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THE ETALE COHOMOLOGY OF p-TORSION SHEAVES. I

WILLIAM ANTHONY HAWKINS, JR.

ABSTRACT. This paper generalizes a formula of Grothendieck, Ogg, and Shafarevich that expresses the Euler-Poincaré characteristic of a constructible sheaf of F_l-modules on a smooth, proper curve, over an algebraically closed field k of characteristic $p > 0$, as a sum of local and global terms, where $l \neq p$. The primary focus is on removing the restriction on l. We begin with calculations for p-torsion sheaves trivialized by p-extensions, but using etale cohomology to give a unified proof for all primes l.

In the remainder of this work, only p-torsion sheaves are considered. We show the existence on X_{et}, X a scheme of characteristic p, of a short exact sequence of sheaves, involving the tangent space at the identity of a finite, flat, height 1, commutative group scheme, and the subsheaf fixed by the pth power endomorphism; the latter turns out to be an etale group scheme. A corollary gives complete results on the Euler-Poincaré characteristic of a constructible sheaf of F_p-modules on a smooth, proper curve, over an algebraically closed field k of characteristic $p > 0$, when the generic stalk has rank p.

Explicit computations are given for the Euler characteristics of such p-torsion sheaves on P^1 and a result on elliptic surfaces is included. A study is made of the comparison of the p-ranks of abelian extensions of curves. Several examples of p-ranks for nonhyperelliptic curves are discussed. The paper concludes with a brief sketch of results on certain constructible sheaves of F_q-modules, $q = p^r$, $r \geqslant 1$.

Introduction. This paper generalizes a formula of Grothendieck, Ogg, and Shafarevich [**15**] that expresses the Euler-Poincaré characteristic of a constructible sheaf of F_l-modules on a smooth, proper curve, over an algebraically closed field k of characteristic $p > 0$, as a sum of local terms and a global term, where $l \neq p$. The primary focus is on removing the restriction on l. The previously known results were limited largely to calculations for p-torsion sheaves trivialized by p-extensions. We begin §I with a result similar to these, namely Theorem 1.1, but using etale cohomology to give a unified proof for all primes l. The other results are corollaries to this theorem.

In the remainder of this work, only p-torsion sheaves are considered. The main theoretical results occur in §II. They are Theorems 2.1 and 2.7. The latter gives complete results on the Euler-Poincaré characteristic of a constructible sheaf of F_p-modules on a smooth, proper curve, over an algebraically closed field k of characteristic $p > 0$, when the generic stalk has rank p. Theorem 2.1 shows the existence of a certain short exact sequence of sheaves on X_{et}, X a scheme of characteristic p, making possible the proof of 2.7 in the presence of Lemma 2.5. The

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sequence involves the tangent space at the identity of a finite, flat, height 1, commutative group scheme and the subsheaf fixed by the pth power endomorphism; the latter turns out to be an etale group scheme. We conjecture the form of a general result for all constructible sheaves of F_p-modules over integral schemes in 2.4.

§III contains explicit computations of Euler characteristics using 2.7 for the case of sheaves over P^1. The key fact is Theorem 3.1, which will be important for later calculations involving the p-ranks of curves over P^1 in §IV. A result on elliptic surfaces is included here.

The topic of §IV is the comparison of the p-ranks of abelian extensions of curves. The direct image of the constant sheaf F_p for certain cyclic extensions of curves is determined in Theorem 4.1 in terms of known sheaves, whose cohomology can be computed using Corollary 3.2. A result of Manin is recovered in Corollary 4.3. Corollary 4.4 is a slight generalization of 4.1 to certain abelian extensions of curves. Examples of p-ranks for several curves over P^1 complete this section.

In the Appendix, one finds a brief sketch of results on constructible sheaves of F_q-modules, $q = p^r$, $r \geqslant 1$. The significance of F_q-cohomology is brought out by Corollary AI.2. The modified versions of Theorems 3.1 and 4.1 are given, but the calculations of p-ranks depend more on the new forms of Corollaries 3.2 and 4.2. This approach provides another generalization of the original Theorem 4.1. Any cyclic extension of curves can be handled by one of these two methods.

Notation and conventions. All rings are commutative, Noetherian with 1; all schemes are locally Noetherian. If R is a ring, then R^* will denote its group of units. A variety is an integral, separated scheme of finite type over a field k. A curve is a variety of dimension 1. A proper variety over k is also called complete. For a field k, we denote the separable closure of k by k_s. If K/k is Galois, then $\mathrm{Gal}(K/k)$ denotes the Galois group. The finite field of l elements is denoted F_l, l any prime. For an integral scheme X, we write $k(X)$ for the field of rational functions of X. The set of points of dimension 0, i.e., the closed points, is written X^0. The etale (Zariski) site $X_{\mathrm{et}}(X_{\mathrm{Zar}})$ means the small etale (Zariski) site on the scheme X. The Cartier dual of a group scheme G is denoted G^D.

References will be enclosed in brackets [ ], except for internal cross-references. In general, references will be numbered to coincide with the corresponding bibliographic item. The symbol // will indicate the end of a proof.

I. l-torsion sheaves and l-extensions. This section concerns the Euler characteristic, on a complete smooth curve X, of a constructible sheaf F of F_l-modules, l any prime, trivialized by an l-extension. Some of the results are not new (see the Remark following Corollary 1.4) but the main theorem (1.1) gives a unified statement and proof using etale cohomology. This theorem contains a global term involving the F_l-cohomology of X and the F_l-dimension of the generic stalk of F; there are local terms involving the F_l-dimension of the stalks of F at all closed points of X as well as its generic stalk. As a corollary, we compute the F_l-cohomology of a certain curve lying over X, with the local terms now relating to ramification. The other corollaries give reinterpretations of the foregoing in terms of algebraic function fields in one variable and Artin-Schreier extensions.

Let X be a smooth curve over an algebraically closed field k of characteristic $p > 0$. Let F be a constructible sheaf of F_l-modules on X_{et}, l a prime not necessarily distinct from p. Let $K = k(X)$ and $g\colon \eta = \operatorname{Spec} K \to X$ be the inclusion of the generic point. Let K_s be the separable closure of K. $F_{\bar{\eta}}$ is a $\Gamma_s = \operatorname{Gal}(K_s/K)$-module. Since $F_{\bar{\eta}}$ is finite, the action factors through a finite quotient $\Gamma = \Gamma_s/H_s$. Let $\chi(X, F)$ denote the sum $\Sigma(-1)^i \dim H^i(X_{\text{et}}, F)$ ($\dim = \dim_{F_l}$). When X is complete and $l \neq p$, this number is finite by [**5**, V.2.1]; for complete X with $l = p$, it is finite by [**5**, VI.2.8].

If X is not complete, it is affine, say $X = \operatorname{Spec} R$, by [**4**, Exercise IV.1.4]. The case $l \neq p$ is uninteresting. Although $\chi(X, F)$ is finite by the argument of [**5**, V.2.4(a)], the vanishing of $H^2(X, F_l)$ shows how much this deviates from the usual result for complete, smooth curves. The case $l = p$ is pathological. Here $H^1(X, F_p) = R/\mathbf{p}R$ where $\mathbf{p}\colon R \to R$ is the map $x \to x^p - x$. Since R has transcendence degree 1 over k, we have $k[T] \subseteq R$. The image of $k[T]/\mathbf{p}k[T]$ in $R/\mathbf{p}R$ is infinite, which shows $H^1(X, F_p)$ does not have finite dimension over F_p.

THEOREM 1.1. *Let Γ be an l-group. Then*

$$\chi(X, F) = \chi(X, F_l)(\dim F_{\bar{\eta}}) - \sum_{x \in X^0} (\dim F_{\bar{\eta}} - \dim F_{\bar{x}}),$$

for X complete.

The proof will require several steps.

Step 1. Let Y be the normalization of X in $L = (K_s)^{H_s}$. We do not require that Γ be an l-group or that X be complete. Let C be the category of all constructible sheaves F on X_{et} satisfying

(i) $F \cong g_* g^* F$ and

(ii) $F_{|Y}$ is constant on an open set.

Let D be the category of finite Γ-modules. Then the functor from C to D that sends F to $F_{\bar{\eta}}$ is a category equivalence.

PROOF. Let E be the category of sheaves on η_{et}. We know that E is equivalent to the category of $\Gamma_s = \operatorname{Gal}(K_s/K)$-modules by [**5**, II.1.9], via the functor sending F_0 to $(F_0)_{\bar{\eta}}$. It is clear that E is also equivalent to the category of sheaves F on X_{et} satisfying (i). Let $\pi\colon Y \to X$ be the finite morphism corresponding to L/K. To complete the proof, we need only show that the action of Γ_s factors through Γ if and only if π^*F becomes constant on an open subset of Y, i.e., that condition (ii) holds.

If π^*F becomes constant on an open subset of Y, then it follows immediately that the action factors. Conversely, let V be an open subset of Y such that $\pi_{|V}$ is etale. We can find such a V as follows. Choose an open subset U' of Y disjoint from the support of $\Omega_{Y/X}$ and let $Z' = Y - U'$. Then the map $\pi\colon Z' \to Z = \pi(Z')$ is finite and so Z is closed. Let $U = X - Z$ and $j\colon U \to X$ be the inclusion. Then $V = \pi^{-1}(U) = U \times_X Y$ is etale over U by [**5**, I.3.21], since $V \subseteq U'$. In addition, V is finite over U by base change [5, I.1.3]. Since $k(V) = L$ and $k(U) = K$, we see that V is Galois over U with group Γ. Let $W \to Y$ be etale and factor through V, where we assume W is connected. Then

$$\pi^*F(W) \cong F(W) \cong g_*g^*F(W) \cong g^*F(\operatorname{Spec} k(W)) \cong F_{\bar{\eta}},$$

which shows that π^*F is constant. This completes the proof of Step 1. //

For the remainder of the proof, we assume X is complete.

Step 2. Let $j\colon U \to X$ be the inclusion of an open subscheme. Then

$$\chi(X, F) = \chi(X, j_!j^*F) + \sum_{x \in X-U} \dim F_{\bar{x}}.$$

PROOF. By [5, II.3.13], there is an exact sequence

$$0 \to j_!j^*F \to F \to i_*i^*F \to 0$$

of sheaves on X_{et}, where $i\colon Z = X - U \to X$ is the inclusion. Since F is constructible, both $j_!j^*F$ and i_*i^*F are constructible. So $\chi(X, F)$, $\chi(X, i_*i^*F)$, and $\chi(X, j_!j^*F)$ are all defined and finite. Thus,

$$\chi(X, F) = \chi(X, j_!j^*F) + \chi(X, i_*i^*F).$$

Z is a finite set and so

$$H^s(X, i_*i^*F) \cong H^s(Z, i^*F) \cong H^s\Big(\coprod_{x \in Z} x, i^*F\Big) \cong \prod_{x \in Z} H^s(x, i^*F) = 0$$

for $s > 0$. We know $H^0(x, i^*F) = F_{\bar{x}}$. Hence,

$$\chi(X, F) = \chi(X, j_!j^*F) + \sum_{x \in X-U} \dim F_{\bar{x}}. \quad //$$

REMARK. Let F_1 and F_2 be two sheaves on X_{et} which satisfy $j^*F_1 \cong j^*F_2$ for some open subscheme U of X. Then

$$\chi(X, F_1) - \sum_{x \in X-U} \dim(F_1)_{\bar{x}} = \chi(X, j_!j^*F_1) = \chi(X, j_!j^*F_2)$$

$$= \chi(X, F_2) - \sum_{x \in X-U} \dim(F_2)_{\bar{x}}.$$

Step 3. Let $j\colon U \to X$ be the inclusion of an open subscheme such that $\pi\colon V = \pi^{-1}(U) \to U$ is etale. Let F be a sheaf on U_{et} with $F_{|V}$ constant and $F \cong g_*g^*F$ for $g\colon \eta \to U$ the inclusion of the generic point. Assume Γ is an l-group. Then

$$\chi(X, j_!F) = \chi(X, j_!F_l)(\dim F_{\bar{\eta}}).$$

PROOF. The Γ-module $F_{\bar{\eta}}$ has a composition series, all of whose quotients are isomorphic to F_l with Γ acting trivially, by [8, p. 139].

