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SOME WEIGHTED NORM INEQUALITIES FOR THE FOURIER TRANSFORM OF FUNCTIONS WITH VANISHING MOMENTS

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ABSTRACT. Weighted L^p norm inequalities are derived between a function and its Fourier transform in case the function has vanishing moments up to some order. For weights of the form $|x|^\gamma$, the results concern values of γ which are outside the range which is normally considered.

1. Introduction. Weighted norm inequalities for the Fourier transform with power weights have natural constraints on the exponents, as indicated in Pitt's theorem **[6]**, which asserts for example that

$$\int_{-\infty}^{\infty} |\hat{f}(x)|^p |x|^{-\gamma+p-2}\,dx \leqslant C\int_{-\infty}^{\infty} |f(x)|^p |x|^\gamma\,dx \tag{1}$$

if $1 < p < \infty$ and $\max\{0, p-2\} \leqslant \gamma < p-1$. The result fails for γ outside this range. We will show, however, that (1) holds for $\gamma > p-1$, $\gamma \neq kp-1$ for $k = 1, 2, \ldots$, provided that enough moments of f vanish. For example, an immediate consequence of Theorem 1 below is that (1) is valid for $p-1 < \gamma < 2p-1$ for all f having mean value zero (cf. **[2]**, where analytic functions in the unit circle are considered). The case $\gamma = p-1$ is excluded, even with this restriction on f, as shown by the counterexample in §5.

We work with functions in $\mathscr{S}_{0,0}$, the class of Schwartz functions whose Fourier transforms have compact support not containing the origin. Note that all the moments of a function in $\mathscr{S}_{0,0}$ vanish:

$$\int_{-\infty}^{\infty} f(x)x^j\,dx = 0, \qquad j = 0, 1, 2, \ldots, \; f \in \mathscr{S}_{0,0}.$$

$\mathscr{S}_{0,0}$ is dense in all the weighted spaces that we will consider, and the Fourier transform operator has a natural extension to functions (not necessarily locally integrable) in these spaces: see §4.

In what follows, if $1 < p < \infty$, A_p stands for the class of nonnegative, locally integrable functions w on $\mathbf{R}^1$ such that

$$\left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leqslant A < \infty$$

for all intervals $I \subset \mathbf{R}^1$.

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THEOREM 1. *If* $1 < p < \infty$ *and* $w \in A_p$, *then*

$$\int_{-\infty}^{\infty} |\hat{f}(x)|^p \frac{1}{|x|} w\left(\frac{1}{x}\right) \frac{dx}{|x|} \leqslant C \int_{-\infty}^{\infty} |f(x)|^p |x|^p w(x)\, dx \tag{2}$$

for all f *with* $\int_{-\infty}^{\infty} f\, dx = 0$, *where* C *is independent of* f.

Inequality (1) for $p - 1 < \gamma < 2p - 1$ follows from (2) since $|x|^{\gamma - p} \in A_p$ if $-1 < \gamma - p < p - 1$. Counterexamples for $\gamma = p - 1$, etc., are discussed in §5.

More generally, we have

THEOREM 1a. *If* $1 < p \leqslant q < \infty$, $w^{q/p} \in A_{1+q/p'}$, $1/p + 1/p' = 1$, *and* k *is a positive integer*, *then*

$$\left(\int_{-\infty}^{\infty} \left|\frac{\hat{f}(x)}{x^{k-1}}\right|^q \left(\frac{1}{|x|} w\left(\frac{1}{x}\right)\right)^{q/p} \frac{dx}{|x|}\right)^{1/q} \leqslant C\left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{kp} w(x)\, dx\right)^{1/p} \tag{3}$$

for all $f \in \mathscr{S}_{0,0}$, *with* C *independent of* f.

The condition on w above is easily seen to be equivalent to

$$\left(\frac{1}{|I|}\int_I w^{q/p}\, dx\right)^{p/q} \left(\frac{1}{|I|}\int_I w^{-1/(p-1)}\, dx\right)^{p-1} \leqslant A < \infty. \tag{4}$$

Moreover, it is equivalent to assuming that $w \in A_p \cap \mathrm{RH}_{q/p}$, where $w \in \mathrm{RH}_r$, $r > 1$, means that w satisfies the reverse Hölder condition

$$\left(\frac{1}{|I|}\int_I w^r\, dx\right)^{1/r} \leqslant C\left(\frac{1}{|I|}\int_I w\, dx\right),$$

with C independent of I.

The proof of these results is extremely simple and based only on Hardy's inequalities and some properties of A_p weights. As a consequence, the results hold under considerably weaker hypotheses than $w^{q/p} \in A_{1+q/p'}$. For example, Theorem 1a together with the density of $\mathscr{S}_{0,0}$ in $L^p(|x|^{kp} w)$ are valid if w is locally integrable and both of the following hold:

$$\begin{aligned} &\left(\int_{|x|>s} \frac{w(x)^{q/p}}{|x|^{1+q/p'}}\, dx\right)^{p/q} \left(\int_{|x|<s} w(x)^{-1/(p-1)}\, dx\right)^{p-1} \leqslant C, \\ &\left(\int_{|x|<s} |x|^{q/p-1} w(x)^{q/p}\, dx\right)^{p/q} \left(\int_{|x|>s} \frac{w(x)^{-1/(p-1)}}{|x|^{p'}}\, dx\right)^{p-1} \leqslant C \end{aligned} \tag{5}$$

for all $s > 0$. This will follow for Theorem 1a by combining the comments in Remarks 1a, 2a of §2. For the density, it follows from **[5]**; see the end of §4.

