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ON ROOT INVARIANTS OF PERIODIC CLASSES IN $\mathrm{Ext}_A(\mathbf{Z}/2, \mathbf{Z}/2)$

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ABSTRACT. We prove that if a class in the cohomology of the mod 2 Steenrod algebra is v_n-periodic in the sense of [10], then its root invariant must be v_{n+1}-periodic, where v_n denotes the nth generator of $\pi_*(\mathrm{BP})$.

1. Introduction and statement of results. This work is motivated by a desire to understand the relationship between two ideas of recent interest in stable homotopy theory. The first concept is that of v_i-periodicity in stable homotopy, which has been extensively studied in the setting of the Novikov spectral sequence [**11, 12**]. In [**2**], a start was made toward studying this phenomenon in the setting of the classical Adams spectral sequence (clASS). This study was continued in [**10**], where complete definitions were given of the notion of v_i-periodicity in $\mathrm{Ext}_A(\mathbf{Z}/2, \mathbf{Z}/2)$. The second idea is that of the root invariant. This invariant is defined using W.-H. Lin's theorem, which relates the stable homotopy of spheres with that of projective spaces. Complete definitions can be found in [**9** and **12**].

To state our results, we need to review the definitions and earlier results involved. The following theorems and definitions can be found in [**10**]. We recall that a class $x \in R$, a commutative ring, is said to be a nonzero divisor if $rx^n \neq 0$ for all nonzero $r \in R$, and all $n \in \mathbf{N}$. Here A denotes the Steenrod algebra at the prime 2, and A_i denotes the Hopf subalgebra generated by $\{\mathrm{Sq}^0, \mathrm{Sq}^1, \mathrm{Sq}^2, \ldots, \mathrm{Sq}^{2^i}\}$. Let Q_i denote the ith Milnor generator and $E(Q_i)$ be the exterior algebra over $\mathbf{Z}/2$.

THEOREM (1.1). *For each $i \geqslant 1$, there exists a unique nonzero divisor $w_i \in \mathrm{Ext}_{A_i}^{2^{i+1}, 2^{i+1}(2^{i+1}-1)}(\mathbf{Z}/2, \mathbf{Z}/2)$ such that w_i restricts nontrivially to $\mathrm{Ext}_{E(Q_i)}(\mathbf{Z}/2, \mathbf{Z}/2)$ and corresponds to the class $v_i^{2^{i+1}} \in \pi_*(\mathrm{BP})$.*

We hereafter use the notation $v_i^{2^{i+1}} \in \mathrm{Ext}_{A_i}(\mathbf{Z}/2, \mathbf{Z}/2)$. We henceforth suppress the second module in $\mathrm{Ext}(M, N)$ whenever it is $\mathbf{Z}/2$.

For $k > i$, there is also some power of $v_i^{2^{i+1}}$ present in $\mathrm{Ext}_{A_k}(\mathbf{Z}/2)$. In fact, we have the following result.

THEOREM (1.2). *For k any positive integer, there exist positive integers $N_1, N_2, \ldots, N_k$ such that*

$$\mathbf{Z}/2\left[h_0, v_1^{4N_1}, v_2^{8N_2}, \ldots, v_i^{2^{i+1}N_i}, \ldots, v_k^{2^{k+1}N_k}\right] \subset \mathrm{Ext}_{A_k}(\mathbf{Z}/2).$$

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Note that the integer N_i also depends upon k. Note also that N_k can be chosen to be 1 by Theorem 1.1. It should be mentioned that although $v_i^{2^{i+1}}$ and its multiples are classes in $\mathrm{Ext}_{A_i}(\mathbf{Z}/2)$, $v_i^{2^{i+1}N_i}$ is a coset in $\mathrm{Ext}_{A_k}(\mathbf{Z}/2)$, for $k > i$. Theorem (1.2) follows easily from a theorem of Lin [**6**]. In particular, Theorem (1.2) implies that for all $k \geqslant i$, $\mathbf{Z}/2[v_i^{2^{i+1}N_i}] \subset \mathrm{Ext}_{A_k}(\mathbf{Z}/2)$. For each $k \geqslant i$, we localize $\mathrm{Ext}_{A_k}(\mathbf{Z}/2)$ with respect to v_i. Since $\mathrm{Ext}_A(\mathbf{Z}/2) = \varprojlim_k \mathrm{Ext}_{A_k}(\mathbf{Z}/2)$ this gives a map

$$f_i\colon \mathrm{Ext}_A^{s,t}(\mathbf{Z}/2) \to \varprojlim_k \left[\mathrm{Ext}_{A_k}^{s,t}(\mathbf{Z}/2)(v_i^{-1})\right],$$

which enables us to define the following concept.

DEFINITION (1.3). A class $x \in \mathrm{Ext}_A(\mathbf{Z}/2)$ is v_i-periodic if $f_i(x) \neq 0$, and is v_i-torsion otherwise.

Notice that this definition is equivalent to the following: if $q_k^*\colon \mathrm{Ext}_A(\mathbf{Z}/2) \to \mathrm{Ext}_{A_k}(\mathbf{Z}/2)$ denotes the usual restriction then $x \in \mathrm{Ext}_A(\mathbf{Z}/2)$ is v_i-periodic if there exists a $K > 0$ such that $q_k^*(x)(v_i^{2^{i+1}N_i})^s \neq 0$ for all $s \geqslant 0$ and $k \geqslant K$.

The main result of [**10**] is

THEOREM (1.4). *If $x \in \mathrm{Ext}_A(\mathbf{Z}/2)$ is v_n-periodic, then x is also v_{n+k}-periodic for all $k \geqslant 0$.*

Equivalently, if $x \in \mathrm{Ext}_A(\mathbf{Z}/2)$ is v_n-torsion, then x is also v_k-torsion, for all k such that $0 \leqslant k \leqslant n$.

In the setting of $\mathrm{BP}_*\mathrm{BP}$-comodules, this result is due to Johnson and Yosimura [**3**]. Theorem (1.4) allows one to prove

COROLLARY (1.5). *There is a filtration, which we call the chromatic filtration,*

$$\mathrm{Ext}_A(\mathbf{Z}/2) = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_i \subset \cdots$$

such that $F_i - F_{i+1}$ is the set of classes that are v_{i+1}-periodic but v_k-torsion for all $k \leqslant i$.