We can write

$$F_{\bar{\eta}} = M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} = 0$$

such that $M_i/M_{i+1} \cong F_l$ with trivial Γ-action. As in Step 1, we can find sheaves F_i on U_{et} satisfying $(F_i)_{\bar{\eta}} \cong M_i$. We get a short exact sequence

$$(*) \qquad 0 \to M_{i+1} \to M_i \to F_l \to 0.$$

Applying g_* to the corresponding sequence of sheaves on η_{et}, we have a sequence of sheaves

$$(**)\qquad 0 \to F_{i+1} \to F_i \to F_l \to 0$$

on U_{et}. We show it is exact. Looking at stalks and using $F_i \cong g_*g^*F_i$, we get a sequence of abelian groups

$$0 \to (F_{i+1})_{\bar\eta}^{I_x} \to (F_i)_{\bar\eta}^{I_x} \to (F_l)_{\bar\eta}^{I_x} \to 0,$$

where I_x is the decomposition subgroup of Γ_s corresponding to an embedding $O_{\bar x} \to K_s$. Since the Γ_s-action factors through Γ, we may replace I_x by $\Gamma_{x'}$, the decomposition subgroup of Γ at some closed point x' mapping to x. The hypothesis that $V \to U$ is etale implies $\Gamma_{x'} \subseteq H_s$ (so $\Gamma_{x'}$ acts trivially on $F_{\bar\eta}$) and the sequence becomes

$$0 \to (F_{i+1})_{\bar\eta} \to (F_i)_{\bar\eta} \to (F_l)_{\bar\eta} \to 0,$$

which is just sequence $(*)$. We conclude that the desired sequence $(**)$ of sheaves on U_{et} is exact. Since $j_!$ is an exact functor, applying it to $(**)$ gives an exact sequence

$$(***)\qquad 0 \to j_!F_{i+1} \to j_!F_i \to j_!F_l \to 0$$

of sheaves on X_{et}. The exactness of the sequences $(*)$ and $(***)$ shows that $\dim(M_i) = \dim(M_{i+1}) + 1$, $\chi(X, j_!F_i) = \chi(X, j_!F_{i+1}) + \chi(X, j_!F_l)$, and (by induction) $\dim F_{\bar\eta} = n + 1$, $\chi(X, j_!F) = \chi(X, j_!F_l)(n + 1)$. Thus,

$$\chi(X, j_!F) = \chi(X, j_!F_l)(\dim F_{\bar\eta}). \quad //$$

REMARK. In particular, when F is a sheaf on X_{et},

$$\chi(X, j_!j^*F) = \chi(X, j_!j^*F_l)(\dim F_{\bar\eta});$$

this follows from $(j^*F)_{\bar\eta} = F_{\bar\eta}$ and $j^*F_l = F_l$.

Step 4. Assume Γ is an l-group. If F is a constructible sheaf of F_l-modules on X_{et}, then

$$\chi(X, F) = \chi(X, F_l)(\dim F_{\bar\eta}) - \sum_{x \in X^0} (\dim F_{\bar\eta} - \dim F_{\bar x}).$$

PROOF. Choose an open subscheme U of X such that $\pi\colon V = \pi^{-1}(U) \to U$ is etale and $F_{|U} = (g_*g^*F)_{|U}$. Let $j\colon U \to X$ be the inclusion. We know from Step 2 that we must compute $\chi(X, j_!j^*F)$; from Step 3, this is

$$\chi(X, j_!F_l)(\dim F_{\bar\eta}).$$

Again from Step 2, we have

$$\chi(X, F_l) = \chi(X, j_!j^*F_l) + \sum_{x \in X-U} \dim(F_l)_{\bar x}.$$

Since $j^*F_l = F_l$, this shows that

$$\chi(X, j_!F_l) = \chi(X, F_l) - \sum_{x \in X-U} \dim(F_l)_{\bar x}.$$

Substituting and using Steps 2 and 3, we get

$$\begin{aligned}\chi(X,F) &= \chi(X, j_! j^* F) + \sum_{x\in X-U} \dim F_{\bar{x}}\\ &= \chi(X, j_! F_l)(\dim F_{\bar{\eta}}) + \sum_{x\in X-U} \dim F_{\bar{x}}\\ &= \chi(X, F_l)(\dim F_{\bar{\eta}}) - \sum_{x\in X-U} (\dim F_{\bar{\eta}} - \dim F_{\bar{x}}).\end{aligned}$$

Since $\dim F_{\bar{x}} = \dim F_{\bar{\eta}}$ for all $x \in U$, we have $\sum_{x\in X-U}(\dim F_{\bar{\eta}} - \dim F_{\bar{x}}) = \sum_{x\in X^0}(\dim F_{\bar{\eta}} - \dim F_{\bar{x}})$. This completes the proof of Step 4 and Theorem 1.1. //

COROLLARY 1.2. *Let Γ have order l^n and Y be the normalization of X in L. Let π: $Y \to X$ be the corresponding finite morphism. For $i = 1,\dots,n$, denote by t_i the number of points where the ramification index is l^i. Then*

$$\chi(Y, F_l) = l^n \cdot \chi(X, F_l) - \sum_{i=1}^{n} t_i(l^n - l^{n-i}).$$

PROOF. Let F_l be the constant sheaf on Y and $F = \pi_* F_l$. Using [5, II.3.5(c)], we compute $F_{\bar{x}}$. If $x \in X^0$, then $F_{\bar{x}} = \prod_{x'\to x}(F_l)_{\bar{x}'}$. If $x = \eta$, then $F_{\bar{\eta}} = (F_l)_{\bar{\eta}'}^{l^n}$, where $\eta' = \operatorname{Spec} L$. Moreover,

$$l^n = [L:K]_s = \sum_{x'\to x} e(x')[k(x'):k(x)]_s = e_x[k(x'):k(x)]_s(\operatorname{order} Y_x),$$

where e_x is the constant value of the ramification index $e(x')$ for all x' mapping to x.

We have $\dim F_{\bar{\eta}} = l^n \cdot \dim F_l = l^n$. For $x \in X^0$, $\dim F_{\bar{x}} = l^{n-i} \cdot \dim F_l = l^{n-i}$, when $e_x = l^i$ for $i = 1,\dots,n$. Note that t_n corresponds to the points of total ramification. Since $\chi(X, F) = \chi(X, \pi_* F_l) = \chi(Y, F_l)$, the result now follows. //

REMARK. (1) When $l \neq$ characteristic k, Theorem 1.1 is a special case of [5, V.2.12].

(2) Let l be any prime. For a smooth curve X over k, the l-rank of X is the dimension over F_l of the points of $J(k)$ of order l, where J is the Jacobian of X [6, p. 64]. If K is a field of algebraic functions in one variable over k, then the divisor (ideal, null) class group of K is isomorphic to the points of $J(k)$, J the Jacobian of the complete smooth curve X over k corresponding to K.

Let σ_X, σ_Y denote the p-ranks of the curves X, Y respectively. For $l =$ characteristic $k = p$, Corollary 1.2 shows

$$1 - \sigma_Y = p^n(1 - \sigma_X) - \sum_{i=1}^{n} t_i(p^n - p^{n-i}).$$

COROLLARY 1.3. *Let K be a field of algebraic functions in one variable over an algebraically closed field k of characteristic $p > 0$. Let L/K be a cyclic extension of degree l, l a prime not necessarily different from p. Let ξ, ρ denote, respectively, the l-ranks of the null class groups of L and K. Then*

$$\xi = l\rho + (t - \delta)(l - 1),$$

where t is the number of ramified primes and δ is 1 or 2 according as l equals p or not.

PROOF. Let X, Y be the complete smooth curves corresponding to the fields K, L respectively. From Kummer and Artin-Schreier theory (see [**5**, pp. 126, 128]), it is known that $\xi = \dim H^1(Y, F_l)$ and $\rho = \dim H^1(X, F_l)$. Also, see [**16**] for the case $l = p$. Writing the formula as

$$\delta - \xi = l(\delta - \rho) - t(l - 1),$$

we see that the result follows easily from the theorem or Corollary 1.2. The values of δ arise from the fact that $\dim H^2(Y, F_l) = \dim H^2(X, F_l) = 0$ or 1 according as l equals p or not. //

COROLLARY 1.4. *Let X/k be a complete smooth curve with function field K, where k is algebraically closed of characteristic $p > 0$. Let $Y \to X$ be the covering corresponding to the Artin-Schreier extension L of K with equation $y^p - y = f$, where $f \in K^*$. Let $S \subseteq X(k)$ be the set of points which ramify in Y. Then the p-ranks σ_X and σ_Y are related by the formula*

$$(\sigma_Y - 1 + \#S) = p(\sigma_X - 1 + \#S).$$

PROOF. This is a special case of Corollary 1.2, with $n = 1$. The group Γ has order p, and $t_1 = \#S$. (The formula is written explicitly in the Remark immediately following Corollary 1.2.) //

REMARK. Corollary 1.3 is due to Madan [**13**], who proved it using Galois cohomology. Corollary 1.4 is due to Subrao [**17**], who used the Cartier operator. Each of these authors proves a result like Corollary 1.2.

II. p-torsion sheaves. The etale core of A is determined by the exact sequence of Theorem 2.1 which involves the tangent space at the identity of a finite, flat, height 1, commutative group scheme A, and the subsheaf fixed by the pth power endomorphism. This subsheaf is the etale core, which is any etale group scheme. The F_p-dimensions of the cohomology groups of a complete projective variety X, over an algebraically closed field k of characteristic $p > 0$, with coefficients in an etale core, are computed in Corollary 2.3. The idea will be to use the cohomology with appropriate etale core coefficients to help calculate the cohomology with coefficients in a given constructible sheaf. In 2.4, we conjecture a result as to the conditions under which such an etale core can be found.

The rest of this section is basically a proof of the conjecture and its consequences in the case that the constructible sheaf has its generic stalk of rank p. After discussing finite etale group schemes of rank p over an integral scheme and its generic point, we prove an extension lemma (2.5). For curves, we give complete results via 2.6 and 2.7 on the Euler-Poincaré characteristic.

Let X be a scheme of characteristic $p > 0$. Let $\mathbf{Lie}(A)$ denote the tangent space at the identity of a group scheme A over X. All group schemes are commutative and killed by p. (See [**10**] for an elementary discussion.)

THEOREM 2.1. *Let A be a finite, flat, height 1 group scheme on X. Then $G = \mathbf{Hom}(A^D, F_p)$ is an etale group scheme on X, and there is a short exact sequence*

$$0 \to G \to \mathbf{Lie}(A) \xrightarrow{[p]-1} \mathbf{Lie}(A) \to 0$$

of sheaves on X_{et}. Here $[p]$ denotes the pth power map.

PROOF By [5, II.2.18], there is an exact sequence

$$0 \to F_p \to \mathbf{G}_a \overset{FR-1}{\to} \mathbf{G}_a \to 0$$

of sheaves on X_{et}, where $\mathbf{G}_a$ is the sheaf given by $\mathbf{G}_a(U) = \Gamma(U, \mathbf{O}_U)$ and FR is the map $a \to a^p$. We apply the function $\mathbf{Hom}(A^D, -)$ to this sequence and obtain the exact sequence

$$(*) \qquad 0 \to \mathbf{Hom}\big(A^D, F_p\big) \to \mathbf{Hom}\big(A^D, \mathbf{G}_a\big) \to \mathbf{Hom}\big(A^D, \mathbf{G}_a\big).$$

We can identify $\mathbf{Hom}(A^D, \mathbf{G}_a)$ with $\mathbf{Lie}(A)$ as follows. Let $X_\varepsilon = \mathbf{Spec}\,\mathbf{O}_X[\varepsilon]$, where $\varepsilon^2 = 0$. Let π: $X_\varepsilon \to X$ be the structure morphism. Then there is an exact sequence

$$(**) \qquad 0 \to \mathbf{G}_a \to \pi_* \mathbf{G}_m \to \mathbf{G}_m \to 0$$

of sheaves on X_{et}, where $\mathbf{G}_m$ is the sheaf given by $\mathbf{G}_m(U) = \Gamma(U, \mathbf{O}_U)^*$. If $U = \operatorname{Spec} R$ is an open affine of X, then the first map is $R \to R[\varepsilon]^*$, with $a \to 1 + a\varepsilon$; the second map is $R[\varepsilon]^* \to R^*$, with $c + d\varepsilon \to c$. Since an element $c + d\varepsilon$ of $R[\varepsilon]$ is invertible if and only if c is invertible, it follows that $R[\varepsilon]^* \to R^*$ is surjective with kernel isomorphic to R, for any ring R (in particular, for R local). Hence the sequence $(**)$ is exact as a sequence of sheaves for the Zariski (or any finer) topology.