Theorem 1a also has translated versions. To obtain these, apply Theorem la to the function $e^{ixb} f(x + a)$ and the weight $w(x + a)$, noting that the condition $w^{q/p} \in A_{1+q/p'}$ is translation invariant. Then translating the integral which arises on the right of (3) by a and the integral which arises on the left by b, we obtain

$$(6)\qquad \left(\int_{-\infty}^{\infty}\left|\frac{\hat{f}(x)}{x-b}\right|^q\left[\frac{1}{x-b}w\left(a+\frac{1}{|x-b|}\right)\right]^{q/p}\frac{dx}{|x-b|}\right)^{1/q} \leqslant C\left(\int_{-\infty}^{\infty}|f(x)|^p|x-a|^{kp}w(x)\,dx\right)^{1/p}$$

with C independent of f, a, and b, provided that $w^{q/p}\in A_{1+q/p'}$ and all the moments of $e^{ixb}f(x)$ vanish. Instead of assuming that $w^{q/p}\in A_{1+q/p'}$, we could assume only that (5) holds for $w(x+a)$.

In n dimensions, we have the following result:

THEOREM 2. *Let $n\geqslant 1$, $1<p\leqslant q<\infty$, $\beta=(n-1)p-n$, and w be a function on $\mathbf{R}^1$ such that the function $\tilde{w}(\rho)=|\rho|^{-(\beta+1)}w(\rho)$ satisfies $\tilde{w}^{q/p}\in A_{1+q/p'}(\mathbf{R}^1)$. If k is a positive integer and $f\in\mathscr{S}_{0,0}$, then*

$$(7)\qquad \left(\int_{\mathbf{R}^n}\left(\frac{|\hat{f}(x)|}{|x|^{k-1}}\right)^q\left[|x|^\beta w\left(\frac{1}{|x|}\right)\right]^{q/p}\frac{dx}{|x|^n}\right)^{1/q}\leqslant C\left(\int_{\mathbf{R}^n}|f(x)|^p|x|^{kp}w(|x|)\,dx\right)^{1/p}$$

with C independent of f.

Again, the conclusion of Theorem 2 holds under a weaker assumption on $\tilde{w}$; it is enough to assume that $\tilde{w}$ satisfies the analogue of (5) with all integrations restricted to positive values of the variable of integration, i.e., to assume

$$(8)\qquad \begin{aligned}&\left(\int_s^\infty\frac{\tilde{w}(\rho)^{q/p}}{\rho^{1+q/p'}}\,d\rho\right)^{p/q}\left(\int_0^s\tilde{w}(\rho)^{-1/(p-1)}\,d\rho\right)^{p-1}\leqslant C,\\ &\left(\int_0^s\rho^{q/p-1}\tilde{w}(\rho)^{q/p}\,d\rho\right)^{p/q}\left(\int_s^\infty\frac{\tilde{w}(\rho)^{-1/(p-1)}}{\rho^{p'}}\,d\rho\right)^{p-1}\leqslant C\end{aligned}$$

for all $s>0$.

In §2, we list some auxiliary weighted inequalities used in the proofs of Theorems 1 and 1a; the proofs themselves are given in §3. In §4, we discuss the extensions by continuity of $\hat{f}$ for general $f\in L^p(|x-a|^{kp}w)$, and in §5 we consider a counterexample for power weights. Theorem 2 is proved in §6.

Throughout the paper, C stands for a constant which may be different at different occurrences, and p' denotes the conjugate index of p: $1/p+1/p'=1$, $1<p<\infty$.

2. Basic inequalities. As mentioned in the introduction, the proofs are based entirely on Hardy's inequality and a few properties of A_p weights. In the next two lemmas, we summarize the facts we shall use.

LEMMA A. *If $w\in A_p(\mathbf{R}^1)$, $1<p<\infty$, and $\alpha\geqslant p$, then*

$$(9)\qquad \int_{|t|>s}t^{-\alpha}w(t)\,dt\leqslant Cs^{-\alpha}\int_{|t|<s}w(t)\,dt,\qquad s>0,$$

with C independent of s.

For a proof, see (2.3) in [**3**].

LEMMA B (HARDY'S INEQUALITIES). *If* $1 < p \leqslant q < \infty$ *and* u *and* v *are nonnegative on* $(0, \infty)$, *then the inequality*

$$\left(\int_0^\infty \left(\int_x^\infty f(t)\,dt\right)^q u(x)\,dx\right)^{1/q} \leqslant C\left(\int_0^\infty f(x)^p v(x)\,dx\right)^{1/p} \tag{10}$$

holds for every $f \geqslant 0$ *if and only if*

$$\sup_{r>0}\left(\int_0^r u(x)\,dx\right)^{1/q}\left(\int_r^\infty v(x)^{-p'/p}\,dx\right)^{1/p'} < \infty. \tag{11}$$

Similarly, the inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)\,dt\right)^q u(x)\,dx\right)^{1/q} \leqslant C\left(\int_0^\infty f(x)^p v(x)\,dx\right)^{1/p} \tag{12}$$

for every $f \geqslant 0$ *is equivalent to*

$$\sup_{r>0}\left(\int_r^\infty u(x)\,dx\right)^{1/q}\left(\int_0^r v(x)^{-p'/p}\,dx\right)^{1/p'} < \infty. \tag{13}$$

For a proof, see **[1]**.

