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THE *bo*-ADAMS SPECTRAL SEQUENCE

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ABSTRACT. Due to its relation to the image of the J-homomorphism and first order periodicity (Bott periodicity), connective real K-theory is well suited for problems in 2-local stable homotopy that arise geometrically. On the other hand the use of generalized homology theories in the construction of Adams type spectral sequences has proved to be quite fruitful provided one is able to get a hold on the respective E_2-terms. In this paper we make a first attempt to construct an algebraic and computational theory of the E_2-term of the *bo*-Adams spectral sequence. This allows for some concrete computations which are then used to give a proof of the bounded torsion theorem of **[8]** as used in the geometric application of **[2]**. The final table of the E_2-term for π_*^S in dim $\leqslant 20$ shows that the statement of this theorem cannot be improved. No higher differentials appear in this range of the *bo*-Adams spectral sequence. We observe, however, that such a differential has to exist in dim 30.

In this paper we analyze the E_1- and E_2-terms of the Adams spectral sequence based on real connective K-theory *bo*. As can be seen from applications **[2, 8, 10]** and our sample calculations, this spectral sequence is quite powerful. It converges to the 2-local stable homotopy groups $\pi_*^s(X)_{(2)}$.

Unfortunately its E_2-term lacks computability due to the fact that an algebraic description is not yet known. In this paper we show that the E_2-term can be embedded into a long exact sequence of which at least one of the other two terms does not have this disadvantage: in most interesting cases it can be described algebraically (as a certain Ext-functor) and it is completely computable in examples like spheres or stunted (real) projective spaces. Tables for $X = S^0$ suggest that in dimensions $\leqslant 45$ nearly all of the classes found in this way detect in fact homotopy classes.

We now give a more detailed account of the contents of the individual chapters.

In §1 we fix some notations and describe the algebra of operations in *bo* and *bsp* (up to torsion). This was done additively in **[13]**. The starting point of the whole analysis is the splitting of *bo*-module spectra

$$bo \wedge bo \simeq \bigvee_{n \geqslant 0} \Sigma^{4n} bo \wedge B(n),$$

where $B(n)$ denotes an integral Brown-Gitler spectrum **[8]**.

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Two different constructions of this homotopy equivalence are available. The one in [8] gives complete control in $\mathbf{Z}/2$-homology, whereas the one in [13] allows complete control in integral homology modulo torsion. We use the second approach. We recollect the details and compute the effect of the splitting maps in homotopy modulo torsion in §2. In §3 we expand on [7] to produce a fairly good controlled splitting of $bo \wedge \overline{bo}^s$. This enables us to determine the differential d_1 of the bo-Adams spectral sequence up to torsion operations factorizing through $\mathbf{Z}/2$-Eilenberg-Mac Lane spectra. For any X let $KV_s(X)$ denote the maximal $\mathbf{Z}/2$-Eilenberg-Mac Lane spectrum splitting from $bo \wedge \overline{bo}^s \wedge X$. It is shown in §4 that

$$\pi_t(KV_s(X)) \subset E_1^{s,t}(X;\, bo) = \pi_t\big(bo \wedge \overline{bo}^{\,s} \wedge X\big)$$

is a subcomplex with respect to d_1. The study of the quotient complex $(\mathscr{C}^{s,t}(X);\, d)$ and its homology is the main theme of this paper. We first give an algebraic interpretation in case X has the property that the $H\mathbf{Z}/2$-Adams spectral sequence for $\pi_*(bo \wedge \overline{bo}^s \wedge X)$ is trivial for all s. To do this, we derive from $bo_*(X)$ and bsp_*X a comodule over a divided polynomial (Hopf)-algebra over $\mathbf{Z}_{(2)}$ in one variable given by the dual of the relevant part of the operation algebra. Both this comodule and the Hopf algebra are filtered (by $H\mathbf{Z}/2$-Adams filtration). Under the hypothesis mentioned above, $H(\mathscr{C}^{*,*}(X);\, d)$ can then be interpreted as an Ext-functor on the appropriate abelian category of filtered comodules and filtration-preserving homomorphisms.

We remark, however, that this interpretation is not explicitly used in the computational part of the paper. Nevertheless there are several computational aspects present in this approach. See Remarks 4.9 and 4.10 for more hints on these.

The second part of the paper deals with techniques for an effective computation of these Ext-groups. A spectral sequence is introduced in §5 and its E_1-term is computed.

To get the more concrete results needed in applications we restrict our attention in §6 to $X = S^0$, $B(1)$, and projective space P_1^∞. This enables us to settle also the higher differentials of the auxiliary spectral sequence from §5 and thus to give complete results in these cases.

§7 is devoted to a detailed proof of the "bounded torsion theorem" (Theorem 7.1) for various X. This theorem was first stated in [8; 9, Theorem 1.1.c] and in some more generality in [2; 3, Theorem 3.6]. It asserts that any d_1-cycle

$$x \in E_2^{s,t}(X;\, bo) \cong \pi_t\big(bo \wedge \overline{bo}^{\,s} \wedge X\big), \qquad s \geqslant 2,$$

which has $H\mathbf{Z}/2$-Adams filtration $\geqslant 2$ is in fact a boundary.

Applying this to the map given by multiplication with 2 implies that $E_2^{s,t}(X;\, bo)$ is for $s \geqslant 2$ at most a $\mathbf{Z}/4$-module, hence the name. The bounded torsion theorem is a quite powerful tool in obstruction theory. Typically it will be applied in conjunction with some sort of vanishing line theorem for the bo-Adams spectral sequence using the following kind of argument: Suppose a self-map f of a finite complex X (as in 7.1) is given together with a homotopy class $\alpha\colon S^t \to X$. Suppose further that f is of $H\mathbf{Z}/2$-Adams filtration 1 and α can be lifted to $\alpha_s\colon S^{t+s} \to \overline{bo}^{\wedge s} \wedge X$. Then two-fold composition of f with α is represented in $E_1^{s,t+s}(X;\ bo)$ by a cycle of $H\mathbf{Z}/2$-Adams filtration at least two and hence a boundary. Therefore $f \circ f \circ \alpha$ has bo-Adams filtration at least $s+1$. In favorable cases the vanishing line will eventually imply the equation $f^{\circ n} \circ \alpha \cong 0$ for $n \gg 0$ (see [**8** or **2**] for concrete examples in this vein).

Unfortunately, the proof of the theorem as stated in [**8, 9**] and [**2, 3**] has been found to be incomplete in case the Adams operation ψ^3 is operating nontrivially on $bo_* X$. Using our very detailed knowledge of $H(\mathscr{C}^{*,*}(X))$ for $X = S^0$, $B(1)$ from §6, we are able to complete the proof for X equal to S^0, $B(1)$, stunted real projective spaces, and all spectra which appear in minimal $H\mathbf{Z}/2$-Adams resolution of these. This suffices at least for the known applications of the theorem, especially to the geometric dimension of vector bundles in [**2**]. It is likely that the class of spectra satisfying the bounded torsion theorem is larger than only those mentioned above, how big it really is we do not know.

The final §8 contains a sample table of $H(\mathscr{C}^{*,*}(S^0))$ for $t - s \leqslant 50$ together with a table of the full E_2-term of the bo-Adams spectral sequence for π_*^s in dimensions $t - s \leqslant 20$.

This table differs in several aspects from the ones derived with help of other homology theories such as $H\mathbf{Z}/2$ or Brown-Peterson homology BP, the most striking difference being the lack of higher differentials in this range of dimensions.

Moreover the image of the J-homomorphism is completely concentrated in filtration 0 or 1, depending on whether an element is detected by the d- or e-invariant.

The tables may also be used to show that the hypothesis of the bounded torsion theorem cannot be improved: both κ and $\eta\kappa$ have bo-filtration 3, hence $\eta\kappa$ must be represented (in E_1) with $H\mathbf{Z}/2$-Adams filtration 1. Similarly the class $\bar{\kappa}$ supports a $\mathbf{Z}/4 \subset E_2^{4,24}(S^0, bo)$. It is finally possible to use the table of $H(\mathscr{C}^*(S^0))$ together with known information on π_*^s to produce the first known nontrivial higher differential in the bo-Adams spectral sequence. This occurs in dimension 30 and is needed to render $\nu^3\bar{\kappa}$ zero in π_{29}^s.

The tables were calculated by hand in the obvious (and, hence, tedious) way before the computational tools of this paper became available. Using these together with a computer, computations of the bo-Adams spectral sequence for S^0 should be possible up to considerably higher dimensions.

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PART 1. GENERAL PROPERTIES OF THE bo-ADAMS SPECTRAL SEQUENCE

1. Operations in bo **and** bsp. Let bo and bsp denote connective real or symplectic K-theory. It is well known that $H^*(bo, \mathbf{Z}/2) \cong \mathfrak{A}/\mathfrak{A}(1)$, where $\mathfrak{A}(1)$ denotes the subalgebra of the Steenrod algebra $\mathfrak{A}$ generated by Sq^1 and Sq^2. A convenient way to compute $bo_* X$ is then given by the **HZ**/2-Adams spectral sequence [**1**], which, using a standard change of rings theorem has E_2-term

$$E_2^{s,t}(bo \wedge X;\ \mathbf{HZ}/2) \cong \mathrm{Ext}_{\mathfrak{A}(1)}^{s,t}\big(H^*(X), \mathbf{Z}/2\big).$$

Using the homotopy equivalence of bo-module spectra $bsp \sim bo \wedge B(1)$ [**8**], we get a similar spectral sequence

$$\operatorname{Ext}^{s,t}_{\mathfrak{A}(1)}(H^*(B(1) \wedge X); \mathbf{Z}/2) \Rightarrow bsp_{t-s}X.$$

For $X = S^0$, these are given by the charts in Figure 1, where a dot denotes $\mathbf{Z}/2$, a vertical line is multiplication by h_0 representing multiplication with 2, a slanting line to the right multiplication by h_1 representing η, and everything is repeated periodically to the right with period $(s, t-s) = (4, 8)$.

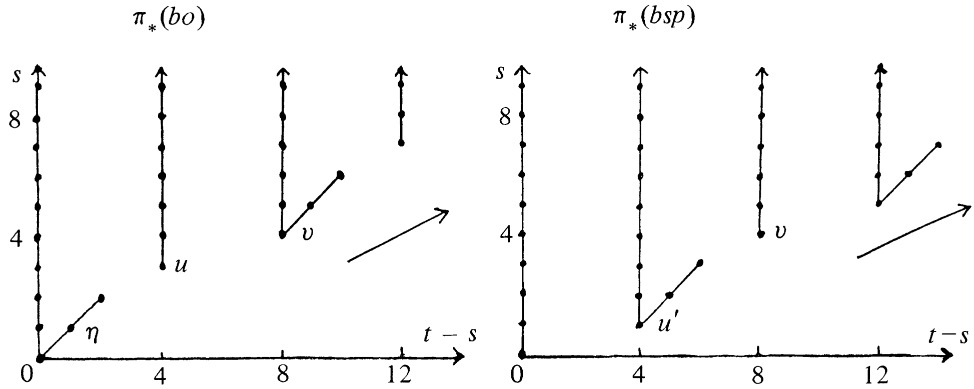


FIGURE 1

Let u, u', and v denote the generators of $\pi_4 bo$, $\pi_4 bsp$, and $\pi_8 bo \cong \pi_8 bsp$ respectively and denote by π both of the standard maps $bo \to bsp \simeq bo \wedge B(1)$ and $bsp \to bo$. These have Adams filtration 0 and 2 respectively and degree 1 and 4 on the bottom cell. Moreover $\pi^2 = 4 \cdot \mathrm{id}$. Using the bo-module structure of bsp, u: $S^4 \to bsp$ induces a bo-module map p: $\Sigma^4 bo \to bsp$ of Adams filtration 1. To construct a similar map $\Sigma^4 bsp \to bo$, we first consider

$$p \wedge 1\colon \Sigma^4 bsp \simeq \Sigma^4 bo \wedge B(1) \to bsp \wedge B(1) \simeq bo \wedge B(1) \wedge B(1).$$

Since the latter spectrum is equivalent to the second term in a minimal Adams resolution of bo [**8, 2**] (see also Corollary 3.6), we may compose with the canonical projection to bo to get a map $\Sigma^4 bsp \to bo$ of Adams filtration 3, which will also be denoted by p.

In what follows, we need to deal with bo and bsp simultaneously. It is therefore convenient to denote by $b_{*,*}(-)$ the $\mathbf{Z} \times \mathbf{Z}/2$-graded groups defined by

$$b_{n,\varepsilon}(X) = \begin{cases} bo_n X, & \text{if } \varepsilon = 0, \\ bsp_n X, & \text{if } \varepsilon = 1. \end{cases}$$

This is a $\mathbf{Z}$-graded multiplicative homology theory, since the second degree does not change under suspension. It has coefficients $b_{*,*}(S^0) \simeq bo_* \oplus bsp_*$.

We want to describe the algebra of operations for $b_{*,*}(-)$ up to torsion. As usual, we restrict our attention to homogeneous operations (in both degrees). Also any operation $bo \to \Sigma^k bo$ induces an operation $bsp \to \Sigma^k bsp$ by smashing with $B(1)$ and similarly vice versa. It is therefore natural to consider only those (additive) operations which commute with π. This algebra will be denoted by $\mathcal{O}^{*,*}$. Let ψ^3: $bo \to bo$ be the (stable) Adams operation. It induces an operation ψ^3: $b \to b$ in the canonical way. The following lemma is well known (cf. [**11**, §2]).

LEMMA 1.1. *There exists a unique operation* ϕ: $b \to \Sigma^4 b$ *such that* $(\psi^3 - \mathrm{id}) = p \cdot \phi$. □

Using the standard conventions, we have bidegree$(p) = (-4, 1)$, bidegree$(\phi) = (4, 1)$, and bidegree$(\pi) = (0, 1)$.

THEOREM 1.2. *Modulo torsion, the algebra* $\mathcal{O}^{*,*}$ *is isomorphic to the algebra* $\mathbf{Z}_{(2)}[\pi]/(\pi^2 - 4\mathrm{id})\langle\langle p, \phi\rangle\rangle$ *of homogeneous power series in p and* ϕ *with coefficients in* $\mathbf{Z}_{(2)}[\pi]/(\pi^2 - 4\mathrm{id})$ *under multiplication. The generators p and* ϕ *are noncommuting, and the relations are generated by*

$$[\pi, p] = 0 = [\pi, \phi], \qquad [\phi, p] = 8 \cdot (\mathrm{id} + p\phi).$$

Moreover, all torsion operations either factorize through $\mathbf{Z}/2$*-Eilenberg-Mac Lane spectra or have dimensions* (s, ε) *with* $s \not\equiv 0\ (4)$.

REMARKS 1.3. (i) It may be shown that the (additive) generators ϕ^i of (1.2) differ from those produced in [**13**] only by units in $\mathbf{Z}_{(2)}$ (see also the remarks on the proof of (1.2)).

(ii) For later use it is important to know the Adams-filtration of the operation ϕ. This is given by

$$\mathrm{AF}(\phi\colon bo \to \Sigma^4 bsp) = 0, \quad \mathrm{AF}(\phi\colon bsp \to \Sigma^4 bo) = 2, \quad \mathrm{AF}(\phi^{2i}) = 4k - \alpha(i),$$

where $\alpha(i)$ denotes the number of 1's in the dyadic expansion of i (cf. Theorem B in [**13**], where the $2n - \alpha(n)$'s should be changed to $4n - \alpha(n)$'s).

(iii) Using the above theorem, $bo^*bo/\mathrm{Tors} \cong bsp^*bsp/\mathrm{Tors}$ and $bsp^*bo/\mathrm{Tors} \simeq bo^*bsp/\mathrm{Tors}$ are easily deduced by homogeneity considerations. For example we have

$$bo^*bo/\mathrm{Tors} \cong \mathbf{Z}_{(2)}[u, v]/(u^2 - 4v)\langle\langle \phi_1, \phi_2\rangle\rangle/(\phi_1^2 - 4\phi_2),$$

where the notation is as above and $u = \pi p$, $v = p^2$, $\phi_1 = \pi\phi$, $\phi_2 = \phi^2$. This implies relations

$$\phi_1^2 = 4\phi_2, \quad [\phi_1, \phi_2] = 0, \quad [\phi_1, u] = 8(4\mathrm{id} + u\phi).$$

A similar but more complicated relation holds for ϕ_2.