If we now apply the functor $\mathbf{Hom}(A^D, -)$ to this exact sequence, we get the exact sequence

$$0 \to \mathbf{Hom}\big(A^D, \mathbf{G}_a\big) \to \mathbf{Hom}\big(A^D, \pi_* \mathbf{G}_m\big) \to \mathbf{Hom}\big(A^D, \mathbf{G}_m\big).$$

By adjointness, we have $\mathbf{Hom}(A^D, \pi_* G_m) \cong \mathbf{Hom}(\pi^* A^D, \mathbf{G}_m)$, which is $A(X_\varepsilon)$. We also know that $\mathbf{Hom}(A^D, \mathbf{G}_m) = A(X)$. Thus, we get the exact sequence

$$0 \to \mathbf{Hom}\big(A^D, \mathbf{G}_a\big) \to A(X_\varepsilon) \to A(X).$$

It is now clear that $\mathbf{Hom}(A^D, \mathbf{G}_a) \cong \mathbf{Lie}(A)$.

This result, together with the fact that the pth power map on $\mathbf{Lie}(A)$ is just FR_*, [6, p. 138], shows that we have exactness at all points of the desired sequence $(*)$, except for the surjectivity of $[p] - 1$. Viewing $\mathbf{Lie}(A)$ as a vector group, we see, by [5, II.2.19], that it suffices to show $[p] - 1$ is an etale morphism (this is equivalent to G being an etale group scheme); if so, then $[p] - 1$ is surjective as a morphism of sheaves on X_{et}. This is the content of the following lemma.

LEMMA 2.2. *Let f: $V \to V$ be a p-linear morphism of vector groups over a scheme X of characteristic $p > 0$. Then $f - 1$ is an etale morphism.*

PROOF. Since the result is local on the base, we may assume $X = \operatorname{Spec} R$ is affine and $V = A_R^n$ is affine n-space over R for some n. Let $e_1, \dots, e_n$ be an R-basis and $(x_1, \dots, x_n)$ denote the coordinates of a point $x \in V$. If $y = f(x)$ has coordinates $(y_1, \dots, y_n)$, then

$$y_i = \sum_{j=1}^{n} A_{ij} x_j^p,$$

where $A = (A_{ij})$ is the matrix of f with respect to the given basis.

We use the criterion of [**5**, I.3.16] to show that $f-1$ is etale. The partial derivative of $\Sigma_{j=1}^{n} A_{ij} x_j^p - x_i$ with respect to x_k is -1, for $i=k$, and 0, otherwise. Hence the Jacobian of the matrix of $f-1$ in this basis is $-I$, where I is the $n \times n$ identity matrix.

This completes the proof of the lemma and Theorem 2.1. //

DEFINITION. The etale group scheme G of Theorem 2.1 will be called the *etale core* of **Lie**(A) or A. (This terminology is suggested by [**1**, Exercise I.1.23].)

REMARK. (1) If $G_x = G \times_X \operatorname{Spec} k(x)$ for $x \in X$, then G_x is finite etale when A_x^D is etale. Otherwise, $G_x = \operatorname{Spec} k(x)$, since this is true for geometric fibers over algebraically closed fields [**7**, p. II.14-2]. In fact, let U be the largest open subset of X such that the structure map $\pi^{-1}(U) \to U$ is etale, for π: $A^D \to X$. Then $A_U^D = A^D \times_X U$ is etale over U. U is precisely the set of all x where G_x is finite etale.

(2) Every finite etale group scheme G killed by p occurs as the etale core of some height 1 group scheme A. First, G^D has height 1, when G is a finite etale group scheme on X such that $pG = 0$. We can see this as follows. The Verschiebung [**3**, VIIA.4.3] followed by the Frobenius is multiplication by p, hence kills G. Since the Frobenius is an isomorphism, G is killed by the Verschiebung. Cartier duality interchanges the Verschiebung and the Frobenius, so G^D is killed by the Frobenius.

Let $G^* = \mathbf{Hom}(G, F_p)$ be the etale core of G^D. Then G^* is a finite etale group scheme killed by p and may be viewed as an "etale" dual of G. Moreover, $G^{**} \cong G$ since this clearly holds for geometric fibers. We conclude that G is the etale core of $A = (G^*)^D$.

Concepts associated with p-linear, additive endomorphisms of finite-dimensional vector spaces will be very important in this and later sections. These include semisimple subspaces and the stable rank of a matrix. We need some notation.

NOTATION. If V is a vector space of dimension d over an algebraically closed field k of characteristic $p > 0$, let V_{ss} denote the semisimple subspace of V under a p-linear, additive endomorphism f. Then $V_{\mathrm{ss}} = \bigcap_{m \geqslant 1} \operatorname{Im}(f^m)$ and $\bigcup_{m \geqslant 1} \operatorname{Ker}(f^m)$ is denoted by V_{n}.

The k-dimension of V_{ss} equals the F_p-dimension of V^f, the set of all v in V such that $f(v) = v$. If B is the matrix of f with respect to any basis, then the k-dimension of V_{ss} equals the rank of the matrix $BB^{(p)} \cdots B^{(p^{d-1})}$, by [**12**], and is sometimes called the stable rank of the matrix B.

COROLLARY 2.3. *Let X be a complete projective variety over an algebraically closed field k of characteristic $p > 0$. Then*

$$\dim_{F_p}\big(H^i(X_{\mathrm{et}}, G)\big) = \dim_k H^i(X_{\mathrm{et}}, \mathbf{Lie}(A))_{\mathrm{ss}}$$

for all i.

PROOF. The exact sequence of sheaves on X_{et} of Theorem 2.1 leads to a long exact sequence of abelian groups:

$$0 \to H^0(G) \to H^0(\mathbf{Lie}(A)) \to H^0(\mathbf{Lie}(A)) \to \cdots \to H^i(\mathbf{Lie}(A))$$
$$\to H^i(\mathbf{Lie}(A)) \to H^{i+1}(G) \to \cdots .$$

Since $\mathbf{Lie}(A)$ is a locally free sheaf of $\mathbf{O}_X$-modules of finite rank, $H^i(X_{\text{et}}, \mathbf{Lie}(A))$ is a finite-dimensional k-vector space, with a p-linear endomorphism induced by $[p]$. By **[5**, III.4.13], $[p]-1$ is surjective on $H^i(X_{\text{et}}, \mathbf{Lie}(A))$ for each i. So the long exact sequence breaks up into short exact sequences

$$0 \to H^i(G) \to H^i(\mathbf{Lie}(A)) \to H^i(\mathbf{Lie}(A)) \to 0$$

for each i. We conclude $\dim_{F_p}(H^i(X_{\text{et}}, G)) = \dim_k H^i(X_{\text{et}}, \mathbf{Lie}(A))_{\text{ss}}$ for all i. //

REMARK. In particular, this result provides a method of calculating $\chi(X_{\text{et}}, G)$ for any finite etale group scheme G killed by p, when X is a complete projective variety over an algebraically closed field k of characteristic $p > 0$.

DEFINITION. Let X be a curve over an algebraically closed field k of characteristic $p > 0$ and let F be a constructible sheaf of F_p-modules on X_{et}. The *tame conductor* of F at $x \in X$ is

$$t_x(F) = \dim F_{\bar{\eta}} - \dim F_{\bar{x}}.$$

CONJECTURE 2.4. *Let X be an integral scheme and F a constructible sheaf of F_p-modules on X_{et}. Then there exists a finite, flat, height* 1 *group scheme A on X whose etale core G has as generic fiber the finite etale group scheme corresponding to $F_{\bar{\eta}}$.*

REMARK. (1) If X is a smooth proper curve over k, then the remark following Step 2, Theorem 1.1, shows (modulo the proof of Conjecture 2.4) that

$$\chi(X, F) = \chi(X, G) + \sum_{x \in X^0} (t_x(G) - t_x(F)).$$

(2) This conjecture is true when the finite etale group scheme corresponding to $F_{\bar{\eta}}$ has rank p (see 2.5–2.7 below). Thus it may be possible to prove the conjecture by induction. Since the proof for rank p requires the existence of an ample invertible sheaf L, it may be necessary to assume the existence of an ample vector bundle E on X and then to exploit the relationship between the canonical line bundle on the projective space bundle $P(E)$ and the vector bundle E on X.

(3) There may be a best choice of A and hence G, but A is not unique, since it is not unique for rank p.

We now turn to the problem of applying these ideas to calculating cohomology with coefficients in a constructible sheaf whose generic stalk has rank p. We first discuss group schemes of rank p over an integral scheme and its generic point. We also describe the etale cores of rank p group schemes which are height 1.

For X integral, let $K = k(X)$ and $\alpha \in K^*$. Then $G^K_{\alpha,0} = \operatorname{Spec}(K[T]/(T^p - \alpha T))$ is a finite etale group scheme of rank p over $\eta = \operatorname{Spec} K$. Every finite etale group scheme of rank p over η has this form. By **[18]**, rank p group schemes over a scheme of characteristic $p > 0$ are classified by triples (L, a, b). Here L is an invertible $\mathbf{O}_X$-module, a is a section of $\Gamma(X, L^{p-1})$, and b is a section of $\Gamma(X, L^{1-p})$ with $a \otimes b = 0$. The group scheme is denoted by $G^L_{a,b}$. If T is any X-scheme, then the T-valued points of $G^L_{a,b}$ can be viewed as the set of all $x \in \Gamma(T, L \otimes_{\mathbf{O}_X} \mathbf{O}_T)$ such that $x^p = a \otimes x$. The Cartier dual of $G^L_{a,b}$ is $G^M_{b,a}$ for $M = L^{-1}$. When $G^L_{a,b}$ is etale, we have $b = 0$ since a is invertible. When the Frobenius is the 0-morphism from G to $G^{(p)}$, it follows that $a = 0$ and the pth power map is given by $f \to b \otimes f^p$ for f

in $\mathbf{Lie}(G) = L$. This identifies the pth power map for height 1 group schemes of rank p. In fact, this motivated the search for the sequence of Theorem 2.1.

Let $A^D = G^L_{a',0}$ extend $G^K_{\alpha^{-1},0}$ and G be the etale core of $A = G^{L^{-1}}_{0,a'}$. The section a' of $\Gamma(X, L^{p-1}) \cong \mathrm{Hom}_X(X, L^{p-1})$, where L^{p-1} is viewed as a vector group over X, gives an element a'_x of $\mathrm{Hom}_x(x, (L^{p-1})_x) \cong \Gamma(x, (L^{p-1})_x)$ by base change:

$$\begin{array}{ccc} (L^{p-1})_x & \to & L^{p-1} \\ \downarrow \uparrow a'_x & & \downarrow \uparrow a' \\ x & \to & X \end{array}$$

So $([p]-1)_x$ is the map $y \to a'_x \otimes y^p - y$ and the sequence of 2.1 becomes

$$0 \to G_x \to \mathrm{Spec}(k'[T]) \overset{P(T)}{\to} \mathrm{Spec}(k'[T]) \to 0;$$

here $P(T) = cT^p - T$ with c the element of $k' = k(x)$ corresponding to a'_x. The element a'_x is in $\Gamma(x, (L^{p-1})_x) \cong L^{p-1}_x$. If m_x is the maximal ideal of $\mathbf{O}_x$, then c is the image of a'_x in $L^{p-1}_x / m_x L^{p-1}_x \cong k'$. Therefore, $G_x = \mathrm{Spec}(k'[T]/(cT^p - T))$.

