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ON SCRAMBLED SETS FOR CHAOTIC FUNCTIONS

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ABSTRACT. Some recent research has raised questions concerning the possible sizes of scrambled sets for chaotic functions. We answer these questions by showing that a scrambled set can have full measure, but cannot be residual although a scrambled set can be second category in every interval. We also indicate relationships that exist between chaotic functions and transitive functions.

1. Introduction. A number of problems involving the way a process evolves with time give rise to rather simple looking difference equations. This may happen, for example, when one wishes to estimate the size of some future generation of a population purely on the basis of its present size. The model $x_{n+1} = f(x_n)$ indicating the size x_{n+1} of the $(n+1)$st generation in terms of the nth can be studied in terms of a function f mapping the interval $[0,1]$ onto itself and analyzing the behavior of the iterates $x, f(x), f(f(x)), \ldots$. Often f vanishes at the end-points of the interval and achieves a single maximum at some interior point of $[0,1]$. The interested reader may consult [**LY** and **VSK**] for discussions of such problems and for extensive reference lists.

An investigator dealing with such problems may hope for essential stability of the iterates; that is, that the sequence $x, f(x), f(f(x)), \ldots$ converges to some fixed point of f, indicating that the population will stabilize at some size independent of x $(x \neq 0, 1)$. If that fails, the investigator would at least hope for predictability: if the initial population is x_0, which the investigator estimates reasonably accurately as y_0, he would hope that the sequence of iterates remain close together. In that case, a slight error in the estimate for the initial population will result in no more than a slight error in the estimates of future generations. Unfortunately, it often happens that neither of these wishes is satisfied. The investigator usually observes various sorts of chaotic behavior, all of which are predictable from the model.

In recent years, a number of authors have studied various sorts of chaotic behavior of functions of the type we mentioned. They have also asked whether certain even more chaotic behavior is possible. In the present paper we answer some of their questions, and discuss other related types of behavior. In §2, we provide the necessary definitions, review some of the recent work that motivated this paper and provide some results that we shall need in the sequel. In §3, we obtain affirmative answers to questions posed in [**S2**]. These questions relate to the size (in terms of Lebesgue measure) possible of so-called scrambled sets. Then in §4, we study the

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analogues of these questions when the size of a set is measured by (Baire) category rather than measure. We also indicate the relationships that exist between two types of behavior, "chaotic" and "nomadic" (or "transitive").

2. Preliminaries. Let f be a function mapping the interval $I_0 = [0, 1]$ onto itself. Let $x \in I_0$. By the *orbit* of x (under f), we mean the sequence $\{f^n(x)\}_{n=0}^{\infty}$, where $f^0(x) = x$, $f^1(x) = f(x)$, and in general $f^{n+1}(x) = f(f^n(x))$, $n = 0, 1, \ldots$. We shall use the notation $O(x)$ for the set that is the range of the sequence $\{f^n(x)\}$. If $I \subseteq I_0$ is an interval, we will denote by $O(I)$ the orbit of the interval I (under f); i.e., $O(I) = \bigcup_{n=0}^{\infty} f^n(I)$. Similarly, the *inverse orbits* of x and of I (under f) are defined by $O^{-1}(x) = \{y \colon f^n(y) = x \text{ for some } n = 0, 1, 2, \ldots\}$ and $O^{-1}(I) = \{O^{-1}(x) \colon x \in I\}$.

We shall be concerned primarily with so-called chaotic functions and their scrambled sets. While these notions have been studied by a number of authors, no entirely consistent use of the terms has evolved. The definitions we give below represent a strong form of chaotic behavior suitable for our purposes.

Let f be a continuous function mapping I_0 onto I_0.

DEFINITION 2.1. A set $S \subseteq I_0$ is called *scrambled* for f provided for every $x, y \in S$ with $x \neq y$,

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| = 1 \tag{1}$$

and

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0. \tag{2}$$

DEFINITION 2.2. The function f is called *chaotic* if there is an uncountable scrambled set for f.

