

THE BINARY MATROIDS WITH NO 4-WHEEL MINOR

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ABSTRACT. The cycle matroids of wheels are the fundamental building blocks for the class of binary matroids. Brylawski has shown that a binary matroid has no minor isomorphic to the rank-3 wheel $M(\mathcal{W}_3)$ if and only if it is a series-parallel network. In this paper we characterize the binary matroids with no minor isomorphic to $M(\mathcal{W}_4)$. This characterization is used to solve the critical problem for this class of matroids and to extend results of Kung and Walton and Welsh for related classes of binary matroids.

1. Introduction. The purpose of this paper is to study the class of binary matroids with no minor isomorphic to $M(\mathcal{W}_4)$, the cycle matroid of the rank-4 wheel. The motivation for this study derives from the fact that for every 3-connected binary matroid M with at least four elements, there is a sequence $M_0, M_1, M_2, \dots, M_n$ of 3-connected matroids with $M_n = M$ such that each matroid in the sequence is a single-element deletion or contraction of its successor and, for some $r \geq 3$, $M_0 \cong M(\mathcal{W}_r)$. This result, a consequence of Tutte's wheels and whirls theorem [18], establishes the wheels as the fundamental nontrivial building blocks for the class of 3-connected binary matroids. Indeed, since every matroid that is not 3-connected is a direct sum or a 2-sum of two matroids on fewer elements (Theorem 1.2), these building blocks are fundamental to the whole class of binary matroids. It is natural then to consider which binary matroids can arise when, for some r , $M(\mathcal{W}_r)$ is excluded as a minor. For $r = 3$, this question was answered by Brylawski [3] who identified the class of such matroids as the class of series-parallel networks. In this paper we characterize the corresponding class of matroids when $r = 4$ by listing its 3-connected members. For larger values of r it appears that a similar characterization will be much more difficult to obtain.

The matroid terminology used here will in general follow Welsh [21]. The ground set and rank of the matroid M will be denoted by $E(M)$ and $\text{rk } M$, respectively. If $T \subseteq E(M)$, $\text{rk } T$ will denote the rank of T . The deletion and contraction of T from M will be denoted by $M \setminus T$ and M/T , respectively. If Z is an n -element circuit of M , then we shall call Z an n -circuit; Z is an *odd* circuit if n is odd and an *even* circuit otherwise.

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A matroid M is *3-connected* if it is connected and $E(M)$ cannot be partitioned into subsets X and Y each having at least two elements such that $\text{rk } X + \text{rk } Y - \text{rk } M = 1$. It is routine to verify that M is 3-connected if and only if its dual M^* is 3-connected.

If M_1 and M_2 are matroids on the sets S and $S \cup e$ where $e \notin S$, then M_2 is an *extension* of M_1 if $M_2 \setminus e = M_1$, and M_2 is a *lift* of M_1 if M_2^* is an extension of M_1^* . We call M_2 a *nontrivial extension* of M_1 if e is neither a loop nor a coloop of M_2 and e is not in a 2-circuit of M_2 . Likewise, M_2 is a *nontrivial lift* of M_1 if M_2^* is a nontrivial extension of M_1^* . The following result is well known (see, for example, [12, Lemma 2.1]).

(1.1) LEMMA. *Let N be a 3-connected matroid having at least three elements and M be an extension of N . Then M is 3-connected if and only if M is a nontrivial extension of N . \square*

We shall assume familiarity with the operation of parallel connection of matroids; a detailed discussion of this operation and its properties can be found in [3]. For matroids M_1 and M_2 such that $E(M_1) \cap E(M_2) = \{p\}$, we shall denote the parallel connection of M_1 and M_2 with respect to the basepoint p by $P((M_1, p), (M_2, p))$. The following basic link between 3-connection and parallel connection was proved by Seymour [15, (2.6)].

(1.2) THEOREM. *A connected matroid M is not 3-connected if and only if there are matroids M_1 and M_2 each of which has at least three elements and is isomorphic to a minor of M such that $M = P((M_1, p), (M_2, p)) \setminus p$ where p is not a loop or a coloop of M_1 or M_2 . \square*

When M decomposes as in this theorem, it is called the *2-sum* of M_1 and M_2 .

If $\{x, y\}$ is a circuit of the matroid M , we say that x and y are *in parallel* in M . If, instead $\{x, y\}$ is a cocircuit of M , then x and y are *in series* in M . The matroid M' is a *parallel extension* of M if $M = M' \setminus T$ and every element of T is in parallel with some element of M' not in T . *Series extensions* are defined analogously. A matroid in which each connected component is obtained from a single-element matroid by a sequence of operations each of which is either a series or parallel extension is called a *series-parallel network*. A detailed investigation of the properties of such matroids can be found in [3].

If A is a matrix with entries in a field F , then the matroid on the set of columns of A that is induced by linear independence over F will be called the *dependence matroid* $D(A)$ of A . A basic tool in this paper is the well-known fact (see, for example, [5, Theorem 3.7]) that binary matroids are uniquely representable, that is, if A and A' are $r \times n$ matrices over $\text{GF}(2)$ such that the map which, for all i in $\{1, 2, \dots, n\}$, takes the i th column of A to the i th column of A' is an isomorphism from $D(A)$ to $D(A')$, then A' can be transformed into A by a sequence of operations each of which consists of interchanging two rows or adding one row to another.

Another result that we shall rely on heavily is the following easy consequence of Seymour's splitter theorem [15, (7.3)]. For $r \geq 2$, \mathcal{W}^r will denote the whirl of rank r [21, p. 81].