Now $c = 0$ if and only if $a'_x \in m_x L^{p-1}_x$. Then $G_x \cong \mathrm{Spec}\, k'$. If $a'_x \notin m_x L^{p-1}_x$, then $c \neq 0$ and $G_x = G^{k'}_{c,0} \cong G^{k'}_{1,0} \cong (F_p)_{k'}$. (It is known that $\mathrm{Lie}(\alpha_p) \cong k'$ with $[p] =$ 0-map and $\mathrm{Lie}(\mu_p) \cong k'$ with basis e such that $[p](e) = e$.) If U is the open set of all $x \in X$ such that $a'_x \notin m_x L^{p-1}_x$, then we see $G_{|U}$ is a finite etale group scheme of rank p and $G_{|X-U}$ is trivial.

We note that $G^K_{\alpha,0} \cong \mathbf{Hom}(G^K_{\alpha^{-1},0}, F_p)$. It suffices to show there is a nondegenerate pairing $G^K_{\alpha^{-1},0} \times G^K_{\alpha,0} \to F_p$. Indeed, since a sheaf on η_{et} is determined by its stalk as a Γ_s-module, we need only exhibit the pairing for global sections over K_s. We know $G^K_{\alpha^{-1},0}(K_s)$ is the set of all b in K_s such that $b^p = \alpha^{-1}b$ and $G^K_{\alpha,0}(K_s)$ is the set of all d in K_s such that $d^p = \alpha d$. It follows that $(bd)^p = bd$, so that $bd \in F_p$. As this is just the pairing $F_p \times F_p \to F_p$ of F_p with its dual vector space, it is clearly nondegenerate.

We need to know when a finite etale group scheme of rank p over the generic point extends to the integral scheme itself. This is detailed in the following lemma, whose proof uses a portion of the proof of [**4**, II.7.6].

LEMMA 2.5. *The finite etale K-group scheme $G^K_{\alpha,0}$ extends to a finite flat group scheme $G^L_{a,0}$ on any quasi-compact integral scheme X with an ample divisor.*

PROOF. Let D_1 be an ample divisor on X. Then $L_1 = L(D_1)$ is an ample invertible sheaf on X_{Zar}. The element $\alpha \in K^*$ can be represented by a regular function s defined on some open set U, so we can regard s as a section of $\mathbf{O}_U$ over U. We may assume $U = \mathrm{Spec}\, A$ is affine and $L_{1|U}$ is free on U; for some $P \in U$, $s_P \neq 0$.

Let Y be the closed set $X - U$ with the reduced induced scheme structure and let I_Y be its sheaf of ideals. Then I_Y is a coherent sheaf on X_{Zar}, so for some $n > 0$, $I_Y \otimes L_1^n$ is generated by global sections. In particular, for the point $P \in U$, there is a section $f \in \Gamma(X, I_Y \otimes L_1^n)$ such that $f_P \notin m_P(I_Y \otimes L_1^n)_P$.

Now $I_Y \otimes L_1^n$ is a subsheaf of $\mathbf{O}_X \otimes L_1^n \cong L_1^n$, so we can think of f as an element of $\Gamma(X, L_1^n)$. If X_f is the open set of all $Q \in X$ such that $f_Q \notin m_Q(L_1^n)_Q$, then it follows from our choice of f that $P \in X_f$ and $X_f \subseteq U$ (when $Q \in Y$, $f_Q \in m_Q(L_1^n)_Q$

since $(I_Y)_Q \otimes (L_1^n)_Q$ is a submodule of $m_Q \otimes (L_1^n)_Q)$. The set U is affine and $L_{1|U}$ is trivial, so under the isomorphism W: $L_{1|U} \xrightarrow{\approx} \mathbf{O}_U$, f induces an element $g = W^n(f_{|U}) \in \Gamma(U, \mathbf{O}_U)$, and then $X_f = U_g$ is also affine.

Under restriction, $s \in \Gamma(U, \mathbf{O}_U)$ maps to a section $t \in \Gamma(X_f, \mathbf{O}_{X_f}) = \Gamma(U_g, \mathbf{O}_{U_g}) = A_g$. Since $\Gamma(X_f, \mathbf{O}_{X_f}) = \Gamma(X_f, \mathbf{O}_X)$, we can apply [**4**, II.5.14] with L replaced here by L_1^n and F by $\mathbf{O}_X$. The hypotheses of (b) are satisfied. Therefore, given the section $t \in \Gamma(X_f, \mathbf{O}_X)$, the section $t \otimes f^m \in \Gamma(X_f, \mathbf{O}_X \otimes (L_1^n)^m) \cong \Gamma(X_f, (L_1^n)^m)$ extends to a global section of $(L_1^n)^m$ for some $m > 0$. Clearly, for $r(p-1) \geqslant m$, the section $t \otimes f^{r(p-1)} \in \Gamma(X_f, \mathbf{O}_X \otimes (L_1^n)^{r(p-1)}) \cong \Gamma(X_f, (L_1^n)^{r(p-1)})$ extends to a global section a of L^{p-1} for $L = L_1^{nr}$. This determines a flat group scheme $G_{a,0}^L$ on X, which we now show extends $G_{\alpha,0}^K$.

Let $\langle\ \rangle$ denote the rational function corresponding to a local section of $\mathbf{O}_X$. Define B: $\Gamma(X, L) \to \Gamma(X, K) \cong K$ by $B(z) = \langle W^{nr}(z_{|U})\rangle / \langle g^r\rangle$ for $z \in \Gamma(X, L)$. Then we want to choose an isomorphism D: $L \otimes_{\mathbf{O}_X} K \to K$ such that the following diagram commutes:

$$\begin{array}{ccc}\Gamma(X, L \otimes_{\mathbf{O}_X} K) & \overset{D}{\xrightarrow{\approx}} & \Gamma(X, K) \cong K \\ \uparrow & \nearrow_{B} & \\ \Gamma(X, L) & & \end{array}$$

But $f^r \in \Gamma(X, L)$ can be regarded as a nonzero element of $\Gamma(X, L \otimes_{\mathbf{O}_X} K)$, hence as a basis element. That is, D can be defined by $D(f^r) = 1$. Then $B(f^r) = \langle W^{nr}(f^r_{|U})\rangle / \langle g^r\rangle = \langle g^r\rangle/\langle g^r\rangle = 1$ and the diagram commutes.

Clearly, the following diagram commutes

$$\begin{array}{ccc}\Gamma\left(X, (L \otimes_{\mathbf{O}_X} K)^{p-1}\right) & \overset{D^{(p-1)}}{\rightarrow} & \Gamma(X, K^{p-1}) \\ \uparrow \cong & & \downarrow \cong \\ \Gamma\left(X, L^{p-1} \otimes_{\mathbf{O}_X} K\right) & & \Gamma(X, K) \cong K \\ \hookuparrow & \nearrow_{B^{(p-1)}} & \\ \Gamma(X, L^{p-1}) & & \end{array}$$

Here $D^{(p-1)}(f^{r(p-1)}) = 1$ and $B^{(p-1)}(y) = \langle W^{nr(p-1)}(y_{|U})\rangle / \langle g^{r(p-1)}\rangle$ for $y \in \Gamma(X, L^{p-1})$. The fact that

$$\begin{aligned} B^{(p-1)}(a) &= \left\langle W^{nr(p-1)}(a_{|U})\right\rangle / \left\langle g^{r(p-1)}\right\rangle \\ &= \left\langle W_{|X_f}^{nr(p-1)}(a_{|X_f})\right\rangle / \left\langle g^{r(p-1)}\right\rangle \\ &= \left\langle t g^{r(p-1)}\right\rangle / \left\langle g^{r(p-1)}\right\rangle = \langle t\rangle = \alpha \end{aligned}$$

shows that $G_{a,0}^L \times_X \eta \cong G_{\alpha,0}^K$. Therefore, $G_{a,0}^L$ extends $G_{\alpha,0}^K$. //

REMARK. (1) It is clear that the extension is not unique.

(2) Let X be a curve over an algebraically closed field k of characteristic $p > 0$. For $\alpha \in K^*$, the divisor of α is the divisor of zeros of α minus the divisor of poles of α, i.e., $(\alpha) = (\alpha)_0 - (\alpha)_\infty$. Choose $D_1 = (\alpha)_\infty$ and let D be a divisor of minimal degree such that $(p-1)D \geqslant D_1$ (D exists since the set of such divisors is nonempty). Then $(\alpha) \geqslant -(p-1)D$ and α determines a global section a of L^{p-1} for

$L = L(D)$. Clearly, $G^L_{\alpha,0} \times_X \operatorname{Spec} K \cong G^K_{\alpha,0}$. This is the best choice of L, which is still not unique, however.

(3) We see α represents a coset in $K^*/K^{*(p-1)}$, which was already known, of course. With the hypotheses of (2), we may assume α is a coset representative whose divisor of poles is of minimal degree by factoring out $(p-1)$st powers. This is the best choice of α.

In the case of curves, we have the next two results which completely describe etale cohomology with coefficients in a constructible sheaf whose generic stalk has rank p. We assume the following hypotheses. Let X be a smooth proper curve over an algebraically closed field k of characteristic $p > 0$. Let F be a constructible sheaf of F_p-modules on X_{et} such that $G^K_{\alpha,0}$ is the finite etale group scheme corresponding to $F_{\bar{\eta}}$. Let g_X denote the genus of X. Let $A^D = G^L_{a',0}$ extend $G^K_{\alpha^{-1},0}$ and let G be the etale core of $A = G^{L^{-1}}_{0,a'}$.

THEOREM 2.6. *Let $L = L(D)$ for $D > 0$. Then $H^0(X_{\text{et}}, G) = H^2(X_{\text{et}}, G) = 0$ and $\dim H^1(X_{\text{et}}, G) = \sigma$, where σ is the stable rank of the matrix B corresponding to $H^1([p])$.*

PROOF. For a curve X, $L(D)$ is ample if and only if $D > 0$ [**4**, IV.3.3]. Hence $H^0(X_{\text{et}}, L^{-1}) = 0$ where $L^{-1} = L(-D)$. Since $\dim X = 1$, $H^2(X_{\text{et}}, L^{-1}) = 0$ as well. The Riemann-Roch theorem shows $\dim_k H^1(X_{\text{et}}, L^{-1}) = g_X + \deg D - 1$.

Since $L^{-1} = \mathbf{Lie}(A)$, we can apply Corollary 2.3 to determine $H^i(X_{\text{et}}, G)$. The value for σ follows from the discussion before Corollary 2.3. //

THEOREM 2.7. *Let $L = L(D)$ for $D \geqslant 0$ and $n(a')$ be the cardinality of the support of the divisor of zeros $(a')_0$ of $a' \in \Gamma(X, L^{p-1})$. Then*

$$\chi(X, F) = [\chi(X, G) + n(a')] - \sum_{x \in X^0} t_x(F).$$

PROOF. As in the proof of [**4**, II.7.8], we find that the complement of $X_{a'}$, the set of all $x \in X$ such that $a'_x \notin m_x L_x^{(p-1)}$, is the support of the divisor of zeros $(a')_0$ of $a' \in \Gamma(X, L^{p-1})$. It follows that

$$n(a') = \sum_{x \in X^0} (\dim G_{\bar{\eta}} - \dim G_{\bar{x}}) = \sum_{x \in X^0} t_x(G)$$

(see the next-to-last paragraph of the discussion before 2.5).

By Step 2, Theorem 1.1, we can write $\chi(X, F) = \chi_c(V, F) + \sum_{x \in X-V} \dim F_{\bar{x}}$ for any open subscheme V of X; similarly, $\chi(X, G) = \chi_c(V, G) + \sum_{x \in X-V} \dim G_{\bar{x}}$. Since F satisfies $F_{\bar{\eta}} \cong G_{\bar{\eta}} \cong (G^K_{\alpha,0})_{\bar{\eta}}$, we may choose V so that $F_{|V} \cong G_{|V}$ and both F, G are locally constant on V.