REMARK. Other authors have replaced (1) with the weaker requirement that $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$, but at the same time have required something about the behavior of points in their scrambled sets vis-à-vis periodic points. Kan [**K**] uses the term "extremally scrambled" when (1) is satisfied. Because of requirement (1), we restrict ourselves in §§3 and 4 to functions mapping I_0 *onto* I_0. The recent articles [**K, S1, S2**] give examples of chaotic functions whose scrambled sets have positive measure or full outer measure. The question arises naturally [**S2**] whether a scrambled set can have full measure. We shall answer this question affirmatively by an example in §3.

We shall also make use of functions exhibiting a type of related behavior.

DEFINITION 2.3. A function $f \colon I_0 \to I_0$ is called *nomadic* if there exists $x \in I_0$ such that $O(x)$ is dense.

Other authors have used other terms (e.g. "transitive") to express this phenomenon. Since we shall make use of results in [**BCR**], where the term "nomadic" was used, we retain that term.

PROPOSITION 2.4. *The function $f \colon I_0 \to I_0$ is nomadic if and only if $O(I)$ is dense for every interval $I \subseteq I_0$.*

PROOF. If f is nomadic, then it is clear that $O(I)$ is dense for each interval $I \subseteq I_0$.

To prove the converse, observe first that our hypothesis implies that if J is an arbitrary interval in I_0, then $O^{-1}(J)$ is dense. Now let $\{J_k\}_{k=1}^{\infty}$ be a countable base for the topology of $[0,1]$. Let $A = \bigcap_{k=1}^{\infty} O^{-1}(J_k)$. Since $O^{-1}(J_k)$ is dense and open for every k, A is a dense set of type G_δ in $[0,1]$. If $x \in A$, then $O(x)$ is dense. Thus f is nomadic.

3. Scrambled sets of full measure. We turn now to a construction of a function g possessing a scrambled set of full measure. The function g is strictly increasing on $[0, \frac{1}{2}]$ and strictly decreasing on $[\frac{1}{2}, 1]$. It provides affirmative answers to two questions of Smital [**S2**].

We shall make use of a function which has been studied by several authors. We begin with a brief review of this function, indicating the properties of this function which we shall need. Define $f\colon I_0 \to I_0$ by

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

It is convenient to consider the binary expansion of points in $[0,1]$. One can then verify easily (see [**BCR**]) that:

(1) If $x = .0x_2x_3x_4\cdots$ (binary expansion), then $f(x) = .x_2x_3x_4\cdots$.

(b) If $x = .1x_2x_3x_4\cdots$ (binary expansion), then $f(x) = .y_2y_3y_4\cdots$, where $y_i = 0$ if $x_i = 1$ and $y_i = 1$ if $x_i = 0$.

(c) If every finite sequence of 0's and 1's appears in the binary expansion of x, then x has a dense orbit. Thus, almost every $x \in [0,1]$ has a dense orbit.

Smital [**S1**] has shown that f has a scrambled set of full outer measure, but that any *measurable* scrambled set for f must have zero measure.

Our construction of a function g possessing a scrambled set of full measure will involve a number of steps. We first show that the function f has a scrambled set S of Borel type $G_{\delta\sigma}$ that has cardinality of the continuum in every interval. A suitable transformation then results in the function g. The main problem in obtaining S arises from the following considerations. If S is a scrambled set for f and $x \neq y$ are in S, then the binary expansions for x and y must contain arbitrarily long strings of 0's and 1's (see (a) and (b) above). These strings must be positioned properly so that the definition of scrambled set is satisfied for x and y. These requirements must be achieved by *every* pair (x,y) in S. We shall achieve this by representing certain points in I_0 by "wedges."

THEOREM 3.1. *There exist a continuous function $g\colon I_0 \to I_0$ such that g is increasing on $[0, \frac{1}{2}]$ and g is decreasing on $[\frac{1}{2}, 1]$, and a corresponding scrambled set T with $T \in F_\sigma$ and $\lambda T = 1$.*

PROOF. We shall prove the theorem by the following five major steps:

(1) Let L be the lattice of pairs of positive integers and $0 < \xi < 1$ be a fixed irrational number. Let Λ be the family of positive, finite slopes of all rays emanating from $(0, \xi)$, lying in the first quadrant, and containing no points of L. If $r, s \in \Lambda$ with $r < s$, let $W_{rs} = \{(x,y) \in L\colon rx + \xi < y < sx + \xi\}$. Let L be enumerated diagonally, $l_1, l_2, \ldots$, as follows:

$$(1,1), (1,2), (2,1), (1,3), (2,2), (3,1), (1,4), (2,3), (3,2), \ldots.$$

Let $\mathcal{W}$ be the system of a finite union of wedges. Thus an element of $\mathcal{W}$ consists of the union of a finite number of "wedges,"

$$W_{r_1s_1}, W_{r_2s_2}, \ldots, W_{r_ns_n}, \qquad r_1 < s_1 < r_2 < s_2 < \cdots < r_n < s_n.$$

Now to each $W \in \mathcal{W}$, we associate that number $x_W \in [0,1]$ whose binary expansion has a 1 in the ith position if and only if the ith lattice point l_i is in W. Let $S = \{x_W : W \in \mathcal{W}\}$. Then

(i) $\operatorname{Card} S = c$.

(ii) For any finite set of integers $i_1 < i_2 < \cdots < i_n$, there exists some $W \in \mathcal{W}$ containing $l_{i_1}, l_{i_2}, \ldots, l_{i_n}$ corresponding to $i_1, i_2, \ldots, i_n$ but no other lattice point l_j with $j \le i_n$. It follows that S is dense in $[0,1]$. Since a slight variant of W in (ii) leads to a different $W^* \in \mathcal{W}$ which still meets the requirement in (ii), S is actually c-dense; i.e., each interval $I \subseteq [0,1]$ contains c points of S.

(iii) If $W_1, W_2 \in \mathcal{W}$ with $W_1 \neq W_2$ and $x = x_{W_1}$, $y = x_{W_2}$, then there exist arbitrary long finite sequences of consecutive integers $n, n+1, \ldots, N$ and $m, m+1, \ldots, M$ such that $x_i = y_i = 0$ for $i = n, \ldots, N$ and $x_i = 1$ with $y_i = 0$ (or $x_i = 0$ with $y_i = 1$) for $i = m, \ldots, M$, and such that x_{m-1}, y_{m-1} are either both 0 or both 1.

(2) Let

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

be the function we discussed in the beginning of this section. Properties (a) and (b) there together with (iii) imply that

$$|f^n(x) - f^n(y)| \le \frac{1}{2^{N-n}} \quad \text{and} \quad |f^m(x) - f^m(y)| \ge 1 - \frac{1}{2^{M-m-1}}.$$

It follows that S is a scrambled set for f.

(3) Let S_1 consist of those $x \in S$ requiring no more than one wedge in their representations; i.e., $x \in S_1$ if there exists $r, s \in \Lambda$ such that $x = x_{W_{rs}}$. Let $A_1 = (\Lambda \times \Lambda) \cap \{(r,s) : r < s\} = \{(r,s) \in \Lambda \times \Lambda : r < s\}$. Then A_1 is a G_δ of cardinality c in the complete space R_2 and is therefore an absolute G_δ [**O**]. Then $(r_n, s_n) \to (r,s)$ in A_1 if and only if $x_n \to x \in S_1$, where $x_n = x_{W_{r_n, s_n}}$ and $x = x_{W_{r,s}}$. Thus S_1 is homeomorphic to A_1 and is therefore an absolute G_δ [**O**]. Similarly, if

$$\begin{aligned} S_n = \{x : x = x_W,\ &\text{where } W = W_{r_1 s_1} \cup \cdots \cup W_{r_n s_n}, \\ &\text{with } r_1 < s_1 < r_2 < \cdots < r_n < s_n, \text{ consists of } n \text{ wedges}\}, \end{aligned}$$

then S_n is homeomorphic to $A_n = \{(r_1, s_1, r_2, \ldots, r_n, s_n) : r_1 < s_1 < \cdots < r_n < s_n,$ and $r_k, s_k \in \Lambda$ for $k = 1, 2, \ldots, n\}$, and thus $S_n \in G_\delta$. Since $S = \bigcup_{n=1}^\infty S_n$, $S \in G_{\delta\sigma}$. Note each S_n is nowhere dense: an x with long strings of 0's and 1's alternating in binary expansion cannot be in S_n. Hence S is a scrambled, c-dense first category $G_{\delta\sigma}$ subset of $[0,1]$.