(1.3) THEOREM. *Let M and N be 3-connected matroids such that N is a minor of M , $|E(N)| \geq 4$, and if $N \cong M(\mathcal{W}_k)$, M has no $M(\mathcal{W}_{k+1})$ -minor, while if $N \cong \mathcal{W}^k$, M has no \mathcal{W}^{k+1} -minor. Then there is a sequence $M_0, M_1, M_2, \dots, M_n$ of 3-connected matroids such that $M_0 \cong N$, $M_n = M$ and, for all i in $\{1, 2, \dots, n\}$, M_i is an extension or lift of M_{i-1} . \square*

In §2 of this paper, we state and prove the main theorem of the paper, a characterization of the binary 3-connected matroids with no $M(\mathcal{W}_4)$ -minor. We also characterize a somewhat larger class of 3-connected binary matroids and, in doing so, use the following well-known result of Tutte [17]. The Fano matroid will be denoted F_7 .

(1.4) THEOREM. *A binary matroid is regular if and only if it has no minor isomorphic to F_7 or F_7^* . \square*

In §3, we use the main theorem to determine a best-possible upper bound on the number of elements in a rank- r simple matroid with no $M(\mathcal{W}_4)$ -minor. We then use this bound to extend a result of Kung [9]. In addition, we extend a result of Murty [11].

Let M be a loopless matroid that is isomorphic to the dependence matroid $D(A)$ of some matrix A over $\text{GF}(q)$. If A has r rows, then $r \geq \text{rk } M$ and the set S of distinct columns of A is a subset of $V(r, q)$. The *critical exponent* $c(M; q)$ of M is the least number k of hyperplanes H_1, H_2, \dots, H_k of $V(r, q)$ such that $(\bigcap_{i=1}^k H_i) \cap S = \emptyset$, this number being independent of the particular matrix A representing M [6]. The problem of determining $c(M; q)$ is known as the *critical problem* [6, Chapter 16] for M . In the fourth and final section of this paper, we use our main theorem to solve the critical problem for the class of binary matroids with no $M(\mathcal{W}_4)$ -minor. We also use results from §3 to solve the critical problem for a related class of binary matroids thereby extending two results of Walton and Welsh [20].

2. The main result. In this section we state and prove the main result of the paper, a characterization of all binary 3-connected matroids having no $M(\mathcal{W}_4)$ -minor. We shall denote by Θ the class of all such matroids. It follows from Theorem 1.2 that one can construct every binary matroid with no $M(\mathcal{W}_4)$ -minor by beginning with the members of Θ and repeatedly using the operations of 2-sum and direct sum.

Let r be an integer exceeding two and A_r be the following $r \times (2r + 1)$ matrix over $\text{GF}(2)$:

$$A_r = \left[\begin{array}{cccc|cccc|c} a_1 & a_2 & \cdots & a_r & b_1 & b_2 & b_3 & \cdots & b_r & c_r \\ & & & & 0 & 1 & 1 & \cdots & 1 & 1 \\ & & & & 1 & 0 & 1 & \cdots & 1 & 1 \\ & & & & 1 & 1 & 0 & \cdots & 1 & 1 \\ & & I_r & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & 1 & 1 & 1 & \cdots & 0 & 1 \end{array} \right]$$

Let Z_r be the dependence matroid of A_r . Then $Z_3 \cong F_7$, the Fano matroid. Moreover, $Z_r^* \cong Z_{r+1} \setminus b_{r+1}$, c_{r+1} for all $r \geq 3$. Hence, in particular, $F_7^* \cong Z_3^* \cong Z_4 \setminus b_4, c_4$. We also note that $Z_4 \setminus c_4 \cong AG(3, 2)$. The following is the main result of the paper.

(2.1) THEOREM. *Let M be a binary matroid. Then M is 3-connected and has no $M(\mathcal{W}_4)$ -minor if and only if*

- (i) $M \cong Z_r$, Z_r^* , $Z_r \setminus c_r$ or $Z_r \setminus b_r$ for some $r \geq 4$; or
- (ii) $M \cong F_7$, F_7^* or $M(\mathcal{W}_3)$; or
- (iii) $M \cong U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$ or $U_{2,3}$.

The only pairs of matroids in the above list that have the same rank and corank are the self-dual matroids $Z_r \setminus c_r$ and $Z_r \setminus b_r$. For $r \geq 4$, these matroids are not isomorphic since the latter has a 3-circuit whereas the former does not. We conclude that all the matroids listed are nonisomorphic.

To prove Theorem 2.1, we shall first characterize a somewhat larger class Ψ of 3-connected binary matroids, namely those such matroids having no P_9 - or P_9^* -minor. Here P_9 denotes the extension of $M(\mathcal{W}_4)$ for which a binary representation is shown in Figure 1. The same figure also shows a Euclidean representation for P_9 . In addition to the planes shown, there is one further 4-point plane. It contains the four circled points.

(2.2) THEOREM. *Let M be a binary matroid. Then M is 3-connected having no minor isomorphic to P_9 or P_9^* if and only if*

- (i) M is regular and 3-connected;
- (ii) $M \cong Z_r$, Z_r^* , $Z_r \setminus c_r$ or $Z_r \setminus b_r$ for some $r \geq 4$; or
- (iii) $M \cong F_7$ or F_7^* .