If we use the remark following Step 2, Theorem 1.1, then we have

$$\chi(X, F) - \sum_{x \in X-V} \dim F_{\bar{x}} = \chi(X, G) - \sum_{x \in X-V} \dim G_{\bar{x}}$$

and so

$$\begin{aligned} \chi(X, F) &= \chi(X, G) - \sum_{x \in X-V} \dim G_{\bar{x}} + \sum_{x \in X-V} \dim F_{\bar{x}} \\ &= \chi(X, G) + n(a') - \sum_{x \in X-V} t_x(F). \end{aligned}$$

The choice of V insures that

$$-\sum_{x\in X-V}\left(\dim F_{\bar\eta}-\dim F_{\bar x}\right)=-\sum_{x\in X-V}t_x(F)=-\sum_{x\in X^0}t_x(F).\quad //$$

REMARK. This does not require the best choice of L or a', but for the best choice of L, we have $(a')_0=(\alpha^{-1})_0=(\alpha)_\infty$, the divisor of poles of α. In this case, we denote $n(a')$ by $n(\alpha^{-1})$.

III. Calculations on P^1. The purpose of this section is two-fold. One is that calculating the cohomology of P^1 with coefficients in a constructible sheaf whose generic stalk has rank p can be done quite explicitly and serves to illustrate the results of §II, especially Theorems 2.6 and 2.7. The other is that, once the results of §IV are established on comparing F_p-cohomology for certain extensions of curves, the p-ranks of certain curves can be calculated by viewing these curves as lying over P^1. In Theorem 3.1, we prove the general result for coefficients in a constructible sheaf. Corollary 3.2 calculates the Euler characteristic when the coefficient sheaf is the direct image of a finite etale group scheme of rank p over the generic point. A result on elliptic surfaces is included here also.

Let F be a constructible sheaf of F_p-modules such that $F_{\bar\eta}\cong G^K_{\alpha,0}(K_s)$, where $K=k(P^1)$ and $\alpha\in K^*$. Choose homogeneous coordinates X_0, X_1 on P^1 and let $T=X_1/X_0$. Then $K=k(T)$. We assume $\alpha^{-1}\in k[T]$.

NOTATION. Let $f\in k[T]$ be a polynomial. Choose $m\geqslant 1$ minimal such that $m(p-1)\geqslant \deg f$. If $f=\Sigma d_iT^i$, then $A(f)$ will denote the matrix

$$\begin{pmatrix} d_{p-1} & d_{2p-1} & \cdots & d_{(m-1)p-1}\\ d_{p-2} & d_{2p-2} & \cdots & d_{(m-1)p-2}\\ \cdot & & & \\ \cdot & & & \\ d_{p-m+1} & d_{2p-m+1} & \cdots & d_{(m-1)p-m+1}\end{pmatrix}.$$

Let $\sigma(f)$ be the stable rank of the $(m-1)\times(m-1)$ matrix A, where $A=A(f)$. (Set $A(f)$ and $\sigma(f)$ both equal to 0 for $m=1$.)

THEOREM 3.1. *Under the above hypotheses,*

$$\chi(P^1,F)=d-\sigma(\alpha^{-1})+\#-\sum_{x\in X^0}t_x(F),$$

where $d=1$ if $\deg\alpha^{-1}=0$, and $d=0$ otherwise. Here $\#$ is the number of distinct zeros and poles of α^{-1}, less 1 when $(p-1)\,|\deg\alpha^{-1}\neq 0$.

PROOF. Suppose $\deg\alpha^{-1}\neq 0$ and let $\alpha^{-1}=\Sigma d_iT^i$. Setting $d_i=0$ for $i>\deg\alpha^{-1}$, it is clear that $a'=\Sigma_{i=0}^N d_iX_0^{N-i}\cdot X_1^i$ is a global section to $\mathbf{O}(m(p-1))$, if $N=m(p-1)>\deg\alpha^{-1}$. The isomorphism D: $\mathbf{O}(m(p-1))\otimes K\to K$ of Lemma 2.5 given by $X_0^N\to 1$ maps the image of a' in $\mathbf{O}(m(p-1))\otimes K$ to α^{-1}. Let G be the etale core of $G_{0,a'}^{\mathbf{O}(-m)}$. Then $\chi(P^1,G)=-\sigma(\alpha^{-1})$ by Theorem 2.6, provided we show the matrix of $H^1([p])$ with respect to the canonical basis of $H^1(\mathbf{O}(-m))$ is $A(\alpha^{-1})$.

Now $H^0(\mathbf{O}(m-2))$ has a canonical basis X_0^{m-2}, $X_0^{m-3}\cdot X_1, \dots, X_0^{m-2-i}\cdot X_1^i, \dots, X_1^{m-2}$. The k-vector space $H^1(\mathbf{O}(-m)) \cong H^0(\mathbf{O}(m-2))^*$ has dual basis $X_0^{-m+1}\cdot X_1^{-1}$, $X_0^{-m+2}\cdot X_1^{-2}, \dots, X_0^{-m+1+i}\cdot X_1^{-1-i}, \dots, X_0^{-1}\cdot X_1^{-m+1}$. For $i = 0, \dots, m-2$, let $w_{i+1} = X_0^{-m+1+i}\cdot X_1^{-1-i}$. Calculating

$$\begin{aligned} a' \otimes w_{i+1}^p &= \sum_{j=0}^{N} d_j X_0^{N-j}\cdot X_1^j \otimes X_0^{-pm+p+pi}\cdot X_1^{-p-pi} \\ &= \sum_{j=0}^{N} d_j X_0^{p(i+1)-m-j}\cdot X_1^{-p(i+1)+j} \\ &= \sum_{s=0}^{m-2} d_{p(i+1)-(s+1)} X_0^{-m+1+s}\cdot X_1^{-1-s}, \end{aligned}$$

where the substitution $j = p(i+1)-(s+1)$ was made in the last summation. It follows that the matrix of $H^1([p])$ with respect to the canonical basis of $H^1(\mathbf{O}(-m))$ is $A(\alpha^{-1})$.

On the other hand, $n(\alpha^{-1})$ is the number of distinct zeros of α^{-1}, regarded as a global section of $\mathbf{O}(m(p-1))$. This is the same as the number of distinct zeros of α^{-1}, regarded as a rational function, together with the point at infinity, except when $(p-1)\,|\deg\alpha^{-1}$. If x is the point at infinity, and $L = \mathbf{O}(m)$, then $a'_x \in m_x L_x^{p-1}$ if and only if $\operatorname{ord}_x(\alpha^{-1}) \geqslant 1 - m(p-1)$, which is equivalent to $\deg\alpha^{-1} \leqslant m(p-1) - 1$. The choice of m shows x is not a zero of the global section α^{-1} of $\mathbf{O}(m(p-1))$, precisely when $(p-1)\,|\deg\alpha^{-1} \neq 0$.

This case, i.e., $\deg\alpha^{-1} \neq 0$, follows immediately from Theorem 2.7 and the previous discussion.

If $\deg\alpha^{-1} = 0$, then α^{-1} is an element of k; the etale core in this case is just F_p and the sequence of 2.1 coincides with that of [**5**, II.2.18(c)] (see also the second line of the proof of Theorem 2.1). By 2.7,

$$\chi(P^1, F) = \chi\left(P^1, F_p\right) + n(\alpha^{-1}) - \sum_{x\in X^0} t_x(F) = 1 - \sum_{x\in X^0} t_x(F).$$

As $m = 1$, $\sigma(\alpha^{-1}) = 0$; $\# = 0$ as well. This completes the second case and the proof of the theorem. //

THEOREM 3.2. *Let g: $\eta = \operatorname{Spec} K \to P^1$ be the inclusion of the generic point. Under the same hypotheses, we have*

$$\chi\left(P^1, g_* G^K_{\alpha,0}\right) = d + z - \sigma(\alpha^{-1}),$$

where $d = 1$ if $\deg\alpha^{-1} = 0$, and $d = 0$ otherwise. Here z is the number of distinct zeros of α^{-1} whose orders are divisible by $p-1$.

PROOF. For this result, we need only show that $\sum_{x\in X^0} t_x(g_* G^K_{\alpha,0}) = n(\alpha^{-1}) - z$ and then use Theorem 3.1. Let Y be the complete smooth curve corresponding to the field $K(\alpha^{1/(p-1)})$. The proof of Theorem 4.1 shows that $\dim(g_* G^K_{\alpha,0})_{\bar{x}} = 1$, if Y_x is unramified over x, and 0 otherwise. A study of the equation $S^{p-1} - \alpha = 0$ over the completion of K, as in [**11**, III.2.6], shows Y_x is unramified over x if and only if

$(p-1)\,|\,\mathrm{ord}_x(\alpha)$. Since $\alpha^{-1} \in k[T]$ by hypothesis, the ramification points lie over some of the zeros of α^{-1} (since $\mathrm{ord}_x(\alpha^{-1}) = \mathrm{ord}_x(\alpha)$) and possibly the point at infinity. This latter does not occur precisely when $(p-1)\,|\deg \alpha^{-1} \neq 0$. It is easy to see that $n(\alpha^{-1}) - \sum_{x \in X^0} t_x(g_* G^K_{\alpha,0}) = z$, when $\deg \alpha^{-1} \neq 0$ (if $\deg \alpha^{-1} = 0$, all three terms are 0). The rest follows immediately from Theorem 3.1. //

REMARK. (1) It is important to note that neither of these results depends on making the best choice of α, but only on making the best choice of L (see Remark (2) following Lemma 2.5).

(2) It follows from Corollary 3.2 that $g_* G^K_{\alpha,0}$ is representable by the corresponding etale core for the best choice of α.

We will conclude this section with a result on elliptic surfaces. We show how to calculate $H^2(X_{\mathrm{et}}, F_p)$ for a smooth, complete, elliptic surface X over an algebraically closed field k of characteristic $p > 0$. We may assume by [5, V.3.1] that there is a proper flat morphism $\pi\colon X \to P^1$ whose generic fiber is a smooth elliptic curve X_K over $K = k(P^1)$, i.e., a Lefschetz pencil.

Now, consider the sheaf $F = R^1\pi_* F_p$. Since we have that $\pi_* \mathbf{O}_X \cong \mathbf{O}_{P^1}$, we see that $\pi_* \mathbf{G}_a \cong \mathbf{G}_a$ and $\pi_* F_p \cong F_p$. Hence $R^0\pi_* F_p \cong F_p$. By [5, III.1.15], it follows that $F_{\bar\eta} \cong H^1(X_{K_s}, F_p)$, where $X_{K_s} = X_k \otimes_K K_s$. If $\bar K$ denotes the algebraic closure of K and $X_{\bar K} = X_K \otimes_K \bar K$, then $H^1(X_{K_s}, F_p) \cong H^1(X_{\bar K}, F_p)$ by [5, VI.2.6]. This latter group is known to be

$$\mathrm{Ker}\left[H^1(X_{\bar K}, \mathbf{O}_{X_{\bar K}}) \overset{FR-1}{\to} H^1(X_{\bar K}, \mathbf{O}_{X_{\bar K}})\right].$$

So we have $\dim_{F_p}(F_{\bar\eta})$ equal to 0 or 1 depending on the Hasse invariant of X_K. Thus, $F_{\bar\eta} \cong G^K_{\alpha,0}(K_s)$ for some $\alpha \in K$ and the cohomology of P^1 with coefficients in $R^1\pi_* F_p$ can be calculated using Theorem 3.1, when $\alpha \in K^*$.

One can show that $R^1\pi_* F_p$ is an etale group scheme. Since all the fibers have genus 1, we can apply [4, III.12.9] to see that $R^1\pi_* \mathbf{O}_X$ is locally free of finite rank on P^1. Viewing it as a vector group, we see that $R^1\pi_* F_p$ is the kernel of an etale morphism and so it is an etale group scheme by Lemma 2.2 and [5, II.2.19]. Hence, $F_\eta \cong G^K_{\alpha,0}$. The induced p-linear morphism on $R^1\pi_* \mathbf{O}_X$ gives it the structure of a p-Lie algebra whose etale core is $R^1\pi_* F_p$. When there are singular fibers, the sheaf $R^1\pi_* F_p$ has no global sections over P^1 by Theorem 2.6.