The second major concept that we deal with here is the root invariant, first defined in [**8**]. This invariant is constructed using W.-H. Lin's theorem [**4**]. To state this, let $\mathbf{R}P_{-k}$ denote the Thom spectrum of $-k$ times the canonical line bundle over $\mathbf{R}P^\infty$. Let $\mathbf{P}$ denote the direct limit $\varinjlim_{j\to-\infty} H^*(\mathbf{R}P_j; \mathbf{Z}/2)$. Here, $\mathbf{P}$ is isomorphic to the ring of Laurent series $\mathbf{Z}/2[x, x^{-1}]$, where $|x| = 1$. The Steenrod algebra action is given by $\mathrm{Sq}^i x^j = \binom{j}{i} x^{i+j}$.

THEOREM (1.6) (LIN'S THEOREM). *The inverse limit $\varprojlim_{j\to-\infty} \pi_*^s(\mathbf{R}P_j) \cong \pi_*^s(S^{-1})$. Also, $\mathrm{Ext}_A(\mathbf{P}) \cong \mathrm{Ext}_A(\Sigma^{-1}\mathbf{Z}/2)$.*

We use the following ideas to define the root invariant. Let P_m denote the A-module $H^*(\mathbf{R}P_m; \mathbf{Z}/2)$, where m is any integer. We have a map of A-modules $j_m\colon P_m \to \Sigma^m\mathbf{Z}/2$, induced from the map generating $\pi_m(\mathbf{R}P_m)$. There is also a map $k_m\colon P_m \to \mathbf{P}$, given by the system of maps $\mathbf{R}P_{m-k} \to \mathbf{R}P_m$ which collapse the bottom k cells of $\mathbf{R}P_{m-k}$. With these conventions, we define the root invariant as follows: for $a \in \mathrm{Ext}_A^{s,t}(\mathbf{Z}/2)$, we may regard a as living in $\mathrm{Ext}_A^{s,t-1}(\mathbf{P})$. There exists a maximal integer N such that $k_N^*(a) \neq 0$ in $\mathrm{Ext}_A^{s,t-1}(P_N)$. We then define the root invariant of a, $R(a)$, to be the coset given by

$$R(a) = \left\{ y \in \mathrm{Ext}_A^{s,t-1}(\Sigma^N \mathbf{Z}/2) \colon j_N^*(y) = k_N^*(a) \right\}.$$

Note that N will always be negative, for $s > 0$, by the proof of the algebraic Kahn-Priddy theorem [**5**], so that R will preserve the s-filtration and raise the $(t - s)$-filtration of a class. The diagram one should have in mind is

$$(1.7) \qquad \begin{array}{ccc} \mathrm{Ext}_A^{s,t}(\mathbf{Z}/2) & \xrightarrow{\cong} & \mathrm{Ext}_A^{s,t-1}(\mathbf{P}) \\ \wr\downarrow R & & \\ \mathrm{Ext}_A^{s,t-N-1}(\mathbf{Z}/2) & & k_N^* \Bigg\downarrow \begin{array}{l} \text{Largest } N \text{ s.t.} \\ k_N^*(a) \neq 0. \end{array} \\ \updownarrow \cong & & \\ \mathrm{Ext}_A^{s,t-1}(\Sigma^N \mathbf{Z}/2) & \xrightarrow{j_N^*} & \mathrm{Ext}_N^{s,t-1}(P_N) \end{array}$$

For $\alpha \in \pi_*^s(S^0)$, we define the geometric root invariant of α, $R_G(\alpha)$, in a similar manner. This geometric root invariant appears as the "Mahowald invariant" in [**13**]. Calculations of $R(a)$ for $a \in \mathrm{Ext}_A^{s,t}(\mathbf{Z}/2)$, $t - s \leqslant 16$, have appeared in [**9**].

The goal of this paper is to prove the following result, which links these concepts of v_i-periodicity and root invariants.

THEOREM A. *Let $a \in \mathrm{Ext}_A^{s,t}(\mathbf{Z}/2)$ be v_i-periodic in the sense of Definition* (1.3). *Then the root invariant of a, $R(a)$, is v_{i+1}-periodic.*

The geometric version of this result was conjectured by Mahowald and Ravenel, and has been attacked by Hopkins and Wegmann using techniques from the proof of the Nilpotence Theorem. Also, this result seems closely tied into the notion of smooth linear $\mathbf{Z}/p$ actions on exotic spheres, as the work of Schultz and Stolz in [**14** and **16**] points out.

The proof of this theorem uses the machinery of Koszul-type resolutions, presented in [**1**], together with the techniques used in the proof of Lin's theorem, found in [**7**]. The major concept in [**7**] is the following splitting of A-modules, due to Davis and Mahowald:

(1.8)

$$\text{(Davis-Mahowald splitting)}: \quad \gamma_i \colon A \otimes_{A_i} \mathbf{P}/F_m \xrightarrow{\cong} \bigoplus_{k \geqslant m} \Sigma^{k2^{i+1}-1}\left(A \otimes_{A_{i-1}} \mathbf{Z}/2\right)$$

where F_m is the A_i-submodule generated by $\{x^j \in \mathbf{P} \colon j < m\}$. This gives a splitting in Ext, after the change of rings isomorphism and taking the limit as m goes to minus infinity:

$$(1.9) \qquad \gamma_i^* \colon \bigoplus_{k \in \mathbf{Z}} \Sigma^{k2^{i+1}-1} \mathrm{Ext}_{A_{i-1}}(\mathbf{Z}/2) \xrightarrow{\cong} \mathrm{Ext}_{A_i}(\mathbf{P}).$$

We use this to define the ith root invariant

$$R_i \colon \mathrm{Ext}_{A_{i-1}}(\mathbf{Z}/2) \rightsquigarrow \mathrm{Ext}_{A_i}(\mathbf{Z}/2)$$

by including $\mathrm{Ext}_{A_{i-1}}(\Sigma^{-1}\mathbf{Z}/2)$ in as the -1 summand in $\mathrm{Ext}_{A_i}(\mathbf{P})$. By taking inverse limits, we can now analyze the root invariant

$$R\colon \mathrm{Ext}_A^{s,t}(\mathbf{Z}/2) \rightsquigarrow \mathrm{Ext}_A^{s,t-N-1}(\mathbf{Z}/2)$$

in terms of these R_i's. Davis and Mahowald have completely calculated

$$R_2\colon \mathrm{Ext}_{A_1}(\mathbf{Z}/2) \rightsquigarrow \mathrm{Ext}_{A_2}(\mathbf{Z}/2)$$

in [**1**]. Recalling that for all $i > 0$, we have $v_i^{2^{i+1}} \in \mathrm{Ext}_{A_i}(\mathbf{Z}/2)$, with all of its powers nonzero, we prove the following result.