PROOF. Evidently neither F_7 nor F_7^* has a minor isomorphic to P_9 or P_9^* . The fact that no regular matroid has such a minor follows, using Theorem 1.4 and duality, from the observation that the contraction of $(1, 0, 0, 1)^T$ from P_9 has an F_7 -minor. We now show that none of the matroids listed in (ii) has a P_9 - or a P_9^* -minor. Since both P_9 and P_9^* have $M(\mathcal{W}_4)$ as a minor, it suffices to show that none of the matroids in (ii) has an $M(\mathcal{W}_4)$ -minor. We shall use the following:

(2.3) LEMMA. *The automorphism group of Z_r is transitive on $\{a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_r\}$. Moreover, for $r \geq 4$, every automorphism of Z_r fixes c_r .*

PROOF. Interchanging rows i and j of A_r induces an automorphism of Z_r that swaps a_i with a_j and b_i with b_j . Moreover, the automorphism of Z_r that is induced by replacing row i of A_r by the sum of rows i , 1 and r for each i in $\{2, 3, \dots, r-1\}$ swaps a_1 with b_r and a_r with b_1 . We conclude that the automorphism group of Z_r is transitive on $\{a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_r\}$. The fact that, for $r \geq 4$, every automorphism fixes c_r follows immediately from the observation that, for such r , c_r is the unique element of Z_r that is in r 3-circuits. \square

(2.4) LEMMA. *Z_r has no $M(\mathcal{W}_4)$ -minor.*

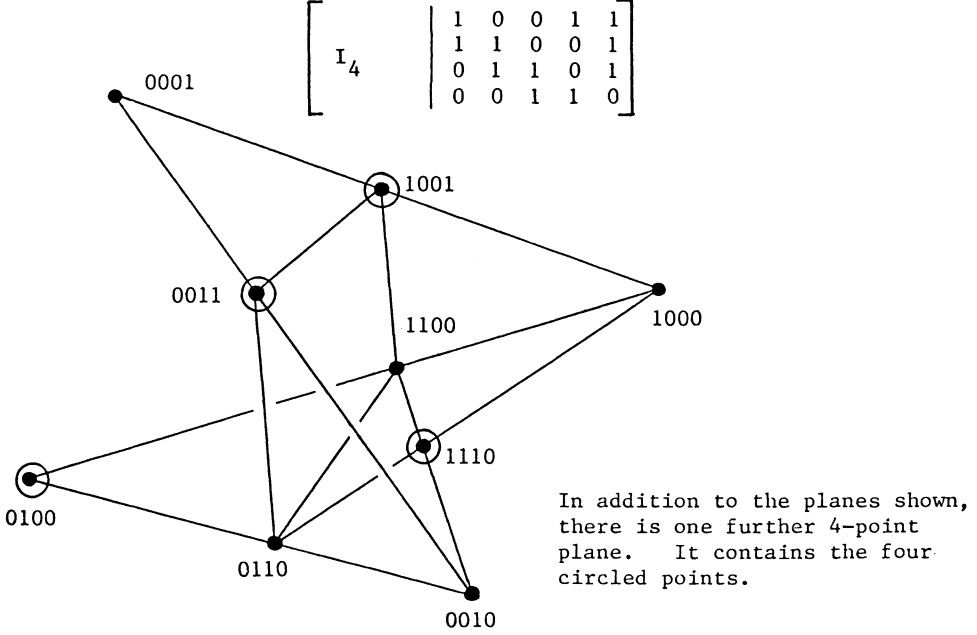


FIGURE 1

PROOF. This is immediate for $r = 3$. Now suppose that $r = 4$. If $M(\mathcal{W}_4)$ is a minor of Z_4 , it is a single-element deletion of Z_4 . But, by the preceding lemma, Z_4 has precisely two nonisomorphic single-element deletions, namely $Z_4 \setminus b_4$ and $Z_4 \setminus c_4$. As $Z_4 \setminus b_4, c_4 \cong F_7^*$, neither $Z_4 \setminus b_4$ nor $Z_4 \setminus c_4$ is isomorphic to $M(\mathcal{W}_4)$. Thus the lemma holds for $r \leq 4$.

Now assume the lemma holds for Z_r and consider Z_{r+1} where $r \geq 4$. Suppose that Z_{r+1} has an $M(\mathcal{W}_4)$ -minor. Then, since $\text{rk } Z_{r+1} > \text{rk } M(\mathcal{W}_4)$, the Scum Theorem [6] implies that for some element z of Z_{r+1} , Z_{r+1}/z has an $M(\mathcal{W}_4)$ -minor. Now Z_{r+1}/a_{r+1} is a parallel extension of Z_r and therefore has no $M(\mathcal{W}_4)$ -minor. Hence, $z \neq a_{r+1}$. Therefore, by the preceding lemma, $z \notin \{a_1, a_2, \dots, a_{r+1}, b_1, b_2, \dots, b_{r+1}\}$. Thus $z = c_{r+1}$. It is routine to check that $Z_{r+1}/c_{r+1} \cong M(C_{r+1}^2)$ where C_{r+1}^2 is the graph obtained by replacing every element of an $(r+1)$ -circuit by two elements in parallel. Since $M(C_{r+1}^2)$ clearly has no $M(\mathcal{W}_4)$ -minor, we have a contradiction. The lemma now follows by induction. \square