We will use Lemmas A and B to prove the next four lemmas, which are the specific inequalities needed for the theorems in the introduction. Since the case $q = p$ is somewhat simpler, we will consider it separately.

LEMMA 1. *If* $1 < p < \infty$ *and* $w \in A_p$, *then*

$$\int_{-\infty}^\infty \left(\int_{|t|>|x|} g(t)\,dt\right)^p |x|^{p-2} w\left(\frac{1}{x}\right) dx \leqslant C\int_{-\infty}^\infty g(x)^p |x|^{2p-2} w\left(\frac{1}{x}\right) dx \tag{14}$$

for $g \geqslant 0$.

PROOF. The expression on the left side of (14) is at most 2^p times

$$\int_0^\infty \left(\int_x^\infty\right)^p + \int_0^\infty \left(\int_{-\infty}^{-x}\right)^p + \int_{-\infty}^0 \left(\int_{-x}^\infty\right)^p + \int_{-\infty}^0 \left(\int_{-\infty}^{x}\right)^p = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.$$

We want to show that each of these is bounded by the term on the right side of (14). We have

$$\mathrm{I} + \mathrm{III} = \int_0^\infty \left(\int_x^\infty g(t)\,dt\right)^p x^{p-2}\left[w\left(\frac{1}{x}\right) + w\left(-\frac{1}{x}\right)\right] dx.$$

By Hardy's inequality (10) with $q = p$, this will be bounded by

$$C\int_0^\infty g(x)^p x^{2p-2} w\left(\frac{1}{x}\right) dx$$

provided that (11) is satisfied for

$$u(x) = x^{p-2}\left[w\left(\frac{1}{x}\right) + w\left(-\frac{1}{x}\right)\right], \qquad v(x) = x^{2p-2} w\left(\frac{1}{x}\right).$$

To check (11), note that by changing x into $1/t$,

$$\left(\int_0^r x^{p-2}\left[w\left(\frac{1}{x}\right)+w\left(-\frac{1}{x}\right)\right]dx\right)^{1/p}\left(\int_r^\infty \left[x^{2p-2}w\left(\frac{1}{x}\right)\right]^{-p'/p}dx\right)^{1/p'}$$

$$\leqslant \left(\int_{|x|<r}|x|^{p-2}w\left(\frac{1}{x}\right)dx\right)^{1/p}\left(\int_{|x|>r}|x|^{-2}w\left(\frac{1}{x}\right)^{-p'/p}dx\right)^{1/p'}$$

$$= \left(\int_{|t|>1/r}|t|^{-p}w(t)\,dt\right)^{1/p}\left(\int_{|t|<1/r}w(t)^{-p'/p}\,dt\right)^{1/p'}, \tag{15}$$

which by Lemma A for $\alpha = p$ and the fact that $w \in A_p$ with constant A is bounded by

$$\left(C\left(r^p\int_{|t|<1/r}w(t)\,dt\right)^{1/p}\right)\left(\int_{|t|<1/r}w(t)^{-p'/p}\,dt\right)^{1/p'} \leqslant CA^{1/p}.$$

Similarly,

$$\mathrm{II}+\mathrm{IV} = \int_0^\infty\left(\int_x^\infty g(-t)\,dt\right)^p x^{p-2}\left(w\left(\frac{1}{x}\right)+w\left(-\frac{1}{x}\right)\right)dx$$

is bounded by

$$\int_0^\infty g(-x)^p x^{2p-2}w\left(-\frac{1}{x}\right)dx = \int_{-\infty}^0 g(x)^p|x|^{2p-2}w\left(\frac{1}{x}\right)dx,$$

since (11) is satisfied for

$$u(x) = x^{p-2}\left(w\left(\frac{1}{x}\right)+w\left(-\frac{1}{x}\right)\right) \quad\text{and}\quad v(x) = x^{2p-2}w\left(-\frac{1}{x}\right),$$

and the lemma follows.

REMARK 1. The hypothesis in Lemma 1 that $w \in A_p$ is unnecessarily strong. As shown in the proof (see (15)), it is enough to assume that

$$\left(\int_{|t|>s}\frac{w(t)}{|t|^p}\,dt\right)\left(\int_{|t|<s}w(t)^{-1/(p-1)}\,dt\right)^{p-1} \leqslant C$$

for all $s > 0$. It follows, for example, that the conclusion of Lemma 1 holds if $w(1/x)$ is replaced by $w(1/|x|)$, assuming only that

$$\left(\int_s^\infty\frac{w(t)}{t^p}\,dt\right)\left(\int_0^s w(t)^{-1/(p-1)}\,dt\right)^{p-1} \leqslant C$$

for $s > 0$.

More generally, we have

LEMMA 1a. *If $1 < p \leqslant q < \infty$ and $w^{q/p} \in A_{1+q/p'}$, then*

$$\left(\int_{-\infty}^\infty\left\{|x|^{1/p'-1/q}w\left(\frac{1}{x}\right)^{1/p}\int_{|t|>|x|}g(t)\,dt\right\}^q dx\right)^{1/q}$$

$$\leqslant C\left(\int_{-\infty}^\infty g(x)^p|x|^{2p-2}w\left(\frac{1}{x}\right)dx\right)^{1/p}$$

for all $g(x) \geqslant 0$.