THEOREM B. *For the class* $v_{i-1}^{2^{i+1}} \in \mathrm{Ext}_{A_{i-1}}(\mathbf{Z}/2)$, *we have* $R_i(v_{i-1}^{2^{i+1}}) = v_i^{2^{i+1}} \in \mathrm{Ext}_{A_i}(\mathbf{Z}/2)$. *Also,* $R_1(h_0^4) = v_1^4 \in \mathrm{Ext}_{A_1}(\mathbf{Z}/2)$.

This is proved using the Koszul spectral sequence, a tool first presented in [**1**]. A brief summary is given in §2.

The paper is organized as follows: in §2, we analyze the Koszul spectral sequence used to calculate $\mathrm{Ext}_{A_i}(\mathbf{P})$. In §3 we prove Theorem B. Finally we prove Theorem A in §4. Throughout the paper, we use homology and cohomology with $\mathbf{Z}/2$ coefficients. By "spectrum", we mean a connective spectrum localized at the prime 2. The author would like to thank Stewart Priddy, Don Davis, and especially Mark Mahowald for many helpful conversations that helped to produce this paper, and also the referee for his helpful comments. The main results of this paper form part of the author's Ph.D. thesis, completed at Northwestern University in 1984 under the direction of Mark Mahowald [**15**].

2. Calculation of the Koszul spectral sequence for $\mathrm{Ext}_{A_i}(\mathbf{P}, \mathbf{Z}/2)$. In this section, we explicitly calculate the Koszul spectral sequence for $\mathrm{Ext}_{A_i}(\mathbf{P}, \mathbf{Z}/2)$ in terms of the components given by the Davis-Mahowald splitting (1.9). We first briefly review the construction of the Koszul spectral sequence, which first appeared in [**1**]. For complete details, see [**10**].

The Koszul spectral sequence (hereafter abbreviated KSS) is a tool which can be used to calculate $\mathrm{Ext}_{A_i}(M)$, where M is any A_i-module, in terms of $\mathrm{Ext}_{A_{i-1}}(\_\_, \mathbf{Z}/2)$ of certain modules. We construct an exact complex of A_i-modules and apply the functor $\mathrm{Ext}_{A_i}(\_\_, \mathbf{Z}/2)$ to it.

To construct this complex, we exploit the following fact about the mod 2 Steenrod algebra: $(A_i \otimes_{A_{i-1}} \mathbf{Z}/2)^* \cong E(\zeta_1^{2^i}, \zeta_2^{2^{i-1}}, \ldots, \zeta_{i+1})$, both as algebras and as left A-modules, where $E(\ )$ denotes an exterior algebra over the field $\mathbf{F}_2$. Here ζ_k is $\chi(\xi_k)$, the conjugate of the kth Milnor generator. The Steenrod algebra action is given on the right by $(\zeta_{i+1-j}^{2^k})\mathrm{Sq}^{2^k} = \zeta_{i-j}^{2^{k+1}}$ and $(\zeta_1^{2^i})\mathrm{Sq}^{2^i} = 1$, extended by the Cartan formula. For convenience, we denote $(A_i \otimes_{A_{i-1}} \mathbf{Z}/2)^* \cong (A_i//A_{i-1})^*$ by $E(i)$. $E(i)$ is an A_i-module but not an A-module. We decompose $E(i)$ as an $\mathbf{F}_2$ vector space into a direct sum $E(i) \cong \oplus_{\sigma \geqslant 0} E_\sigma(i)$, where $E_\sigma(i)$ is the $\mathbf{F}_2$ vector space spanned by the monomials of length σ in $\{\zeta_1^{2^i}, \zeta_2^{2^{i-1}}, \ldots, \zeta_{i+1}\}$. Each of these $E_\sigma(i)$'s is closed under the A_{i-1}-action inherited from $E(i)$, so the decomposition holds as an A_{i-1}-module.

We resolve this exterior algebra by using pieces of a polynomial algebra. Let $R(i) = \mathbf{Z}/2[\zeta_1^{2^i}, \zeta_2^{2^{i-1}}, \ldots, \zeta_{i+1}]$. This is a right A-module, with the same action as $E(i)$. We can decompose $R(i)$ as an A_{i-1}-module $R(i) \cong \oplus_{\sigma \geqslant 0} R_\sigma(i)$, where $R_\sigma(i)$

is the $\mathbf{F}_2$ vector space spanned by monomials of length s in $\{\zeta_1^{2^i}, \zeta_2^{2^{i-1}}, \dots, \zeta_{i+1}\}$. Each of the R_σ's is an A-module.

To construct the resolution, form the tensor product $E_r(i) \overset{\triangle}{\otimes}_{\mathbf{Z}/2} R_s(i)$, a right A-module with the action given by the Cartan formula. (We will abbreviate this by $E_r \otimes R_s$.) Define $k_{r,s}$: $E_r \otimes R_s \to E_{r-1} \otimes R_{s+1}$ by

$$k_{r,s}(x_1 x_2 \cdots x_r \otimes p) = \sum_{j=1}^{r} x_1 x_2 \cdots \hat{x}_j \cdots x_r \otimes x_j p, \quad \text{for all } r \geqslant 1,\ s \geqslant 0,$$

where each x_k is an element of $\{\zeta_1^{2^i}, \zeta_2^{2^{i-1}}, \dots, \zeta_{i+1}\}$ and p is a polynomial of these. Each $k_{r,s}$ is an A_i-module map. Composing these we get an exact sequence

$$0 \to E_{i+1} \otimes R_s \to E_i \otimes R_{s+1} \to \cdots \to E_0 \otimes R_{s+i+1} \to 0.$$