The Leray spectral sequence [5, III.1.18] shows that $H^i(P^1_{\mathrm{et}}, R^j\pi_* F_p) \rightrightarrows H^{i+j}(X_{\mathrm{et}}, F_p)$. There is an associated exact sequence

$$\begin{aligned} 0 &\to H^1(P^1, F_p) \to H^1(X, F_p) \to H^0(P^1, F) \to H^2(P^1, F_p) \\ &\to \mathrm{Ker}\left[H^2(X, F_p) \to H^0(P^1, R^2\pi_* F_p)\right] \to H^1(P^1, F) \to H^3(P^1, F_p); \end{aligned}$$

see [5, Appendix B]. We know $H^i(P^1, F_p) = 0$ for $i \geqslant 1$. Since $(R^2\pi_* F_p)_{\bar s} = H^2(X_{\bar s}, F_p) = 0$ ($X_{\bar s}$ is a curve) for all $s \in P^1$, we see that $R^2\pi_* F_p = 0$. Thus $H^i(X, F_p) \cong H^{i-1}(P^1, F)$ for $i = 1, 2$. This shows $\chi(X, F_p) = 1 - \chi(P^1, F)$.

If X_s is nonsingular, then $\dim_{F_p}(F_{\bar s}) = \dim_{F_p}(F_{\bar\eta})$ and $t_s(F) = 0$ in Theorem 3.1. Hence we may compute $\chi(X, F_p)$ using only singular fibers, where $H^1(X, F_p) = 0$. With the notation of 3.1, we have proven the following theorem.

THEOREM 3.3. *Notation as above. Let X be a smooth, complete elliptic surface with a Lefschetz pencil. Assume the generic fiber is smooth of Hasse invariant* 1. *If T is the set of points s of P^1 with $\pi^{-1}(s) = X_s$ singular, then*

$$\dim_{F_p} H^2(X_{\mathrm{et}}, F_p) = \sigma(\alpha^{-1}) - \# + \sum_{s \in T} t_s\left(R^1\pi_* F_p\right). \quad //$$

IV. The p-ranks of abelian extensions. The concept of the p-rank of a complete, smooth curve X was first discussed in §I. The result discussed there, Corollary 1.2, involves p-extensions, i.e., the action of the Galois group of the separable closure K_s over $K = k(X)$ on the generic stalk factors through a finite quotient which is a p-group. This section concerns cyclic and abelian extensions of order dividing $p - 1$, Theorem 4.1 and Corollary 4.4 respectively. Although 4.4 generalizes it, Theorem 4.1 is the most important, its proof containing all the significant details. For Corollary 4.2, we take the cohomology of the sheaves in 4.1. In Corollary 4.3, we give a new proof of a result due to Manin on hyperelliptic curves. Some examples are included.

Thoughout this section, let X be a complete smooth curve over an algebraically closed field k of characteristic $p > 0$. Let $K = k(X)$ and $g\colon \eta \to X$ be the inclusion of the generic point. Let K_s denote the separable closure of K and $\Gamma_s = \mathrm{Gal}(K_s/K)$. We will denote $G^K_{\beta,0}$ by $G_{\beta,0}$ for $\beta \in K^*$.

THEOREM 4.1. *Let* $\Gamma = \mathrm{Gal}(L/K)$ *be a cyclic group of order* $n\,|\,(p-1)$, *with* $L = K(f^{1/n})$ *for some* $f \in K^*$. *Let* $\pi\colon Y \to X$ *be the corresponding finite morphism. Then*

$$\pi_* F_p \cong \bigoplus_{i=0}^{n-1} g_* G_{\alpha^i,0}$$

for $\alpha = f^{(p-1)/n}$.

PROOF. The cyclic group $\Gamma = \langle s \rangle$ has n distinct irreducible representations of degree 1. Let $C_f\colon \Gamma \to (F_p)^*$ be defined by $s \to C_f(s)$, with $s(f^{1/n}) = C_f(s) f^{1/n}$. Here $C_f(s) \in \mu_n(K) \subseteq (F_p)^*$. The n distinct irreducible characters C_f^i correspond to simple $F_p[\Gamma]$-modules $V_i \cong F_p$ for $i = 0, \ldots, n-1$. The group Γ acts on V_i by $sv = C_f^i(s)v$ for $v \in V_i$.

The category equivalences of [5, II.1.9] and Step 1 of Theorem 1.1 show there exist finite etale group schemes G_i of rank p over K such that $G_i(K_s) \cong V_i$ as a Γ_s-module. Since the action factors through Γ, we see $G_i(L) \cong V_i$ as a Γ-module. We want to show that $G_i \cong G_{\alpha^i,0}$. For this, consider $G_{\alpha^i,0}(L)$ which is the set of all $c \in L$ such that $c^p - \alpha^i c = 0$. Then the elements of $G_{\alpha^i,0}(L)$ are 0, $f^{i/n}$, $2f^{i/n}, \ldots$, $(p-1)f^{i/n}$. Calculating, $s(f^{i/n}) = (s(f^{1/n}))^i = (C_f(s)f^{1/n})^i = C_f^i(s) f^{i/n}$. Therefore, $G_i \cong G_{\alpha^i,0}$.

There is a commutative diagram:

$$\begin{array}{cccc} \mathrm{Spec}\, L = & \eta' & \xrightarrow{g'} & Y \\ & \downarrow \pi' & & \downarrow \pi \\ \mathrm{Spec}\, K = & \eta & \xrightarrow{g} & X \end{array}$$

The natural morphism $F_p \to g'_*g'^*F_p$ of sheaves on Y_{et} induces a morphism $\pi_*F_p \to \pi_*g'_*g'^*F_p$; this last sheaf is isomorphic to $g'\pi'_*g'^*F_p$ by commutativity. By adjunction, we get a morphism $g^*\pi_*F_p \to \pi'_*g'^*F_p$. Since the first of these morphisms induces an isomorphism of generic stalks, it is easy to see that the second and third do also. The category equivalence of [**5**, II.1.9] (together with [**5**, II.3.1(e)]) shows $\pi'_*F_p = (\pi'_*g'^*F_p)_{\bar{\eta}}$, where $\pi'_*F_p = \pi'_*(g'^*F_p)_{\bar{\eta}'}$. We find that the Γ-modules $(\pi_*F_p)_{\bar{\eta}}$ and π'_*F_p are isomorphic; use V for either. We can write $V = \oplus_{i=0}^{n-1}W_i$, where W_i is the set of all $v \in V$ such that $sv = C_f^i(s)v$. This is the canonical decomposition [**9**, p. 21].

Each W_i is irreducible if and only if no $W_i = 0$. However, $V = \pi'_*F_p$ is the induced representation of the trivial representation on the trivial subgroup $\mathrm{Gal}(L/L)$, whence $V = F_p \otimes_{F_p} F_p[\Gamma]$ is the regular representation. From [**9**, p. 18], it follows that each W_i is irreducible and so $W_i = V_i$ for each i. Thus,

$$\left(\pi_*F_p\right)_{\bar{\eta}} \cong \bigoplus_{i=0}^{n-1} V_i \quad \text{and} \quad g^*\pi_*F_p \cong \bigoplus_{i=0}^{n-1} G_{\alpha^i,0}.$$

It remains to show that $\pi_*F_p \cong g_*g^*\pi_*F_p$. We now know that $g_*g^*\pi_*F_p \cong \oplus_{i=0}^{n-1}g_*G_{\alpha^i,0}$. Hence there is a natural map $\pi_*F_p \to \oplus_{i=0}^{n-1}g_*G_{\alpha^i,0}$. We will show it is an isomorphism by checking stalks. First, note that the kernel of the induced map on stalks is $H_x^0(X, \pi_*F_p)$ for any $x \in X^0$. By [**5**, III.1.28], this is $H_x^0(\operatorname{Spec}\mathbf{O}_{\bar{x}}, \pi_*F_p)$, which is the kernel of the restriction map $\pi_*F_p(\operatorname{Spec}\mathbf{O}_{\bar{x}}) \to \pi_*F_p(\operatorname{Spec}K_{\bar{x}})$, where $K_{\bar{x}} = K_s^{I_x}$ for I_x the inertia group corresponding to $\mathbf{O}_{\bar{x}} \to K_s$. Let $X' = \operatorname{Spec}\mathbf{O}_{\bar{x}}$ and $Y' = X' \times_X Y$. Then this is the map

$$F_p\ (\text{closed fiber of } Y'/X') \to F_p\ (\text{generic fiber of } Y'/X'),$$

which is clearly injective since F_p is the constant sheaf on Y'. Thus, the map on stalks is injective, and the same is true for the map of the sheaves themselves.

If $x \in X$ and $r > 1$ points of Y lie over x, i.e., Y_x consists of the points $y_1, \dots, y_r$, then $(\pi_*F_p)_{\bar{x}} \cong (F_p)^r$. On the other hand, it is true that $\oplus_{i=0}^{n-1}(g_*G_{\alpha^i,0})_{\bar{x}} \cong \oplus_{i=0}^{n-1}(G_{\alpha^i,0})_{\bar{\eta}}^{I_x}$. Since $\Gamma_y = \Gamma \cap I_x$ for some $y \to x$, we find $(G_{\alpha^i,0})_{\bar{\eta}}^{I_x} = (G_{\alpha^i,0})_{\bar{\eta}}^{\Gamma_y}$. But the right hand side is F_p if Γ_y acts trivially and 0 otherwise.

The decomposition group Γ_y is a subgroup of the cyclic group $\Gamma = \langle s \rangle$ and the ramification index at y is n/r, so the order of Γ_y is n/r and $\Gamma_y = \langle s^r \rangle$. For $v \in V_i$, we get $s^r v = C^{ir}v$, where we abbreviate $C_f(s)$ by C. This means we want C^{ir} to be the identity, more precisely, $i = 0,\ n/r, 2n/r, \dots, (r-1)n/r\ (= n - n/r < n - 1)$. We conclude that

$$\bigoplus_{i=0}^{n-1} \left(G_{\alpha^i,0}\right)_{\bar{\eta}}^{\Gamma_y} = \bigoplus_{k=0}^{r-1} \left(G_{\alpha^{kn/r},0}\right)_{\bar{\eta}}^{\Gamma_y} \cong \left(F_p\right)^r.$$

Since the stalks have the same dimension and we know the map of sheaves is injective, it is an isomorphism, as desired. //

COROLLARY 4.2. *Let σ_X, σ_Y denote the p-ranks of X, Y respectively. Then*

$$1 - \sigma_Y = 1 - \sigma_X + \sum_{i=1}^{n-1} \chi(X, g_*G_{\alpha^i,0}).$$

PROOF. Take cohomology in the theorem and use that $\chi(X_{\text{et}}, \pi_* F_p) = \chi(Y_{\text{et}}, F_p) = 1 - \sigma_Y$, $\chi(X_{\text{et}}, F_p) = 1 - \sigma_X$. Also see the statement below about $g_* G_{1,0}$. //

COROLLARY 4.3. *Let $X = P^1$ and Y be a hyperelliptic curve with π: $Y \to X$ the corresponding morphism of degree 2. Let $L = k(Y) = K(f^{1/2})$ for $f \in K^*$ and $\alpha = f^{(p-1)/2}$, where the characteristic of k is greater than 2. Then the Hasse-Witt matrix of Y is $A(\alpha)$.*

PROOF. The Hasse-Witt matrix is the matrix of the endomorphism of $H^1(Y, \mathbf{O}_Y)$ induced by the Frobenius morphism of Y. By Theorem 4.1, we have $\pi_* F_p = g_* G_{1,0} \oplus g_* G_{\alpha,0}$. It is clear that $g_* G_{1,0}$, which is $g_* F_p$, is the constant sheaf F_p. Hence there is a short exact sequence

$$0 \to F_p \to \pi_* F_p \to g_* G_{\alpha,0} \to 0$$

of sheaves on X_{et}.