PROOF. The proof is the same as the previous one, using Hardy's inequality for the pair of weights u, v defined by

$$u(x) = x^{q/p'-1}\left[w\left(\frac{1}{x}\right)^{q/p} + w\left(-\frac{1}{x}\right)^{q/p}\right], \qquad v(x) = x^{2p-2}w\left(\pm\frac{1}{x}\right).$$

To verify (11) for this choice of u and v, we use the fact that Lemma A holds for $w^{q/p}$ with $\alpha = 1 + q/p'$ since $w^{q/p} \in A_{1+q/p'}$. Moreover, as noted in the introduction, (4) holds.

REMARK 1a. As the proof shows, the conclusion of Lemma 1a holds if w merely satisfies

$$\left(\int_{|t|>s} \frac{w(t)^{q/p}}{|t|^{1+q/p'}}\,dt\right)^{p/q}\left(\int_{|t|<s} w(t)^{-1/(p-1)}\,dt\right)^{p-1} \leqslant C$$

for $s > 0$. Thus, the conclusion holds with $w(1/x)$ replaced by $w(1/|x|)$, assuming only that

$$\left(\int_s^\infty \frac{w(t)^{q/p}}{t^{1+q/p'}}\,dt\right)^{p/q}\left(\int_0^s w(t)^{-1/(p-1)}\,dt\right)^{p-1} \leqslant C.$$

LEMMA 2. *If* $1 < p < \infty$ *and* $w \in A_p$, *then*

$$\int_{-\infty}^{\infty}\left(\int_{|t|<|x|} h(t)\,dt\right)^p |x|^{-2}w\left(\frac{1}{x}\right)dx \leqslant C\int_{-\infty}^{\infty} h(x)^p|x|^{p-2}w\left(\frac{1}{x}\right)dx \tag{16}$$

for $h(x) \geqslant 0$.

PROOF. The argument is similar to that in the proof of Lemma 1. The integral on the left of (16) is at most 2^p times

$$\int_0^\infty\left(\int_0^x\right)^p + \int_0^\infty\left(\int_{-x}^0\right)^p + \int_{-\infty}^0\left(\int_0^{-x}\right)^p + \int_{-\infty}^0\left(\int_x^0\right)^p = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},$$

and

$$\mathrm{I} + \mathrm{III} = \int_0^\infty\left(\int_0^x h(t)\,dt\right)^p x^{-2}\left[w\left(\frac{1}{x}\right) + w\left(-\frac{1}{x}\right)\right]dx,$$

$$\mathrm{II} + \mathrm{IV} = \int_0^\infty\left(\int_0^x h(-t)\,dt\right)^p x^{-2}\left[w\left(\frac{1}{x}\right) + w\left(-\frac{1}{x}\right)\right]dx.$$

Therefore, in order to prove (16), hypothesis (13) has to be checked for the pair u, v defined by

$$u(x) = x^{-2}\left[w\left(\frac{1}{x}\right) + w\left(-\frac{1}{x}\right)\right] \quad\text{and}\quad v(x) = x^{p-2}w\left(\pm\frac{1}{x}\right).$$

By changing x into $1/t$.

$$\left(\int_r^\infty x^{-2}\left[w\left(\frac{1}{x}\right) + w\left(-\frac{1}{x}\right)\right]dx\right)^{1/p}\left(\int_0^r\left[x^{p-2}w\left(\pm\frac{1}{x}\right)\right]^{-p'/p}dx\right)^{1/p'}$$

$$\leqslant\left(\int_{|t|<1/r} w(t)\,dt\right)^{1/p}\left(\int_{|t|>1/r}|t|^{-p'}w(t)^{-p'/p}\,dt\right)^{1/p'}. \tag{17}$$

Since the condition that $w \in A_p$ is equivalent to $w^{-p'/p} \in A_{p'}$, we see by Lemma A applied to $w^{-p'/p}$ and p', with $\alpha = p'$, that the last term is bounded by

$$C\left(\int_{|t|<1/r} w(t)\,dt\right)^{1/p}\left(r^{p'}\int_{|t|<1/r} w(t)^{-p'/p}\,dt\right)^{1/p'} \leqslant CA^{1/p},$$

and the lemma follows.

REMARK 2. The conclusion of Lemma 2 holds if instead of assuming that $w \in A_p$, we assume (see (17))

$$\left(\int_{|t|<s} w(t)\,dt\right)\left(\int_{|t|>s} \frac{w(t)^{-1/(p-1)}}{|t|^{p'}}\,dt\right)^{p-1} \leqslant C$$

for all $s > 0$. Hence, the conclusion holds with $w(1/x)$ replaced by $w(1/|x|)$ assuming only that

$$\left(\int_0^s w(t)\,dt\right)\left(\int_s^\infty \frac{w(t)^{-1/(p-1)}}{t^{p'}}\,dt\right)^{p-1} \leqslant C, \qquad s > 0.$$

LEMMA 2a. *If* $1 < p \leqslant q < \infty$ *and* $w^{q/p} \in A_{1+q/p'}$, *then*

$$\left(\int_{-\infty}^{\infty}\left\{|x|^{-1/p-1/q} w\left(\frac{1}{x}\right)^{1/p}\int_{|t|<|x|} h(t)\,dt\right\}^q dx\right)^{1/q}$$
$$\leqslant C\left(\int_{-\infty}^{\infty} h(x)^p |x|^{p-2} w\left(\frac{1}{x}\right) dx\right)^{1/p}$$

for all $h(x) \geqslant 0$.