The last lemma completes the proof that none of the matroids listed in (2.2) (i)–(iii) has a P_9 - or a P_9^* -minor. We now prove that if M has no such minor and is 3-connected and binary, then M is listed in (i)–(iii). Suppose that M is not regular. Then, by Theorem 1.4, M has a minor isomorphic to F_7 or F_7^* . Thus, by Theorem 1.3, there is a chain M_0, M_1, \dots, M_n of 3-connected matroids such that $M_0 \cong F_7$ or F_7^* , $M_n = M$, and for all i in $\{0, 1, 2, \dots, n-1\}$, M_i is a single-element deletion or contraction of M_{i+1} . For the rest of the proof of Theorem 2.2, we shall be concerned with the members of this chain. If $M \cong F_7$ or F_7^* , then M is listed under (iii). Thus assume that this does not occur. We shall first suppose that $M_0 \cong F_7^*$. Then, since F_7^* has no nontrivial binary lifts, M_1 is a nontrivial extension of F_7^* . Seymour [16]

has noted that F_7^* has precisely two nonisomorphic binary nontrivial extensions, these two matroids being represented by the matrices X_4 and Y_4 shown below. Evidently these two matroids are

$$X_4 = \left[I_4 \left| \begin{array}{ccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{array} \right. \right], \quad Y_4 = \left[I_4 \left| \begin{array}{ccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right. \right]$$

isomorphic to $Z_4 \setminus c_4$ and $Z_4 \setminus b_4$. Thus, if $M_0 \cong F_7^*$, then $M_1 \cong Z_4 \setminus c_4$ or $Z_4 \setminus b_4$. But both of the last two matroids are isomorphic to their duals and therefore, if $M_0 \cong F_7$, we again get that $M_1 \cong Z_4 \setminus c_4$ or $Z_4 \setminus b_4$. We shall treat the cases when $M_1 \cong Z_4 \setminus b_4$ and when $M_1 \cong Z_4 \setminus c_4$ separately. First suppose that $M_1 \cong Z_4 \setminus c_4$.

(2.5) LEMMA. *Every nontrivial binary extension of $Z_4 \setminus c_4$ is isomorphic to Z_4 .*

PROOF. Let N be a nontrivial binary extension of $Z_4 \setminus c_4$. Then N can be represented by the matrix obtained by adjoining the column $(x_1, x_2, x_3, x_4)^T$ to X_4 where each of x_1, x_2, x_3 and x_4 is in $\{0, 1\}$. Evidently exactly two or exactly four of x_1, x_2, x_3 and x_4 are equal to 1. Moreover, by the symmetry of X_4 , all six matroids that arise by adjoining a column with exactly two ones are isomorphic. In addition, each of these matroids is isomorphic to the matroid Z_4 which is obtained when $(x_1, x_2, x_3, x_4) = (1, 1, 1, 1)$. To see this, observe that by adding row 3 to each of rows 1 and 2 in the matrix A_4 , we get the following matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

On deleting the last column of this and suitably reordering the first eight columns, we get the matrix X_4 . \square

We now suppose that $M_1 \cong Z_4 \setminus b_4$. The next lemma is stronger than we need for the proof of Theorem 2.2. We shall use this additional strength in the proof of Theorem 2.1.

(2.6) LEMMA. *The only column that can be adjoined to the matrix Y_4 to give a representation of a 3-connected binary matroid with no $M(\mathcal{W}_4)$ -minor is $(1, 1, 1, 0)^T$. Thus, the unique extension of $Z_4 \setminus b_4$ in Θ is Z_4 . Moreover, every other nontrivial binary extension of $Z_4 \setminus b_4$ is isomorphic to P_9 .*

PROOF. Assume that the column $(x_1, x_2, x_3, x_4)^T$ is adjoined to Y_4 to give a matrix Y'_4 over GF(2) representing a 3-connected matroid N . If $(x_1, x_2, x_3, x_4) = (1, 1, 1, 0)$, then clearly $N \cong Z_4$. We may now assume that (x_1, x_2, x_3, x_4) has exactly two ones. Moreover, by the symmetry of Y_4 , we can suppose that (x_1, x_2, x_3, x_4) is one of $(1, 1, 0, 0)$ and $(1, 0, 0, 1)$. In each case, it is not difficult to check that the matrix obtained by deleting the sixth column from Y'_4 represents a matroid isomorphic to $M(\mathcal{W}_4)$ and that Y'_4 itself represents a matroid isomorphic to P_9 . \square

As an immediate consequence of the preceding lemma, we have the following corollary.

(2.7) COROLLARY. *Every nontrivial binary extension of Z_4 has a P_9 -minor and hence has an $M(\mathcal{W}_4)$ -minor. Hence, for all $r \geq 4$, every nontrivial binary extension of Z_r has a P_9 -minor. \square*

Now either (a) M_2 is an extension of M_1 , or (b) M_2 is a lift of M_1 . By Lemmas 2.5 and 2.6 and Corollary 2.7, in case (a), $M_2 \cong Z_4$ and M_3 is a lift of M_2 . By duality, in case (b), $M_2 \cong Z_4^*$ and M_3 is an extension of M_2 . In general, Z_r^* is represented by the matrix

$$A_r^* = \left[\begin{array}{cccccc|cccccc} b_1 & b_2 & \cdots & b_r & c_r & a_1 & a_2 & a_3 & \cdots & a_r \\ & & & & & 0 & 1 & 1 & \cdots & 1 \\ & & & & & 1 & 0 & 1 & \cdots & 1 \\ & & I_{r+1} & & & 1 & 1 & 0 & \cdots & 1 \\ & & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & & 1 & 1 & 1 & \cdots & 0 \\ & & & & & 1 & 1 & 1 & \cdots & 1 \end{array} \right]$$

The next lemma consider the nontrivial binary extensions of Z_r^* .

