}, \frac{1}{\lambda p u}\right) < 1,$$

$$\lim_{\sigma \to 1/\lambda} t\left[1 - \left(1 - \left(\varepsilon + \frac{1}{2}\right)\right)r\right] = u\left(1 - \frac{p}{2}\right) < 1$$

and

$$\lim_{\sigma \to 1/\lambda} \left(\varepsilon + \frac{1}{2}\right)rt = \lambda p u.$$

Consequently, if σ is chosen sufficiently close to $1/\lambda$, we have $1 < r < \infty$, $2 < \sigma \leqslant s$, $\max(1, 1/r(\varepsilon + \frac{1}{2})) < t < \infty$ and $t[1 - (1 - (\varepsilon + \frac{1}{2}))r] < 1$. Furthermore, since $U(x)^t = W(x)^u \in A_b$ for some $b < \lambda pu$, we also have $U(x)^t \in A_{(\varepsilon+1/2)rt}$ for σ close enough to $1/\lambda$. Then with such a σ apply Theorem (3.2) with $s = 2$, $m(x)$ replaced by $m(iv, x)$, p replaced by r, λ by $\varepsilon + \frac{1}{2}$ and u by t. This, (8.6), Theorem (7.2) and Theorem (2.2) imply that for f in $L^2_{U(x)^{-2/r}} \cap L^r$ that

$$\int_{-\infty}^{\infty} |T(iv)f(x)|^r \, dx \leqslant C(1 + v^2)^r \int_{-\infty}^{\infty} |f(x)|^r \, dx,$$

where

$$T(z)f(x) = U(x)^{(1-z)/r}\Big[m(z,x)\big(f(x)U(x)^{(z-1)/r}\big)^{\wedge}\Big]^{\vee}.$$

Because of (8.5), we also have

$$\int_{-\infty}^{\infty} |T(1 + iv)f(x)|^2 \, dx \leqslant C(1 + v^2)^2 \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

for f in L^2. Complex interpolation, Theorem 4.1, p. 205 of **[21]** then implies for f in $L^p \cap L^2_{U(x)^{4/\sigma r}}$ that

$$\int_{-\infty}^{\infty} \left| T\left(1 - \frac{2}{\sigma}\right) f(x) \right|^p dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p \, dx.$$

Since $2pu/\sigma rt = 1$, this is equivalent to

$$\int_{-\infty}^{\infty} \left| \left[\hat{f}(x) \sum_{l=-N}^{N} m_l(x) \right]^{\vee} \right|^p W(x) \, dx \leqslant C \int_{-\infty}^{\infty} |f(x)|^p W(x) \, dx.$$

Letting $N \to \infty$ completes the proof if $B(m, s, \lambda) = 1$; the general case follows from the fact that $B(\gamma m, s, \lambda) = |\gamma| B(m, s, \lambda)$.

THEOREM (8.7). *If* $1 < p < \infty$, $2 < s \leqslant \infty$, $\max(\frac{1}{s}, |\frac{1}{p} - \frac{1}{2}|) < \lambda \leqslant \frac{1}{2}$, $m \in M(s, \lambda)$, $W(x) = (1 + |x|)^a \prod_{j=1}^{J} |x - b_j|^{a_j}$, *where the* b_j*'s are real and distinct,* $a_0 = a + \sum_{j=1}^{J} a_j$, $\max(-p\lambda, -1 + p(\frac{1}{2} - \lambda)) < a_j < \min(p\lambda, -1 + p(\lambda + \frac{1}{2}))$ *for* $0 \leqslant j \leqslant J$, $|a_j - a_k| \leqslant p\lambda$ *for* $1 \leqslant j$, $k \leqslant J$ *and* $f \in \mathscr{S}$, *then* $\|(m\hat{f})^{\vee}\|_{p,W} \leqslant CB(m, s, \lambda)\|f\|_{p,W}$ *with C independent of f.*