There is also an exact sequence

$$0 \to F_p \to \mathbf{O}_Y \xrightarrow{FR-1} \mathbf{O}_Y \to 0$$

of sheaves on Y_{et}. The locally free sheaf $\pi_* \mathbf{O}_Y$ is, in fact, decomposable of rank two on X_{et} and hence $\pi_* \mathbf{O}_Y = \mathbf{O}(s) \oplus \mathbf{O}(t)$ for some integers s, t by [**4**, V.2.14]. Calculation of the cohomology groups of this sheaf on X_{et} shows that $s = 0$ and $t = -u < 0$, and then $\dim_k H^1(X_{\text{et}}, \mathbf{O}_X \oplus \mathbf{O}(-u)) = g_Y$ implies $u = g_Y + 1$. As a result, we get a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & FR_X - 1 & \downarrow & & \\
0 & \to & F_p & \to & \mathbf{O}_X & \to & \mathbf{O}_X & \to & 0 \\
 & & \downarrow & & \downarrow & \pi_*(FR_Y - 1) & \downarrow & & \\
0 & \to & \pi_* F_p & \to & \pi_* \mathbf{O}_Y & \to & \pi_* \mathbf{O}_Y & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H & \to & \mathbf{O}(-g-1) & \to & \mathbf{O}(-g-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Therefore, $H = g_* G_{\alpha,0}$.

Since $\alpha^2 = f^{(p-1)}$ is a $(p-1)$st power in K^*, we see that α and α^{-1} determine the same coset of $K^*/K^{*(p-1)}$, so $G_{\alpha,0} \cong G_{\alpha^{-1},0}$. For the standard choice of f as a polynomial of degree $2g_Y + 1$ or $2g_Y + 2$ with no multiple roots, the best choice of L is clearly $\mathbf{O}(g+1)$. We find from Corollary 3.2 that $\chi(X_{\text{et}}, g_* G_{\alpha^{-1},0}) = -\sigma(\alpha)$.

From Remark (2) following Corollary 3.2, we see that $g_* G_{\alpha^{-1},0}$ is the etale core of $\mathbf{O}(-g-1)$. Thus, the Hasse-Witt matrix of Y is the matrix of $H^1([p])$ with respect to the canonical basis of $H^1(\mathbf{O}(-g-1))$; as in the proof of Theorem 3.1, this is just $A(\alpha)$. //

REMARK. (1) Corollary 4.3 is due to Manin [14], who proved it using the Cartier operator. His matrix is $A(\alpha)^t$, since he worked essentially with the k-linear dual of $H^1(Y, \mathbf{O}_Y)$.

(2) If $n \nmid (p-1)$, let $d = \gcd(n, p-1)$. If $d \neq 1$, then Theorem 4.1 goes through for the cyclic subgroup of Γ of order d. More precisely, with the same hypotheses as Theorem 4.1 except that $n \nmid (p-1)$, let $\Delta \subseteq \Gamma$ be a cyclic subgroup of order d and $M = L^\Delta$. Then $\Delta = \mathrm{Gal}(L/M)$ and there is a corresponding finite morphism μ: $Y \to Z$ of degree d (so that π factors through Z). It follows that $\mu_* F_p$ can be calculated via Theorem 4.1.

Alternatively, there is a minimal r such that $n \mid (q-1)$, where $q = p^r$. Hence the above theory can be applied to sheaves of F_q-modules. This will be discussed in the appendix.

Let L/K be an abelian extension of degree $n \mid (p-1)$ and $\Gamma = \mathrm{Gal}(L/K)$. Let π: $Y \to X$ be the corresponding finite morphism. The irreducible representations of Γ are homomorphisms C: $\Gamma \to Q/Z$, where $C(\Gamma)$ is a subgroup of Z/nZ, the unique cyclic subgroup of Q/Z of order n. There is a tower of fields $K \leqslant K^C = L^{\mathrm{Ker}\, C} \leqslant L$. Then $\mathrm{Ker}\, C = \mathrm{Gal}(L/K^C)$ and $C(\Gamma) = \mathrm{Gal}(K^C/K)$ is a cyclic group of order $d_C \mid n$. Moreover, we have $K^C = K(f_C^{1/d_C})$ for some $f_C \in K^*$.

COROLLARY 4.4. *Let π: $Y \to X$ be the morphism corresponding to L/K. Then*

$$\pi_* F_p = \bigoplus_{C \in \Gamma^\vee} g_* G^{n_C}_{f_C, 0},$$

where $\Gamma^\vee$ is the gruop $\mathrm{Hom}_Z(\Gamma, Q/Z)$.

PROOF. Let V_C be the simple $F[\Gamma]$-module corresponding to the homomorphism C. Then Theorem 4.1 shows V_C is the stalk of the etale K-group scheme $G^{n_C}_{f_C,0}$ for $n_C = (p-1)/d_C$. If Ch_f: $C(\Gamma) \to F_p^*$ is defined by $s_C \to Ch_f(s_C)$, then Γ acts on V_C by $s_C v = Ch_f(s_C) v$, where s_C is the generator of $C(\Gamma)$.

Since $(\pi_* F_p)_{\bar\eta}$ is the regular representation of Γ, we get $(\pi_* F_p)_{\bar\eta} \cong [\oplus_{C \in \Gamma^\vee} G^{n_C}_{f_C,0}]_{\bar\eta}$. As in Theorem 4.1, it follows that we have $\pi_* F_p = \oplus_{C \in \Gamma^\vee} g_* G^{n_C}_{f_C,0}$. //

EXAMPLES. We conclude by computing the p-ranks of several curves over P^1 to illustrate Theorem 4.1 and its corollaries.

(1) Let Y be the projective completion of the elliptic curve with affine equation

$$y^2 = x(x-1)(x-\lambda),$$

for $\lambda \in k^*$, k an algebraically closed field of characteristic $p > 2$. For $f(x) = x(x-1)(x-\lambda)$ and $\alpha = f^{(p-1)/2}$, we can use $A(\alpha)$ as in the proof of 4.3. We have $A(\alpha) = d_{p-1}$, the coefficient of x^{p-1} in α. This is just the polynomial $h_p(\lambda)$ of [4, IV.4.22], giving the classical Hasse invariant for the elliptic curve Y.

(2) Let $p = 7$ and L/K be Galois of degree 3. Choosing $l \neq p$ and applying Corollary 1.2, we see that the number t of ramification points of the corresponding curve Y is $g_Y + 2$, since Y is totally ramified. The matrix $A(\alpha^{-i})$ is an $(m-1) \times (m-1)$ matrix, where $m(p-1) \geqslant \deg(\alpha^{-i})$. Assume the ground field k is algebraically closed of characteristic p.

(a) Let $g_Y = 4$ and assume Y is the projective completion of the curve with affine equation

$$y^3 = [h(x)]^{-1}.$$

Here $h(x) = \prod_{j=1}^{6}(x - a_j)$ for distinct $a_j \in F_7$. Then $\alpha^{-i} = [h(x)]^{2i}$ for $i = 0, 1, 2$. For $i = 1$, we get $m = 2$ and $A(\alpha^{-1}) = d_6$. For $i = 2$, the value of m is 4. The 3×3 matrix $A(\alpha^{-2})$ has entries

$$\begin{pmatrix} d_6 & d_{13} & d_{20} \\ d_5 & d_{12} & d_{19} \\ d_4 & d_{11} & d_{18} \end{pmatrix}.$$

Using a Fortran program, a computer generated the following table with 7 rows (Table 1). In all these cases, $\sigma_Y = 4 = g_Y$ and Y is ordinary, i.e., the p-rank and genus of Y coincide.

(b) Let $g_Y = 4$ and the affine equation be

$$y^3 = (1 - x^5)^{-1}.$$

The sizes of the matrices are the same as in (a). For $i = 1$, the entry $d_6 = 0$ and $\sigma(\alpha^{-1}) = 0$. For $i = 2$, the only nonzero entries are $d_5 = 3$ and $d_{20} = 1$. Thus $\sigma(\alpha^{-2}) = 0$. We find $\sigma_Y = 0 < g_Y$ and Y is supersingular, i.e., the p-rank of Y is 0.

Now let the affine equation be

$$y^3 = [l(x)]^{-1},$$

where $l(x) = \prod_{j=1}^{5}(x - a_j)$ for distinct $a_j \in F_7$. The matrix sizes are unchanged. The following 21-row table (Table 2) was also computer-generated. When $A(\alpha^{-1}) = 0$, we get $\sigma_Y = 3 < g_Y$ and Y is not ordinary. Otherwise, $\sigma_Y = 4$ and Y is ordinary.

The 3×3 matrices $A(\alpha^{-2})$ whose entries are given in the two tables below have the following properties: (i) $A^3 = 6I_3 = -I_3 \pmod 7$ for all $A = A(\alpha^{-2})$; (ii) all A^2 are distinct.

TABLE 1

DEGREE 6

	a_1	a_2	a_3	a_4	a_5	a_6	$A(\alpha^{-1})$	d_6	d_5	d_4	d_{13}	d_{12}	d_{11}	d_{20}	d_{19}	d_{18}
1.	0	1	2	3	4	5	5	3	3	1	0	6	1	0	0	3
2.	0	1	2	3	4	6	5	3	6	4	0	6	2	0	0	3
3.	0	1	2	3	5	6	5	3	2	2	0	6	3	0	0	3
4.	0	1	2	4	5	6	5	3	5	2	0	6	4	0	0	3
5.	0	1	3	4	5	6	5	3	1	4	0	6	5	0	0	3
6.	0	2	3	4	5	6	5	3	4	1	0	6	6	0	0	3
7.	1	2	3	4	5	6	5	3	0	0	0	6	0	0	0	3

TABLE 2

DEGREE 5

	a_1	a_2	a_3	a_4	a_5	$A(\alpha^{-1})$	d_6	d_5	d_4	d_{13}	d_{12}	d_{11}	d_{20}	d_{19}	d_{18}
1.	0	1	2	3	4	1	5	4	4	3	2	6	1	2	5
2.	0	1	2	3	5	0	1	1	2	4	4	5	1	5	1
3.	0	1	2	3	6	0	2	6	1	5	1	2	1	1	2
4.	0	1	2	4	5	1	3	4	2	5	4	6	1	1	3
5.	0	1	2	4	6	1	6	4	1	6	1	6	1	4	6
6.	0	1	2	5	6	2	1	0	4	0	3	0	1	0	1
7.	0	1	3	4	5	0	4	6	4	6	2	2	1	4	4
8.	0	1	3	4	6	2	2	0	2	0	6	0	1	0	2
9.	0	1	3	5	6	1	6	3	1	1	1	1	1	3	6
10.	0	1	4	5	6	0	2	1	1	2	1	5	1	6	2
11.	0	2	3	4	5	2	4	0	1	0	5	0	1	0	4
12.	0	2	3	4	6	0	4	1	4	1	2	5	1	3	4
13.	0	2	3	5	6	1	3	3	2	2	4	1	1	6	3
14.	0	2	4	5	6	0	1	6	2	3	4	2	1	2	1
15.	0	3	4	5	6	1	5	3	4	4	2	1	1	5	5
16.	1	2	3	4	5	5	3	0	0	1	6	0	1	3	3
17.	1	2	3	4	6	5	5	0	0	2	3	0	1	6	5
18.	1	2	3	5	6	5	6	0	0	3	5	0	1	2	6
19.	1	2	4	5	6	5	6	0	0	4	5	0	1	5	6
20.	1	3	4	5	6	5	5	0	0	5	3	0	1	1	5
21.	2	3	4	5	6	5	3	0	0	6	6	0	1	4	3

Appendix I: Sheaves of F_q-modules. Here we briefly sketch results on constructible sheaves of F_q-modules that can be obtained by the above methods. We begin with a discussion of the group scheme F_q and q-linear maps. The relationship between F_p- and F_q-cohomology is given in Corollary AI.2. Following a version of Theorem 2.1, we consider finite etale group schemes of rank q over the generic point of an integral scheme and modifications of Theorems 2.6, 2.7. We repeat the appropriate versions of 3.1, 3.2 and 4.1, 4.2. We conclude with several examples of the use of F_q-cohomology to compute p-ranks via the new versions of 3.2, 4.2.

Let X be a scheme of characteristic $p > 0$ and set $q = p^r$, $r \geqslant 1$. Then $F_q = \mathbf{Spec}(\mathbf{O}_X[T]/(T^q - T))$ is a finite etale group scheme of rank q over X. There is a short exact sequence

$$0 \to F_q \to \mathbf{G}_a \overset{(FR)^r - 1}{\to} \mathbf{G}_a \to 0$$

of sheaves on X_{et}. Recall that FR is the map $a \to a^p$. (To show exactness, one can view $\mathbf{G}_a$ as a vector group and apply the obvious variant of Lemma 2.2.)