(2.8) LEMMA. *Suppose that $r \geq 4$ and that $x = (x_1, x_2, \dots, x_{r+1})^T$ is a column that is adjoined to A_r^* to give a matrix B over $\text{GF}(2)$ representing a 3-connected matroid N . Then either (i) N has a P_9 -minor, or (ii) $x_1 = x_2 = \cdots = x_r = 1$.*

PROOF. As $(x_1, x_2, \dots, x_{r+1})^T$ is distinct from every column of A_r^* , it has at least two ones. We may also assume it has at least two zeros otherwise (ii) holds. We now choose i_1, i_2, i_3 and i_4 to be distinct elements of $\{1, 2, \dots, r\}$ such that $x_{i_1} = 1$, $x_{i_2} = 0$ and $x_{i_3} = 1 + x_{r+1}$, the sum here being taken modulo 2. Then exactly two of $x_{i_1}, x_{i_2}, x_{i_3}$ and x_{r+1} are 1. To construct a matrix representing a P_9 -minor of N , we proceed as follows. First delete all the rows of B except i_1, i_2, i_3 and $r + 1$, and arrange the remaining rows in the order listed. Then delete all the columns of the resulting matrix except $b_{i_1}, b_{i_2}, b_{i_3}, c_r, a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}$ and x , and arrange the remaining columns in the order listed. The matrix obtained by this construction is Y_4 with one column adjoined, this extra column having two ones and two zeros. We conclude, using Lemma 2.6, that N has a P_9 -minor. \square

On adjoining $(1, 1, 1, \dots, 1, 0)^T$ to Z_r^* , we obtain a matrix representing $Z_{r+1} \setminus c_{r+1}$; on adjoining $(1, 1, 1, \dots, 1, 1)^T$ instead, or adjoining both these columns, we get representations for $Z_{r+1} \setminus b_{r+1}$ and Z_{r+1} , respectively. It follows from this, using a straightforward induction argument and Lemma 1.1, that for every $r \geq 4$, all of $Z_r, Z_r^*, Z_r \setminus b_r$ and $Z_r \setminus c_r$ are 3-connected. The following result is an easy consequence of these observations and the last lemma.

(2.9) COROLLARY. *For $r \geq 4$, Z_r^* has precisely two nonisomorphic extensions in Ψ , these being isomorphic to $Z_{r+1} \setminus b_{r+1}$ and $Z_{r+1} \setminus c_{r+1}$. Moreover, the unique extension of each of the last two matroids that is in Ψ is Z_{r+1} . \square*

The next result finishes the proof of Theorem 2.2.

(2.10) COROLLARY. *Let M_0, M_1, M_2, \dots be a sequence of binary 3-connected matroids each of which is an extension or lift of its predecessor. Suppose also that $M_0 \cong F_7$ or F_7^* and that none of M_1, M_2, \dots has a P_9 - or P_9^* -minor. Then, for all $j \geq 1$, $M_{2j-1} \cong Z_{j+3} \setminus b_{j+3}$ or $Z_{j+3} \setminus c_{j+3}$ and $M_{2j} \cong Z_{j+3}$ or Z_{j+3}^* .*

PROOF. We argue by induction on j . We noted earlier that $M_1 \cong Z_4 \setminus b_4$ or $Z_4 \setminus c_4$ and that $M_2 \cong Z_4$ or Z_4^* . Thus the result holds for $j = 1$. Assume it holds for $j < k$ and let $j = k$. Then $M_{2k-2} \cong Z_{k+2}$ or Z_{k+2}^* . In the latter case, by Corollaries 2.7 and 2.9, M_{2k-1} is an extension of M_{2k-2} and is isomorphic to $Z_{k+3} \setminus b_{k+3}$ or $Z_{k+3} \setminus c_{k+3}$. Since the last two matroids are isomorphic to their duals, if $M_{2k-2} \cong Z_{k+2}$, then again $M_{2k-1} \cong Z_{k+3} \setminus b_{k+3}$ or $Z_{k+3} \setminus c_{k+3}$. Now using Corollary 2.9 and duality, we get that $M_{2k} \cong Z_{k+3}$ or Z_{k+3}^* and the theorem follows by induction. \square

It is now straightforward to complete the proof of Theorem 2.1 and we omit the details.

3. A bound on the number of elements. Dirac [7] proved that, for all $n \geq 3$, a simple n -vertex graph with no subgraph homeomorphic from \mathcal{W}_3 has at most $2n - 3$ edges. As every binary matroid having no $M(\mathcal{W}_3)$ -minor is a series-parallel network [3, Theorem 7.6] and hence is graphic, Dirac's result gives that, for all $r \geq 2$, a simple binary matroid of rank r having no $M(\mathcal{W}_3)$ -minor has at most $2r - 1$ elements. A similar linear bound on the number of elements in a rank- r simple ternary matroid having no $M(\mathcal{W}_3)$ -minor was proved in [13, Theorem 5.1]. By making the obvious modifications to the proof of that result and using the main theorem of this paper, we get the next theorem. The details of the proof are omitted.