(iv) It is a corollary of the proof that all operations are (mod torsion) uniquely determined by their action in homotopy or (equivalently) rational homology.

REMARKS ON THE PROOF OF (1.2). The two cases of bo^*bo and bsp^*bo are handled separately. Additively these groups are known from [**13**]. There it is also shown that mod torsion the operations are uniquely determined by their effect in homotopy. It is then not difficult to see that our generators ϕ^i differ from the generators produced in [**13**] only by units in $\mathbf{Z}_{(2)}$. (See [**5**] for a similar but more detailed computation in the odd primary case.) □

2. The structure of $bo \wedge bo$. Let X be any spectrum and denote by $X^{\langle n\rangle}$ the nth term in a minimal $H\mathbf{Z}/2$-Adams resolution of X:

$$\begin{array}{ccccccccccc} X & & \leftarrow & & X^{\langle 1\rangle} & & \leftarrow & & X^{\langle 2\rangle} & & \leftarrow & \cdots \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\ & & H_0 & & & & H_1 & & & & H_2 & \end{array}$$

Here H_i are $\mathbf{Z}/2$-Eilenberg-Mac Lane spectra, the map $H_i \to X^{\langle i+1\rangle}$ is of degree 1, and $X \to H_0 \to H_1 \to \cdots$ induces a minimal resolution of H^*X as an $\mathfrak{A}$-module.

Let B_n denote the nth integral Brown-Gitler spectrum with bottom cell in dimension $4n$ [**8**].

THEOREM 2.1 [**8, 2**]. *There exists a homotopy equivalence of bo-module spectra*

$$bo \wedge B_n \simeq KV_n \vee \begin{cases} \Sigma^{4n} bo^{\langle 2n-\alpha(n)\rangle}, \\ \Sigma^{4n} bsp^{\langle 2n-1-\alpha(n)\rangle}. \end{cases}$$

Here KV_n is the Eilenberg-Mac Lane spectrum associated to the $\mathbf{Z}/2$-vector-space V_n and $\alpha(n)$ is as in 1.3(*i*). □

Theorem 2.1 asserts that modulo operations $bo \to KV_n$ any operation $bo \to \Sigma^j(bo \wedge B_n)$ may be obtained as an operation $bo \to \Sigma^{j+4n} bo$ of suitable filtration. Write

$$\left(\binom{m}{n}\right) = \frac{(9^m-1)\cdots(9^{m-n+1}-1)}{(9^n-1)\cdots(9-1)}$$

[**4**] and let $\nu_2(n)$ be defined by $n = 2^{\nu_2(n)}\cdot$ odd. Since $\nu_2(9^k-1) = 3+\nu_2(k)$, we have $\nu_2((\binom{m}{n})) = \nu_2\binom{m}{n}$. Similarly, let

$$n!! = \left(\binom{1}{1}\right)\cdot\left(\binom{2}{1}\right)\cdots\left(\binom{n}{1}\right) = 2^{-3n}\cdot(9^n-1)\cdots(9-1).$$

Then $\nu_2(n!!) = n - \alpha(n)$ and $((\binom{m}{n})) = m!!/n!!(m-n)!!$. Denote by

$$\pi_n\colon \begin{cases} bo^{\langle 2n-\alpha(n)\rangle} \to bo, & n \text{ even}, \\ bsp^{\langle 2n-\alpha(n)-1\rangle} \to bsp, & n \text{ odd}, \end{cases}$$

the unique bo-module maps of homology degree $2^n\cdot n!!$ for n even and $2^{n-1}\cdot n!!$ for n odd on the bottom cell. Then ϕ^n lifts by 1.3(i) through π_n and induces maps

$$\phi_n\colon bo \to \begin{cases} \Sigma^{4n} bo^{\langle 2n-\alpha(n)\rangle}, & n \text{ even}, \\ \Sigma^{4n} bsp^{\langle 2n-1-\alpha(n)\rangle}, & n \text{ odd}. \end{cases}$$

Recall from (2.1) that $bo \wedge B_n$ contains the target spectrum of ϕ_n with cofactor KV_n.

THEOREM 2.2 [**13, 8**]. *ϕ_n can be extended to $\tilde{\phi}_n$: $bo \to bo \wedge B_n$, such that*

$$\tilde{\phi} = \bigvee_{n\geq 0} \tilde{\phi}_n\colon bo \wedge bo \xrightarrow{\vee 1\wedge\phi_n} \bigvee_{n\geq 0} bo \wedge bo \wedge B_n \xrightarrow{\mu\wedge 1} \bigvee_{n\geq 0} bo \wedge B_n$$

is a homotopy equivalence of bo-module spectra. □

Observe that the action of $\tilde{\phi}_n$ is completely known in $\pi_*(-)/\mathrm{Tors}$, since this is true for ϕ^n and π_n.

Let η_L, η_R: $bo \to bo \wedge bo$ denote the canonical right and left unit maps and write $u_1 = \eta_R(u)$ and $u_0 = \eta_L(u) \in \pi_4(bo \wedge bo)$. It is then easy to see that

$$t_1 = \frac{u_1 - u_0}{8} \in \pi_4(bo \wedge bo).$$

There are no obstructions to extend t_1: $S^4 \to bo \wedge bo$ to ψ_1: $B_1 \to bo \wedge bo$. Let

$$(2.3) \quad t_n = t_n(u_0, u_1) = \frac{u_1 - u_0}{8} \cdot \frac{u_1 - 9u_0}{8} \cdots \frac{u_1 - 9^{n-1}u_0}{8} \in \pi_{4n}(bo \wedge bo).$$

Since $\nu_2(9^{k-1}) = 3 + \nu_2(k) \geqslant 3$, the element t_n is well defined. Observe that, via π: $bo \to bsp$, the element t_n may also be viewed as an element of $\pi_{4n}(bo \wedge bsp)$. (We write $t_n \cdot 1_{bsp}$ in this case.)

PROPOSITION 2.4.

$$(1 \wedge \phi^n)(t_s) = \begin{cases} \left(\binom{s}{n}\right) n!!2^n t_{s-n}, & n \text{ even}, \\ \left(\binom{s}{n}\right) n!!2^{n-1} t_{s-n} \cdot 1_{bsp}, & n \text{ odd}. \end{cases}$$

PROOF. One first computes $(1 \wedge p\phi)(t_s) = (1 \wedge (\psi^3 - 1))(t_s)$. Using $\psi^3(u_1) = 9u_1$ and multiplicativity, we get

$$\begin{aligned} (1 \wedge p\phi)(t_s) &= \frac{9u_1 - u_0}{8} \cdot \frac{9u_1 - 9u_0}{8} \cdots \frac{9u_1 - 9^{s-1}u_0}{8} - t_s \\ &= \left[\frac{9^{s-1}(9u_1 - u_0)}{8} - \frac{u_1 - 9^{s-1}u_0}{8}\right] t_{s-1} \\ &= \frac{9^s - 1}{9 - 1} u_1 \cdot t_{s-1}. \end{aligned}$$

Using naturality with respect to π: $bo \to bsp$ and the equations $\pi(1_{bo}) = 1_{bsp}$, $\pi(u_1) = 4u_1'$, one gets similarly

$$(1 \wedge p\phi)(t_s \cdot 1_{bsp}) = \frac{9^s - 1}{9 - 1} 4u_1' \cdot t_{s-1}.$$

Since $p(1_{bsp}) = u_1$ and $p(1_{bo}) = u_1'$, this proves

$$(1 \wedge \phi)t_s = \left(\binom{s}{1}\right) t_{s-1} 1_{bsp}, \qquad (1 \wedge \phi)t_s \cdot 1_{bsp} = \left(\binom{s}{1}\right) \cdot 4t_{s-1}.$$

The proposition follows by an easy induction. □

COROLLARY 2.5. (a)

$$\tilde{\phi}_{n*}(t_s) = \begin{cases} 0, & s \neq n, \\ 1_{4n} \in \pi_{4n}(bo \wedge B_n) \cong \mathbf{Z}_{(2)}, & s = n. \end{cases}$$

(b) t_n: $S^{4n} \to bo \wedge bo$ *may be extended to* ψ_n: $B_n \to bo \wedge bo$, *such that*

$$\bigvee_{n \geqslant 0} \tilde{\psi}_n: \bigvee_{n \geqslant 0} bo \wedge B_n \xrightarrow{1 \wedge \psi_n} bo \wedge bo \wedge bo \xrightarrow{\mu \wedge 1} bo \wedge bo$$

is the inverse map to $\tilde{\phi}$.

PROOF. (a) follows easily from the fact that $\mu_*(u_1) = \mu_*(u_0) = u$, so $\mu_*(t_{s-n}) = 0$ if $s - n \neq 0$.

(b) follows from (a) and the fact that $\tilde{\phi}$ is a homotopy equivalence. $\square$

REMARK 2.6. Since $\pi_*(bo \wedge B_n) \otimes \mathbf{Q}$ is generated by 1_{4n}: $S^{4n} \to bo \wedge B_n$ as a $bo_* \otimes \mathbf{Q}$-module, formula (2.4) describes completely the effect of $\bigvee_{n \geq 0} \tilde{\psi}_n$ in rational homotopy.

COROLLARY 2.7. *For all $n > 0$, the map $\mu \circ \psi_n$: $B_n \to bo \wedge bo \to bo$ is trivial.*

PROOF. By (2.5) and (2.6) the map $bo \wedge B_n \to bo$ induced from $\mu\psi_n$ is trivial. Since $H^*bo \cong \mathfrak{A}//\mathfrak{A}(1)$ is monogenic over $\mathfrak{A}$, $\mu \circ \psi_n$ is trivial in $H^*(-; \mathbf{Z}/2)$ too. Use the Adams spectral sequence for bo^*B_n to see that this implies the assertion. (See [13] for a computation of E_2 of the Adams spectral sequence.) $\square$

3. The differential d_1 of the *bo*-Adams spectral sequence. We first recall the general technique to analyse

$$d_1\colon bo \wedge \overline{bo}^{s} \xrightarrow{\mathrm{pr} \wedge 1} bo \wedge \overline{bo}^{s} \xrightarrow{\iota \wedge 1} bo \wedge \overline{bo}^{s+1}.$$

This technique is discussed for example in [7], in event the spectrum which plays the role of bo is a Thom spectrum. As opposed to our case, the map d_1 is then induced by a particularly nice map between the base spaces.

Let Y be an associative ring spectrum with multiplication μ, unit ι: $S^0 \to Y$, and canonical cofibration $S^0 \xrightarrow{\iota} Y \xrightarrow{\mathrm{pr}} \overline{Y}$. Then the cofibration $Y \wedge S^0 \xrightarrow{1 \wedge \iota} Y \wedge Y \xrightarrow{1 \wedge \mathrm{pr}} Y \wedge \overline{Y}$ of Y-module spectra has the following property.

LEMMA 3.1. *There exists a unique Y-module map*

$$r\colon Y \wedge \overline{Y} \to Y \wedge Y$$

such that $\mathrm{id}_{Y \wedge Y} = r \circ (1 \wedge \mathrm{pr}) - (1 \wedge \iota) \circ \mu$. The map r is natural with respect to maps of ring spectra. $\square$

PROOF. Since $\mathrm{id} = \mu \circ (1 \wedge \iota)$: $Y = Y \wedge S^0 \to Y \wedge Y \to Y$, the following sequence is split short exact:

$$0 \to [Y \wedge \overline{Y}, Y \wedge Y] \xrightarrow{(1 \wedge \mathrm{pr})^*} [Y \wedge Y, Y \wedge Y] \underset{\mu^*}{\overset{(1 \wedge \iota)^*}{\rightleftarrows}} [Y, Y \wedge Y] \to 0.$$

Let r be such that $(1 \wedge \mathrm{pr})^*(r) = \mathrm{id} - (1 \wedge \iota) \circ \mu$ and suppose r is not a Y-module map. We may then consider the module map $\tilde{r}$ induced from r:

$$\tilde{r}\colon Y \wedge \overline{Y} = Y \wedge S^0 \wedge \overline{Y} \xrightarrow{1 \wedge \iota \wedge 1} Y \wedge Y \wedge \overline{Y} \xrightarrow{1 \wedge r} Y \wedge Y \wedge Y \xrightarrow{\mu \wedge 1} Y \wedge Y.$$

Since $\mathrm{id} - (1 \wedge \iota)\mu$ is a (left) Y-module map it coincides with its induced module map. Therefore

$$(1 \wedge \mathrm{pr})^*(\tilde{r}) = \mathrm{id} - (1 \wedge \iota)\mu = (1 \wedge \mathrm{pr})^*(r)$$

and, by injectivity, $r = \tilde{r}$ is a module map itself. To see that r is also natural, suppose we have a map f: $Y \to Z$ of ring spectra inducing $\bar{f}$: $\overline{Y} \to \overline{Z}$. We then need to show the commutativity of:

$$\begin{array}{ccc} Y \wedge \overline{Y} & \xrightarrow{r_Y} & Y \wedge Y \\ \downarrow f \wedge \bar{f} & & \downarrow f \wedge f \\ Z \wedge \overline{Z} & \xrightarrow{r_Z} & Z \wedge Z \end{array}$$

Using the easy to establish equation $(1 \wedge \mathrm{pr}) \circ r_Y = \mathrm{id}_{Y \wedge \bar{Y}}$ we have

$$\begin{aligned}(f \wedge f) \circ r_Y &= (f \wedge f) \circ (r_Y \circ (1 \wedge \mathrm{pr}) \circ r_Y)\\ &= (f \wedge f) \circ (\mathrm{id}_{Y \wedge Y} - (1 \wedge \iota_Y)\mu_Y) \circ r_Y\\ &= (\mathrm{id}_{Z \wedge Z} - (1 \wedge \iota_Z)\mu_Z)(f \wedge f) \circ r_Y\\ &= r_Z \circ (1 \wedge \mathrm{pr}_Z) \circ f \wedge f \circ r_Y\\ &= r_Z \circ f \wedge \bar{f} \circ \mathrm{pr}_Y \circ r_Y = r_Z \circ f \wedge \bar{f}. \qquad \square\end{aligned}$$

Define $r_s\colon Y \wedge \bar{Y}^s \to Y \wedge Y^s$ as the composition

$$(3.2) \qquad r_s\colon Y \wedge \bar{Y}^s \xrightarrow{1 \wedge (r \circ (\iota \wedge 1))^s} Y \wedge (Y \wedge Y)^s \xrightarrow{\mu \wedge \cdots \wedge \mu \wedge 1} Y \wedge Y^s.$$

Then r_s is a natural Y-module map and splits the canonical projection

$$Y \wedge Y^s \to Y \wedge \bar{Y}^s.$$

r_s is identical to the composition

$$Y \wedge \bar{Y}^s \xrightarrow{r \wedge 1} Y \wedge Y \wedge \bar{Y}^{s-1} \xrightarrow{1 \wedge r \wedge 1} Y \wedge Y \wedge Y \wedge \bar{Y}^{s-2} \to \cdots \to Y \wedge Y^s.$$

LEMMA 3.3. *Let*

$$d^s = (\iota \wedge 1) \circ (\mathrm{pr} \wedge 1)\colon Y \wedge \bar{Y}^s \to Y \wedge \bar{Y}^{s+1}$$

be the standard boundary type map and let

$$\theta_i^s\colon Y \wedge Y^s \to Y \wedge Y^{s+1}$$

for $0 \leqslant i \leqslant s+1$ *be given by the unit map* $\iota\colon S^0 \to Y$ *into the ith factor of* $Y \wedge Y^{s+1}$. *Then the following diagram is commutative*:

$$\begin{array}{ccc} Y \wedge \bar{Y}^s & \xrightarrow{d^s} & Y \wedge \bar{Y}^{s+1}\\ \downarrow r_s & & \downarrow r_{s+1}\\ Y \wedge Y^s & \xrightarrow[\Sigma(-1)^i\theta_i^s]{} & Y \wedge Y^{s+1}\end{array}$$

PROOF. The proof is by induction over s. For $s = 0$ we have

$$\begin{aligned} r \circ d^0 &= r \circ (\iota \wedge 1) \circ \mathrm{pr} = r \circ (1 \wedge \mathrm{pr}) \circ (\iota \wedge 1)\\ &= \left[\mathrm{id}_{y \wedge y} - (1 \wedge \iota)\mu\right] \circ (\iota \wedge 1) = (\iota \wedge 1) - (1 \wedge \iota).\end{aligned}$$