(4) Let $\{I_n\}$ be an enumeration of the rational intervals in $[0,1]$. For each n, the set $S \cap I_n$ is c-dense in I_n and is of type $G_{\delta\sigma}$. Thus, there exists a nonempty perfect set $P_n \subseteq S \cap I_n$. Let $P = \bigcup_{n=1}^\infty P_n$. Then P is of type F_σ and is c-dense in $[0,1]$. Furthermore, since $P \subseteq S$, P is a scrambled set for f.

(5) Let h be a homeomorphism of I_0 onto itself such that $h(P) = T$ has measure one, and $h(\frac{1}{2}) = \frac{1}{2}$ (such a homeomorphism exists by [**G**]). Let $g = h \circ f \circ h^{-1}$. Then T is a scrambled set for g and $\lambda T = 1$. Furthermore g increases on $[0, \frac{1}{2}]$ and decreases on $[\frac{1}{2}, 1]$.

REMARKS. The homeomorphism h can be chosen to be arbitrarily close to the identity. (One only needs to subdivide I_0 into sufficiently small subintervals and require that h map each subinterval onto itself.) Thus g can be chosen uniformly close to f.

Let S, T be the scrambled sets appearing in Theorem 3.1. If $x \in S$, then 0 and $1/2^n$, $n = 0, 1, \ldots$, are the only limit points of $\{f^n(x)\}$. In particular, S contains no points of dense orbit under f. Similarly T contains no points of dense orbit of g. Let D_f and D_g denote sets of points having dense orbits under f and g respectively. Then $\lambda S = 0$, $\lambda D_f = 1$, $\lambda T = 1$, $\lambda D_g = 0$.

Furthermore, note that f must be one-to-one on a scrambled set. Similarly, f^n must be one-to-one. Thus our example shows that it is possible for a continuous function and all its iterates to be one-to-one on a set of full measure without being one-to-one on I_0.

4. Category analogues. We say in §3 that there exists a function g possessing a scrambled set of full measure. The function g we defined is nomadic. Two questions arise naturally:

(1) Does the category analogue of Theorem 3.1 hold? I.e. does there exist a continuous function possessing a scrambled set that is residual in I_0?

(2) What relationships exist between nomadic functions and chaotic functions?

We address both these questions in this section. First we show that if a scrambled set has the Property of Baire, then it must be first category. Since the Property of Baire is the category analogue of a set being measurable, this raises the question of whether a scrambled set can be second category if it lacks the Property of Baire. We show this is possible by showing that *every* continuous function f with f^2 nomadic possesses a scrambled set that is second category in every interval. (This statement fails if one weakens the requirement that f^2 be nomadic to the requirement that f be nomadic.) Finally, we observe in Corollary 4.9 that a continuous f is chaotic if and only if f^2 is nomadic.

PROPOSITION 4.1. *Let f be continuous on an interval $[a, b]$ and let S be a residual subset of $[a, b]$. If f is one-to-one on S, then $f(S)$ is residual in $f([a, b])$.*

PROOF. Let H be a dense set of type G_δ contained in S. Then $f(H)$ is a Borel set, being the one-to-one continuous image of a set of type G_δ. Thus, if $f(H)$ is not residual in $f([a, b])$, then there exists an open interval J such that $f(H) \cap J$ is first category. Let I be a component interval of the open set $f^{-1}(J)$ and let $H^* = H \cap I$. Since $f(H^*)$ is a first category subset of J, there exist nowhere dense sets $B_1, B_2, \ldots$ such that

$$f(H^*) = \bigcup_{k=1}^{\infty} B_k \subseteq \bigcup_{k=1}^{\infty} \overline{B}_k \subseteq \overline{J}.$$

Then $H^* \subseteq \bigcup_{k=1}^{\infty}(f^{-1}(\overline{B}_k) \cap \overline{I})$. Since H^* is residual in I, the same is true of $\bigcup_{k=1}^{\infty}(f^{-1}(\overline{B}_k) \cap I)$. This implies that there is a k such that the closed set $f^{-1}(\overline{B}_k) \cap \overline{I}$ contains an interval L. But H^* is residual in L and f is one-to-one on H^*, so the set $f(L)$ is a nondegenerate interval. This is impossible since $f(L) \subset \overline{B}_k$, a nowhere dense set. Thus the proof is complete.