(3.1) THEOREM. *Let M be a simple binary matroid of rank r having no $M(\mathcal{W}_4)$ -minor. Then*

$$|E(M)| \leq \begin{cases} 3r - 2, & \text{if } r \text{ is odd,} \\ 3r - 3, & \text{if } r \text{ is even.} \end{cases}$$

Moreover, the following list includes all the matroids that attain this bound:

- (i) $r = 1$, $U_{1,1}$;
- (ii) $r = 2$, $U_{2,3}$;
- (iii) $r = 3$, F_7 ;
- (iv) $r = 4$, Z_4 , $P(F_7, U_{2,3})$;
- (v) $r = 2t + 1$ for $t \geq 2$, all matroids that can be formed from t copies of F_7 using $t - 1$ parallel connections;
- (vi) $r = 2t$ for $t \geq 3$, all matroids that can be formed from $t - 1$ copies of F_7 and one copy of $U_{2,3}$ using $t - 1$ parallel connections, and all matroids that can be formed from $t - 2$ copies of F_7 and one copy of Z_4 using $t - 2$ parallel connections. \square

We now consider extending this result. We shall use some additional terminology. For a class \mathcal{M} of matroids which contains a simple matroid of every nonzero rank, Kung [9] has defined the *size function* $h(\mathcal{M}, r)$ to be the function on \mathbf{Z}^+ given by

$$h(\mathcal{M}, r) = \max\{|E(M)| : M \in \mathcal{M}, \text{rk } M = r \text{ and } M \text{ is simple}\}.$$

If N_1, N_2, \dots, N_m are matroids, then $\text{EX}(N_1, N_2, \dots, N_m)$ will denote the class of *binary* matroids having no minor isomorphic to any of N_1, N_2, \dots, N_m . The following is an immediate consequence of the last theorem.

(3.2) COROLLARY.

$$h(\text{EX}(M(\mathcal{W}_4)), r) = \begin{cases} 3r - 2, & \text{if } r \text{ is odd,} \\ 3r - 3, & \text{if } r \text{ is even.} \end{cases} \quad \square$$

The next result extends this corollary.

(3.3) THEOREM.

$$h(\text{EX}(M(K_5), P_9, P_9^*), r) = \begin{cases} 3r - 2, & \text{if } r \text{ is odd.} \\ 3r - 3, & \text{if } r \text{ is even.} \end{cases}$$

To prove this theorem we shall use both Corollary 3.2 and the next theorem. The latter is one of the several results of Kung [9, 10] on the size functions of various classes of binary matroids.

(3.4) THEOREM [9, THEOREM 9.1].

$$h(\text{EX}(M(K_5), F_7^*), r) = \begin{cases} 3r - 2, & \text{if } r \text{ is odd.} \\ 3r - 3, & \text{if } r \text{ is even.} \end{cases} \quad \square$$

PROOF OF THEOREM 3.3. By [9, Lemma 10.1], since $\text{EX}(M(\mathcal{W}_4))$ and $\text{EX}(M(K_5), F_7^*)$ have the same size function, $\text{EX}(M(\mathcal{W}_4)) \cup \text{EX}(M(K_5), F_7^*)$ also has this size function. To establish the theorem, we shall first show that

$$(3.5) \quad \text{EX}(M(\mathcal{W}_4)) \cup \text{EX}(M(K_5), F_7^*) \subseteq \text{EX}(M(K_5), P_9, P_9^*).$$

Since the class $\text{EX}(M(\mathcal{W}_4)) \cup \text{EX}(M(K_5), F_7^*)$ is closed under minors, it can certainly be characterized by excluded minors. It is straightforward to check that $M(K_5)$ is a minimal such excluded minor. The following lemma completes the proof of (3.5). We omit its routine proof.

(3.6) LEMMA. *Both P_9 and P_9^* are minor-minimal matroids not in $\text{EX}(M(\mathcal{W}_4)) \cup \text{EX}(M(K_5), F_7^*)$.* \square

The next lemma completes the proof of Theorem 3.3.

(3.7) LEMMA. *Let M be a simple matroid that is in $\text{EX}(M(K_5), P_9, P_9^*)$ but is not in $\text{EX}(M(\mathcal{W}_4)) \cup \text{EX}(M(K_5), F_7^*)$. Then*

$$|E(M)| \leq \begin{cases} 3 \text{ rk } M - 2, & \text{if } \text{rk } M \text{ is odd,} \\ 3 \text{ rk } M - 3, & \text{if } \text{rk } M \text{ is even.} \end{cases}$$

PROOF. Assume that the lemma is not true and let N be a minor-minimal counterexample. It is straightforward to show, using Theorem 1.2, that N must be 3-connected. Moreover, since $N \notin \text{EX}(M(K_5), F_7^*)$ but $N \in \text{EX}(M(K_5), P_9, P_9^*)$, N has an F_7^* -minor. Thus, by Theorem 1.3, there is a chain $N_0, N_1, N_2, \dots, N_n$ of 3-connected matroids such that $N_0 \cong F_7^*$, $N_n = N$ and, for each i in $\{0, 1, 2, \dots, n-1\}$, N_i is a single-element deletion or contraction of N_{i+1} . By Theorem 3.1, $N = N_n \notin \text{EX}(M(\mathcal{W}_4))$. But $N_0 \in \text{EX}(M(\mathcal{W}_4))$. Let

$$m = \min\{i: N_i \notin \text{EX}(M(\mathcal{W}_4))\}.$$

Then $N_{m-1} \in \text{EX}(M(\mathcal{H}_4))$ and, by Theorem 2.1, N_{m-1} is isomorphic to $Z_r, Z_r^*, Z_r \setminus b_r$ or $Z_r \setminus c_r$ for some $r \geq 4$.