For $s > 0$ we may use the case of $s = 0$ to show the commutativity of

$$\begin{array}{ccc} Y \wedge \bar{Y}^s & \xrightarrow{d^s} & Y \wedge \bar{Y} \wedge Y^s\\ & \searrow^{\iota \wedge 1 \wedge 1 - 1 \wedge \iota \wedge 1} & \downarrow r \wedge 1\\ & & Y \wedge Y \wedge \bar{Y}^s\end{array}$$

It therefore suffices to show that the following diagram is commutative:

$$\begin{array}{ccc} Y \wedge \bar{Y}^s & \xrightarrow{1 \wedge \iota \wedge 1} & Y \wedge Y \wedge \bar{Y}^s\\ \downarrow r_s & & \downarrow 1 \wedge 1 \wedge r_s\\ Y \wedge Y^s & \xrightarrow[1 \wedge \Sigma(-1)^i\theta_i^s]{} & Y \wedge Y \wedge Y^s\end{array}$$

This is done inductively by chasing the following diagram:

$$
\begin{tikzcd}
Y \wedge \overline{Y}^s \arrow[d, equal] \arrow[rrdd, "1 \wedge \iota \wedge 1"] & & \\
Y \wedge \overline{Y} \wedge \overline{Y}^{s-1} \arrow[d, "r \wedge 1"] & & \\
Y \wedge Y \wedge \overline{Y}^{s-1} \arrow[r, "1 \wedge \mathrm{pr} \wedge 1"] \arrow[rr, bend right, "1 \wedge d^{s-1}"'] \arrow[d, "1 \wedge r_{s-1}"] & Y \wedge \overline{Y} \wedge \overline{Y}^{s-1} \arrow[r, "1 \wedge \iota \wedge 1"] & Y \wedge Y \wedge \overline{Y}^s \arrow[d, "1 \wedge r_s"] \\
Y \wedge Y \wedge Y^{s-1} \arrow[rr, "1 \wedge \Sigma(-1)^i \theta_i^{s-1}"'] & & Y \wedge Y \wedge Y^s
\end{tikzcd}
$$

Let $\psi = \bigvee_{n \geqslant 0} \psi_n \colon \bigvee_{n \geqslant 0} B_n \to bo \wedge bo$ be as in (2.5). Define

$$\psi^{(s)} \colon \Big(\bigvee_{n \geqslant 0} B_n \Big)^{\wedge s} \to bo \wedge bo$$

as the composition

$$\Big(\bigvee_{n \geqslant 0} B_n \Big)^{\wedge s} \xrightarrow{\psi^{\wedge s}} (bo \wedge bo)^{\wedge s} \xrightarrow{1 \wedge \mu \wedge \cdots \wedge \mu \wedge \mu} bo \wedge bo^s.$$

The following lemma is an easy consequence of definitions and (2.7).

LEMMA 3.4. *There exists a commutative diagram*:

$$
\begin{tikzcd}
\Big(\bigvee_{n > 0} B_n \Big)^{\wedge s} \arrow[r, "\psi^{(s)}"] \arrow[d, hook] & bo \wedge \overline{bo}^{\,s} \arrow[d, "r_s"] \\
\Big(\bigvee_{n \geqslant 0} B_n \Big)^{\wedge s} \arrow[r, "\psi^{(s)}"] & bo \wedge bo^s
\end{tikzcd}
$$

Moreover, the bo-module map

$$\tilde{\psi}^{(s)} \colon bo \wedge \Big(\bigvee_{n > 0} B_n \Big)^{\wedge s} \to bo \wedge \overline{bo}^{\,s}$$

induced from $\psi^{(s)}$ is a homotopy equivalence. □

REMARK 3.5. By construction, $\psi^{(s)}$ restricted to the bottom cell of $B_{n_1} \wedge \cdots \wedge B_{n_s}$ is given by the homotopy class

$$t_{n_1}(u_0, u_1) \cdot t_{n_2}(u_1, u_2) \cdots t_{n_s}(u_{s-1}, u_s) \in \pi_{4n}(bo \wedge bo^s),$$

where $u_i \in \pi_4(bo \wedge bo^s)$ "lives" in the ith factor $(0 \geqslant i \geqslant s)$ and t_n is as in (2.3).

Given $\mathbf{n} = (n_1, \ldots, n_s) \in \mathbf{N}^s$, we write $|\mathbf{n}| = \Sigma_i n_i$ and $\alpha(\mathbf{n}) = \Sigma_i \alpha(n_i)$, and set $bo \wedge B_{\mathbf{n}} = bo \wedge B_{n_1} \wedge \cdots \wedge B_{n_s}$. Then (2.1) implies

COROLLARY 3.6. *There exists a homotopy equivalence of bo-module spectra*

$$bo \wedge B_{\mathbf{n}} \simeq KV_{\mathbf{n}} \vee \begin{cases} \Sigma^{4|\mathbf{n}|} bo\langle 2|\mathbf{n}| - \alpha(\mathbf{n}) \rangle, & |\mathbf{n}| \ \textit{even}, \\ \Sigma^{4|\mathbf{n}|} bsp\langle 2|\mathbf{n}| - 1 - \alpha(\mathbf{n}) \rangle, & |\mathbf{n}| \ \textit{odd}. \end{cases} \quad \square$$

We abbreviate the second summand in (3.6) by $bo_{\langle n\rangle}$.

Let $\mathbf{m}=(m_1,\ldots,m_{s+1})$. We call $\mathbf{m}$ a successor of $\mathbf{n}=(n_1,\ldots,n_s)$ if $\mathbf{m}=(n_1,\ldots,n_{i-1},j,n_i-j,n_{i+1},\ldots,n_s)$ for some j. In this case $|\mathbf{m}|=|\mathbf{n}|$ and

$$2|\mathbf{n}|-\alpha(\mathbf{n})-\left(2|\mathbf{m}|-\alpha(\mathbf{m})\right)=\alpha(j)+\alpha(n_i-j)-\alpha(n_i)=\nu_2\left(\binom{n_i}{j}\right).$$

Let $\pi_{\mathbf{n},\mathbf{m}}\colon bo\wedge B_{\mathbf{n}}\to bo\wedge B_{\mathbf{m}}$ denote any map induced from the canonical map $bo_{\langle\mathbf{n}\rangle}\to bo_{\langle\mathbf{m}\rangle}$ of degree $(\binom{n_i}{j})$ on the bottom cell.

THEOREM 3.7. *Modulo torsion operations factorizing through $\mathbf{Z}/2$-Eilenberg-Mac Lane spectra, the components*

$$d^s_{\mathbf{n},\mathbf{m}}\colon bo\wedge B_{\mathbf{n}}\overset{\tilde\psi^{(s)}}{\hookrightarrow} bo\wedge\overline{bo}^{\,s}\overset{d^s}{\to} bo\wedge\overline{bo}^{\,s+1}\xrightarrow{\mathrm{pr}\circ(\tilde\psi^{(s+1)})^{-1}} bo\wedge B_{\mathbf{m}}$$

of the differential d^s are given by

$$d^s_{\mathbf{n},\mathbf{m}}=\begin{cases}\phi_{n_0}\wedge\mathrm{id}\colon bo\wedge B_{\mathbf{n}}\to bo\wedge B_{n_0}\wedge B_{\mathbf{n}} & \text{if } \mathbf{m}=(n_0,\mathbf{n}),\\ (-1)^i\pi_{\mathbf{n},\mathbf{m}}\colon bo\wedge B_{\mathbf{n}}\to bo\wedge B_{\mathbf{m}} & \text{if } \mathbf{m} \text{ succeeds } \mathbf{n},\\ 0 & \text{elsewhere}.\end{cases}$$

PROOF. As it is with operations $bo\to\Sigma^k bo$, any operation $bo\wedge B_{\mathbf{n}}\to bo\wedge B_{\mathbf{m}}$ is determined up to the torsion operations mentioned in (3.7) by its effect in homotopy. Using (3.5), (3.3), and (2.4) (in this order) this effect can be computed. The result follows.

4. The bo-essential part of the E_1-term and its algebraic interpretation. We write

(4.1) $$bo\wedge\overline{bo}^{\,s}\wedge X\simeq\bigvee_{\mathbf{n}\in\mathbf{N}^s}(bo\wedge B_{\mathbf{n}}\wedge X)\simeq KV_s(X)\vee\bigvee_{\mathbf{n}}(bo\wedge X)_{\langle\mathbf{n}\rangle}.$$

Here $KV_s(X)$ is a maximal $\mathbf{Z}/2$-Eilenberg-Mac Lane spectrum and

$$(bo\wedge X)_{\langle\mathbf{n}\rangle}=\begin{cases}\Sigma^{4|\mathbf{n}|}(bo\wedge X)^{\langle 2|\mathbf{n}|-\alpha(\mathbf{n})\rangle}, & \mathbf{n}\text{ even},\\ \Sigma^{4|\mathbf{n}|}(bsp\wedge X)^{\langle 2|\mathbf{n}|-1-\alpha(\mathbf{n})\rangle}, & \mathbf{n}\text{ odd}.\end{cases}$$

By Margolis's theorem [**12**] one may think of $V_s(X)$ as the $\mathbf{Z}/2$-vector space spanned by an $\mathfrak{A}$-basis of a maximal $\mathfrak{A}$-free submodule of $H^*(bo\wedge\overline{bo}^s\wedge X)$ or, equivalently, by an $\mathfrak{A}_1$-basis of a maximal $\mathfrak{A}_1$-free submodule of $H^*(\overline{bo}^s\wedge X)$. We call $\bigvee_{\mathbf{n}}(bo\wedge X)_{\langle\mathbf{n}\rangle}$ the "bo-essential" part of the bo-resolution.

LEMMA 4.2. *The $\mathbf{Z}/2$-vector spaces*

$$V_s(X)=\pi_*KV_s(X)\subset E_1^{s,*}(X;\,bo)\cong\pi_*\left(bo\wedge\overline{bo}^{\,s}\wedge X\right)$$

constitute a subcomplex of $(E_1^{*,*}(X;\,bo);\,d_1)$.

PROOF. The minimality condition imposed onto $(bo\wedge X)_{\langle\mathbf{n}\rangle}$ above implies that any map $\Sigma^r K\mathbf{Z}/2\to(bo\wedge X)_{\langle\mathbf{n}\rangle}$ has to be trivial on the bottom cell, hence in homotopy. □

Let $(\mathscr{C}^{*,*}(X),d)$ denote the quotient complex

$$\mathscr{C}^{s,*}(X)=E_1^{s,*}(X;\,bo)/V_s(X).$$

Then this "bo-essential complex" $\mathscr{C}^{*,*}(X)$ may be computed from (4.1) and (3.7).

COROLLARY 4.3. *Under the natural isomorphism*

$$\mathscr{C}^{s,t}(X) \cong \bigoplus_{\mathbf{n}\in\mathbf{N}^s} \left(\pi_t(bo \wedge X)_{\langle\mathbf{n}\rangle}\right),$$

the differential d has components

$$d_{\mathbf{n},\mathbf{m}} = \begin{cases} \phi_{n_0}\colon bo \wedge X_{\langle\mathbf{n}\rangle} \to bo \wedge X_{\langle n_0,\mathbf{n}\rangle} & \text{if } \mathbf{m} = (n_0, \mathbf{n}), \\ (-1)^i \pi_{\mathbf{n},\mathbf{m}}\colon bo \wedge X_{\langle\mathbf{n}\rangle} \to bo \wedge X_{\langle\mathbf{m}\rangle} & \text{if } \mathbf{m} \text{ succeeds } \mathbf{n}, \\ 0 & \text{elsewhere}. \end{cases}$$

PROOF. The only point in question is the possible operation of the torsion operations of (3.7) on $\mathscr{C}^{s,t}(X)$. As in (4.2) this is prohibited by minimality of $(bo \wedge X)_{\langle\mathbf{m}\rangle}$. $\square$

To construct an algebraic functor describing the homology of the *bo*-essential complex we need spectra X fulfilling the following property.

DEFINITION 4.4 [**14**]. A spectrum X is called (bo, H)-prime if the $H\mathbf{Z}/2$-Adams sectral sequence for $\pi_*(bo \wedge \overline{bo}^s \wedge X)$ converges and is trivial from E_2 for all s.

From (3.4) and (3.6) we have immediately

LEMMA 4.5. *A spectrum X is (bo, H)-prime if the Adams spectral sequences for $bo_* X$ and $bsp_* X$ converge and are trivial from E_2.* $\square$

Examples of (bo, H)-primary spectra X are given by the Brown-Gitler spectra $B(n)$, stunted projective spaces P_{2k+1}^{2l}, arbitrary products between these, their Spanier-Whitehead duals, and also their coverings $X^{\langle i\rangle}$ in an $H\mathbf{Z}/2$-Adams resolution. This may easily be seen from a computation of the $\mathfrak{A}(1)$-module structure of $H^*(X)$ [**8**] (see also Chapter 7) together with the fact that any $\mathfrak{A}(1)$-free submodule of $H^*(X)$ splits of a $K\mathbf{Z}/2$ from $bo \wedge X$. This follows from [**12**].

Suppose X is (bo, H)-prime. Then obviously

$$\pi_*(bo \wedge X)_{\langle\mathbf{n}\rangle} \xrightarrow{\pi_\mathbf{n}} \begin{cases} bo_{*-4|\mathbf{n}|}(X), & |\mathbf{n}| \text{ even}, \\ bsp_{*-4|\mathbf{n}|}(X), & |\mathbf{n}| \text{ odd}, \end{cases}$$

is injective and onto elements of suitable Adams filtration. Here $\pi_\mathbf{n}$ is defined similar to π_n in §2 as induced by the canonical map of homology degree $2^{|\mathbf{n}|} \cdot n_1!!n_2!! \cdots n_s!!$ for n even and $2^{|\mathbf{n}|-1} \cdot n_1!! \cdot n_2!! \cdots n_s!!$ for n odd.

Using this identification we may derive a convenient notation: Let $\Gamma(t) \cong \operatorname{Hom}_{\mathbf{Z}_{(2)}}(\mathbf{Z}_{(2)}[t], \mathbf{Z}_{(2)})$ denote a 1-dimensional divided polynomial Hopf-algebra over $\mathbf{Z}_{(2)}$ with $(\mathbf{Z}_{(2)})$-generators t_i of dimension $4i$, product $t_i t_j = \binom{i+j}{i} t_{i+j}$, and coproduct $\psi(t_i) = \Sigma t_j \otimes t_{i-j}$.

LEMMA 4.6. *Suppose X is (bo, H)-prime. Then, via the identification maps $\pi_\mathbf{n}$, the elements of $\mathscr{C}^{s,t}(X)$ can be written uniquely as*

$$\sum_{\mathbf{n}\in\mathbf{N}} x_\mathbf{n}\left[t_{n_1}| \cdots |t_{n_s}\right],$$

where $x_\mathbf{n} \in bo_{t-4|\mathbf{n}|}(X)$ for $|\mathbf{n}|$ even, or $x_\mathbf{n} \in bsp_{t-4|\mathbf{n}|}(X)$ for $|\mathbf{n}|$ odd satisfy:

$$\text{Adams filtration}(x_\mathbf{n}) \geqslant \begin{cases} 2|\mathbf{n}| - \alpha(\mathbf{n}), & |\mathbf{n}| \text{ even}, \\ 2|\mathbf{n}| - \alpha(\mathbf{n}) - 1, & |\mathbf{n}| \text{ odd}. \end{cases}$$

The differential d is then given by the formula

$$d\left(x_{\mathbf{n}}\left[t_{n_1}|\cdots|t_{n_s}\right]\right) = \sum_{n_0} \phi^{n_0} x_{\mathbf{n}}\left[t_{n_0}|\cdots|t_{n_s}\right]$$
$$+\sum_{i,j}(-1)^i x_{\mathbf{n}}\left[t_{n_1}|\cdots|t_{n_{i-1}}|t_j|t_{n_i-j}|\cdots|t_{n_s}\right]\cdots$$
$$+x_{\mathbf{n}}\left[t_{n_1}|\cdots|t_{n_s}|1\right]. \qquad \square$$

Observe that if $\mathbf{m}$ succeeds $\mathbf{n}$ (as in §3), then $\pi_{\mathbf{m}} \circ \pi_{\mathbf{n},\mathbf{m}} = \pi_{\mathbf{n}}$. This explains the lack of binomial coefficients $(\binom{n_i}{j})$ in the formula of (4.6).