This is proved using Theorem (8.1) in essentially the same way that Theorem (3.3) was proved using Theorem (3.2). Given $f \in \mathscr{S}$, the functions f_1 and f_2 are chosen as in the proof of Theorem (3.3) and the proof is reduced to proving (3.12)–(3.14). In the proof of (3.12), the requirement (3.15) is replaced by

$$\max_{1 \leqslant j \leqslant J}\left(0, -a_j, 1 - \frac{p}{2}\right) < \frac{1}{u} < \min_{1 \leqslant j \leqslant J}\left(\lambda p - a_j, \lambda p, \lambda p + 1 - \frac{p}{2}\right).$$

The hypotheses insure the existence of such a u, and it follows easily that $u > \max(2/(2p\lambda + 2 - p), 1/\lambda p)$, $u(1 - p/2) \leqslant 1$ and $-1 < a_j u < \lambda pu - 1$ for $1 \leqslant j \leqslant J$. Theorem (8.1) then implies (3.16) with $V(x) = [1 + |x|]^{-a_0} W(x)$ and (3.12) follows.

In the proof of (3.13), define a_j^*, a^* and $V(x)$ as in the proof of (3.13) for Theorem (3.3). The requirement (3.17) is replaced by

$$\max\left(0, 1 - \frac{p}{2}\right) < \frac{1}{u} < \min_{0 \leqslant j \leqslant J}\left(1 + \lambda p - \frac{p}{2}, \lambda p - a_j^*\right).$$

The hypotheses imply the existence of u, $u > \max(2/(2p\lambda + 2 - p), 1/\lambda p)$, $u(1 - p/2) \leqslant 1$ and $-1 < ua_j^* < \lambda pu - 1$ for $0 \leqslant j \leqslant J$. Theorem (8.1) then implies (3.18) and (3.13) follows as before.

The proof of (3.14) is the same as in the proof of Theorem (3.3). This completes the proof of Theorem (8.7).

9. The periodic case. Periodic results are needed for the applications to Jacobi polynomials in [**15**]; for this the weights considered are usually $(1 - x)^\alpha(1 + x)^\beta$ and Theorem (9.17) is the one that will be used. The periodic theorems for multipliers are similar to the nonperiodic ones. In some cases the proofs are the same, and in other cases they are simpler. We will sketch the theory and point out the major differences.

For this section, we will assume that all functions have period 2π and will define

$$\|f\|_{p,W} = \left[\int_{-\pi}^{\pi} |f(x)|^p W(x)\,dx\right]^{1/p}.$$

As usual, define l^p to be the set of sequences g with $\|g\|_p = [\Sigma|g(k)|^p]^{1/p} < \infty$. For a sequence g in l^1, define $\check{g}(x) = \Sigma g(k)e^{ikx}$, and for integrable f, let $\hat{f}(k) = (2\pi)^{-1}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx$. For a sequence g in l^1 and $\lambda > -1$, we define

$$\Delta^\lambda g(k) = \left[\check{g}(x)[1 - e^{-ix}]^\lambda\right]^\wedge(k), \tag{9.1}$$

where $[1 - e^{-ix}]^\lambda$ is defined to have argument equal to $\lambda \arg(1 - e^{-ix})$ with $\arg(1 - e^{-ix})$ taken between $-\pi/2$ and $\pi/2$. We will need the following equivalent formulation.

LEMMA (9.2). *If $\lambda > -1$ and g is in l^1, then*

$$\Delta^\lambda g(k) = \sum_{j=0}^{\infty} \binom{\lambda}{j}(-1)^j g(j + k).$$

Lemma (9.2) is proved by starting with the identity

$$[1 - re^{-ix}]^\lambda = \sum_{j=0}^{\infty} \binom{\lambda}{j}(-r)^j e^{-ijx},$$

which holds for $|r| < 1$. Since $|\binom{\lambda}{j}| \leqslant C_\lambda j^{-\lambda-1}$, it follows that

$$\left[\check{g}(x)[1 - re^{-ix}]^\lambda\right]^\wedge(k) = \sum_{j=0}^{\infty} \binom{\lambda}{j}(-r)^j g(j + k)$$

for $|r| < 1$. Since both sides of this identity are continuous functions of r for $0 \leqslant r \leqslant 1$, the conclusion follows by letting $r \to 1^-$.