Summing these sequences over a constant s, we obtain

$$0 \to \mathbf{Z}/2 \to E(i) \otimes R_0(i) \xrightarrow{d_0} E(i) \otimes R_1(i) \xrightarrow{d_1} \cdots$$

which is an exact sequence. The differential is given by

$$d_\sigma[(x_1 x_2 \cdots x_r) \otimes p] = \sum_{j=1}^{r} (x_1 x_2 \cdots \hat{x}_j \cdots x_r) \otimes x_j p.$$

Denote the dual of $R_\sigma(i)$ by $N_\sigma(i)$. Then, dualizing the exact sequence above, we have

THEOREM (2.1). *For $i > 0$ there exists a family of A-modules, $N_\sigma(i)$, $\sigma \geqslant 0$, defined above, and A_i-module maps δ_σ: $A_i \otimes_{A_{i-1}} N_{\sigma+1}(i) \to A_i \otimes_{A_{i-1}} N_\sigma(i)$, such that the sequence*

$$0 \leftarrow \mathbf{Z}/2 \leftarrow A_i \otimes_{A_{i-1}} N_0(i) \leftarrow A_i \otimes_{A_{i-1}} N_1(i) \leftarrow \cdots \leftarrow A_i \otimes_{A_{i-1}} N_\sigma(i) \leftarrow \cdots$$

is exact as a sequence of A_i-modules.

We refer to this as the Koszul-type resolution of $\mathbf{Z}/2$ over A_i ($KR_i(\mathbf{Z}/2)$ or just KR if i is understood).

Applying the functor $\mathrm{Ext}_{A_i}^{s-\sigma,t}(\underline{\ \ })$ to the complex, we obtain

THEOREM (2.2). *For i any positive integer, there is a family of A-modules, $N_\sigma(i)$, $\sigma \geqslant 0$, defined above, such that for any A_i-module M there is a trigraded spectral sequence converging to* $\mathrm{Ext}_{A_i}^{s,t}(M)$, *with* $E_1^{\sigma,s,t} \cong \mathrm{Ext}_{A_{i-1}}^{s-\sigma,t}(N_\sigma(i) \otimes M)$.

This is called the Koszul spectral sequence for M over A_i ($\mathrm{KSS}_i(M)$). Note that a trigraded spectral sequence is a family of spectral sequences, one for each positive integer t.

We use this KSS to calculate part of $\mathrm{Ext}_{A_i}(\mathbf{P})$. We recall that the Davis-Mahowald splitting of $A_i \otimes_{A_{i-1}} \mathbf{P}/F$ (1.8) yields, after change of rings and limits,

$$\text{(1.9)} \qquad \gamma_i^*\colon \bigoplus_{k \in \mathbf{Z}} \Sigma^{k2^{i+1}-1} \mathrm{Ext}_{A_{i-1}}(\mathbf{Z}/2) \xrightarrow{\cong} \mathrm{Ext}_{A_i}(\mathbf{P}).$$

There are KSS's converging to both sides of this isomorphism. To the right-hand side, we have

$$(2.3)\qquad E_1^{\sigma,s,t} = \operatorname{Ext}_{A_i}^{s-\sigma,t}\left(A_i \otimes_{A_{i-1}} N_\sigma(i) \otimes \mathbf{P}\right) \cong \operatorname{Ext}_{A_{i-1}}^{s-\sigma,t}\left(N_\sigma(i) \otimes \mathbf{P}\right),$$

converging to $E_\sigma^0 \operatorname{Ext}_{A_i}^{s,t}(\mathbf{P})$. On the left-hand side we have

$$(2.4)\qquad \overline{E}_1^{\sigma,s,t} = \operatorname{Ext}_{A_{i-1}}^{s-\sigma,t}\left(A_{i-1} \otimes_{A_{i-2}} N_\sigma(i-1) \otimes \left[\bigoplus_{k\in\mathbf{Z}} \Sigma^{k2^{i+1}-1}\mathbf{Z}/2\right]\right)$$
$$\cong \operatorname{Ext}_{A_{i-2}}^{s-\sigma,t}\left(\bigoplus_{k\in\mathbf{Z}} \Sigma^{k2^{i+1}-1} N_\sigma(i-1)\right),$$

converging to $E_\sigma^0 \operatorname{Ext}_{A_{i-1}}^{s,t}(\oplus_{k\in\mathbf{Z}} \Sigma^{k2^{i+1}-1}\mathbf{Z}/2)$.

We now explicitly relate these two E_1 terms, using the following two lemmas.

LEMMA (2.5). *As an A_{i-2}-module, $N_\sigma(i)$ splits as a direct sum*

$$N_\sigma(i) \cong \bigoplus_{k=0}^{\sigma} \Sigma^{2^i\sigma} N_k(i-1).$$

PROOF. Assume that $N_{\sigma-1}(i)$ splits in this fashion. We recall that the dual of $N_\sigma(i)$, $R_\sigma(i)$, is given as the vector space spanned by monomials of length σ in $\{\zeta_1^{2^i}, \zeta_2^{2^{i-1}}, \ldots, \zeta_{i+1}\}$. Now $R_\sigma(i)$ automatically contains a copy of $\zeta_1^{2^i} \cdot R_{\sigma-1}(i)$, since $\zeta_1^{2^i} \cdot m$ is of length σ if m is of length $\sigma - 1$. Further, considered as an A_{i-2}-module, this copy of $\zeta_1^{2^i} \cdot R_{\sigma-1}(i)$ splits off as a direct sum, since no $\zeta_1^{2^i} \cdot m$ can be a target of Sq^{2^j} for $j \leqslant i-1$. This shows that $R_\sigma(i) \cong \zeta_1^{2^i} \cdot R_{\sigma-1}(i) \oplus M$, where M is given by all monomials having no factor of $\zeta_1^{2^i}$ in them. Thus M is given as $R_\sigma(i-1)$, under the doubling homomorphism $\zeta_n \to \zeta_{n+1}$. This raises dimension by $2^i \cdot \sigma$, so that $R_\sigma(i) \cong \Sigma^{2^i\sigma} R_\sigma(i-1) \oplus \Sigma^{2^i} R_{\sigma-1}(i)$. By our inductive hypothesis, we have the result. The $i = 1$ case that initiates the induction can be readily computed by hand (see [**15**]).