Our way of describing $\mathscr{C}^{*,*}(X)$ is more than just convenient: Let $\Delta\colon b_{*,*}(X) \to b_{*,*}(X) \otimes_{\mathbf{Z}_{(2)}} \Gamma(t)$ be given by $x \to \Sigma\phi^i x \otimes t_i$. Assign bidegree $(-4i, 1) \in \mathbf{Z} \times \mathbf{Z}/2$ to t_i and define a filtration on $b_{*,*}(X)$ and $\Gamma(t)$ by

$$\gamma(x) = \begin{cases} \text{Adams filtr}(x) & \text{for } x \in b_{*,0}(X), \\ \text{Adams filtr}(x) + 1 & \text{for } x \in b_{*,1}(X); \end{cases}$$
$$\gamma(t_i) = -2i + \alpha(i); \qquad \gamma(2) = 1.$$

Then Δ as well as coproduct and product on $\Gamma(t)$ are filtration preserving and may be viewed as structure maps in a suitable *abelian* category of filtered comodules and filtration preserving comodule homomorphisms over the filtered Hopf algebra $\Gamma(t)$.

The comodule $b_{*,*}(X) \otimes_{\mathbf{Z}_{(2)}} \Gamma(t)$ is an extended $\Gamma(t)$ comodule and can be used as an injective envelope of $b_{*,*}(X)$. Thus injective resolutions exist.

Let $\mathrm{Hom}^t_{\mathscr{F}}(-,-)$ denote the group of filtration preserving comodule homomorphisms which raise bidegree by $(t, 0)$. Define $\mathrm{Ext}^{s,t}_{\mathscr{F}}(M, -)$ as usual as the sth derived functor of $\mathrm{Hom}^t_{\mathscr{F}}(M, -)$.

With $\mathbf{Z}_{(2)}$ concentrated in bidegree $(0, 0)$, the above discussion can be summarized in

THEOREM 4.7. *Suppose X is (bo, H)-prime. Then the "bo-essential" homology $H(\mathscr{C}^{s,t}(X), d_1)$ is naturally isomorphic to* $\mathrm{Ext}^{s,t}_{\mathscr{F}}(\mathbf{Z}_{(2)}; b_{*,*}(X))$.

REMARK 4.8. (i) The somehow unpleasant bigrading (on $b_{*,*}(-)$) is forced by the fact that one half on the *bo*-operations take their natural values in *bsp* instead of *bo*. If one is dealing with odd primes or $bo \wedge M_\eta \simeq bu$, no such problems arise and one works simply in the category of graded filtered comodules over a divided polynomial algebra in one variable of degree 2.

(ii) The filtration induces a natural spectral sequence converging to $\mathrm{Ext}^{*,*}_{\mathscr{F}}(-,-)$. In the geometric case, this spectral sequence corresponds to the "geometric May spectral sequence" of [**14**].

REMARK 4.9. From the definition we have a short exact sequence

$$0 \to V_s(X) \to E_1^{s,*}(X; bo) \to \mathscr{C}^{s,*}(X) \to 0.$$

This induces a long exact sequence

$$\cdots \to H^{s,*}(V_*(X); d) \to E_2^{s,*}(X; bo) \to \mathrm{Ext}^{s,*}_{\mathscr{F}}\left(\mathbf{Z}_{(2)}; b_{*,*}(X)\right) \to \cdots$$

valid for all $(bo;\ H)$-primary spectra X. We shall describe computational methods for dealing with $\mathrm{Ext}_{\mathscr{F}}^{*,*}(\mathbf{Z}_{(2)};\ -)$ in the second part of this paper. To deal with $V_*(X)$ observe that $V_s(X) = \pi_*(KV_s(X)) \subset \pi_*(bo \wedge \overline{bo}^s \wedge X)$ is concentrated in $H\mathbf{Z}/2$-Adams filtration 0. Therefore $V_s(X) \subset H_*(\overline{bo}^s \wedge X)$ is precisely the span of all those $\mathfrak{A}(1)^*$-primitives in $H_*(\overline{bo}^s \wedge X)$ which support a full copy of $\mathfrak{A}(1)^*$ (as an $\mathfrak{A}(1)^*$-comodule). It is then easy to see that

$$V_s(X) = \mathrm{im}\Big(\big(\mathrm{Sq}^2\big)^3\colon H_*\big(\overline{bo}^s \wedge X\big) \to H_*\big(\overline{bo}^s \wedge X\big)\Big).$$

The differential on $V_s(X)$ is, under a suitable isomorphism, induced from the standard differential of the bar resolution. Since all these formulae are quite manageable it is possible to pass the problem to a computer. Details will appear elsewhere.

REMARK 4.10. From an algebraic standpoint one can improve slightly on the long exact sequence in Remark 4.9. Consider the short exact sequence

$$\begin{aligned} 0 &\to \mathscr{F}_1\Big(\pi_*\big(bo \wedge \overline{bo}^s \wedge X\big)\Big) \to E_1^{s,t}(X;\ bo) \\ &\to E_1^{s,t}(X;\ bo)/\mathscr{F}_1\Big(\pi_*\big(bo \wedge \overline{bo}^s \wedge X\big)\Big) \to 0. \end{aligned}$$

Here $\mathscr{F}_i(-)$ denotes the submodule of elements of $H\mathbf{Z}/2$-Adams filtration $\geqslant i$. Since all elements of $\pi_*(KV_s(X))$ have Adams filtration 0 it is easily seen that there is a natural isomorphism

$$\mathscr{F}_1\Big(\pi_*\big(bo \wedge \overline{bo}^s \wedge X\big)\Big) \cong \mathscr{C}^{s,*}\big(X^{\langle 1\rangle}\big).$$

On the other hand the quotient complex above for $(bo;\ H)$-primary X is given as

$$\begin{aligned} E_1^{s,*}(X;\ bo)/\mathscr{F}_1\Big(\pi_*\big(bo \wedge \overline{bo}^s \wedge X\big)\Big) &\cong \mathrm{Hom}_{\mathfrak{A}}\Big(H^*\big(bo \wedge \overline{bo}^s \wedge X\big);\ \mathbf{Z}_2\Big) \\ &\cong \mathrm{Hom}_{\mathfrak{A}}\Big(\mathfrak{A} \otimes_{\mathfrak{A}(1)} \overline{\mathfrak{A}} \otimes_{\mathfrak{A}(1)} \cdots \otimes_{\mathfrak{A}(1)} H^*(X);\ \mathbf{Z}/2\Big). \end{aligned}$$

Its homology may be interpreted as a "relative Ext" $\mathrm{Ext}_{\mathfrak{A},\mathfrak{A}(1)}(H^*(X);\ \mathbf{Z}/2)$ in the category of $\mathfrak{A}$-modules where the class of exact sequences is restricted to those which split when viewed as a sequence of modules over $\mathfrak{A}(1)$. (See **[6]** for a more detailed construction in the odd primary case.) We therefore get a natural long exact sequence

$$\begin{aligned} \cdots &\to \mathrm{Ext}_{\mathscr{F}}^{s,*}\Big(\mathbf{Z}_{(2)};\ b_{*,*}\big(X^{\langle 1\rangle}\big)\Big) \\ &\to E_2^{s,*}(X;\ bo) \to \mathrm{Ext}_{\mathfrak{A},\mathfrak{A}(1)}^{s,*}\big(H^*(X);\ \mathbf{Z}/2\big) \to \cdots. \end{aligned}$$

PART 2. COMPUTATIONAL PROPERTIES OF THE bo-ADAMS SPECTRAL SEQUENCE

5. The weight filtration spectral sequence. Recall from (4.3) the isomorphism

$$\mathscr{C}^{s,t}(X) \cong \bigoplus_{\mathbf{n}\in\mathbf{N}^s} \pi_t\big(bo \wedge X_{\langle\mathbf{n}\rangle}\big).$$

Define the *weight-filtration* ω on $\mathscr{C}^{*,*}(X)$ by

$$\omega\Big(\pi_*\big(bo \wedge X_{\langle\mathbf{n}\rangle}\big)\Big) = |\mathbf{n}| = \Sigma n_i.$$

From (4.3) we also know that the components of the differential d are given by

$$d_{\mathbf{n},\mathbf{m}} = \begin{cases} \phi_{n_0}, & \mathbf{m} = (n_0, \mathbf{n}), \\ (-1)^i \pi_{\mathbf{n},\mathbf{m}}, & \mathbf{m} \text{ succeeds } \mathbf{n}, \\ 0, & \text{elsewhere.} \end{cases}$$

This shows that the differential cannot decrease the weight. We therefore get a *weight-filtration spectral sequence* $\{E_r^{\sigma,s,t}(\mathscr{C}(X)); \partial_r\}$ with

$$E_0^{\sigma,s,t}(C(X)) \cong \bigoplus_{\substack{\mathbf{n} \in \mathbf{N}^s \\ |\mathbf{n}| = \sigma}} \pi_t(bo \wedge X_{\langle \mathbf{n} \rangle})$$

and differential $\partial_0 = \Sigma_{\mathbf{m}\text{ succ. }\mathbf{n}}(-1)^i \pi_{\mathbf{n},\mathbf{m}}$.

Suppose now that X is (bo, H)-prime. Then we may write the elements of $\pi_*(bo \wedge X_{\langle \mathbf{n} \rangle})$ as $\Sigma x_{\mathbf{n}}[t_{n_1}| \cdots |t_{n_s}]$ with $x_{\mathbf{n}} \in bo_* X$ or $bsp_* X$ of suitable Adams filtration by (4.6).

Let $d\colon \overline{\Gamma(t)}^{\otimes s} \to \overline{\Gamma(t)}^{\otimes s+1}$ denote the standard differential in the cobar complex of $\Gamma(t)$. Then the differential ∂_0 may be written as

$$\begin{aligned} \partial_0\left(x_{\mathbf{n}}\left[t_{n_1}| \cdots |t_{n_s}\right]\right) &= x_{\mathbf{n}} \sum (-1)^i \left[t_{n_1}| \cdots |t_{n_{i-1}}|t_j|t_{n_i - j}| \cdots |t_{n_s}\right] \\ &= x_{\mathbf{n}} d\left(\left[t_{n_1}| \cdots |t_{n_s}\right]\right). \end{aligned}$$

This linearity in "$x_{\mathbf{n}}$" looks as if one were dealing with cohomology of $\Gamma(t)$ with trivial coefficients $bo_* X$ or $bsp_* X$. This is, however, misleading, since it does not take into account the filtration condition imposed on the $x_{\mathbf{n}}$'s. We illustrate this by the following example.

Consider the differential $d[t_2] = [t_1 | t_1]$. In the summand of $E_0^{2,1,*}(\mathscr{C}(X))$ associated to $[t_2]$ the coefficient $x_{(2)} \in bo_* X$ has Adams filtration $\mathrm{AF}(x_{(2)}) \geqslant 3$. In the summand associated to $[t_1 | t_1]$ the condition on $x_{(1,1)} \in bo_* X$ is $\mathrm{AF}(x_{(1,1)}) \geqslant 2$. This produces homology classes $x[t_1 | t_1]$ for $x \in \mathscr{F}_2(bo_* X)/\mathscr{F}_3(bo_* X)$, where $\mathscr{F}_i(bo_* X)$ denotes the submodule of elements of $H\mathbf{Z}/2$-Adams filtration $\geqslant i$. As we observed in **[3]**, this is the only type of exceptional behavior one encounters in these computations.

PROPOSITION 5.1 **[2]**. *Suppose X is (bo, H)-prime. Then $E_1^{\sigma,s,t}(\mathscr{C}(X))$ is isomorphic to*

(a) $\pi_t(bo \wedge X^{\langle 0 \rangle}) = \pi_t(bo \wedge X)/V_0(X)$, $s = \sigma = 0$,

(b) $\pi_{t-4}(bsp \wedge X^{\langle 0 \rangle})$, $s = \sigma = 1$,

(c)

$$\bigoplus_{0 \leqslant e_1 \leqslant \cdots \leqslant e_{s-2} < e_{s-1}} \frac{\mathscr{F}_{2\sigma - s}(\pi_{t-4\sigma}(bo \wedge X))}{\mathscr{F}_{2\sigma - s+1}(\pi_{t-4\sigma}(bo \wedge X))} d[t_{2^{e_1}} \cdots t_{2^{e_{s-1}}}], \qquad \begin{array}{l} s \geqslant 2, \\ \sigma = \Sigma 2^{e_i} \textit{ even}, \end{array}$$

$$\bigoplus_{0 \leqslant e_1 \leqslant \cdots \leqslant e_{s-2} < e_{s-1}} \frac{\mathscr{F}_{2\sigma - s-1}(\pi_{t-4\sigma}(bsp \wedge X))}{\mathscr{F}_{2\sigma - s}(\pi_{t-4\sigma}(bsp \wedge X))} d\left[t_{2^{e_1}}| \cdots |t_{2^{e_{s-1}}}\right], \qquad \begin{array}{l} s \geqslant 2, \\ \sigma = \Sigma 2^{e_i} \textit{ odd}, \end{array}$$

(d) 0 *in all other cases.*

REMARK 5.2. Observe that it follows from the hypothesis that the groups in (c) are isomorphic to

$$\mathrm{Ext}_{\mathfrak{A}_1}^{2\sigma-2,t-2\sigma-2}(H^*X;\mathbf{Z}/2) \quad \text{and} \quad \mathrm{Ext}_{\mathfrak{A}_1}^{2\sigma-s-1,t-2\sigma-s-1}(H^*(X\wedge B(1));\mathbf{Z}/2)$$

respectively.

PROOF OF 5.1. To compute the homology of ∂_0, we use a spectral sequence induced from the Adams filtration on $\pi_*(bo \wedge X_{\langle \mathbf{n}\rangle})$.

Assign the filtration γ to $bo_*(X) \otimes \overline{\Gamma(t)}^{\otimes s}$ by $\gamma(t_i) = -2i + \alpha(i)$, $\gamma(2) = 1$, and $\gamma(x) = \mathrm{AF}(x)$ for $x \in bo_*X$, $\gamma(x) = \mathrm{AF}(x) + 1$ for $x \in bsp_*X$. Then $E_0^{*,*,*}(\mathscr{C}(X))$ is generated by all classes

$$x_{\mathbf{n}}\left[t_{n_1}|\cdots|t_{n_s}\right] \in \begin{Bmatrix} bo_*X \\ bsp_*X \end{Bmatrix} \otimes \overline{\Gamma(t)}^{\otimes s}$$

such that $\gamma(x_{\mathbf{n}}[t_{n_1}|\cdots|t_{n_s}]) \geqslant 0$. Since $\gamma(d[t_i]) \geqslant \gamma([t_i])$, the differential ∂_0 is filtration preserving and we get a spectral sequence converging to $E_1^{*,*,*}(\mathscr{C}(X))$. Denote its differentials by δ_r.

Consider first the case of coefficients $\mathbf{Z}_{(2)}$ instead of bo_*X with no restrictions on the filtration imposed. (We are then computing the cohomology of $\Gamma(t)$.) The associated graded algebra to $\Gamma(t)$ is a primitively generated exterior algebra in generators $\{t_{2^i}\,|\,i \geqslant 0\}$ over a polynomial algebra $\mathbf{Z}/2[a_0]$, where a_0 corresponds to $2 \in \mathbf{Z}_{(2)}$. Therefore the E_1-term of the γ-filtration spectral sequence in this case is a polynomial algebra over $\mathbf{Z}/2[a_0]$ in generators $\{[t_{2^i}]\,|\,i \geqslant 0\}$.

To compute the E_2-term, observe that $\delta_1[t_{2^i}] = [t_{2^{i-1}}\,|\,t_{2^{i-1}}]$. It was shown in **[2]** that a basis for the submodules of boundaries and cycles for δ_1 can then be written down in the following way (boundaries = cycles for $s \geqslant 2$):

	boundaries	cycles
$s = 0$	0	$\{[\ ]\}$
1	0	$\{[t_1]\}$
$\geqslant 2$	$\left\{d\left[t_2e_1\right|\cdots\left|t_2e_{s-1}\right]\,\middle|\,e_1 \leqslant \cdots \leqslant e_{s-2} < e_{s-1}\right\}$	

Moreover, δ_1 restricted to the submodule spanned by $\{[t_2e_1|\cdots|t_2e_{s-1}]|e_1 \leqslant \cdots \leqslant e_{s-2} < \varepsilon_{s-1}\}$ is injective. (This certainly exhibits $H^*\Gamma(t)$ as an exterior algebra in one generator $[t_1]$ over $\mathbf{Z}_{(2)}$, as is well known.)