We will define the periodic class $M^*(s, \lambda)$ of multipliers as follows. As in the nonperiodic case, let ψ be a C^∞ function with support in $\frac{1}{2} < |x| < 2$ satisfying $\sum_{j=-\infty}^{\infty} \psi(2^{-j}x) \equiv 1$, and given a sequence $m(k)$, define $m_j(k) = \psi(2^{-j}k)m(k)$. For $\lambda > -1$ and $1 \leqslant s \leqslant \infty$, let

$$B^*(m, s, \lambda) = \|m\|_\infty + \sup_j 2^{j(\lambda - 1/s)}\left\|\Delta^\lambda m_j(k)\right\|_s$$

and define $M^*(s, \lambda)$ to be the set of all sequences m such that $B^*(m, s, \lambda)$ is finite.

For λ an integer, it follows immediately that $\Delta^\lambda g(k)$ is $(-1)^\lambda$ times the usual difference operator and, as a result, $M^*(s,\lambda)$ for λ an integer is the traditional multiplier space. Note that if Δ^λ were defined with $[1-e^{-ix}]^\lambda$ replaced by $|1-e^{-ix}|^\lambda$, then $M^*(s,\lambda)$ is unchanged for $1<s<\infty$ by the following reasoning. Let

$$g(x)=\frac{|1-e^{-ix}|^\lambda}{[1-e^{-ix}]^\lambda};$$

then $g(x)=e^{i\lambda(x-\pi)/2}$ for $0<x<2\pi$, and for λ not an even integer, $\hat{g}(k)=2\sin(\pi\lambda/2)/\pi(\lambda-2k)$. The boundedness of the discrete Hilbert transform on l^s shows the spaces are the same. If λ is an even integer, $\hat{g}(k)=(-1)^{\lambda/2}$ for $k=\lambda/2$ and $\hat{g}(k)=0$ for $k\neq\lambda/2$ and the equivalence is trivial. Similar reasoning shows that replacing $[1-e^{-ix}]^\lambda$ by $[1-e^{ix}]^\lambda$ in the definition of Δ^λ would also produce the same spaces for $1<s<\infty$; this shows that if $1<s<\infty$, then $m(k)$ is in $M^*(s,\lambda)$ if and only if $m(-k)$ is in $M^*(s,\lambda)$. Since $\Delta^\lambda m_j(k)$ for $k\geqslant 0$ does not depend on values of $m(i)$ for $i<0$, it also follows that if $1<s<\infty$ then m is in $M^*(s,\lambda)$ if and only if $m\chi_{[0,\infty)}$ and $m\chi_{(-\infty,0)}$ are in $M^*(s,\lambda)$.

Other definitions have been used for discrete multiplier spaces by Gasper and Trebels in **[9]** and Connett and Schwartz in **[6]** for sequences defined on the positive integers. Gasper and Trebels define the weak bounded variation space $\mathrm{wbv}_{s,\lambda}$ essentially as the set of m for which

$$\|m\|_{s,\lambda;w}=\|m\|_\infty+\sup_{j\geqslant 0}2^{j(\lambda-1/s)}\left[\sum_{k=2^j}^{2^{j+1}}|\Delta^\lambda m(k)|^s\right]^{1/s}<\infty$$

for $\lambda>1$ and $1\leqslant s\leqslant\infty$. Connett and Schwartz define $s(s,\lambda)$ for $1<s<\infty$ and $\lambda>1/s$ to be sequences $m(k)$ for which there is a continuous function $f(x)$ in $M(s,\lambda)$ with $f(k)=m(k)$ for k a positive integer. They prove in **[6**, pp. 48–55], that $s(s,\lambda)$ is the same as $\mathrm{wbv}_{s,\lambda}$. That these spaces are the same as $M^*(s,\lambda)$ is shown by the following lemma.