PROOF. As usual, it is enough to check (13) for

$$u(x) = x^{-q/p-1}\left[w\left(\frac{1}{x}\right)^{q/p} + w\left(-\frac{1}{x}\right)^{q/p}\right] \quad \text{and} \quad v(x) = x^{p-2} w\left(\pm\frac{1}{x}\right).$$

In fact, changing x into $1/t$,

$$\text{(18)}\quad \left(\int_r^\infty u(x)\,dx\right)^{1/q}\left(\int_0^r v(x)^{-p'/p}\,dx\right)^{1/p'}$$
$$\leqslant \left(\int_{|t|<1/r} |t|^{q/p-1} w(t)^{q/p}\,dt\right)^{1/q}\left(\int_{|t|>1/r} |t|^{-p'} w(t)^{-p'/p}\,dt\right)^{1/p'}.$$

The first factor on the right is at most

$$\left(r^{-q/p+1}\int_{|t|<1/r} w(t)^{q/p}\,dt\right)^{1/q}$$

since $q/p \geqslant 1$, and the second is bounded by

$$C\left(r^{p'}\int_{|t|<1/r} w(t)^{-p'/p}\,dt\right)^{1/p'}$$

by Lemma A applied to $w^{-p'/p} \in A_{p'}$ with $\alpha = p'$. (Recall that $w^{q/p} \in A_{1+q/p'}$ means that $w \in A_p \cap \mathrm{RH}_{q/p}$; in particular, $w \in A_p$, and so $w^{-p'/p} \in A_{p'}$.) Thus, (18) is bounded by

$$C\left(r\int_{|t|<1/r} w(t)^{q/p}\,dt\right)^{1/q}\left(r\int_{|t|<1/r} w(t)^{-p'/p}\,dt\right)^{1/p'},$$

which is at most a constant independent of r since $w^{q/p} \in A_{1+q/p'}$.

Remark 2a. The conclusion of Lemma 2a holds if

$$\left(\int_{|t|<s} |t|^{q/p-1} w(t)^{q/p}\,dt\right)^{p/q}\left(\int_{|t|>s} \frac{w(t)^{-1/(p-1)}}{|t|^{p'}}\,dt\right)^{p-1} \leqslant C, \qquad s>0,$$

as can be seen from (18). In particular, the conclusion holds with $w(1/x)$ replaced by $w(1/|x|)$ if

$$\left(\int_0^s t^{q/p-1} w(t)^{q/p}\,dt\right)^{p/q}\left(\int_s^\infty \frac{w(t)^{-1/(p-1)}}{t^{p'}}\,dt\right)^{p-1} \leqslant C, \qquad s>0.$$

3. Proofs of Theorems 1 and 1a. We first prove Theorem 1. For an integrable f with $\int_{-\infty}^{\infty} f = 0$, we can write

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y)(e^{ixy}-1)\,dy = \int_{|xy|<1} + \int_{|xy|>1} = \mathrm{I} + \mathrm{II}.$$

Using the estimate $|e^{ixy}-1| \leqslant \mathrm{Min}\{|xy|, 2\}$ and letting $y = 1/t$, we obtain

$$|\mathrm{I}| \leqslant \int_{|y|<1/|x|} |xy|\,|f(y)|\,dy = |x|\int_{|t|>|x|} |t^{-3} f(1/t)|\,dt$$

and

$$|\mathrm{II}| \leqslant 2\int_{|y|>1/|x|} |f(y)|\,dy = 2\int_{|t|>|x|} |t^{-2} f(1/t)|\,dt.$$

If we estimate $|\hat{f}(x)|$ in (2) by the sum of these expressions, apply Lemmas 1 and 2 with $g(t) = |t^{-3} f(1/t)|$ and $h(t) = |t^{-2} f(1/t)|$, respectively, and then make the change of variables $y = 1/x$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\hat{f}(x)|^p |x|^{-2} w\left(\frac{1}{x}\right) dx \\
&\quad\leqslant C\int_{-\infty}^{\infty}\left(\int_{|t|>|x|} g(t)\,dt\right)^p |x|^{p-2} w\left(\frac{1}{x}\right) dx \\
&\quad\quad + C\int_{-\infty}^{\infty}\left(\int_{|t|<|x|} h(t)\,dt\right)^p |x|^{-2} w\left(\frac{1}{x}\right) dx \\
&\quad\leqslant C\int_{-\infty}^{\infty} g(y)^p |y|^{2p-2} w\left(\frac{1}{y}\right) dy + C\int_{-\infty}^{\infty} h(y)^p |y|^{p-2} w\left(\frac{1}{y}\right) dy \\
&\quad= C\int_{-\infty}^{\infty} |f(x)|^p |x|^p w(x)\,dx.
\end{aligned}$$

This completes the proof.