If, instead of $\mathbf{Z}_{(2)}$, we now introduce coefficients bo_*X and bsp_*X, respectively, together with the filtration condition, we see that nothing changes in the first step of the argument (δ_0): Since δ_0 does not change the γ-filtration on $\overline{\Gamma(t)}^{\otimes s}$ the condition on the Adams filtration of the coefficient is the same in source and target of δ_0. Therefore the E_1-term of the γ-filtration spectral-sequence is given as

$$\mathrm{span}\left\{x\left[t_{2^{e_1}}\right|\cdots\left|t_{2^{e_s}}\right]\,\middle|\,e_1 \leqslant \cdots \leqslant e_s;\ \gamma\left(x\left[t_{2^{e_1}}\right|\cdots\left|t_{2^{e_s}}\right] \geqslant 0\right)\right\}.$$

For δ_1, however, we have $\gamma(\delta_1[t_{2^i}]) = \gamma[t_{2^i}] + 1$, so in the target space of δ_1 elements of Adams filtration one less than in the source are allowed, and this may occur for all possible cycles

$$\left\{d\left[t_{2^{e_1}}\right|\cdots\left|t_{2^{e_{s-1}}}\right]\,\middle|\,e_1 \leqslant \cdots \leqslant e_{s-2} < e_{s-1}\right\}. \quad \square$$

The result as stated follows.

From the proof of 5.1 one easily extracts the following observations:

REMARK 5.3. Let $x \in E_0^{\sigma,s,t}(\mathscr{C}(X))$, $s \geqslant 2$, be any ∂_0-cycle such that x is represented in $H\mathbf{Z}/2$-Adams filtration $\geqslant 1$ (as an element of $\oplus_{\mathrm{n}} \pi_*(bo \wedge X_{\langle \mathrm{n} \rangle})$. Then x is actually a ∂_0-boundary.

REMARK 5.4. Suppose ϕ operates trivially on $bo_* X$ and $bsp_* X$. Then $E_1^{\sigma,s,t}(\mathscr{C}(X)) = E_\infty^{\sigma,s,t}(\mathscr{C}(X))$ and we have computed $H(\mathscr{C}^{s,t}(X))$. An important example is $X = M_{2\iota}$ or $X = M_{2\iota} \wedge M_\eta$.

6. Computation of the weight filtration spectral sequence for various X. The first case of interest is $X = S^{0\langle i \rangle}$. Recall the Adams spectral sequence charts for $\pi_* bo$ and $\pi_* bsp$ from §1 and observe that the corresponding chart for $\pi_* bo^{\langle i \rangle}$ or $\pi_* bsp^{\langle i \rangle}$ is constructed out of these by deleting all rows below filtration $s = i$. A typical example is $\pi_* bsp^{\langle 3 \rangle}$ as shown in Figure 2.

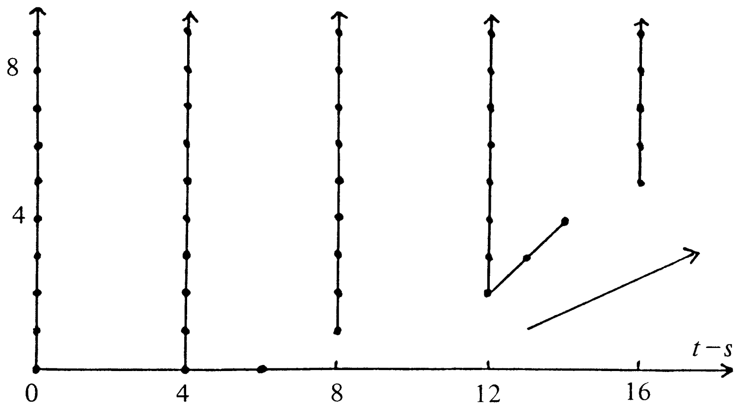


FIGURE 2

With u, u' and v as in §1, the operation of ϕ is given by

$$\phi(v^m) = (9^{2m} - 1)\begin{Bmatrix} u' \\ u \end{Bmatrix} v^{m-1} \quad \text{and} \quad \phi\begin{Bmatrix} u \\ u' \end{Bmatrix} v^m = (9^{2m+1} - 1) v^m.$$

In particular this implies

$$\phi^n(\eta^\varepsilon v^m) = 0 = \phi^n(\eta^\varepsilon u' v^m) \quad \text{for all } m \geqslant 0, 0 < \varepsilon \leqslant 2.$$

As a consequence, the differential d_1 in $\mathscr{C}^{*,*}(S^{0\langle i \rangle})$ operates on classes $\eta^\varepsilon v^m$ or $\eta^\varepsilon u' v^m$ only through the projections $\pi_{\mathrm{n,m}}$ considered in the last section. Since ϕ^n shifts dimensions only by numbers $\equiv 0$ (4), no operation ϕ^n can hit these classes either. It follows that

$$E_1^{\sigma,s,t}(\mathscr{C}(S^{0\langle i \rangle})) = E_\infty^{\sigma,s,t}(\mathscr{C}(S^{0\langle i \rangle})) \quad \text{for } t \not\equiv 0\ (4).$$

This implies the following corollary to Proposition 5.1.

COROLLARY 6.1. (a)

$$E_\infty^{0,0,t}(\mathscr{C}(S^{0\langle i \rangle})) \cong \mathbf{Z}/2 \quad \textit{for } t \equiv 1, 2 \bmod 8,\ t \geqslant 2i - 2,$$

(b)

$$E_\infty^{1,1,t}(\mathscr{C}(S^{0\langle i \rangle})) \cong \mathbf{Z}/2 \quad \textit{for } t \equiv 1, 2 \bmod (8),\ t \geqslant 2i + 4,$$

(c) *for any sequence* $E = (e_1, \ldots, e_{s-1})$ *s.th.* $0 \leqslant e_1 \leqslant \cdots \leqslant e_{s-2} < e_{s-1}$ *and* $i - s \equiv 1, 2 \bmod 4$, *there is exactly one nontrivial* $E_\infty^{\sigma,s,t}(\mathscr{C}(S^{0\langle i\rangle})) \cong \mathbf{Z}/2$ *with* $t \not\equiv 0$ (4). *This occurs with*

$$\sigma = \sum_i 2^{e_i} \quad \text{and} \quad t = \begin{cases} 8\sigma + 2i - 2s - 1, & i - s \equiv 1 \bmod 4, \\ 8\sigma + 2i - 2s - 2, & i - s \equiv 2 \bmod 4. \end{cases}$$

In these cases

$$t - s \equiv \begin{cases} 2 - i \bmod 4, & \text{if } i - s \equiv 1 \bmod 4, \\ -i \bmod 4, & \text{if } i - s \equiv 2 \bmod 4. \end{cases}$$

(d) *If* (σ, s, t) *is none of the above and* $t \not\equiv 0$ (4) *we have* $E_\infty^{\sigma,s,t} = 0$. □

To deal with $E^{\sigma,s,t}(\mathscr{C}(S^{0\langle i\rangle}))$ for $t \equiv 0$ (4), we introduce the following notational conventions.

Let $\tau_n = 2^n n!! t_n \in \Gamma(t)$, so τ_n equals τ_1^n up to multiplication by a unit in $\mathbf{Z}_{(2)}$. Let $\varepsilon = \varepsilon(n_1, \ldots, n_s) = \varepsilon(\mathbf{n})$ denote either a generator of $\pi_0(bo) \cong \mathbf{Z}_{(2)}$ for $|\mathbf{n}| = \sum n_i$ even or 2^{-1} (generator of $\pi_0 bsp$) for $|\mathbf{n}|$ odd. Finally let $w \in H_4(bo; \mathbf{Z}_{(2)})/\text{Torsion}$ be a generator. Then

$$w^m \varepsilon\left[\tau_{n_1}| \cdots |\tau_{n_s}\right] = 2^{|\mathbf{n}|} n_1!! \cdots n_s!! w^m \varepsilon\left[t_{n_1}| \cdots |t_{n_s}\right]$$

can be identified with a generator of

$$H_{4m+4|\mathbf{n}|}\left(bo_{\langle \mathbf{n}\rangle}; \mathbf{Z}_{(2)}\right)/\text{Tors} \supset \pi_{4m+4|\mathbf{n}|}\left(bo_{\langle n\rangle}\right)/\text{Tors}$$

via the canonical map

$$\pi_{\mathbf{n}}\colon bo_{\langle \mathbf{n}\rangle} \to \Sigma^{4|\mathbf{n}|}\begin{cases} bo \\ bsp. \end{cases}$$

Figure 3 explains this notation ($bo_{\langle 1,2\rangle} = \Sigma^{12} bsp^{\langle 3\rangle}$).

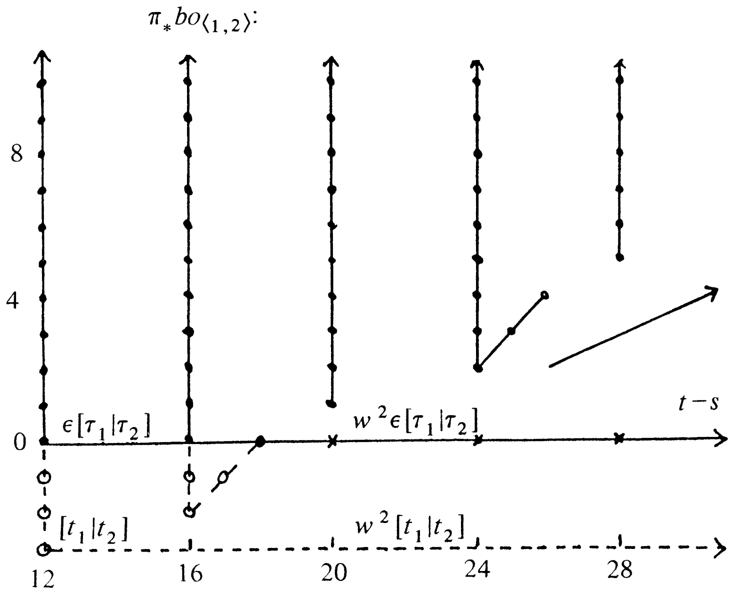


FIGURE 3

Let $[k]$ denote the largest integer $\leqslant k$ and define a function $\rho_i\colon \mathbf{N} \to \{0,1\}$ by

$$\rho_i(n) = \begin{cases} 1, & n \equiv i\ (4), \\ 0, & \text{elsewhere.} \end{cases}$$

Then one can easily compute

LEMMA 6.2.

$$\mathscr{C}^{*,*}(S^0)/\mathrm{Tors} = \operatorname{span}_{\mathbf{Z}_{(2)}}\Big\{ aw^m\varepsilon\big[\tau_{n_1}|\cdots|\tau_{n_s}\big] \,|\, a \in \mathbf{Z}_{(2)},\ m \leqslant |\mathbf{n}| + \left[\frac{\nu_2(a) - \alpha(\mathbf{n})}{2}\right] - \rho_2(\nu_2(a) - \alpha(\mathbf{n}))\Big\}.$$

For $a \in \mathbf{Z}_{(2)}$ the differential takes the form

$$\begin{aligned} d\big(aw^m\varepsilon\big[\tau_{n_1}|\cdots|\tau_{n_s}\big]\big) &= \sum_j \left(\!\binom{m}{j}\!\right) aw^{m-j}\varepsilon\big[\tau_j|\tau_{n_1}|\cdots|\tau_{n_s}\big] \\ &+ \sum_{i,j} (-1)^i \left(\!\binom{n_i}{j}\!\right) aw^m\varepsilon\big[\tau_{n_1}|\cdots|\tau_{n_i-j}|\tau_j|\cdots|\tau_{n_s}\big] \\ &+ (-1)^{s+1} aw^m\varepsilon\big[\tau_{n_1}|\cdots|\tau_{n_s}|1\big]. \qquad \square \end{aligned}$$

For the remaining part of the paper we abbreviate $\mathscr{C}^{*,*}(S^0)/\mathrm{Tor} = \mathscr{C}^{*,*}$ and, similarly, let $\mathscr{F}_i\mathscr{C}^{*,*} = \mathscr{C}^{*,*}(S^{0\langle i\rangle})/\mathrm{Tors}$ be the $\mathbf{Z}_{(2)}$-submodule spanned by

$$(6.3)\qquad \Big\{ aw^n\varepsilon_i\big[\tau_{n_1}|\cdots|\tau_{n_s}\big] \,|\, a \in \mathbf{Z}_{(2)},\ m \leqslant |\mathbf{n}| + \left[\frac{\nu_2(a) - \alpha(\mathbf{n}) + i}{2}\right] - \rho_2(\nu_2(a) - \alpha(\mathbf{n}) + i)\Big\},$$

where $\varepsilon_i = 2^i\varepsilon$. Write $d = D_0 + D_1$ with

$$\begin{aligned} D_0\big(aw^m\varepsilon\big[\tau_{n_1}|\cdots|\tau_{n_s}\big]\big) &= \sum_{j,((\binom{m}{j}))\equiv 0\,(2)} a\left(\!\binom{m}{j}\!\right) w^{m-j}\varepsilon\big[\tau_j|\tau_{n_1}|\cdots|\tau_{n_s}\big] \\ &+ \sum_{i,j,((\binom{n_i}{j}))\equiv 0\,(2)} \alpha\left(\!\binom{n_i}{j}\!\right) w^m\varepsilon\big[\tau_{n_1}|\cdots|\tau_{n_i-j}|\tau_j|\cdots|\tau_{n_s}\big]. \end{aligned}$$

Finally, let $h_i = [\tau_2 i]$ and, for $i = (i_0, \dots, i_t)$, $h^I = h_0^{i_0} \cdots h_t^{i_t} = [\tau_1|\cdots|\tau_1|\tau_2|\cdots|\tau_2|\cdots|\tau_{2^t}]$, where i_j is the number of τ_{2^j}'s. With $\|I\| = \Sigma i_j 2^j$ and $s = \Sigma i_j$ we have from (6.3)

$$(6.4)\qquad aw^m\varepsilon_i h^I \in \mathscr{F}_i\mathscr{C}^{*,*} \Leftrightarrow m \leqslant \|I\| + \left[\frac{i - s + \nu_2(a)}{2}\right] - \rho_2(i - s + \nu_2(a)).$$

The following proposition is a technical result which is needed for the computation of $H(\mathscr{F}_i\mathscr{C}^{*,*})$. Its proof is postponed until after this computation. Let $\Delta_k = (0, \dots, 0, 1, 0, \dots)$, with "1" in slot k.

PROPOSITION 6.5. (i) *Suppose $I = (i_0, \dots, i_t)$, $\Sigma i_j = s - 1$, and $m \leqslant \|I\| + [(i-s)/2] - \rho_2(i-s)$. Then $D_0(w^m\varepsilon_i h^I)/2 \in \mathscr{F}_i\mathscr{C}^{*,*}$.*

(ii) *If moreover $i_0 = \cdots = i_{k-1} = 0$, $i_k > 0$, and $m \equiv 0\ (2^{k+1})$, then $dD_1(w^{m+2^k}\varepsilon_1 h^{I-\Delta_k})/2 \in \mathscr{F}_i\mathscr{C}^{*,*}$.*

THEOREM 6.6. *The groups $H^s(\mathscr{F}_i\mathscr{C}^{*,*})$ are vector spaces over $\mathbf{Z}/2$ for $s \geqslant 2$ and in degree $\equiv 0\ (4)$. A basis is given by*

$$\left\{ \tfrac{1}{2} dD_1 w^{m+2^k} \varepsilon_i h^{I-\Delta_k} \right\},$$

where $I = (0, \ldots, 0, i_k, \ldots, i_{t-1}, 1)$, $i_k > 0$, $\sum_{j=0}^{t} i_j = s-1$, and m satisfies the conditions $m \equiv 0\ (2^{k+1})$ and

$$\|I\| + \left[\frac{i-s}{2}\right] + \rho_3(i-s) - 2^{k+1} < m \leqslant \|I\| + \left[\frac{i-s}{2}\right] - \rho_2(i-s).$$

REMARK 6.7. Observe that there may not be an "m" satisfying the conditions if $(i-s) \equiv 2, 3 \bmod 4$. In the remaining cases there is exactly one such m. See Table 8.1 for a concrete computation up to $t-s = 50$.