LEMMA (9.3). *If $m(k)=0$ for $k<0$, $1<s<\infty$ and $\lambda>0$, then there is a C, independent of m, such that*

$$\|m\|_{s,\lambda;w}\leqslant CB^*(m,s,\lambda)\quad\textit{and}\quad B^*(m,s,\lambda)\leqslant C\|m\|_{s,\lambda;w}.$$

To prove the first of these, observe that

$$\left|\binom{\lambda}{l}\right|\leqslant C_\lambda l^{-\lambda-1}\tag{9.4}$$

for $l\geqslant 1$ and $|m(l+k)-\sum_{n=j}^{j+2}m_n(l+k)|$ is bounded by $\|m\|_\infty$ for all l and k and equals 0 for $2^j\leqslant l+k\leqslant 2^{j+2}$. These facts show that $\|m\|_{s,\lambda,w}$ is equivalent to

$$\|m\|_\infty+\sup_j 2^{j(\lambda-1/s)}\left[\sum_{k=2^j}^{2^{j+1}}\left|\sum_{l=0}^{\infty}(-1)^l\binom{\lambda}{l}\sum_{n=j}^{j+2}m_n(l+k)\right|^s\right]^{1/s}.\tag{9.5}$$

Minkowski's inequality and the definition of $B^*(m,s,\lambda)$ then show that (9.5) is bounded by $CB^*(m,s,\lambda)$.

For the opposite inequality, start with the facts that for $k > 2^{j+1}$ we have $\Delta^\lambda m_j(k) = 0$, and for $k \leqslant 2^{j-2}$ we can use (9.4) to show that

$$\left|\Delta^\lambda m_j(k)\right| \leqslant \sum_{l=2^{j-1}}^{2^{j+1}} \left|m(l)\binom{\lambda}{l-k}\right| \leqslant C2^j \|m\|_\infty [2^{j-1}-k]^{-\lambda-1}.$$

From this,

$$\sum_{k=-\infty}^{2^{j-2}} \left|\Delta^\lambda m_j(k)\right|^s \leqslant C2^{j(1-s\lambda)}\|m\|_\infty^s.$$

If j satisfies $2^{j+2} \leqslant \lambda + 1$, it also follows from (9.4) that

$$\sum_{k=2^{j-2}}^{2^{j+1}} \left|\Delta^\lambda m_j(k)\right|^s \leqslant C2^{j(1-s\lambda)}\|m\|_{s,\lambda,w}^s. \tag{9.6}$$

To complete the proof that $B^*(m,s,\lambda) \leqslant C\|m\|_{s,\lambda,w}$, we must, therefore, prove (9.6) for j satisfying $2^{j+2} > \lambda + 1$.

To prove (9.6) for $2^{j+2} > \lambda + 1$, define $N = 1 + [\lambda]$ and use Taylor's theorem to write

$$\psi(2^{-j}(k+l)) = \sum_{n=0}^{N-1} \frac{2^{-jn}}{n!}\psi^{(n)}(2^{-j}k) + O(2^{-jN}l^N).$$

Using Lemma (9.2) and this on the left side of (9.6), we see that (9.6) can be proved by showing that

$$\sum_{k=2^{j-2}}^{2^{j+1}} \left|2^{-jn}\psi^{(n)}(2^{-j}k)\sum_{l=0}^{2^{j+2}}(-1)^l\binom{\lambda}{l}l^n m(k+l)\right|^s \tag{9.7}$$

for $0 \leqslant n \leqslant N-1$ and

$$\sum_{k=2^{j-2}}^{2^{j+1}} \left[2^{-jN}\sum_{l=0}^{2^{j+2}}\left|\binom{\lambda}{l}\right| \left|l^N\right| m(k+l)\right|\right]^s \tag{9.8}$$

are bounded by the right side of (9.6).