LEMMA (2.6). *Let M be any finite A-module. Then*

$$\operatorname{Ext}_{A_{i-1}}\left(\bigoplus_{k\in\mathbf{Z}} \Sigma^{k2^{i+1}-1} M\right) \xrightarrow{\cong} \operatorname{Ext}_{A_i}\left(M \otimes_{\mathbf{Z}/2} \mathbf{P}\right).$$

PROOF. If we tensor the Davis-Mahowald splitting (1.8) with M we get

$$\gamma_i\colon \bigoplus_{k\geqslant m} \Sigma^{k2^{i+1}-1}\left(A \otimes_{A_{i-1}} \mathbf{Z}/2\right) \otimes M \xrightarrow{\cong} A \otimes_{A_i} \mathbf{P}/F_m \otimes M.$$

Applying $\operatorname{Ext}_A(\underline{\quad})$, the change of rings theorem gives us the result. Note that it is necessary that M be an A-module, not just an A_i-module. This is the case for the $N_\sigma(i)$'s which we shall use.

We now examine the relationship between the two E_1 terms given earlier. On the RHS:

$$(2.7)\qquad E_1^{\sigma,s,t} \cong \operatorname{Ext}_{A_{i-2}}^{s-\sigma,t}\left(\bigoplus_{m\in\mathbf{Z}} \Sigma^{m2^i-1} N_\sigma(i)\right) \quad \text{by Lemma (2.6)}$$
$$\cong \operatorname{Ext}_{A_{i-2}}^{s-\sigma,t}\left(\bigoplus_{m\in\mathbf{Z}} \Sigma^{m2^i-1}\left[\bigoplus_{j=0}^{\sigma} \Sigma^{2^i\sigma} N_j(i-1)\right]\right) \quad \text{by Lemma (2.5)}.$$

So the E_1 term for the $\mathrm{Ext}_{A_i}(\mathbf{P})$ contains all of the $\bar{E}_1$ term for $\mathrm{Ext}^{s,t}_{A_{i-1}}(\oplus_{k\in\mathbf{Z}}\Sigma^{k2^{i+1}-1}\mathbf{Z}/2)$, plus a considerable amount of excess. The diagram that one should have in mind is

(2.8)

$$\begin{array}{lcr}
E_1 = \bigoplus_\sigma \mathrm{Ext}^{s-\sigma,t}_{A_i}\left(A_i \otimes_{A_{i-1}} N_\sigma(i) \otimes \mathbf{P}\right) & \overset{\mathrm{KSS}}{\Rightarrow} & \mathrm{Ext}^{s,t}_{A_i}(\mathbf{P}) \\
j \updownarrow \cong \text{ by Lemmas (2.5) and (2.6)} & & \\
\bigoplus_\sigma \mathrm{Ext}^{s-\sigma,t}_{A_{i-2}}\left(\bigoplus_{m\in\mathbf{Z}} \Sigma^{m2^i-1}\left[\bigoplus_{j=0}^{\sigma} \Sigma^{2^i\sigma} N_j(i-1)\right]\right) & & \gamma_i^* \Big\downarrow \cong \\
\updownarrow i & & \\
E_1 = \bigoplus_\sigma \mathrm{Ext}^{s-\sigma,t}_{A_{i-1}}\Big(A_{i-1} \otimes_{A_{i-2}} N_\sigma(i-1) & & \\
\qquad \otimes \left[\bigoplus_{k\in\mathbf{Z}} \Sigma^{k2^{i+1}-1}\mathbf{Z}/2\right]\Big) & \overset{\mathrm{KSS}}{\Rightarrow} & \mathrm{Ext}^{s,t}_{A_{i-1}}\left(\bigoplus_{k\in\mathbf{Z}} \Sigma^{k2^{i+1}-1}\mathbf{Z}/2\right).
\end{array}$$

In particular, by Lemma (2.5), $N_{2^{i+1}}(i) \cong \Sigma^{2^i 2^{i+1}} N_{2^{i+1}}(i-1) \oplus (\text{other terms})$, where $\zeta_{i+1}^{2^{i+1}}$ corresponds to $\Sigma^{2^i 2^{i+1}}\zeta_i^{2^{i+1}}$, as the top cell in each module. Let g_k denote the class in $\mathrm{Ext}^0_{A_i}(\mathbf{P})$, so that g_k is nonzero for $k \equiv -1 \pmod{2^{i+1}}$. Let ι_k denote the analogous class in $\mathrm{Ext}^0_{A_{i-1}}(\oplus\Sigma\mathbf{Z}/2)$, also nonzero for $k \equiv -1 \pmod{2^{i+1}}$. Then in diagram (2.8) one observes that $i(\zeta_i^{2^{i+1}} \cdot \iota_{2^i 2^{i+1}-1}) = \zeta_{i+1}^{2^{i+1}} \cdot g_{-1}$ in Ext^0.

3. Calculation of $R_i(v_{i-1}^{2^{i+1}})$. We recall the definition of the ith root invariant, R_i. It uses the Davis-Mahowald splitting (1.9)

$$\gamma_i^*\colon \bigoplus_{k\in\mathbf{Z}} \Sigma^{k2^{i+1}-1}\mathrm{Ext}_{A_{i-1}}(\mathbf{Z}/2) \xrightarrow{\cong} \mathrm{Ext}_{A_i}(\mathbf{P}).$$

This splitting commutes with the natural projections [7, (1.5)]. We use this, together with the maps j_m and k_m defined in §1 (now thoght of as A_i-module maps), to define R_i as follows:

$$(3.1)\qquad \begin{array}{ccc}
\mathrm{Ext}^{s,t}_{A_{i-1}}(\Sigma^{-1}\mathbf{Z}/2) & \to & \mathrm{Ext}^{s,t}_{A_i}(\mathbf{P}) \\
\wr\downarrow R_i & & \\
\mathrm{Ext}^{s,t-N}_{A_i}(\mathbf{Z}/2) & & k_N^* \Big\downarrow \text{largest } N \\
\updownarrow\cong & & \\
\mathrm{Ext}^{s,t}_{A_i}(\Sigma^N\mathbf{Z}/2) & \xrightarrow{j_N^*} & \mathrm{Ext}^{s,t}_{A_i}(P_N)
\end{array}$$

Thus $R_i\colon \mathrm{Ext}_{A_{i-1}}(\mathbf{Z}/2) \rightsquigarrow \mathrm{Ext}_{A_i}(\mathbf{Z}/2)$ is given by $R_i(a) = \{b \in \mathrm{Ext}_{A_i}(\Sigma^N\mathbf{Z}/2)\colon j_N^*(b) = k_N^*(a)$, where N is the maximal integer s.t. $k_N^*(a) \neq 0\}$.