To prove Theorem 1a, note that for $f \in \mathscr{S}_{0,0}$ and any fixed positive integer k,

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y)\left[e^{ixy} - \sum_{j=0}^{k-1} \frac{(ixy)^j}{j!}\right] dy = \int_{|xy|<1} + \int_{|xy|>1} = \mathrm{I} + \mathrm{II}.$$

Again, setting $y = 1/t$, we have

$$|\mathrm{I}| \leqslant C\int_{|xy|<1} |xy|^k |f(y)|\, dy = C|x|^k \int_{|t|>|x|} |t^{-k-2} f(1/t)|\, dt,$$

$$|\mathrm{II}| \leqslant C\int_{|xy|>1} |xy|^{k-1} |f(y)|\, dy = C|x|^{k-1} \int_{|t|<|x|} |t^{-k-1} f(1/t)|\, dt.$$

The theorem follows from these estimates as before, except that instead of Lemmas 1 and 2 we now use Lemmas 1a and 2a with $g(t) = |t^{-k-2} f(1/t)|$ and $h(t) = |t^{-k-1} f(1/t)|$, respectively.

4. Extensions. Let $W(x) = |x|^{kp} w(x)$, where k is a positive integer and w is a weight such that $w \in A_p$, $1 < p < \infty$. If we define

$$\mathscr{F}f(x) = \int_{-\infty}^{\infty} f(y)\left[e^{ixy} - \sum_{j=0}^{k-1} \frac{(ixy)^j}{j!}\right] dy,$$

then the proof above shows in particular that the integral defining $\mathscr{F}f$ converges absolutely almost everywhere for any $f \in L^p(W)$. We will now show that $\mathscr{F}f(x)$ converges absolutely for *all* x for every $f \in L^p(W)$, with W as above. If $x = 0$, the integrand is zero. For any fixed $x \neq 0$, there are constants $C = C_x > 0$ such that

$$\begin{aligned}|\mathscr{F}f(x)| &\leqslant C\int_{|y|<C} |y^k f(y)|\, dy + C\int_{|y|>C} |y^{k-1} f(y)|\, dy \\ &\leqslant C\|f\|_{L^p(W)}\left\{\left(\int_{|y|<C} w^{-p'/p}\, dy\right)^{1/p'} + \left(\int_{|y|>C} |y|^{-p'} w^{-p'/p}\, dy\right)^{1/p'}\right\}\end{aligned}$$

by Hölder's inequality. Thus, $|\mathscr{F}f(x)| \leqslant C\|f\|_{L^p(W)} < \infty$, $C = C_{x,w}$.

Furthermore, since $\mathscr{S}_{0,0}$ is dense in such $L^p(W)$ (see **[5]**), if $\{f_n\} \subset \mathscr{S}_{0,0}$ converges to f in $L^p(W)$, then $\mathscr{F}f(x) = \lim_{n\to\infty} \hat{f}_n(x)$ for every $x \in \mathbf{R}$: in fact, by the previous argument,

$$\begin{aligned}|\mathscr{F}f(x) - \hat{f}_n(x)| &= \left|\int_{-\infty}^{\infty} [f(y) - f_n(y)]\left[e^{ixy} - \sum_{j=0}^{k-1} \frac{(ixy)^j}{j!}\right] dy\right| \\ &\leqslant C\|f - f_n\|_{L^p(W)}, \qquad C = C_{x,w},\end{aligned}$$

and the right side tends to 0 as $n \to \infty$. Thus, by Fatou's lemma, we obtain under the hypotheses of Theorem 1a that

$$\left(\int_{-\infty}^{\infty} \left|\frac{\mathscr{F}f(x)}{x^{k-1}}\right|^q \left[\frac{1}{|x|} w\left(\frac{1}{x}\right)\right]^{q/p} \frac{dx}{|x|}\right)^{1/q} \leqslant C\left(\int_{-\infty}^{\infty} |x^k f(x)|^p w(x)\, dx\right)^{1/p} \tag{19}$$

for any $f \in L^p(|x|^k w)$, with C independent of f.

For weights of the form $W(x) = |x - a|^{kp}w(x)$ (see the right side of (6)), it is necessary to use a variant of $\mathscr{F}f$ in order to obtain an inequality like (19). In fact, if for given a we define

$$\mathscr{F}_a f(x) = \int_{-\infty}^{\infty} f(y)\left[e^{ixy} - \mathscr{P}_{k-1,a,x}(y)\right] dy,$$

where

$$\mathscr{P}_{k-1,a,x}(y) = \sum_{j=0}^{k-1} \frac{(ix)^j e^{ixa}}{j!}(y-a)^j$$

is the Taylor polynomial around a of e^{ixy} as a function of y, it follows by changing variables that if $w \in A_p \cap \mathrm{RH}_{q/p}$, then

$$\left(\int_{-\infty}^{\infty} \left|\frac{\mathscr{F}_a f(x)}{x^{k-1}}\right|^q \left[\frac{1}{|x|} w\left(a + \frac{1}{x}\right)\right]^{q/p} \frac{dx}{|x|}\right)^{1/q} \leqslant C\left(\int_{-\infty}^{\infty} |f(x)|^p |x-a|^{kp} w(x)\, dx\right)^{1/p} \tag{20}$$

with C independent of a and f.