To prove (9.7), start with the identity $l^n = \sum_{i=0}^n C_{n,i}\binom{l}{i}$, where the constants $C_{n,i}$ are independent of l. Multiplying by $\binom{\lambda}{l}$ gives $l^n\binom{\lambda}{l} = \sum_{i=0}^n C_{n,i}\binom{\lambda}{i}\binom{\lambda-i}{l-i}$. Using this and the fact that $|\psi^{(n)}|$ is bounded, it is sufficient to show that

$$\sum_{k=2^{j-2}}^{2^{j+1}} 2^{-jns}\left|\sum_{l=0}^{2^{j+2}}(-1)^l\binom{\lambda-i}{l-i}m(k+l)\right|^s \tag{9.9}$$

is bounded by the right side of (9.6) for $0 \leqslant i \leqslant n \leqslant N-1 = [\lambda]$. To do this, use the fact that $2^{j+2} > N > i$ and (9.4) to show that

$$\sum_{k=2^{j-2}}^{2^{j+1}} \left[2^{-jn}\sum_{l=1+2^{j+2}}^{\infty}\left|\binom{\lambda-i}{l-i}m(k+l)\right|\right]^s \leqslant C2^{j(1-s\lambda)}\|m\|_\infty^s$$

if $i < \lambda$, and note that this inequality is trivial if $i = \lambda$. Therefore, since $\binom{\lambda-i}{l-i} = 0$ for $l < i$, we can prove the asserted bound for (9.9) by showing that

$$\sum_{k=2^{j-2}}^{2^{j+1}} \left|2^{-jn}\Delta^{\lambda-i}m(k+i)\right|^s \leqslant C2^{j(1-s\lambda)}\|m\|_{s,\lambda;w}^s.$$

This follows from the fact that $\|m\|_{s,\mu;w} \leqslant C\|m\|_{s,\lambda;w}$ for $0 < \mu < \lambda$ and $1 \leqslant s \leqslant \infty$ proved in Lemma 1 of [**9**]. Finally, replacing $m(k+l)$ by $\|m\|_\infty$ and using (9.4) shows that (9.8) is bounded by $C2^{j(1-s\lambda)}\|m\|_\infty^s$. This completes the proof of Lemma (9.3).

Lemma (9.3) can also be proved using Lemma 2 of [**9**].

The periodic analogues of the results in [**18**] are easily proved; in most cases the proofs are simpler. Lemma (9.2) serves as the replacement for Lemma (2.6) of [**18**], and the analogue of Corollary (2.10) of [**18**] follows immediately. The conclusion of Lemma (2.11) of [**18**] becomes

$$\Delta^\alpha f(k) = \sum_{i=k}^{\infty} \binom{\alpha-\lambda}{i-k}(-1)^{i-k}\Delta^\lambda f(i)$$

for $\lambda > 0$, $\lambda - 1 < \alpha < \lambda$ and f in l^1. This is proved by writing the right side as

$$\sum_{i=k}^{\infty}(-1)^{i-k}\binom{\alpha-\lambda}{i-k}\left[\sum_{j=i}^{\infty}(-1)^{j-i}\binom{\lambda}{j-i}f(j)\right]. \tag{9.10}$$

Using (9.4) and the fact that f is an l^1 sequence shows that the double sum in (9.10) is absolutely convergent. Therefore, the summation order can be reversed to give

$$\sum_{j=k}^{\infty}(-1)^{j-k}f(j)\left[\sum_{i=k}^{j}\binom{\alpha-\lambda}{i-k}\binom{\lambda}{j-i}\right],$$

which is equivalent to the assertion.

The periodic version of Theorem (2.1), which is Theorem (2.12) of [**18**], is proved as before. The analogue of Lemma (2.15) of [**18**],

$$\Delta^\alpha f(k) = -\Delta\left[\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}f(k+i)\right],$$

for f in l^1 and $0 < \alpha < 1$, follows easily from Lemma (9.2). The periodic version of the rest of the results in §2 of [**18**] are proved as they were before.