Next let X be a complete projective variety over an algebraically closed field k of characteristic p. The finite-dimensional k-vector spaces $H^i(X_{\text{et}}, \mathbf{G}_a)$ have a q-linear endomorphism induced by $(FR)^r$. By a q-linear endomorphism of a finite-dimensional k-vector space V, we mean an additive homomorphism $f\colon V \to V$ such that $f(\alpha v) = \alpha^q f(v)$ for $\alpha \in k$, $v \in V$.

Modifying the results of **[12]**, we find that any such vector space V has a direct-sum decomposition into a semisimple subspace $V_{\text{ss}}^{(q)}$ $(= \bigcap_{m \geqslant 1} \operatorname{Im} f^m)$ on which f is bijective and a subspace $V_{\text{n}}^{(q)}$ $(= \bigcup_{m \geqslant 1} \operatorname{Ker} f^m)$ on which it is nilpotent. (In the discussion before Corollary 2.3, we denote $V_{\text{ss}}^{(p)}$ by V_{ss}, $V_{\text{n}}^{(p)}$ by V_{n}.) Here the k-dimension of $V_{\text{ss}}^{(q)}$ equals the F_q-dimension of V^f. If we define the q-stable rank of the matrix B of f with respect to any basis to be the rank of $BB^{(q)} \cdots B^{(q^{d-1})}$ $(d = \dim_k V)$, then the q-stable rank of B equals the k-dimension of $V_{\text{ss}}^{(q)}$. We see that $H^i(X_{\text{et}}, F_q)$ is a finite-dimensional F_q-vector space for all i.

Proposition AI.1. *Let V be a finite-dimensional vector space over an algebraically closed field k of characteristic $p > 0$. Let f be a q-linear endomorphism of V with corresponding decomposition $V = V_{\text{ss}}^{(q)} \oplus V_{\text{n}}^{(q)}$. Then f^s gives the same decomposition of V for all $s \geqslant 1$.*

Proof. For simplicity, denote the original decomposition by $V = W_{\text{ss}} \oplus W_{\text{n}}$ and the decomposition corresponding to f^s by $V = U_{\text{ss}} \oplus U_{\text{n}}$. Then f is bijective on W_{ss} implies f^s is also; hence $W_{\text{ss}} \subseteq U_{\text{ss}}$. If $f^s \circ g$ is the identity on U_{ss}, then $f \circ f^{s-1}g$ is also; then $U_{\text{ss}} \subseteq W_{\text{ss}}$. Moreover, f is nilpotent on a subspace if and only if f^s is nilpotent on this same subspace. //

Let

$$\chi(X, F_q) = \sum (-1)^i \dim_{F_q} H^i(X_{\text{et}}, F_q)$$

and

$$\chi(X, F_p) = \sum (-1)^i \dim_{F_p} H^i(X_{\text{et}}, F_p).$$

Corollary AI.2. $\chi(X, F_q) = \chi(X, F_p)$.

Proof. Apply the proposition to $V = H^i(X_{\text{et}}, \mathbf{G}_a)$ and $f = FR$. //

Remark. If F_p-cohomology is known, we see that F_q-cohomology provides no new information. However, we will use F_q-cohomology to calculate F_p-cohomology when this latter cannot be obtained by our usual methods.

We now restate some of the results of earlier sections in this context.

Theorem 2.1 (bis). *Let X be a scheme of characteristic $p > 0$. Let M be a locally-free sheaf of finite rank on X_{Zar} with q-linear endomorphism f. Denote by H the subsheaf of M fixed by f. Then H is an etale group scheme on X and*

$$0 \to H \to M \xrightarrow{f-1} M \to 0$$

is a short exact sequence of sheaves on X_{et}.

PROOF. Use the obvious variant of Lemma 2.2. //

REMARK. With the hypotheses of Theorem 2.1, let $'G = \mathbf{Hom}(A^D, F_q)$ and $f = [p]'$. Then $'G$ is the subsheaf of $\mathbf{Lie}(A)$ fixed by f. This is a special case of the above result.

For X integral, $K = k(X)$ and $\alpha \in K^*$, let $H_\alpha^{(q)} = \operatorname{Spec}(K[T]/(T^q - \alpha T))$. (We used $G_{\alpha,0}^K$ for $H_\alpha^{(p)}$ earlier.) Then $H_\alpha^{(q)}$ is a finite etale group scheme of rank q over $\eta = \operatorname{Spec} K$. Every twisted form [**5**, p. 134] of F_q over η can be described this way by Kummer theory [**5**, p. 125], since $\mathbf{Aut}(F_q) \cong Z/(q-1)Z$ (see [**2**, III.5.3.3]). If X is quasi-compact and has an ample divisor, the proof of Lemma 2.5 shows there is an invertible sheaf L and a global section a of L^{q-1} extending α.

Suppose L is an invertible sheaf and a' is a global section of L^{q-1} extending α^{-1}. Define a q-linear morphism f: $L^{-1} \to L^{-1}$ by $f(x) = a' \otimes x^q$. Let H be the subsheaf of L^{-1} fixed by f. We can apply Theorem 2.1 (bis) to H. The discussion on group schemes of rank p before Lemma 2.5 can easily be modified to show that $H_{|U}$ is a finite etale group scheme of rank q and $H_{|X-U}$ is trivial, where U is the open set of all $x \in X$ such that $a'_x \notin m_x L_x^{q-1}$. If $L_x^{q-1}/m_x L_x^{q-1} \cong k'$, then $H_x \simeq \operatorname{Spec} k'$ for $x \notin U$ and $H_x \cong (F_q)_{k'}$ for $x \in U$. Finally, $H_\alpha^{(q)} \cong \mathbf{Hom}(H_{\alpha^{-1}}^{(q)}, F_q)$.

Let X be a smooth proper curve over an algebraically closed field k of characteristic $p > 0$. Let F be a constructible sheaf of F_q-modules on X_{et} such that $H_\alpha^{(q)}$ is the finite etale group scheme corresponding to $F_{\bar\eta}$. Let L be an invertible sheaf and a' a global section of L^{q-1} extending α^{-1}. Let H be the subsheaf of L^{-1} fixed by the q-linear morphism of the previous paragraph. Then Theorem 2.6 goes through with G replaced by H and σ replaced by σ_q (= the q-stable rank of the matrix corresponding to $H^1(f)$. Denote the q-tame conductor at x of a constructible sheaf F of F_q-modules by

$$t_x^{(q)}(F) \underset{\text{def}}{=} \dim_{F_q}(F_{\bar\eta}) - \dim_{F_q}(F_{\bar x}).$$

Clearly, Theorem 2.7 holds with $n(a')$ renamed by $n_q(a')$, L^{p-1} replaced by L^{q-1} and $t_x(F)$ replaced by $t_x^{(q)}(F)$.

At the beginning of §III, let F be a constructible sheaf of F_q-modules such that $F_{\bar\eta} \cong H_\alpha^{(q)}(K_s)$, where $K = k(P^1)$ and $\alpha \in K^*$. Denote by $A_q(f)$ the matrix $A(f)$ with q substituted for p. Let $\sigma_q(f)$ denote its q-stable rank.

THEOREM 3.1 (BIS). *Under the above hypotheses,*

$$\chi(P^1, F) = d - \sigma_q(\alpha^{-1}) + \#_q - \sum_{x \in X^0} t_x^{(q)}(F),$$

where $d = 1$ if $\deg \alpha^{-1} = 0$, and $d = 0$ otherwise. Here $\#_q$ is the number of distinct zeros and poles of α^{-1}, less 1 when $(q-1) \mid \deg \alpha^{-1} \neq 0$. //

The modification of the proof is straightforward.

COROLLARY 3.2 (BIS). *Let g: $\eta = \operatorname{Spec} K \to P^1$ be the inclusion of the generic point. Under the same hypotheses, we have*

$$\chi(P^1, g_* H_\alpha^{(q)}) = d + z_q - \sigma_q(\alpha^{-1}),$$

where $d = 1$ if $\deg \alpha^{-1} = 0$, and $d = 0$ otherwise. Here z_q is the number of distinct zeros of α^{-1} whose orders are divisible by $q - 1$. //

Again, the modification of the proof is easy.

For §IV, let r be minimal such that $n\,|\,(q-1)$, where $q = p^r$. Fix this value of q. Denote $H_\alpha^{(q)}$ by H_α.

THEOREM 4.1 (BIS). *Let $\Gamma = \mathrm{Gal}(L/K)$ be a cyclic group of order $n\,|\,(q-1)$, where $L = K(f^{1/n})$ for some $f \in K^*$. Let $\pi\colon Y \to X$ be the corresponding finite morphism. Then*

$$\pi_* F_q \cong \bigoplus_{i=0}^{n-1} g_* H_{\alpha^i},$$

for $\alpha \in f^{(q-1)/n}$. //

The proof is easy to change.

COROLLARY 4.2 (BIS). *Let σ_X, σ_Y denote the p-ranks of X, Y respectively. Then*

$$1-\sigma_Y = 1-\sigma_X + \sum_{i=1}^{n-1} \chi(X, g_* H_{\alpha^i}).$$

PROOF. Take cohomology in 4.1 (bis) and use Corollary AI.2. Note that $g_* H_1 = g_* F_q \cong F_q$. //

REMARK. The above result provides the method of addressing the case $n \nmid (p-1)$ indicated in Remark (2) following Corollary 4.3.

In the paragraph immediately preceding Corollary 4.4, replace $p-1$ by $q-1$. In the statement of Corollary 4.4, replace F_p by F_q and $g_* G_{f_C,0}^{n_C}$ by $g_* H_{f_C}^{n_C}$, where $n_C = (q-1)/d_C$. The proof presents no difficulty.

EXAMPLES. We now compute the p-ranks of several curves over P^1 to illustrate Theorem 4.2 (bis).

Let $p = 5$ and L/K be Galois of degree $n = 3$, so $n\,|\,(q-1)$ for $q = p^2 = 25$. The number t of ramification points of the corresponding curve Y is $g_Y + 2$. The matrix $A_q(\alpha^{-i})$ is an $(m-1)\times(m-1)$ matrix, where $m(q-1) \geqslant \deg(\alpha^{-i})$. Assume the ground field k is algebraically closed of characteristic p.

(1) Let $g_Y = 3$ and assume Y is the projective completion of the curve with affine equation

$$y^3 = (1-x^4)^{-1};$$

it is ramified over the point at infinity. Then $\alpha^{-i} = (1-x^4)^{8i}$ for $i = 0, 1, 2$. For $i = 1$, we get $m = 2$ and $A_q(\alpha^{-1}) = d_{24} = 3$. Thus, $\sigma_q(\alpha^{-1}) = 1$. For $i = 2$, the value of m is 3. The 2×2 matrix $A_q(\alpha^{-2})$ has entries

$$\begin{pmatrix} d_{24} & d_{49} \\ d_{23} & d_{48} \end{pmatrix}.$$

The only nonzero entry is $d_{24} = 3$ and $\sigma_q(\alpha^{-2}) = 1$. Hence $\sigma_Y = 2 < g_Y$ and Y is not ordinary.

(2) Let $g_Y = 4$ and the affine equation be

$$y^3 = (1-x^6)^{-1},$$

which is unramified over the point at infinity. We have $\alpha^{-i} = (1 - x^6)^{8i}$ for $i = 0, 1, 2$. If $i = 1$, then $m = 2$ and $A_q(\alpha^{-1}) = d_{24} = 0$; this gives $\sigma_q(\alpha^{-1}) = 0$. The value of m is 4 when $i = 2$. The 3×3 matrix $A_q(\alpha^{-2})$ has entries

$$\begin{pmatrix} d_{24} & d_{49} & d_{74} \\ d_{23} & d_{48} & d_{73} \\ d_{22} & d_{47} & d_{72} \end{pmatrix}.$$

It is the 0-matrix so $\sigma_q(\alpha^{-2}) = 0$. This gives $\sigma_Y = 0 < g_Y$ and Y is supersingular.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF THE DISTRICT OF COLUMBIA, WASHINGTON, D. C. 20008