To calculate $R_i(v_{i-1}^{2^{i+1}})$, we need to recall the construction of the classes $v_i^{2^{i+1}} \in \mathrm{Ext}_{A_i}(\mathbf{Z}/2)$, as given in [**10**, §2]. In the Koszul-type resolution (2.2), the top class $(\zeta_{i+1}^{2^{i+1}})^* \in N_\sigma(i)$, for $\sigma = 2^{i+1}$, can be split off by A_i-module maps

$$\Sigma^t\mathbf{Z}/2 \xrightarrow{h} N_\sigma(i) \xrightarrow{g} \Sigma^t\mathbf{Z}/2, \quad \text{where } t = 2^{i+1}(2^{i+1}-1).$$

This leads to a splitting of complexes

(3.2)

$$\begin{array}{ccccc} & & A_i \otimes_{A_{i-1}} N_0(i) & \leftarrow & A_i \otimes_{A_{i-1}} N_1(i) & \leftarrow \cdots \\ & & \downarrow \hat{h} & & \downarrow h_1 & \\ 0 \leftarrow \mathbf{Z}/2 \leftarrow A_i \otimes_{A_{i-1}} N_0(i) \leftarrow \cdots & \leftarrow & A_i \otimes_{A_{i-1}} N_\sigma(i) & \leftarrow & A_i \otimes_{A_{i-1}} N_{\sigma+1}(i) & \leftarrow \cdots \\ & & \downarrow \hat{g} & & \downarrow g_1 & \\ & & A_i \otimes_{A_{i-1}} N_0(i) & \leftarrow & A_i \otimes_{A_{i-1}} N_1(i) & \leftarrow \cdots \end{array}$$

The map $\hat{g}$ corresponds to a class $g' \in \operatorname{Hom}'_{A_i}(A_i \otimes_{A_{i-1}} N_\sigma(i)) = \operatorname{Ext}^{0,t}_{A_{i-1}}(N_\sigma(i))$, where $\sigma = 2^{i+1}$ and $t = 2^{i+1}(2^{i+1} - 1)$. This is the $E_1^{\sigma, 2^{i+1}, t}$ term of the KSS. In [10], it is shown that this class is a nonbounding cycle in the KSS and projects to the class $w_i \in \operatorname{Ext}^{2^{i+1}, 2^{i+1}(2^{i+1}-1)}_{A_i}(\mathbf{Z}/2)$ of Theorem (1.1).

Recall that

$$g_{k2^{i+1}-1} \in \operatorname{Ext}^{0,2^{i+1}-1}_{A_i}(\mathbf{P}), \quad \iota_{k2^{i+1}-1} \in \bigoplus_{k \in \mathbf{Z}} \operatorname{Ext}^{0,2^{i+1}-1}_{A_{i-1}}\left(\Sigma^{k2^{i+1}-1}\mathbf{Z}/2\right)$$

denote the appropriate nonzero classes. There is a Yoneda product in $\operatorname{Ext}_{A_i}(\mathbf{P})$ given by the pairing $\operatorname{Ext}^{s,t}_{A_i}(\mathbf{Z}/2) \otimes \operatorname{Ext}^{s',t'}_{A_i}(\mathbf{P}) \to \operatorname{Ext}^{s+s',t+t'}_{A_i}(\mathbf{P})$. In particular, there is a class given by $v_i^{2^{i+1}} \otimes g_{-1}$ which we will denote by $v_i^{2^{i+1}} g_{-1} \in \operatorname{Ext}^{2^{i+1}, t-1}_{A_i}(\mathbf{P})$ where t is as above. This class is nonzero. In fact this class can be constructed by tensoring the above diagram (3.2) with the A_i-module $\mathbf{P}$.

$$\begin{array}{ccccccc} & & A_i \otimes_{A_{i-1}} N_0(i) \otimes P & \leftarrow & A_i \otimes_{A_{i-1}} N_1(i) \otimes P & \leftarrow \cdots \\ & & \downarrow & & \downarrow & \\ (3.3) & \cdots \leftarrow & A_i \otimes_{A_{i-1}} N_\sigma(i) \otimes P & \leftarrow & A_i \otimes_{A_{i-1}} N_{\sigma+1}(i) \otimes P & \leftarrow \cdots \\ & & \downarrow & & \downarrow & \\ & & A_i \otimes_{A_{i-1}} N_0(i) \otimes P & \leftarrow & A_i \otimes_{A_{i-1}} N_1(i) \otimes P & \leftarrow \cdots \end{array}$$

Thus, the nonzero class in $\operatorname{Ext}^{0,-1}_{A_{i-1}}(\mathbf{P})$ corresponds to a nonzero class in

$$\operatorname{Ext}^{0,-1}_{A_{i-1}}(N_\sigma(i) \otimes \mathbf{P}),$$

where $\sigma = 2^{i+1}$ as before. This class must be a nonbounding cycle in the $\mathrm{KSS}_i(P)$ by the proof of Theorem A in [10]. This nontrivial class is exactly $v_i^{2^{i+1}} g_{-1} \in \operatorname{Ext}_{A_i}(\mathbf{P})$, as one can check by simply chasing the Yoneda pairing on the E_1 term of the KSS. We now calculate $\gamma_i^*(v_i^{2^{i+1}} g_{-1})$.

THEOREM (3.4). *For* $i \geqslant 1$, $\gamma_i^*(v_i^{2^{i+1}} g_{-1}) = v_{i-1}^{2^{i+1}} \iota_{2^i 2^{i+1}-1}$.