The results of §§3–5 of [**18**] are also carried over easily to the periodic case. Various modifications are needed; some are obvious, such as replacing $M(s,\lambda)$ and $B(m,s,\lambda)$ by $M^*(s,\lambda)$ and $B^*(m,s,\lambda)$ and changing intervals of integration from $(-\infty,\infty)$ to $[-\pi,\pi]$. Powers of $|x|$ are replaced by powers of $|1-e^{-ix}|$, and $|x-b|$ is replaced by $|e^{-ib}-e^{-ix}|$. The periodic versions of Lemma (3.1), Theorem (3.2) and Theorem (3.4) of [**18**] have the added restriction $r < \pi$, and the integration set in (3.12) of [**18**] becomes $2|y| < |x| < \pi$. In the theorems of §§4 and 5 of [**18**], the set $\mathscr{S}$ is replaced by the set S of functions f with $\hat{f}(j) = 0$ for all but a finite number of j's. To adapt Theorem (1.2) of [**18**] to the periodic case we need a replacement for $\mathscr{S}_{0,0}$. For this we define S_k to be the subset of S with $\hat{f}(j) = 0$ for $0 \leqslant j \leqslant k$. In the statement of Theorem (1.2) of [**18**], the set $\mathscr{S}_{0,0}$ is replaced by S_{l-1} where $l = \operatorname{int}((\alpha+1)/p)$. The proofs in §§4 and 5 remain essentially unchanged.

Similarly, the results in §§2–6 and §8 of this paper can easily be modified for the periodic case in the same way. The results in §2 are from [**18**] and were mentioned above. The most significant change in §3 is in Theorem (3.3) which has the following periodic version.

THEOREM (9.11). *If* $1 < s \leqslant \infty$, $\max(\frac{1}{s}, \frac{1}{2}) < \lambda < 1$, $m \in M^*(s, \lambda)$, $1 < p < \infty$, $W(x) = \prod_{j=1}^{J}|e^{-ib_j} - e^{-ix}|^{a_j}$, *where the* b_j*'s are real, distinct and lie in* $(-\pi, \pi]$, $\max(-1, -p\lambda) < a_j < \min(p - 1, p\lambda)$ *for* $1 \leqslant j \leqslant J$, $|a_j - a_k| < p\lambda$ *for* $1 \leqslant j, k \leqslant J$ *and* f *is in* S, *then* $\|(m\hat{f})^\vee\|_{p,W} \leqslant CB^*(m, s, \lambda)\|f\|_{p,W}$, *where* C *is independent of* m *and* f.

Theorem (9.11) has a simpler proof than Theorem (3.3) since it is not necessary to write f as a sum of two functions. The proof of Theorem (9.11) is like the proof of (3.12) with $V(x)$ taken to be $W(x)$.

In §4, Lemma (4.11) is true for the periodic case with $[1 - e^{-ix}]^j$ used to replace x^j and $e^{-ib} - e^{-ix}$ to replace $x - b$. In §5, no analogues of Lemmas (5.1) and (5.7) are needed. Lemmas (5.5) and (5.9) are modified in the obvious way with R_0 in Lemma (5.9) taken to be π. The periodic versions of the theorems about general weight functions in §§6 and 8, Theorems (6.1), (6.5) and (8.1), are as follows.