Now N_m is an extension or lift of N_{m-1} . Since $Z_r \setminus b_r$ and $Z_r \setminus c_r$ are both isomorphic to their duals and $Z_r^* \cong Z_{r+1} \setminus b_{r+1}, c_{r+1}$, one of N_m and N_m^* is isomorphic to an extension of N' where N' is isomorphic to $Z_r, Z_r \setminus b_r$ or $Z_r \setminus c_r$ for some $r \geq 4$, or to $Z_r \setminus b_r, c_r$ for some $r \geq 5$. But, by Lemmas 2.5, 2.6 and 2.8 and Corollary 2.7, every nontrivial binary extension of each of the possibilities for N' has a P_9 -minor or is in $\text{EX}(M(\mathcal{H}_4))$. Thus one of N_m and N_m^* has a P_9 -minor. It follows that N has a P_9 - or a P_9^* -minor; a contradiction. \square

To conclude this section, we determine the size function of $\text{EX}(P_9, P_9^*)$ noting that, unlike $h(\text{EX}(M(K_5), P_9, P_9^*), r)$, it is quadratic in r .

(3.8) THEOREM.

$$h(\text{EX}(P_9, P_9^*), r) = \begin{cases} \binom{r+1}{2}, & \text{if } r \neq 3, \\ \binom{r+1}{2} + 1, & \text{if } r = 3. \end{cases}$$

This result extends the following theorem of Murty [11] (see also [1, 2]) which itself is an extension of Heller's result [8] that $h(\text{EX}(F_7, F_7^*), r) = \binom{r+1}{2}$.

(3.9) THEOREM. $h(\text{EX}(F_7), r) = \binom{r+1}{2}$. \square

On combining Heller's result with Theorem 3.8, we get the size function for $\text{EX}(F_7^*)$. This can also be deduced from a result of Walton [19].

(3.10) COROLLARY.

$$h(\text{EX}(F_7^*), r) = \begin{cases} \binom{r+1}{2}, & \text{if } r \neq 3, \\ \binom{r+1}{2} + 1, & \text{if } r = 3. \end{cases} \quad \square$$

PROOF OF THEOREM 3.8. We argue by induction on r . As $PG(2, 2) \cong F_7 \in \text{EX}(P_9, P_9^*)$, it follows easily that $h(\text{EX}(P_9, P_9^*), 3) = 7 = \binom{3+1}{2} + 1$. To see that $h(\text{EX}(P_9, P_9^*), r) \geq \binom{r+1}{2}$ for $r \neq 3$, we note that $M(K_{r+1})$ is a rank- r member of $\text{EX}(P_9, P_9^*)$. Evidently, for $r = 1$ or 2 , $h(\text{EX}(P_9, P_9^*), r) = \binom{r+1}{2}$. Hence, for $r \leq 3$, $h(\text{EX}(P_9, P_9^*), r)$ is as stated in the theorem. Now suppose that M is a simple rank- r member of $\text{EX}(P_9, P_9^*)$ having $h(\text{EX}(P_9, P_9^*), r)$ elements and assume that $r \geq 4$. If $M \in \text{EX}(F_7)$, then, by Theorem 3.9, $h(\text{EX}(P_9, P_9^*), r) = \binom{r+1}{2}$. Thus we can suppose that M has an F_7 -minor. Since $|E(M)| = h(\text{EX}(P_9, P_9^*), r) \geq \binom{r+1}{2}$, it follows from Theorem 2.2 that M is not 3-connected. Then using Theorem 1.2 and the choice of M , we can deduce that $M = P(N_1, N_2)$ where both N_1 and N_2 are simple members of $\text{EX}(P_9, P_9^*)$ having rank at least two. It is now straightforward to apply the induction assumption to N_1 and N_2 to obtain the contradiction that $|E(M)| \leq \binom{r+1}{2} - 1$. \square

4. The critical problem. A loopless binary matroid with no $M(\mathcal{W}_3)$ -minor has critical exponent at most two [3]. In this section we shall show that a loopless binary matroid with no $M(\mathcal{W}_4)$ -minor has critical exponent at most three and explicitly determine the critical exponents of all loopless binary 3-connected matroids with no $M(\mathcal{W}_4)$ -minor. We shall use the following result.

(4.1) LEMMA [4, THEOREM 10.3]. *Let M be a matroid that is represented by the matrix $[I_r | A]$ over $GF(2)$. Then the following statements are equivalent:*

- (i) $c(M; 2) = 1$.
- (ii) M has no odd circuits.
- (iii) Every column of A contains an odd number of nonzero entries. \square

(4.2) THEOREM. *Let M be a loopless binary 3-connected matroid having no $M(\mathcal{W}_4)$ -minor. Then $c(M; 2) = 2$ unless*

- (i) M has rank one, or r is an even integer exceeding 3 and $M \cong Z_r \setminus c_r$ or Z_{r-1}^* ; or
- (ii) r is an odd integer exceeding 2 and $M \cong Z_r$.

In cases (i) and (ii), $c(M; 2) = 1$ and 3, respectively.