PROOF OF 6.8. We use the weight spectral sequence

$$\left(E_r^{\sigma,s,t}(i), \partial_r\right) = E_r^{\sigma,s,t}\left((S^{0\langle i \rangle}), \partial_r\right) \Rightarrow H^{s,t}(\mathscr{F}_i\mathscr{C}).$$

Recall that $d(h_e) = d[\tau_{2^e}] = \sum_{j=1}^{2^e-1}(\binom{2^e}{j})[\tau_j \mid \tau_{2^e-j}]$ and similarly for $d(h^I)$. With these notations we have from (5.1)

$$E_1^{*,*,*}(i) = \operatorname{span}_{\mathbf{Z}/2}\left\{ \frac{1}{2} w^m \varepsilon_i d(h^I) \mid I = (i_0, \ldots, i_{t-1}, 1),\ \sum_{j=0}^{t} i_j = s-1,\right.$$
$$\left. m \leqslant \|I\| + \left[\frac{i-s}{2}\right] - \rho_2(i-s) \right\}.$$

To get the higher differentials we shall prove

CLAIM 6.9. (i) $\frac{1}{2} w^m \varepsilon_i d(h^I) \in E_1^{\sigma,s,*}(i)$ can be represented by $\frac{1}{2} D_0 w^m \varepsilon_i h^I \in \mathscr{F}_1\mathscr{C}^{*,*}$ for I, m as above;

(ii) if moreover $i_0 = \cdots = i_{k-1} = 0$, $i_k > 0$, and $m \equiv 2^e \bmod 2^{e+1}$ for some $e \leqslant k$, the class $\frac{1}{2} D_0 w^m \varepsilon_i h^I$ represents a cycle through $E_{2^e-1}(i)$ and $\partial_{2^e}(\frac{1}{2} D_0 w^m \varepsilon_i h^I) = \frac{1}{2} D_0 w^{m-2^e} \varepsilon_i h_e h^I$;

(iii) if I is as in (ii), but $m \equiv 0\ (2^{k+1})$ and $m \leqslant \|I\| + [(i-s)/2] - \rho_2(i-s)$, then $\frac{1}{2} D_0 w^m \varepsilon_i h^I$ can be represented by $\frac{1}{2} dD_1(w^{m+2^k} \varepsilon_i h^{I-\Delta_k}) \in \mathscr{F}_i\mathscr{C}^{*,*}$ and is an infinite cycle.

ad(i) Since

$$\begin{aligned} D_0 w^m \varepsilon_i h^I &= w^m \varepsilon_i d(h^I) + \sum_{j, (\binom{m}{j}) \equiv 0\ (2)} \left(\binom{m}{j}\right) w^{m-j} \varepsilon_i [\tau_j] h^I \\ &\equiv w^m \varepsilon_i dh^I \quad \text{mod weight } > \|I\|, \end{aligned}$$

the claim follows from (6.5).

ad(ii) Under the conditions of the hypothesis, we have

$$D_1\left(w^m \varepsilon_i h^I\right) \equiv w^{m-2^e} \varepsilon_i h_e h^I \quad \text{mod weight} > \|I\| + 2^e.$$

This implies

$$\begin{aligned} d\left(\tfrac{1}{2} D_0 w^m \varepsilon_i h^I\right) &\equiv d\left(\tfrac{1}{2} D_1 w^m \varepsilon_i h^I\right) \\ &\equiv d\left(\tfrac{1}{2} w^{m-2^e} \varepsilon_i h_e h^I\right) \quad \text{mod weight} > \|I\| + 2^e \\ &\equiv D_0\left(\tfrac{1}{2} w^{m-2^e} \varepsilon_i h_e h^I\right) \quad \text{mod weight} > \|I\| + 2^e. \end{aligned}$$

Therefore $\partial_{2^e}(\frac{1}{2}D_0 w^m \varepsilon_i h^I) = \frac{1}{2}D_0 w^{m-2^e} h_e h^I$.

ad(iii) By (6.5) we have $\frac{1}{2}dD_1(w^{m+2^k}\varepsilon_i h^{I-\Delta_k}) \in \mathscr{F}_i\mathscr{C}^{*,*}$. Since $D_1(w^{m+2^k}\varepsilon_i h^{I-\Delta_k}) \equiv w^m \varepsilon_i h^I$ mod weight $> \|I\|$, and $d(w^m\varepsilon_i h^I) \equiv D_0 w^m \varepsilon_i h^I$ mod weight $> \|I\|$, we can take $\frac{1}{2}dD_1 w^{m+2^k}\varepsilon_i h^{I-\Delta_k}$ as a representative. This clearly is a cycle under d, so an infinite cycle in the spectral sequence.

Observe that any element of the basis of $E_1^{*,*,*}(i)$ is dealt with in case (ii) or (iii) of the claim. Moreover, those of case (ii) are mapped injectively to those of case (iii) by the various differentials ∂_{2^e}, where different differentials take their values in different subvector spaces. It is then easy to see that a case (iii) basis element $\frac{1}{2}D_0 w^m \varepsilon_i h^I$ can be the boundary only of $\frac{1}{2}D_0 w^{m+2^k}\varepsilon_i h^{I-\Delta_k}$ and this happens only if the latter is actually in $\mathscr{F}_i\mathscr{C}^{*,*}$. This is the case exactly if

$$m + 2^k \leqslant \|I\| - 2^k + \left[\frac{i-s+1}{2}\right] - \rho_2(i-s+1)$$

$$\Leftrightarrow m \leqslant \|I\| + \left[\frac{i-s}{2}\right] + \rho_3(i-s) - 2^{k+1}.$$

The theorem follows. $\square$

PROOF OF 6.5. (i) We need to show that

$$D_0(w^m\varepsilon_i h^I)/2 \in \mathscr{F}_i\mathscr{C}^{*,*}$$

if $I = (i_0, i_1, \ldots, i_t)$, $\Sigma i_j = s-1$, and $m \leqslant \|I\| + [(i-s)/2] - \rho_2(i-s)$. The components of $D_0(w^m\varepsilon_i h^I)/2$ are of the form $\frac{1}{2}((^m_j))w^{m-j}\varepsilon_i[\tau^j]h^I$ for $((^m_j)) \equiv 0\ (2)$ and $\frac{1}{2}((^{2^e}_j))w^m\varepsilon_i[\tau_1|\cdots|\tau_1|\cdots|\tau_j|\tau_{2^e-j}|\cdots|\tau_{2^t}]$. For the former we need to show by (6.3)

$$m - j \leqslant \|I\| + j + \left[\frac{\nu_2(^m_j) - 1 - (s-1) - \alpha(j) + i}{2}\right] - \rho',$$

where ρ or ρ' denote the appropriate values of ρ_2. Using $[(a-b)/2] = -b + [(a+b)/2]$ this is equivalent to

$$m \leqslant \|I\| + 2j - \alpha(j) + \left[\frac{i - s + \nu_2(^m_j) + \alpha(j)}{2}\right] - \rho'.$$

Since $2j - \alpha(j) = \nu_2(2^j \cdot j!) \geqslant 1$ for $j \geqslant 1$, this follows from the hypothesis. Similarly, the conditions imposed by the second type of summands are shown to be equivalent to the hypothesis.

(ii) For (6.5)(ii), we need to show that

$$\tfrac{1}{2}dD_1\left(w^{m+2^k}\varepsilon_i h^{I-\Delta_k}\right) \in \mathscr{F}_i\mathscr{C}^{*,*}$$

if $i_0 = i_1 = \cdots = i_{k-1} = 0$, $i_k > 0$, $m \equiv 0\ (2^{k+1})$, and

$$m \leqslant \|I\| + \left[\frac{i-s}{2}\right] - \rho_2(i-s).$$

To that end, let

$$\mathscr{F}_i\mathscr{D}^{*,*} = \operatorname{span}\left\{aw^m\varepsilon_i\left[\tau_{n_1}|\cdots|\tau_{n_s}\right]/a \in \mathbf{Z}_{(2)}\right\}$$

with the boundary induced from $\mathscr{F}_i\mathscr{C}^{*,4}$ by the canonical map $\mathscr{F}_i\mathscr{C}^{*,*} \to \mathscr{F}_i\mathscr{D}^{*,*}$. Let $W_i^{*,*} \subset \mathscr{F}_i\mathscr{C}^{*,*}$ denote the submodule generated by

$$\left\{ aw^m\varepsilon_i\left[\tau_{n_1}|\cdots|\tau_{n_s}\right]/a \in \mathbf{Z}_{(2)},\ m \leqslant |\mathbf{n}| + \left[\frac{i-\alpha(\mathbf{n})}{2}\right] - \rho_2(i-\alpha(\mathbf{n}))\right\}.$$

W_i is precisely the submodule which is mapped directly (i.e. with torsion free cokernel) by the canonical map $\mathscr{F}_i\mathscr{C}^{*,*} \to \mathscr{F}_i\mathscr{D}^{*,*}$. Since $0 = dd = dD_0 + dD_1$, and D_0 is divisible by 2 in $\mathscr{F}_i\mathscr{D}^{*,*}$, so is $dD_1 = D_0D_1 + D_1D_1$.

ASSERTION 6.10. With I, k, and m as above, the following are true:

(a) $\frac{1}{2}D_0D_1(w^{m+2^k}\varepsilon_i h^{I-\Delta_k}) \in \mathscr{F}_i\mathscr{C}^{*,*}$.

(b) $D_1D_1(w^{m+2^k}\varepsilon_i h^{I-\Delta_k}) \in W_i$.

Suppose 6.10 is true. Then from the equation $D_1D_1 = dD_1 - D_0D_1$ we have

$$D_1D_1\left(w^{m+2^k}\varepsilon_i h^{I-\Delta_k}\right) \in W_i^{*,*} \cap 2\mathscr{F}_i\mathscr{D}^{*,*}.$$

Therefore

$$D_1D_1\left(w^{m+2^k}\varepsilon_i h^{I-\Delta_k}\right) \in 2\mathscr{F}_i\mathscr{C}^{*,*}$$

and the proposition follows.

The proof of 6.10(a),(b) is obtained by checking the different components of the elements versus condition (6.3). It is very similar to the proof of 6.5(i) and best left to the reader. □

For the remaining cases $s = 0$ and $s = 1$ we have

PROPOSITION 6.11. (a) $H^{0,4k}(\mathscr{C}^{*,*}) = 0$,

(b) $H^{1,4k}(\mathscr{C}^{*,*}) \cong \mathbf{Z}/2^{3+\nu_2(k)}\mathbf{Z} \cong \operatorname{Im} J_{4k-1}$. *A generator is given by the class of* $(\binom{k+1}{1})^{-1}d(2^{2k}w^{k+1}\varepsilon[\ ])$ *for k even and* $(\binom{k+1}{1})^{-1}d(2^{2k-1}w^{k+1}\varepsilon[\ ])$ *for k odd.*

PROOF. By Proposition (5.1), we have isomorphisms

$$E_1^{0,0,t}(\mathscr{C}(S^0)) \cong \pi_t(bo)\cdot[\ ],$$
$$E_1^{1,1,t}(\mathscr{C}(S^0)) \cong \pi_{t-4}(bsp)\cdot[t_1],$$
$$E_1^{\sigma,1,t}(\mathscr{C}(S^0)) = 0 \quad \text{if } \sigma \neq 1.$$

Since

$$d\left(\left(\binom{k+1}{1}\right)^{-1}2^{2k}w^{k+1}\varepsilon[\ ]\right) \equiv 2^{2k}w^k\varepsilon[\tau_1] \quad \text{mod weight} > 1,$$

and

$$d\left(\left(\binom{k+1}{1}\right)^{-1}2^{2k-1}w^{k+1}\varepsilon[\ ]\right) \equiv 2^{2k-1}w^k\varepsilon[\tau_1] \quad \text{mod weight} > 1,$$

we see that these classes represent generators of $E_1^{1,1,*}(\mathscr{C}(S^0))$ if they exist in $\mathscr{C}^{1,*}$.

Consider the case k odd first; the other is similar. We need to check whether for $j > 2$

$$\left(\binom{k+1}{1}\right)^{-1}2^{2k-1}\left(\binom{k+1}{j}\right)w^{k+1-j}\varepsilon[\tau] \in \mathscr{C}^{1,*}.$$

By (6.2) this is the case if

$$k+1-j \leqslant j + \left[\frac{-\nu_2(k+1) + 2k - 1 + \nu_2\left(\binom{k+1}{j}\right) - \alpha(j)}{2}\right] - \rho_2(\cdots).$$

This is easily checked in the usual way. It follows that all elements of $E_1^{1,1,4k}(\mathscr{C}(S^0))$ are infinite cycles. The only possible differential is ∂_1: $E_1^{0,0,4k} \to E_1^{1,1,4k}$. It is given by ϕ: $bo \to \Sigma^4 bsp$ with cofiber $\mathrm{Im}(J)$. This implies the proposition. □

REMARK 6.12. A similar computation can easily be made for $\mathscr{F}_i\mathscr{C}^{*,*}$.

Similar to the computations for $X = S^{0\langle i\rangle}$ are those for $X = B(1)^{\langle i\rangle}$. We therefore only state the results.

COROLLARY 6.13 (*compare* 6.1). (a) $E_\infty^{0,0,t}(\mathscr{C}(B(1)^{\langle i\rangle})) \cong \mathbf{Z}/2$, *for* $t \equiv 5, 6 \bmod 8$ *and* $t \geqslant 2i$,

(b) $E_\infty^{1,1,t}(\mathscr{C}(B(1)^{\langle i\rangle})) \cong \mathbf{Z}/2$, *for* $t \equiv 5, 6 \bmod 8$ *and* $t \geqslant 2i + 2$,

(c) *for any sequence* $E = (e_1, \ldots, e_{s-1})$ *such that* $0 \leqslant e_1 \leqslant \cdots \leqslant e_{s-2} < e_{s-1}$ *and* $i - s \equiv 2, 3 \bmod 4$ *there is exactly one nontrivial* $E_\infty^{\sigma,s,t}(\mathscr{C}(B(1)^{\langle i\rangle})) \cong \mathbf{Z}/2$ *with* $t \not\equiv 0 \bmod 4$. *This occurs with* $\sigma = \Sigma 2^{e_j}$ *and*

$$t = \begin{cases} 8\sigma + 2i - 2s + 1 & \text{for } i - s \equiv 2 \bmod 4, \\ 8\sigma + 2i - 2s & \text{for } i - s \equiv 3 \bmod 4. \end{cases}$$

In these cases $t - s \equiv 3 - i \bmod 4$ *and* $(1 - i) \bmod 4$ *respectively.*

(d) *If* (σ, s, t) *is none of the above and* $t \equiv 0 \bmod 4$ *we have*

$$E_\infty^{\sigma,s,t}\left(\mathscr{C}\left(B(1)^{\langle i\rangle}\right)\right) \cong 0. \qquad \square$$

Elements in $\mathscr{C}^{s,4k}(B(1)^{\langle i\rangle})$ may be written as

$$\sum a w^m \eta_i \left[\tau_{n_1} | \cdots | \tau_{n_s}\right],$$

where $a \in \mathbf{Z}_{(2)}$ and $\eta_i = \eta_i(\mathbf{n})$ denotes either a generator of $\pi_0 bsp$ (Σn_i even) or 2(generator of $\pi_0 bo$) for Σn_i odd. Furthermore m is restricted by

$$m \leqslant |\mathbf{n}| + \left[\frac{i - s + 1 + \nu_2(a)}{2}\right] - \rho_0(i - s + 1 + \nu_2(a)).$$

As an analogue to Proposition 6.5 and Theorem 6.6 one proves

PROPOSITION 6.14. (i) *Suppose* $I = (i_0, \ldots, i_t)$, $\Sigma i_j = s - 1$, *and*

$$m \leqslant \|I\| + \left[\frac{i - s + 1}{2}\right] - \rho_0(i - s + 1).$$

Then $\frac{1}{2} D_0 w^m \eta_i h^I \in \mathscr{C}^{s,*}(B(1)^{\langle i\rangle})$.