THEOREM (9.12). *Assume that* $1 \leqslant s \leqslant \infty$, $1 < p < \infty$, $\lambda \geqslant 1$, $l \geqslant 0$, $m \in M^*(s, \lambda)$, $V(x) \in A_p$, $g(x)$ *is a trigonometric polynomial of degree* l *and* $W(x) = |g(x)|^p V(x)$. *Let* Q_j *be the least upper bound of all* q *for which*

$$(9.13) \qquad \int_{r \leqslant |x - b_j| \leqslant 2r} W(x)^q\,dx \leqslant Cr^{1-q}\left[\int_{r \leqslant |x - b_j| \leqslant 2r} W(x)\,dx\right]^q$$

holds for $0 < r < \pi/2$, *where* $\{b_j\}_{j=1}^J$ *is the set of distinct real roots of* g *in* $(-\pi, \pi]$. *For* $1 \leqslant j \leqslant J$, *let* $T_j = \min(2, (pQ_j')', s)$, *let* l_j *be the multiplicity of the root* b_j *in* g *and assume that* $\lambda > l_j - 1 + 1/T_j$. *If* $\lambda < l_j + 1/T_j$ *for* $1 \leqslant j \leqslant J$, *assume in addition that* $|e^{-ib_j} - e^{-ix}|^{p(l_j - \lambda + 1/T_j)}V(x) \in A_p$. *Then for every* f *in* S_{l-1} *we have*

$$\|(m\hat{f})^\vee\|_{p,W} \leqslant CB^*(m, s, \lambda)\|f\|_{p,W},$$

where C *is independent of* m *and* f.

The proof is a simplified version of the proof of Theorem (6.1). The analogue of (6.9) holds for $L \leqslant l - 1$ since f is in S_{l-1}. The analogue of (6.10) has limits $-\pi$ and π and is estimated as (6.16) was. The analogue of (6.11) has region of integration $8d < |x - b_i| < \pi$ and is estimated as (6.18) was. The analogue of (6.12) is treated as (6.12) was in the proof of Theorem (6.1).

THEOREM (9.14). *Assume that* $1 \leqslant s \leqslant \infty$, $1 < p < \infty$, $\lambda < 1$, $m \in M^*(s, \lambda)$ *and* $V(x) \in A_p$. *Let* $W(x) = \prod_{j=1}^{J}|e^{-ib_j} - e^{-ix}|^p V(x)$, *where the* b_j*'s are real, distinct and lie in* $(-\pi, \pi]$, *and assume that for every* $m \in M^*(s, \lambda)$ *and* $h \in S$ *that* $\|(m\hat{h})^\vee\|_{p,V_1} \leqslant HB(m, s, \lambda)\|h\|_{p,V_1}$, *where* $V_1(x) = \prod_{j=1}^{J}|e^{-ib_j} - e^{-ix}|^{\alpha_j}V(x)$ *and* H *and the* α_j*'s are constants independent of* f *and* m *with* $\alpha_j \leqslant p$. *Let* Q_j *be the least upper bound of all* q *for which* (9.13) *holds for* $0 < r < \pi/2$; *let* $T_j = \min(2, (pQ_j')', s)$ *and assume that* $\lambda > 1/T_j$ *and* $|e^{-ib_j} - e^{-ix}|^{p(1-\lambda+1/T_j)}V(x) \in A_p$. *Then for every* f *in* S_{J-1} *and every* m *in* $M^*(s, \lambda)$, *we have*

$$\|(m\hat{f})^\vee\|_{p,W} \leqslant C(H + 1)B^*(m, s, \lambda)\|f\|_{p,W},$$

where C *is independent of* m *and* f.

To prove Theorem (9.14), apply the periodic version of Lemma (4.11); the left side of the conclusion is bounded by $\sum_{j=0}^{J}\|(m\hat{f}_j)^{\vee}\|_{p,W}$. For $\|(m\hat{f}_0)^{\vee}\|_{p,W}$, we use the facts that $W \leqslant CV_1$, that $\|(m\hat{f}_0)^{\vee}\|_{p,V_1} \leqslant BH\|f_0\|_{p,V_1}$ and that $V_1 \leqslant CW$ on the support of f_0. For the others, define $K(x,y) = K_N(x-y) - K_N(x-b_j)$ and observe that since $\hat{f}_j(0) = 0$, we have $(K_N * f_j)(x) = \int_{-\pi}^{\pi} K(x,y) f_j(y)\,dy$. Next use Lemmas (5.5) and (5.9) with this kernel, let N approach ∞ and use Fatou's lemma to complete the proof.