Inequality (20) is actually valid assuming only that $w(x + a)$ satisfies (5). This will follow as before if we show that $\mathscr{S}_{0,0}$ is dense in $L^p(|x - a|^{kp}w)$ for such w. It suffices to consider the case $a = 0$. By Theorem (6.19) of **[5]**, since w is locally integrable, we have only to show that

$$\frac{1}{n^p}\int_{|x|<n} w(x)\, dx \to 0 \quad \text{as } n \to \infty.$$

We assume that

$$\int_{|x|>1} \frac{w(x)^{q/p}}{|x|^{1+q/p'}} dx < \infty$$

(see the first factor in the first inequality of (5)). For N large, write

$$\frac{1}{n^p}\int_{|x|<n} w\, dx = \frac{1}{n^p}\int_{|x|<N} w\, dx + \frac{1}{n^p}\int_{N<|x|<n} w\, dx = A + B.$$

Clearly, $A \to 0$ as $n \to \infty$ for any fixed N. By Hölder's inequality,

$$B \leqslant \frac{1}{n^p}\left(\int_{N<|x|<n} w^{q/p}\, dx\right)^{p/q} (2n)^{1-p/q} \leqslant C\left(\int_{|x|>N} \frac{w(x)^{q/p}}{|x|^{1+q/p'}}\, dx\right)^{p/q}.$$

Thus, $B \to 0$ uniformly in n as $N \to \infty$, and the result follows.

5. A counterexample. In this section, we show that Theorem 1 fails for $w(x) = 1/|x|$, i.e., that inequality (1) fails for $\gamma = p - 1$ for the class of f with integral zero. In fact, we will show that no norm inequality of the form

$$\left(\int_{-\infty}^{\infty} |\hat{f}(x)|^q u(x)\, dx\right)^{1/q} \leqslant C\left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{p-1}\, dx\right)^{1/p},$$

$1 < p < \infty$, $0 < q \leqslant \infty$, $u(x) \not\equiv 0$, can hold for all f with integral zero. It will then follow immediately from the density result in Theorem (6.1) of **[5]** that no such inequality can hold for all $f \in \mathcal{S}_{0,0}$. A similar statement can be made if the weight $|x|^{p-1}$ on the right above is replaced by $|x|^{kp-1}$, $k = 1, 2, \ldots$. For simplicity, we consider only $k = 1$.

For $N > 1$, define

$$f(x) = f_N(x) = \frac{1}{x}\{\chi_{(1/N,1)}(x) - \chi_{(1,N)}(x)\},$$

where $\chi_{(a,b)}$ denotes the characteristic function of the interval (a, b). First note that

$$\int_{-\infty}^{\infty} f\,dx = \int_{1/N}^{1} \frac{dx}{x} - \int_{1}^{N} \frac{dx}{x} = \ln N - \ln N = 0,$$

and that

$$\left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{p-1}\,dx\right)^{1/p} = \left(\int_{1/N}^{N} x^{-p} x^{p-1}\,dx\right)^{1/p} = (2\ln N)^{1/p}. \tag{22}$$

Next, we will show that

$$|\hat{f}(x)| \geqslant \ln N - 2|x| - 3/|x| \quad \text{for all } x. \tag{23}$$

To see this, write

$$|\hat{f}(x)| = \left|\int_{1/N}^{1} \frac{e^{ixt}}{t}\,dt - \int_{1}^{N} \frac{e^{ixt}}{t}\,dt\right| = \left|\int_{1/N}^{1} \frac{dt}{t} + \int_{1/N}^{N} \frac{e^{ixt}-1}{t}\,dt - \int_{1}^{N} \frac{e^{ixt}}{t}\,dt\right|$$

$$\geqslant \int_{1/N}^{1} \frac{dt}{t} - \left|\int_{1/N}^{1} \frac{e^{ixt}-1}{t}\,dt\right| - \left|\int_{1}^{N} \frac{e^{ixt}}{t}\,dt\right| = \ln N - A - B, \quad \text{say.}$$

Then

$$A \leqslant \int_{1/N}^{1} \frac{2|x|t}{t}\,dt \leqslant 2|x|,$$

and

$$B = \left|\int_{1}^{N} \frac{1}{t}\frac{d}{dt}\left\{\frac{e^{ixt}}{ix}\right\}dt\right| = \left|\frac{1}{t}\frac{e^{ixt}}{ix}\Big|_{t=1}^{t=N} + \int_{1}^{N} \frac{e^{ixt}}{ixt^2}\,dt\right|$$

$$\leqslant \frac{1}{N|x|} + \frac{1}{|x|} + \frac{1}{|x|}\int_{1}^{\infty} \frac{dt}{t} \leqslant \frac{3}{|x|}.$$

Combining estimates leads immediately to (22).

Now, if $u(x) \geqslant 0$ and $u(x) \not\equiv 0$, pick a, b with $0 < a < b < \infty$ and $\int_{a<|x|<b} u\,dx > 0$. Then pick $N = N_{a,b}$ so large that, by (23),

$$|\hat{f}(x)| \geqslant \ln N - 2|x| - 3/|x| \geqslant \tfrac{1}{2}\ln N \quad \text{for } a < |x| < b.$$

Thus,

$$\left(\int_{-\infty}^{\infty} |\hat{f}(x)|^q u(x)\,dx\right)^{1/q} \geqslant \left(\tfrac{1}{2}\ln N\right)\left(\int_{a<|x|<b} u\,dx\right)^{1/q}.$$

In view of (22) and the fact that $p > 1$, this contradicts (21).

We note in passing that a similar construction is given in **[4]**.