PROOF. $v_{i-1}^{2^{i+1}} \in \operatorname{Ext}_{A_{i-1}}(\mathbf{Z}/2)$ is obtained by splitting off the top cell $(\zeta_i^{2^{i+1}})^*$ in $N_{2^{i+1}}(i-1)$. We now calculate $\gamma_i^*(v_i^{2^{i+1}} g_{-1})$ on the E_1 level of the KSS's. Again, let $t = 2^{i+1}(2^{i+1} - 1)$ and $\sigma = 2^{i+1}$. $v_i^{2^{i+1}} g_{-1}$ corresponds to a class $\{u\} \in \operatorname{Ext}^{0,t-1}_{A_i}(N_\sigma(i) \otimes \mathbf{P})$, given as the top class in $N_\sigma(i)$, tensored with P. But

$$\operatorname{Ext}^{0,t-1}_{A_{i-1}}(N_\sigma(i) \otimes \mathbf{P}) \cong \operatorname{Ext}^{0,t-1}_{A_{i-2}}\left(\bigoplus_{m \in \mathbf{Z}} \Sigma^{m2^i - 1}\left[\bigoplus_{j=0}^{\sigma} \Sigma^{\sigma 2^i} N_j(i-1)\right]\right)$$

by Lemmas (2.5) and (2.6). Now $\{u\}$ lies completely in the $m = 0$ summand of $\mathrm{Ext}_{A_{i-2}}(\oplus_{m\in\mathbf{Z}}\Sigma^{m2^i-1}[\oplus_{j=0}^{\sigma}\Sigma^{\sigma 2^i}N_j(i-1)])$ for dimensional reasons. Since the isomorphism of Lemma (2.6) is given here by $\gamma_i \otimes \mathrm{id}_{N_\sigma}$, we conclude that $\{u\}$ corresponds to the top cell in $\Sigma^{2^i 2^{i+1}}N_\sigma(i-1)$, the top summand in $N_\sigma(i)$. By the observation at the end of §2, this class $\{u\}$ corresponds precisely to $\zeta_i^{2^{i+1}}\cdot\iota_{2^i 2^{i+1}-1}$, which yields $v_{i-1}^{2^{i+1}}\iota_{2^i 2^{i+1}-1}$ in $\mathrm{Ext}_{A_{i-1}}(\oplus\Sigma\mathbf{Z}/2)$. Since this class $\{u\}$ yields $v_i^{2^{i+1}}g_{-1}$ $\in \mathrm{Ext}_{A_i}(\mathbf{P})$ in one KSS and $v_{i-1}^{2^{i+1}}\iota_{2^i 2^{i+1}-1} \in \oplus_{k\in\mathbf{Z}}\mathrm{Ext}_{A_{i-1}}(\Sigma^{k2^{i+1}-1}\mathbf{Z}/2)$ in the other, we conclude that $\gamma_i^*(v_i^{2^{i+1}}g_{-1}) = v_{i-1}^{2^{i+1}}\iota_{2^i 2^{i+1}-1}$. It should be pointed out that we are dealing with classes in Ext_{A_i} and $\mathrm{Ext}_{A_{i-1}}$, not cosets, so that this is actually an equality here. The easiest way to view the calculation is in the following diagram.

$$
(3.5)\quad
\begin{array}{ccc}
\mathrm{Ext}^{0,t-1}_{A_{i-1}}(N_\sigma(i)\otimes P) & \Rightarrow & \mathrm{Ext}^{2^{i+1},t-1}_{A_i}(\mathbf{P}) \\
\cong\updownarrow \text{ by Lemmas (2.5) and (2.6)} & & \\
\mathrm{Ext}^{0,t-1}_{A_{i-2}}\left(\bigoplus_{m\in\mathbf{Z}}\Sigma^{m2^i-1}\left[\bigoplus_{j=0}^{\sigma}\Sigma^{2^i2^{i+1}}N_j(i-1)\right]\right) & & \uparrow \gamma_i^* \\
\uparrow & & \\
\mathrm{Ext}^{0,t-1}_{A_{i-2}}\left(\bigoplus_{k\in\mathbf{Z}}\Sigma^{k2^{i+1}-1}N_\sigma(i-1)\right) & \Rightarrow & \mathrm{Ext}^{2^{i+1},t-1}_{A_{i-1}}\left(\bigoplus_{k\in\mathbf{Z}}\Sigma^{k2^{i+1}-1}\mathbf{Z}/2\right)
\end{array}
$$

From this theorem, we can deduce the following

THEOREM B. *For the class* $v_{i-1}^{2^{i+1}} \in \mathrm{Ext}_{A_{i-1}}(\mathbf{Z}/2)$, *we have* $R_i(v_{i-1}^{2^{i+1}}) = v_i^{2^{i+1}} \in \mathrm{Ext}_{A_i}(\mathbf{Z}/2)$. *Also*, $R_1(h_0^4) = v_1^4 \in \mathrm{Ext}_{A_1}(\mathbf{Z}/2)$.

PROOF. The class $v_i^{2^{i+1}}g_{-1} \in \mathrm{Ext}^{2^{i+1},t-1}_{A_i}(\mathbf{P})$ survives k_{-1}^* but not k_0^*, by construction. Further, $\gamma_i^*(v_i^{2^{i+1}}g_{-1}) = v_{i-1}^{2^{i+1}}\iota_{2^i2^{i+1}-1}$, so that we have

$$
(3.6)\quad
\begin{array}{ccc}
\mathrm{Ext}^{2^{i+1},t-1}_{A_{i-1}}\left(\Sigma^{2^i2^{i+1}-1}\mathbf{Z}/2\right) & \hookrightarrow & \mathrm{Ext}^{2^{i+1},t-1}_{A_i}(\mathbf{P}) \\
\wr\downarrow Q & & k_{-1}^* \downarrow \text{largest } N \\
\mathrm{Ext}^{2^{i+1},t-1}_{A_i}(\Sigma^{-1}\mathbf{Z}/2) & \hookrightarrow & \mathrm{Ext}^{2^{i+1},t-1}_{A_i}(P_{-1})
\end{array}
$$

Here the map $Q = \Sigma^{2^i2^{i+1}-1}R_i$. Desuspending the entire diagram $2^i2^{i+1}-1$ times, together with the fact that $\mathrm{Ext}^{2^{i+1},t}_{A_i}(\mathbf{Z}/2)$ has only one nonzero class, completes the proof of Theorem B.