THEOREM (9.15). *If $1 < p < \infty$, $2 < s \leqslant \infty$, $\max(\frac{1}{s}, |\frac{1}{p} - \frac{1}{2}|) < \lambda \leqslant \frac{1}{2}$, $m \in M(s,\lambda)$, $u \geqslant \max(2/(2p\lambda + 2 - p), 1/\lambda p)$, $u(1-p/2) \leqslant 1$, $W(x)^u \in A_{\lambda p u}$ and f is in S, then $\|(m\hat{f})^{\vee}\|_{p,W} \leqslant CB^*(m,s,\lambda)\|f\|_{p,W}$, where C is independent of m and f.*

This follows from the periodic version of Theorem (3.2) in the same way that Theorem (8.1) was proved from Theorem (3.2).

The periodic version of Theorem (1.4) is as follows.

THEOREM (9.16). *If $1 < p < \infty$, $1 \leqslant s \leqslant \infty$, $\max(\frac{1}{s}, |\frac{1}{p} - \frac{1}{2}|) < \lambda$ or $\lambda = s = 1$, $m \in M^*(s,\lambda)$, $W(x) = \prod_{j=1}^{J}|e^{-ib_j x} - e^{-ix}|^{a_j}$, where the b_j's are real, distinct and lie in $(-\pi,\pi]$, $l = \sum_{i=1}^{J}\operatorname{int}[(1+a_j)/p]$,*

$$\max\big(-1, -p\lambda, -1 + p(-\lambda + \tfrac{1}{2})\big) < a_j < \min\big(p\lambda, -1 + p(\lambda + \tfrac{1}{2}), -1 + p(\lambda + 1 - \tfrac{1}{s})\big)$$

for $1 \leqslant j \leqslant J$, $(a_j+1)/p$ is not an integer for $1 \leqslant j \leqslant J$ and $|a_j - a_k| < p\lambda$ for $1 \leqslant j, k \leqslant J$, then for f in S_{l-1}, we have $\|(m\hat{f})^{\vee}\|_{p,W} \leqslant CB^*(m,s,\lambda)\|f\|_{p,W}$ with C independent of m and f.

Like Theorem (1.4), Theorem (9.16) is an amalgamation of two results: a periodic version of Theorem (6.7) and a periodic version of Theorem (8.7). The periodic version of Theorem (6.7) is proved from Theorems (9.11), (9.12) and (9.14) in the same way that Theorem (6.7) is proved from Theorems (3.3), (6.1) and (6.5). The periodic version of Theorem (8.7) is proved from Theorem (9.15) in the same way that Theorem (8.7) was proved from Theorem (8.1).

The following corollary of Theorem (9.16) will be applied to Jacobi expansions in **[15]**.

THEOREM (9.17). *Assume that $1 \leqslant s \leqslant \infty$, $1 < p < \infty$, $\lambda > \max(\frac{1}{s}, |\frac{1}{p} - \frac{1}{2}|)$ or $\lambda = s = 1$, $m \in M^*(s,\lambda)$, $W(x) = |1 - e^{-ix}|^{a_1}|1 + e^{-ix}|^{a_2}$, $l_j = \operatorname{int}((a_j+1)/p)$, $(a_j+1)/p$ is not an integer,*

$$\max\big(-1, -p\lambda, -1 + p(-\lambda + \tfrac{1}{2})\big) < a_j < \min\big(p\lambda, -1 + p(\lambda + \tfrac{1}{2}), -1 + p(\lambda + 1 - \tfrac{1}{s})\big)$$

for $j = 1$ and $j = 2$, and $|a_1 - a_2| < p\lambda$. Then for f in $S_{l_1 + l_2 - 1}$, we have

$$\|(m\hat{f})^{\vee}\|_{p,W} \leqslant CB^*(m,s,\lambda)\|f\|_{p,W},$$

where C is independent of m and f.

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