(ii) *Suppose that in addition* $i_0 = \cdots = i_{k-1} = 0$, $i_k > 0$, *and* $m \equiv 0 \bmod 2^{k+1}$. *Then*

$$\tfrac{1}{2} dD_1\left(w^{m+2^k}\eta_i h^{I-\Delta k}\right) \in \mathscr{C}^{s,*}\left(B(1)^{\langle i\rangle}\right). \qquad \square$$

THEOREM 6.15. *The groups* $H^s(\mathscr{C}(B(1)^{\langle i\rangle}))$ *are vector spaces over* $\mathbf{Z}/2$ *for* $s \geqslant 2$. *A basis is given by*

$$\left\{\tfrac{1}{2} dD_1 w^{m+2^k}\eta_i h^{I-\Delta_k}\right\},$$

where $I = (0, \ldots, 0, i_k, \ldots, i_{t-1}, 1)$, $i_k \geqslant 1$, $\Sigma_{j=k}^t i_j = s - 1$, *and* m *satisfies the conditions* $m \equiv 0 \bmod 2^{k+1}$ *and*

$$\|I\| + \left[\frac{i - s + 1}{2}\right] + \rho_1(i - s + 1) - 2^{k+1} < m \leqslant \|I\| + \left[\frac{i - s + 1}{2}\right] - \rho_0(i - s + 1). \qquad \square$$

For filtration $s = 0, 1$ we have

PROPOSITION 6.16 (*compare* 6.11). (i)

$$H^{0,4k}(\mathscr{C}(B(1))) \cong \begin{cases} \mathbf{Z}_{(2)} & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

(ii) $H^{1,4k}(\mathscr{C}(B(1)))$ *is cyclic of order* 2^r *where* $r = 1$ *for* $k = 1$ *and* $r = 3 + \nu_2(k)$ *for* $k > 0$. □

We now describe various computations for spaces different from $S^{0\langle i \rangle}$, $B(1)^{\langle i \rangle}$.

Let $\lambda\colon \Sigma P_1^\infty \to S^1$ be any stable map which is η on the bottom cell and denote by R the fiber of λ. Then $S^0 \to R \to \Sigma P_1^\infty$ is a cofibration where $S^0 \hookrightarrow R$ is the inclusion of the bottom cell. It is well known that $bo \wedge R \simeq \bigvee_{n \geqslant 0} \Sigma^{4n} K\mathbf{Z}_{(2)}$ [8]. So $bo_* R$ is generated by w_ι^m, where ι is the inclusion of the bottom cell. We may think of w as a generator of $H_4(bo;\ \mathbf{Z}_{(2)})/\mathrm{Tors} = \frac{1}{8}\pi_4(bo)$. Similarly $bsp \wedge R \simeq \bigvee_{n \geqslant 0} \Sigma^{4n} K\mathbf{Z}_{(2)} \vee \bigvee_{n > 0} \Sigma^{4n+2} K\mathbf{Z}/2$. It follows that $\mathscr{C}^{s,t}(R^{\langle i \rangle})$ is totally concentrated in dimensions $t \equiv 0\ (4)$ and, using the notations from (6.2), we see that

$$\mathscr{C}^{s,t}(R) = \operatorname{span}\left\{ aw^m\left[\tau_{n_1}| \cdots |\tau_{n_s}\right] \mid a \in \mathbf{Z}_{(2)} \right\}.$$

The inclusion of the bottom cell $S^0 \to R$ induces a map $\mathscr{C}^{s,t}(S^0) \to \mathscr{C}^{s,t}(R)$, which is the canonical one suggested by the notations.

Since $\mathscr{C}^{*,*}(R)$ is torsion free, this forces the following differential on $\mathscr{C}^{s,t}(R)$:

$$\begin{aligned} d\left(aw^m\left[\tau_{n_1}| \cdots |\tau_{n_s}\right]\right) = {} & \sum_{j \geqslant 0} \left(\!\!\binom{m}{j}\!\!\right) aw^{m-j}\varepsilon\left[\tau_j | \tau_{n_1}| \cdots |\tau_{n_s}\right] \\ & + \sum_{n,j} (-1)^j \left(\!\!\binom{n_i}{j}\!\!\right) aw^m\varepsilon\left[\tau_{n_1}| \cdots |\tau_{n_i - j}|\tau_j| \cdots |\tau_{n_s}\right] \\ & + (-1)^{s+1} aw^m\varepsilon\left[\tau_{n_1}| \cdots |\tau_{n_s}|1\right]. \end{aligned}$$

It is then obvious that

$$\mathscr{C}^{s,*}(R) \cong \mathbf{Z}_{(2)}[\tau_1] \otimes \overline{\mathbf{Z}_{(2)}[\tau_1]}^{\otimes s}$$

and that the differential is the differential of the cobar-*resolution* of a polynomial algebra over $\mathbf{Z}_{(2)}$ (with the one exception that all binomial coefficients are replaced by their counterparts based on powers of 9). This complex is trivially contractible.

A similar argument for $R^{\langle i \rangle}$ completes the proof of the following lemma.

LEMMA 6.17.

$$H^s(\mathscr{C}^{*,*}(R^{\langle i \rangle}), d_*) = \begin{cases} \mathbf{Z}_{(2)}, & s = t = 0, \\ 0, & \textit{elsewhere}. \end{cases} \quad \square$$

To compute $H^{*,*}(\mathscr{C}(P_1^{\infty\langle i \rangle}))$, we use the cofibration $S^0 \to R \to \Sigma P_1^\infty$. From (6.2) and the discussion above we see that there is a short exact sequence

$$0 \to \mathscr{C}^{s,4k}(S^{0\langle i \rangle}) \to \mathscr{C}^{s,4k}(R^{\langle i \rangle}) \to \mathscr{C}^{s,4k}(\Sigma P_1^{\infty\langle i \rangle}) \to 0$$

and an isomorphism

$$\mathscr{C}^{s,4k+l}(P_1^{\infty\langle i \rangle}) \xrightarrow{\simeq} \mathscr{C}^{s,4k+l+1}(S^{0\langle i+1 \rangle}) \quad \text{for } l = 0, 1, 2.$$

This implies

PROPOSITION 6.18.

$$H^{s,t}\big(\mathscr{C}\big(P_1^{\infty\langle i\rangle}\big)\big) \cong \begin{cases} H^{s+1,t+1}\big(\mathscr{C}\big(S^{0\langle i\rangle}\big)\big), & t \equiv -1\ (4), \\ H^{s,t+1}\big(\mathscr{C}\big(S^{0\langle i+1\rangle}\big)\big), & t \equiv -1\ (4). \quad \square \end{cases}$$

REMARK 6.19. Similar computations are possible for stunted projective spaces $P_{2k+1}^{2l\langle i\rangle}$. They are based on the fact that, as a module over the operation algebra, $bo_*(P_{2k+1}^{2l\langle i\rangle})$ can be computed as the third term in a cofiber sequence whose other two terms are of the form $bo_*(S^{0\langle n\rangle})$ or $bo_*(B(1)^{\langle m\rangle})$ and the map between these is induced by a map of degree 1 on the bottom cells. Using the results of this section this map can be computed in $H^{*,*}(\mathscr{C}(-))$. A suitable long exact sequence provides the results for $(P_{2k+1}^{2l})^{\langle i\rangle}$. (See (7.3), (7.4) for details.)

7. The bounded torsion theorem. In this section we shall prove

THEOREM 7.1. *Let X be in $\{S^{0\langle i\rangle}, B(1)^{\langle i\rangle}, (P_{2k+1}^{2l})^{\langle i\rangle} \mid l \leqslant \infty\}$ and suppose $x \in E_1^{s,t}(X;\ bo)$, $s \geqslant 2$, is a cycle under d_1 and is represented in $\pi_t(bo \wedge \overline{bo}^s \wedge X)$ by an element of $H\mathbf{Z}/2$-Adams filtration $\geqslant 2$. Then x is a boundary under d_1.*

The proof will be quite computational. We use our detailed knowledge of $H(\mathscr{C}^{*,*}(X))$ for $X = S^{0\langle i\rangle}$ and $B(1)^{\langle i\rangle}$ to show that a similar statement is true for $H(\mathscr{C}^{*,*}(X))$ and deduce (7.1) from that.

As for the precise class of spectra X for which (7.1) is true, nothing is known to date.

The hypothesis "$H\mathbf{Z}/2$-Adams filtration $\geqslant 2$" is necessary. A hand computation of $E_2^{s,t}(S^0, bo)$ for $t - s \leqslant 20$ shows that $\kappa \in \pi_{14}^s$ and $\eta\kappa \in \pi_{15}^s$ both have bo-filtration 3 and that $\bar\kappa$ and $2\bar\kappa \in \pi_{20}^s$ both have bo-filtration 4. See §8 for the tables. Let ι: $X^{\langle i\rangle} \to X^{\langle i-1\rangle}$ denote the canonical map. (7.1) will be deduced from

PROPOSITION 7.2. *Let X be as in* (7.1). *Then ι_*: $H^{s,t}(\mathscr{C}(X^{\langle i\rangle})) \to H^{s,t}(\mathscr{C}(X^{\langle i-1\rangle}))$ is trivial for all $i \geqslant 1$ and $s \geqslant 2$.*

PROOF OF 7.1 FROM 7.2. We use the short exact sequence of chain complexes

$$0 \to \pi_t(KV_s(X)) \to E_1^{s,t}(X, bo) \xrightarrow{\mathrm{pr}} \mathscr{C}^{s,t}(X) \to 0.$$

Suppose $y \in E_1^{s,t}(X;\ bo)$ has $\mathrm{AF}(y) \geqslant 1$ and satisfies $\mathrm{pr}(y) = 0 \in \mathscr{C}^{s,t}(X)$. Then $y = 0$ since it cannot be hit by an element of $\pi_t(KV_s(X))$, which splits off the elements of exactly Adams filtration 0. So given any $x \in E_1^{s,t}(X, bo)$, $\mathrm{AF}(x) \geqslant 2$ as in 7.1, there exists $y' \in \mathscr{C}^{s-1,t}(X)$, $\mathrm{AF}(y') \geqslant 1$ such that $dy' = \mathrm{pr}(x)$ by (7.2) for $i = 2$. Therefore $y' = \mathrm{pr}(y)$ with $\mathrm{AF}(y) \geqslant 1$ and hence $d_1 y = x$. $\square$

We now start proving (7.2). We shall first consider the case $X = S^{0\langle i\rangle}$; the case $X = B(1)^{\langle i\rangle}$ is similar and will be omitted. Finally we deal with stunted projective spaces.

Let $x_{m,I} = \frac{1}{2} dD_1 w^{m+2^k} \varepsilon_i h^{I-\Delta_k} \in \mathscr{C}^{s,t}(S^{0\langle i\rangle})$, $t \equiv 0\ (4)$, be any of the generators of $H^{s,t}(\mathscr{C}(S^{0\langle i\rangle}))$. (By (6.6), this means that $I = (0, \ldots, 0, i_k, \ldots, i_{t-1}, 1)$, $\Sigma i_j = s - 1$, $i_k > 0$, $m \equiv 0\ (2^{k+1})$, and

$$\|I\| + \left[\frac{i-s}{2}\right] + \rho_3(i-s) - 2^{k+1} < m \leqslant \|I\| + \left[\frac{i-s}{2}\right] - \rho_2(i-s).\Big)$$

We need to show that

$$\iota_*(x_{m,I}) = dD_1 w^{m+2^k}\varepsilon_{i-1}h^{I-\Delta_k}$$

is a boundary.

This is certainly true if $D_1 w^{m+2^k}\varepsilon_{i-1}h^{I-\Delta_k} \in \mathscr{C}^{s-1,t}(S^{0\langle i-1\rangle})$. A calculation similar to those of §6 gives the result.

If $t \not\equiv 0$ (4), we see from (6.1) that $H^{s,t}(\mathscr{C}(S^{0\langle i\rangle}))$ is concentrated in dimensions $t-s \equiv i \bmod 2$. Therefore ι_* is trivial for dimensional reasons. $\square$

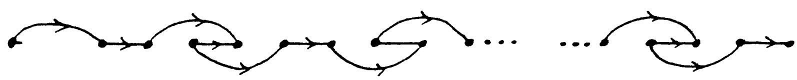
We now proceed to the case of stunted projective spaces $P_{2k+1}^{2l\langle i\rangle}$ with $k < l \leqslant \infty$. To this end we need to describe $bo_*(P_{2k+1}^{2l\langle i\rangle})$ as a module over the algebra of bo-operations, where $\mathbf{Z}/2$'s arising as homotopy of Eilenberg-Mac Lane spectra may be ignored.

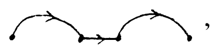
Let $S^0 \xrightarrow{\alpha} S^{0\langle j\rangle}$ and $S^0 \xrightarrow{\alpha} B(1)^{\langle j\rangle}$ be the inclusion of the bottom cell and denote the corresponding cofibers by $X(j)$ and $Y(j)$ respectively ($S^{0\langle\infty\rangle} = B(1)^{\langle\infty\rangle} = R$).

To motivate and illustrate the following lemma, we need to describe the stable $\mathfrak{A}(1)$-structure of some of our modules. Recall from [**8** and **13**] that, as a stable $\mathfrak{A}(1)$-module, $H^*(S^{0\langle j\rangle})$ may be described by the following diagrams. (We have taken $j \equiv 0, 3 \bmod 4$ for simplicity. A dot denotes a $\mathbf{Z}/2$, a curved arrow denotes the operation of Sq^2, and a straight arrow indicates a nontrivial Sq^1.)

$j = 4k$:

$j = 4k + 3$:



(In the first example there are $2k$ copies of "  " and $2k+1$ occur in the second.)

Similarly, $H^*(B(1)^{\langle j\rangle})$ may for $j \equiv 0, 1 \bmod 4$ be described by the diagrams:

$j = 4k$:

$j = 4k + 1$:

(Here $2k$ and $2k+1$ " " occur.)

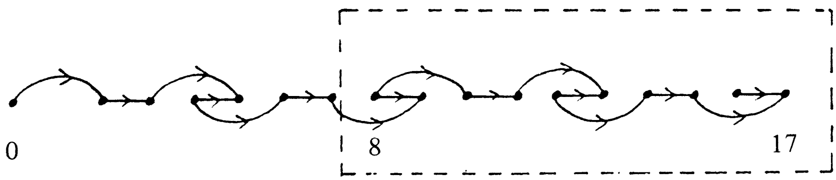
Together with the stable isomorphism

$$H^*(B(1) \wedge B(1)) \underset{\mathfrak{A}(1)}{\cong} H^*(S^{0\langle 2\rangle})$$

one easily deduces the following examples of short exact sequences of stable $\mathfrak{A}(1)$-modules which correspond to the geometric cofibrations we have in mind.

(a) $H^*(\Sigma P_7^{16}) \underset{\mathfrak{A}(1)}{\cong} H^*(X(5)^{\langle 3\rangle}) \underset{\mathfrak{A}(1)}{\cong} \ker[H^*(S^{0\langle 8\rangle}) \xrightarrow{\alpha^*} H^*(S^{0\langle 3\rangle})]$.

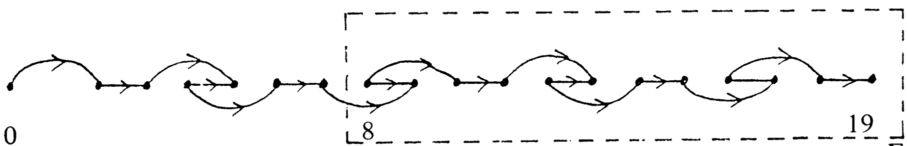
$S^{0\langle 8\rangle}$:



P_7^{16}

(b) $H^*(\Sigma P_7^{18}) \underset{\mathfrak{A}(1)}{\cong} H^*(Y(5)^{\langle 3\rangle}) \cong \ker[H^*(B(1)^{\langle 8\rangle}) \xrightarrow{\alpha^*} H^*(S^{0\langle 3\rangle})]$.

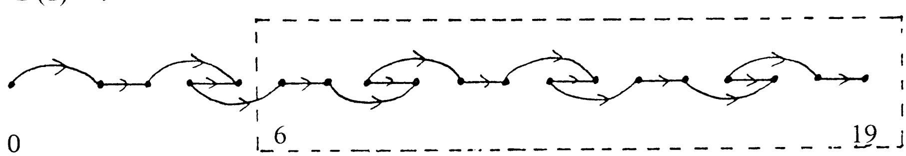
$B(1)^{\langle 8\rangle}$:



ΣP_7^{18}

(c) $H^*(\Sigma P_5^{18}) \underset{\mathfrak{A}(1)}{\cong} H^*(B(1) \wedge X(7)^{\langle 1\rangle}) \underset{\mathfrak{A}(1)}{\cong} \ker[H^*(B(1)^{\langle 8\rangle}) \xrightarrow{\alpha^*} H^*(B(1)^{\langle 1\rangle})]$.