4. On the root invariant of a v_i-periodic class. In this section, we prove the main theorem.

THEOREM A. *Let* $a \in \mathrm{Ext}^{s,t}_A(\mathbf{Z}/2)$ *be* v_i*-periodic in the sense of Definition* (1.3). *Then the root invariant of* a, $R(a)$, *is* v_{i+1}*-periodic.*

We recall that $a \in \mathrm{Ext}_A(\mathbf{Z}/2)$ is v_i-periodic if $f_i(a) \neq 0$, where f_i: $\mathrm{Ext}^{s,t}_A(\mathbf{Z}/2)$ $\to \varprojlim_k[\mathrm{Ext}^{s,t}_{A_k}(\mathbf{Z}/2)(v_i^{-1})]$. We will show that $f_{i+1}(R(a)) \neq 0$.

We also recall that for $k > i$, there are cosets v_i^N and v_{i-1}^M in $\mathrm{Ext}_{A_k}(\mathbf{Z}/2)$.

LEMMA (4.1). *For the kth root invariant*, R_k: $\mathrm{Ext}_{A_{k-1}}(\mathbf{Z}/2) \rightsquigarrow \mathrm{Ext}_{A_k}(\mathbf{Z}/2)$, *we have* $v_i^N \subset R_k(v_{i-1}^N)$, *whenever both* v_{i-1}^N *and* v_i^N *are nonzero in* $\mathrm{Ext}_{A_{k-1}}(\mathbf{Z}/2)$ *and* $\mathrm{Ext}_{A_k}(\mathbf{Z}/2)$ *respectively.*

PROOF. For $k = i$, this is an easy consequence of the proof of Theorem B (the proof works for any power of v_i, not just for the 2^{i+1}st power). For $k > i$, we note that the cosets in question here are of σ-filtration zero in the KSS's. Thus, the calculations actually take place in the KSS's on the $\mathrm{Ext}_{A_{k-2}}$ and $\mathrm{Ext}_{A_{k-1}}$ levels. An easy induction starting at R_i completes the proof.

We can now complete the proof of Theorem A by using the A-module structure preserved by the Davis-Mahowald splitting (1.8). We recall that the splitting is given by

(1.8)

$$\text{(Davis-Mahowald Splitting)}: \gamma_k\colon A \otimes_{A_k} \mathbf{P}/F_m \xrightarrow{\cong} \bigoplus_{j \geq m} \Sigma^{j2^{k+1}-1}\left(A \otimes_{A_{k-1}} \mathbf{Z}/2\right)$$

where F_m is the A_k-submodule generated by $\{x^j \in \mathbf{P}\colon j < m\}$. This gives a splitting in Ext, after the change of rings isomorphism and taking the limit as m goes to minus infinity:

$$\gamma_k^*\colon \bigoplus_{j \in \mathbf{Z}} \Sigma^{j2^{k+1}-1}\,\mathrm{Ext}_{A_{k-1}}(\mathbf{Z}/2) \xrightarrow{\cong} \mathrm{Ext}_{A_k}(\mathbf{P}). \tag{1.9}$$

Since (1.8) is an isomorphism of A-modules, the induced map in $\mathrm{Ext}_A(\underline{\quad})$ must respect Yoneda products with classes from $\mathrm{Ext}_A(\mathbf{Z}/2)$. After change of rings, the induced map in Ext (1.9) must therefore respect Yoneda products with classes $q_k^*(a)$ for $a \in \mathrm{Ext}_A(\mathbf{Z}/2)$. With this in mind, we can prove Theorem A.

PROOF OF THEOREM A. Let $a \in \mathrm{Ext}(\mathbf{Z}/2)$ be v_i-periodic. Let k be large enough so that $q_k^*(a)$ and $q_{k-1}^*(a)$ are nonzero in $\mathrm{Ext}_{A_k}(\mathbf{Z}/2)$ and $\mathrm{Ext}_{A_{k-1}}(\mathbf{Z}/2)$, respectively. For ease of notation, denote $q_k^*(a)$ by $a' \in \mathrm{Ext}_{A_k}(\mathbf{Z}/2)$. Now, since a is v_i-periodic in $\mathrm{Ext}_A(\mathbf{Z}/2)$, we have $v_i^{2^s}a' \neq 0$ in $\mathrm{Ext}_{A_{k-1}}(\mathbf{Z}/2)$, for all s where $v_i^{2^s} \neq 0$ there. Consider the action of the map γ_k^* on this class. Since a' is the projection of a class from $\mathrm{Ext}_A(\mathbf{Z}/2)$, $\gamma_k^*(a' \cdot b\iota_{-1}) = a' \cdot \gamma_k^*(b)g_{-1}$, because the map γ_k is an A-module map.

Now $\gamma_k^*(a'\iota_{-1}) = a'g_{-1}$, and if we consider the map k_N^* on this class we have $k_N^*(a'g_{-1}) = R(a')g_N$ in $\mathrm{Ext}_{A_k}(P_N)$, by the definition of root invariant, where N is the maximal N such that $k_N^*\gamma_k^*(a') \neq 0$.

Recall that $\gamma_k^*(v_i^{2^m}\iota_{-1}) = v_{i+1}^{2^m}g_{q-1}$, where $q = 2^{m+i+1}$, by Theorem (3.4) and the fact that γ_k commutes with the natural projections. (Note that for m sufficiently large, $q \equiv -1 \pmod{2^{k+1}}$.) Thus, $\gamma_k^*(v_i^{2^m}a'\iota_{-1}) = v_{i+1}^{2^m}a'g_{q-1}$, and if we consider k_M^* on this class, we have $k_{q-N}^*(v_{i+1}^{2^m}a'g_{q-1}) = v_{i+1}^{2^m}R(a')g_{q-N}$ in $\mathrm{Ext}_{A_k}(P_{q-N})$, where q and N are as above. Thus $R(a')$ is v_{i+1}-periodic in $\mathrm{Ext}_{A_k}(\mathbf{Z}/2)$, completing the proof of Theorem A.

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