PROOF. If $|E(M)| \leq 4$, then, by Theorem 2.1, $M \cong U_{1,1}, U_{1,2}, U_{1,3}$ or $U_{2,3}$ and it is easy to check that $c(M; 2)$ is as claimed. Now suppose that $|E(M)| > 4$. Then, for some $r \geq 3$, M is isomorphic to one of $Z_r, Z_r^*, Z_r \setminus b_r$ or $Z_r \setminus c_r$. We shall treat the cases when r is even and when r is odd separately. First suppose that r is even. The matroid $Z_r \setminus c_r$ is represented by the matrix A'_r obtained from A_r by deleting the last column. Every column of A'_r has an odd number of nonzero entries and so, by Lemma 4.1, $c(Z_r \setminus c_r; 2) = 1$. It follows that $c(Z_r; 2) \leq 2$ and hence $c(Z_r \setminus b_r; 2) \leq 2$. Moreover, since $Z_{r-1}^* \cong Z_r \setminus c_r, b_r$, $c(Z_{r-1}^*; 2) = 1$. Now consider $Z_r \setminus b_r$. It has an odd circuit, namely $\{a_1, a_2, \dots, a_r, c_r\}$. Therefore, by Lemma 4.1, $c(Z_r \setminus b_r; 2) \geq 2$ and hence $c(Z_r; 2) \geq 2$. We conclude that

$$c(Z_r \setminus b_r; 2) = c(Z_r; 2) = 2.$$

Next we suppose that r is odd. Then Z_{r-1}^* has an odd circuit, namely $\{a_1, b_2, b_3, \dots, b_{r-1}, c_{r-1}\}$. Therefore $c(Z_{r-1}^*; 2) \geq 2$. Moreover, as both $Z_r \setminus b_r$ and $Z_r \setminus c_r$ are extensions of Z_{r-1}^* , both $c(Z_r \setminus b_r; 2)$ and $c(Z_r \setminus c_r; 2)$ are bounded below by 2. Now consider the representations for $Z_r \setminus b_r$ and $Z_r \setminus c_r$ obtained from the matrix A_r by deleting the second last and last columns, respectively. Let H_1, H_2 and H_3 be the hyperplanes of $PG(r-1, 2)$ defined by the equations $x_1 + x_2 + \dots + x_{r-1} = 0$, $x_1 + x_2 + x_r = 0$ and $x_1 + x_r = 0$. Then $H_1 \cap H_2 \cap E(Z_r \setminus b_r) = \emptyset$ and $H_1 \cap H_3 \cap E(Z_r \setminus c_r) = \emptyset$. Thus both $c(Z_r \setminus b_r; 2)$ and $c(Z_r \setminus c_r; 2)$ are bounded above by 2, so both quantities equal 2. Moreover, $c(Z_{r-1}^*; 2) = 2$ and $c(Z_r; 2) \leq 3$. The next result completes the proof of the theorem.

(4.3) LEMMA. *If r is odd and exceeds 2, then $c(Z_r; 2) > 2$.*

PROOF. Consider the matrix A_r representing Z_r . We shall show that $PG(r-1, 2)$ does not have two hyperplanes H and H' such that $H \cap H' \cap E(Z_r) = \emptyset$. Assume that two such hyperplanes do exist. Each hyperplane is defined by an equation of the form $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_r x_r = 0$ where each of $\alpha_1, \alpha_2, \dots, \alpha_r$ is 0

or 1 with at least one α_i being nonzero. Now we may assume that H avoids c_r . Then the equation defining H has an odd number of nonzero coefficients, so we may suppose, without loss of generality, that this equation is $x_1 + x_2 + \cdots + x_{2k+1} = 0$ for some $k \geq 0$. The elements of $E(Z_r)$ which are contained in H include $b_1, b_2, \dots, b_{2k+1}, a_{2k+2}, a_{2k+3}, \dots, a_r$ and these elements form an odd circuit in Z_r . Therefore, by Lemma 4.1, H' cannot avoid all these elements and we have a contradiction. \square

The last theorem can be used to show that every loopless binary matroid having no $M(\mathcal{W}_4)$ -minor has critical exponent at most 3. The proof here follows the same lines as the proof of the corresponding result for loopless ternary matroids with no $M(\mathcal{W}_3)$ -minor [13, Corollary 3.3]. We omit it since we can obtain a stronger result by combining Theorem 3.3 with the following theorem of Kung.

(4.4) THEOREM [9, LEMMA 3.1]. *Let \mathcal{M} be a class of simple matroids representable over $GF(q)$ that is closed under restriction. Assume that there is an integer k so that for all $r \geq 1$, every rank- r member of \mathcal{M} has at most kr elements. Then, for every member M of \mathcal{M} , $c(M; q) \leq k$. \square*

(4.5) COROLLARY. *Let M be a loopless binary matroid having no minor isomorphic to any of $M(K_5)$, P_9 or P_9^* . Then $c(M; 2) \leq 3$. \square*

Since both $\text{EX}(M(K_5), F_7)$ and $\text{EX}(M(K_5), F_7^*)$ are contained in $\text{EX}(M(K_5), P_9, P_9^*)$, a consequence of the last corollary is the following result of Walton and Welsh [20, Theorem 2, (a) and (b)].

(4.6) COROLLARY. *Let M be a loopless binary matroid. If M is in $\text{EX}(M(K_5), F_7)$ or $\text{EX}(M(K_5), F_7^*)$, then $c(M; 2) \leq 3$. \square*

Corollary 4.5 provides a partial answer to Walton and Welsh's question [20, p. 5] as to whether $c(M; 2) \leq 3$ for all loopless binary matroids having no $M(K_5)$ -minor. Kung [10] has given another partial answer to this question by showing that for all such M , $c(M; 2) \leq 8$.

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