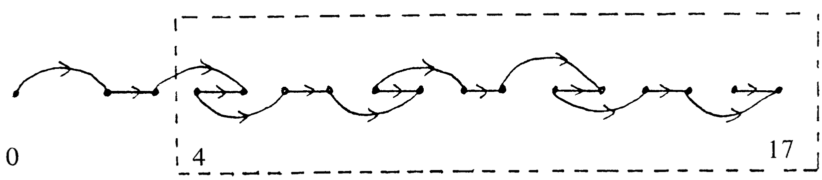
$B(1)^{\langle 8\rangle}$:



ΣP_5^{18}

(d) $H^*(\Sigma P_3^{16}) \cong H^*(B(1) \wedge Y(1)^{\langle 6\rangle}) \cong \ker[H^*(S^{0\langle 8\rangle}) \xrightarrow{\alpha^*} H^*(B(1))]$.

$S^{0\langle 8\rangle}$:



ΣP_3^{16}

LEMMA 7.3. *Modulo a $\mathbf{Z}/2$ vector space arising as the homotopy of Eilenberg-Mac Lane spectra, $bo_*(P_{2k+1}^{2l\langle n\rangle})$ is isomorphic to one of the following*:

$$\pi_*\big(bo \wedge X(j)^{\langle i\rangle}\big), \quad \pi_*\big(bo \wedge Y(y)^{\langle i\rangle}\big), \quad \pi_*\big(bo \wedge B(1) \wedge X(j)^{\langle i\rangle}\big) \text{ or}$$

$$\pi_*\big(bo \wedge B(1) \wedge Y(j)^{\langle i\rangle}\big) \quad \textit{for suitable values of } i, j.$$

These isomorphisms are valid as modules over the algebra of bo-operations.

PROOF. The proof is similar to the one for $k = 0$ and $l = \infty$. It rests on the fact that, similar to the cohomological case studied above, we can represent all the occurring modules naturally as subquotients of bo_*R (in dimensions $\equiv 0 \bmod 4$). Thus everything is determined by the module structure of $bo_*(S^0) \subset bo_*(R)$. Denote by $C(2k)$ and $C(2k+1)$ the $(4k+1)$ and $(4k+3)$ skeleta of R. They trivially have the $\mathfrak{A}(1)$-stable type of either an $S^{0\langle 4k\rangle}$, $S^{0\langle 4k+3\rangle}$, $B(1)^{\langle 4k\rangle}$, or $B(1)^{\langle 4k+1\rangle}$.

Lifting the standard inclusions $S^0 \to R$ and $B(1) \to R$, we get canonical maps $S^{0\langle i\rangle} \to R^{\langle i\rangle}$ and $B(1)^{\langle i\rangle} \to R^{\langle i\rangle}$. The Adams lift of 2^i: $R \to R$ then induces an isomorphism $bo_*(R) \to bo_*(R^{\langle i\rangle})$ (modulo $\mathbf{Z}/2$'s) which in turn generates an isomorphism of $bo_*(C(j))$ with either of $bo_*(S^{0\langle 4k\rangle})$, $bo_*(S^{0\langle 4k+3\rangle})$, $bo_*(B(1)^{\langle 4k\rangle})$, or $bo_*(B(1)^{\langle 4k+1\rangle})$. This is obviously also true as modules over the operations.

To prove the lemma, it therefore suffices to present P_{2k+1}^{2l} as a quotient of $C(j)$'s. This is trivial from the cofibration $S^0 \to R \to P_1^\infty$. $\square$

In view of 7.3 it therefore suffices to prove

PROPOSITION 7.4. *The conclusion of Proposition* (7.2) *holds also for* $X = X(j)$, $Y(j)$, $X(j) \wedge B(1)$, *and* $Y(j) \wedge B(1)$.

PROOF. Let X be one of the spaces mentioned above and suppose $t \not\equiv 0$ (4). Then it is easy to see that there is no possible nontrivial action of ϕ on $\mathscr{C}^{s,t}(X^{\langle i\rangle})$. This shows that the differential ∂_0 of the weight filtration spectral sequence coincides with the total differential d on $\mathscr{C}^{s,t}(X^{\langle i\rangle})$. We have computed $E_1^{\sigma,s,t}(X^{\langle i\rangle})$ in Proposition (5.1). It was shown that any cycle in $\pi_{t-s}(bo \wedge \overline{bo}^s \wedge X^{\langle i\rangle})$, $s \geqslant 2$, which has at least $H\mathbf{Z}/2$-Adams filtration 1 is a boundary (see Remark (5.3)).

For $t \equiv 0$ (4) a more detailed analysis is necessary. We only do the case $X(j)$ here, the others are similar and left to the reader. In this case we have a short exact sequence $0 \to \mathscr{C}^{s,t}(S^{0\langle i\rangle}) \to \mathscr{C}^{s,t}(S^{0\langle i+j\rangle}) \to \mathscr{C}^{s,t}(X(j)^{\langle i\rangle}) \to 0$. It induces a diagram of long exact sequences:

$$\begin{array}{ccccc}
H^{s,t}(\mathscr{C}(S^{0\langle i\rangle})) & \xrightarrow{\alpha_*} & H^{s,t}(\mathscr{C}(S^{0\langle i+j\rangle})) & \xrightarrow{\beta_*} & H^{s,t}(\mathscr{C}(X(j)^{\langle i\rangle})) \\
\iota_*\downarrow = 0 & & \iota_*\downarrow = 0 & & \downarrow \iota_* \\
H^{s,t}(\mathscr{C}(S^{0\langle i-1\rangle})) & \xrightarrow{\alpha_*} & H^{s,t}(\mathscr{C}(S^{0\langle i+j-1\rangle})) & \xrightarrow{\beta_*} & H^{s,t}(\mathscr{C}(X(j)^{\langle i-1\rangle}))
\end{array}$$

$$\begin{array}{ccccc}
\to & H^{s+1,t}(S^{0\langle i\rangle}) & \xrightarrow{\alpha_*} & H^{s+1,t}(\mathscr{C}(S^{0\langle i+j\rangle})) \\
 & \iota_*\downarrow = 0 & & \iota_*\downarrow = 0 \\
\to & H^{s+1,t}(S^{0\langle i-1\rangle}) & \xrightarrow{\beta_*} & H^{s+1,t}(\mathscr{C}(S^{0\langle i+j-1\rangle}))
\end{array}$$

We already know that ι_* is trivial on $H^*(\mathscr{C}(S^{0\langle i\rangle}))$. We therefore only need to show that for any $x \in H^{s+1,t}(S^{0\langle i\rangle})$ such that $\alpha_*(x) = 0$ we can find a preimage $y \in H^{s,t}(X(j)^{\langle i\rangle})$ with $\iota_*(y) = 0$. By (6.6) a basis for $H^{s+1,t}(S^{0\langle i\rangle})$ is given by the classes

$$\tfrac{1}{2} dD_1 w^{m+2^k} \varepsilon_i h^{l-\Delta_k},$$

where $I = (0, \ldots, 0, i_k, \ldots, i_{t-1}, 1)$, $i_k > 0$, $\Sigma i_j = s$, $m \equiv 0\ (2^{k+1})$, and

$$(7.5)\quad \|I\| + \left[\frac{i-s-1}{2}\right] + \rho_3(i-s-1) - 2^{k+1} < m \leqslant \|I\| + \left[\frac{i-s-1}{2}\right] - \rho_2(i-s-1),$$

and similarly for $H^{s+1,t}(S^{0\langle i+j\rangle})$. It also follows from the proof of that theorem that a class of the above form is a boundary if it satisfies all the conditions except that $m \leqslant \|I\| + [(i-s-1)/2] + \rho_2(i-s-1) - 2^{k+1}$. Since $\alpha_*(\frac{1}{2}dD_1 w^{m+2^k}\varepsilon_i h^{I-\Delta_k}) = \frac{1}{2}dD_1 w^{m+2^k}\varepsilon_{i+j}h^{I-\Delta_k}$ we see that $\ker(\alpha_*)$ is spanned by

$$\left\{\tfrac{1}{2}dD_1 w^{m+2^k}\varepsilon_i h^{I-\Delta_k}\right\},$$

where (I, k, m) are as above and moreover

$$(7.6)\quad \|I\| + \left[\frac{i-s-1}{2}\right] + \rho_3(i-s-1) - 2^{k+1} < m \leqslant \|I\| + \left[\frac{i+j-s-1}{2}\right] + \rho_3(i+j-s-1) - 2^{k+1}.$$

In this case $x = \frac{1}{2}dD_1 w^{m+2^k}\varepsilon_{i+j}h^{I-\Delta_k} = d(\frac{1}{2}D_0 w^{m+2^k}\varepsilon_{i+j}h^{I-\Delta_k})$. Therefore $z = \frac{1}{2}D_0 w^{m+2^k}\varepsilon_{i+j}h^{I-\Delta_k} \in \mathscr{C}^{s,t}(S^{0\langle i+j\rangle})$ is mapped under β_* to a preimage y of x.

We now consider $\iota_* z = D_0 w^{m+2^k}\varepsilon_{i+j-1}h^{I-\Delta_k}$. Using (7.6) it follows easily that

$$w^{m+2^k}\varepsilon_{i+j-1}h^{I-\Delta_k} \in \mathscr{C}^{s-1,t}(S^{0\langle i+j-1\rangle}).$$

Therefore $\iota_* z$ and $D_1 w^{m+2^k}\varepsilon_{i+j-1}h^{I-\Delta_k}$ are homologous through

$$d\left(w^{m+2^k}\varepsilon_{i+j-1}h^{I-\Delta_k}\right).$$

Using (7.5) it follows easily that

$$D_1 w^{m+2^k}\varepsilon_{i-1}h^{I-\Delta_k} \in \mathscr{C}^{s-1,t}(S^{0\langle i-1\rangle}).$$

Therefore $\iota_*(z)$ is homologous to an element in $\mathrm{im}(\alpha_*)$ and consequently $\iota_* y = \iota_*\beta_* z$ is homologous to zero. This finishes the proof of (7.4). □

8. Some tables. The following charts display the homology of the bo-essential complex $\mathscr{C}^{*,*}(S^0)$ for $t-s \leqslant 50$ (Table 8.1) and the E_2-term of the bo-Adams spectral sequence for $t-s \leqslant 20$ (Table 8.2).

The notation is as follows:

a dot "$\cdot$" represents a $\mathbf{Z}/2$;

a number "2^n" represents a $\mathbf{Z}/2^n$;

a vertical line represents a nontrivial extension by 2;

a horizontal or slanting line represents a nontrivial extension by $\eta \in \pi_1^s$;

a name "h^I" indicates that the element is represented with leading term h^I, i.e. on $bo \wedge B_1 \wedge \cdots \wedge B_1 \wedge B_2 \wedge \cdots \wedge B_{2^k}$ with i_j copies of B_{2^j}'s.

Observe that the complete $\mathrm{Im}(J)$ is concentrated in filtration 0 or 1, depending on whether the element is detected by the d- or the e-invariant. The class with name h_0^3 in dimension 14 of 8.1 represents $\kappa \in \pi_{14}^s$. Certainly $\eta\kappa = 0$ in $H^{3,18}(\mathscr{C}(S^0))$ since the $H\mathbf{Z}_2$-Adams filtration of its representative is as least 1. Nevertheless we find that

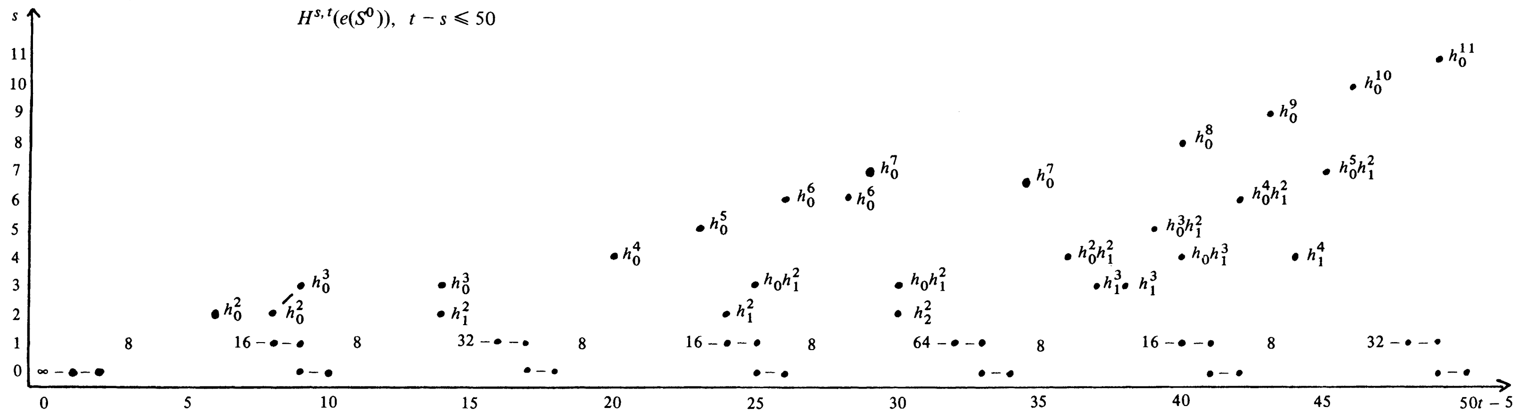


TABLE 8.1

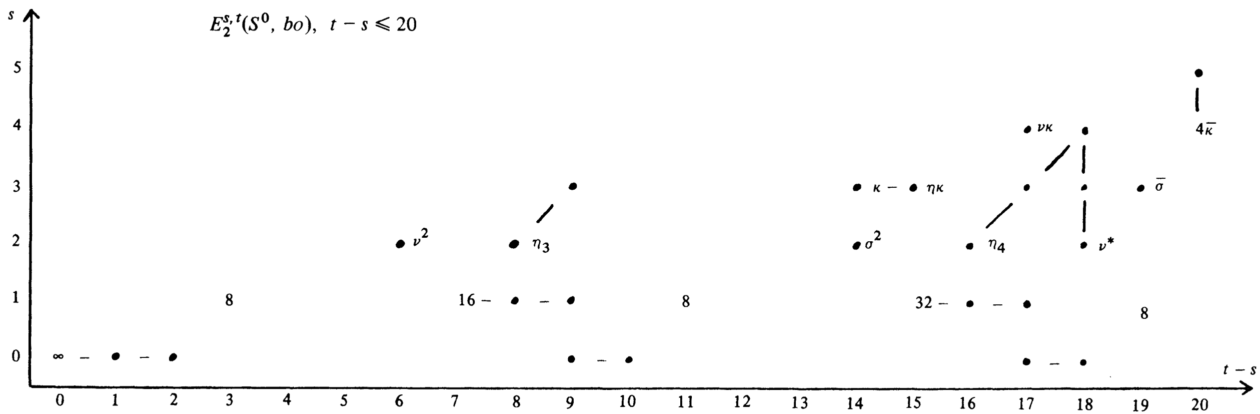


TABLE 8.2

$\eta\kappa \neq 0$ in $E_2^{3,18}(S^0, bo)$. Therefore its representative in $E_1^{3,18}(S^0, bo)$ must be homologous to a class living on the subcomplex given by the Eilenberg-Mac Lane spectra. From this we may deduce that the differential d_1 of the *bo*-resolution really mixes between the quotient complex $\mathscr{C}^{*,*}(S^0)$ and the subcomplex $V_*(S^0)$. A corresponding phenomenon occurs in dimension 20: here $\bar{\kappa}$ is represented with name h_0^4 in Table 8.1, dimension 20, but $2\bar{\kappa} \neq 0$ in $E_2^{4,24}(S^0, bo)$. This produces the first known $\mathbf{Z}/4$ with filtration at least 2 in the *bo*-resolution. We are also able to produce the first known higher differential in the *bo*-Adams spectral sequence: Since $0 \neq \nu^3\bar{\kappa}$ $(= h_0^7)$ in $H^{7,29}(\mathscr{C}(S^0))$, the same is true in $E_2^{7,29}(S^0, bo)$. But $\nu^3\bar{\kappa} = 0$ in π_{29}^s, so the class must be hit by a differential.

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