6. Proof of Theorem 2. Let $f \in \mathscr{S}_{0,0}(\mathbf{R}^n)$, $n > 1$. Using $(x \cdot y)$ to denote the ordinary dot product in $\mathbf{R}^n$ and noting that $(x \cdot y)^j$ is a polynomial in y for $j = 0, 1, \dots,$ we can write

$$\hat{f}(x) = \int_{\mathbf{R}^n} f(y)\left[e^{i(x\cdot y)} - \sum_{j=0}^{k-1} \frac{\{i(x\cdot y)\}^j}{j!}\right] dy$$

$$= \int_{|y|<1/|x|} + \int_{|y|>1/|x|} = \mathrm{I} + \mathrm{II}.$$

Since for the change of variables $y = t/|t|^2$ the Jacobian is bounded in absolute value by a constant times $|t|^{-2n}$, we have the estimates

$$|\mathrm{I}| \leqslant C|x|^k \int_{|y|<1/|x|} |y|^k |f(y)|\,dy \leqslant C|x|^k \int_{|t|>|x|} |t|^{-k-2n}\left|f\left(\frac{t}{|t|^2}\right)\right| dt,$$

$$|\mathrm{II}| \leqslant C|x|^{k-1} \int_{|y|>1/|x|} |y|^{k-1} |f(y)|\,dy \leqslant C|x|^{k-1} \int_{|t|<|x|} |t|^{-k+1-2n}\left|f\left(\frac{t}{|t|^2}\right)\right| dt.$$

Inequality (7) will then follow from showing that both

$$\left(\int_{\mathbf{R}^n}\left(\int_{|t|>|x|} |t|^{-k-2n}\left|f\left(\frac{t}{|t|^2}\right)\right| dt\right)^q \left[|x|^\beta w\left(\frac{1}{|x|}\right)\right]^{q/p} |x|^{q-n}\,dx\right)^{1/q} \tag{24}$$

and

$$\left(\int_{\mathbf{R}^n}\left(\int_{|t|<|x|} |t|^{-k+1-2n}\left|f\left(\frac{t}{|t|^2}\right)\right| dt\right)^q \left[|x|^\beta w\left(\frac{1}{|x|}\right)\right]^{q/p} |x|^{-n}\,dx\right)^{1/q} \tag{25}$$

are bounded by the right side of (7). Note that the right side of (7) is equivalent to

$$\left(\int_{\mathbf{R}^n}\left|f\left(\frac{x}{|x|^2}\right)\right|^p |x|^{-kp-2n} w\left(\frac{1}{|x|}\right) dx\right)^{1/p}. \tag{26}$$

To estimate (24), change to polar coordinates $t = \tau t'$, $\tau = |t|$, and $x = \rho x'$, $\rho = |x|$; in this way, (24) becomes

$$\left(\sigma_{n-1}\int_0^\infty\left(\int_\rho^\infty \int_{|t'|=1}\left|f\left(\frac{t'}{\tau}\right)\right|\tau^{-k-n-1}\,d\tau\,dt'\right)^q \rho^{q+\beta q/p-1} w\left(\frac{1}{\rho}\right)^{q/p} d\rho\right)^{1/q}, \tag{27}$$

where σ_{n-1} is the surface area of the unit ball in $\mathbf{R}^n$. Letting

$$g(\tau) = \int_{|t'|=1}\left|f\left(\frac{t'}{\tau}\right)\right|\tau^{-k-n-1}\,dt',$$

and

$$\tilde{w}(\rho) = |\rho|^{-(p-1)(n-1)} w(\rho),$$

and recalling that $\beta = (n-1)p - n$, we may rewrite (27) as a constant times

$$\left(\int_0^\infty \left[\rho^{1/p'-1/q}\tilde{w}\left(\frac{1}{\rho}\right)^{1/p}\int_\rho^\infty g(\tau)\,d\tau\right]^q d\rho\right)^{1/q}.$$

Since $\tilde{w}^{q/p} \in A_{1+q/p'}(\mathbf{R}^1)$, Lemma 1a implies that this is at most

$$C\left(\int_0^\infty g(\rho)^p\rho^{2p-2}\tilde{w}\left(\frac{1}{\rho}\right)d\rho\right)^{1/p}. \tag{28}$$

Since by Hölder's inequality

$$g(\rho)^p \leqslant \sigma_{n-1}^{p/p'}\int_{|x'|=1}\left|f\left(\frac{x'}{\rho}\right)\right|^p \rho^{-p(k+n+1)}\,dx',$$

it follows that (28) is bounded by

$$C\left(\int_0^\infty \int_{|x'|=1}\left|f\left(\frac{x'}{\rho}\right)\right|^p \rho^{-p(k+n-1)-2}\tilde{w}\left(\frac{1}{\rho}\right)dx'\,d\rho\right)^{1/p}.$$

This, however, is easily seen to be a multiple of (26), and the estimation of (24) is complete.

To estimate (25), we argue similarly, rewriting (25) in the form

$$C\left(\int_0^\infty \left(\rho^{-1/p-1/q}\tilde{w}\left(\frac{1}{\rho}\right)^{1/p}\int_0^\rho h(\tau)\,d\tau\right)^q d\rho\right)^{1/q}$$

where $\tilde{w}$ is as before and

$$h(\tau) = \int_{|t'|=1}\left|f\left(\frac{t'}{\tau}\right)\right|\tau^{-k-n}\,dt'.$$

Thus, by Lemma 2a, (25) is bounded by

$$C\left(\int_0^\infty h(\rho)^p\rho^{p-2}\tilde{w}\left(\frac{1}{\rho}\right)d\rho\right)^{1/p}.$$

Using Hölder's inequality to estimate $h(\rho)^p$ and arguing as before, we see this is at most a constant times (26), and the proof is complete.

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