THEOREM 4.2. *Let $f\colon I_0 \to I_0$ be continuous and let S be a scrambled set for f. If S has the Property of Baire, then S is first category.*

PROOF. We argue by contradiction. Suppose S is a second category scrambled set possessing the Property of Baire. Then there exists an interval J such that S is residual in J. We show first that f must be monotonic on J. If not, there exist points $a < b < c$ in J such that $f(a) = f(c) \neq f(b)$. Without loss of generality, assume $\max\{f(x)\colon x \in [a,b]\} = f(b)$. Since f is one-to-one on S, it follows from Proposition 4.1 that $f(S \cap [a,b])$ and $f(S \cap [b,c])$ are residual subsets of $f([a,b])$. Thus these two sets have a nonempty intersection, which is impossible since f is one-to-one on S. Now let $0 < \delta < \frac{1}{2}$. We claim that f is monotonic on $[\delta, 1-\delta]$. Let $x \neq y$ be points of $S \cap J$. Since S is a scrambled set, there exists n such that $|f^n(x) - f^n(y)| > 1 - \delta$, say $f^n(x) < \delta$ and $f^n(y) > 1 - \delta$. It now follows from Proposition 4.1 that $f^n(S \cap J)$ is residual in $[\delta, 1-\delta]$. If f is not monotonic on $[\delta, 1-\delta]$, then f cannot be one-to-one on the set $f^n(S \cap [\delta, 1-\delta])$; i.e., f^{n+1} is not one-to-one on $S \cap [\delta, 1-\delta]$. This is impossible since $S \cap [\delta, 1-\delta]$ is a scrambled set for f. Thus the claim is proved. Since $0 < \delta < \frac{1}{2}$ is arbitrary, we have shown that f is monotonic on I_0. However, a monotonic function cannot have scrambled sets. This is a contradiction to the hypothesis that S is a scrambled set for f. Thus S must be first category in I_0.

If we drop the requirement that S have the Property of Baire, the conclusion that S is first category no longer follows. Our next theorem shows this, as well as establishing a connection between nomadic functions and chaotic functions. Observe that the requirement that f^2 be nomadic implies also that f is nomadic.

THEOREM 4.3. *Let f be continuous on $I_0 = [0,1]$. Suppose f^2 is nomadic. Assuming the Continuum Hypothesis, there exists a set S of second category in every interval such that if $x, y \in S$ with $x \neq y$, then the set $\{f^n(x) - f^n(y)\}$ is dense in $[-1,1]$.*

The proof of Theorem 4.3 depends on three propositions that may be of interest in themselves. The authors are grateful to the referee for pointing out that the hypotheses for Proposition 4.4 suffice. Originally we had stronger hypotheses which led to weaker versions of the remaining results.

PROPOSITION 4.4. *If f is continuous on $I_0 = [0,1]$ and f^2 is nomadic, then to every $\delta > 0$ and every open interval $I \subset I_0$ there corresponds a positive integer N such that $f^k(I) \supset [\delta, 1-\delta]$ for every $k \geq N$.*

PROOF. Since f is nomadic, there is a fixed point x_0 for f in $(0,1)$. Let J be a nondegenerate closed interval such that x_0 is an end-point of J and $J \subset f([0,x_0]) \cap f([x_0,1]) \cap (0,1)$. (If no such interval existed, then $f^2[0,x_0]$ would be contained in $[0,x_0]$ and f^2 would not be nomadic.) For each positive integer n, let $J_n = f^n(J)$. Each set J_n is an interval containing x_0. Now f is nomadic, so there exists a positive integer M such that $f^M(J) \supset J$. Let $H = J \cup J_1 \cup \cdots \cup J_{M-1}$. Then H is an interval containing J and

$$f(H) = f(J) \cup f(J_1) \cup \cdots \cup f(J_{M-1}) = J_1 \cup \cdots \cup J_M.$$

Since $J_M \supset J$, $f(H) \supset H$.

Now let $H_k = f^k(H)$. Then the sequence $\{H_k\}$ is an expanding sequence of intervals, $H_{k+1} \supset H_k$, $k = 1, 2, 3, \ldots$. Since each such interval contains x_0 and since f is nomadic, there exists M_1 such that $f^k(H) \supset [\delta, 1-\delta]$ for all $k \geq M_1$.

We now show that there exists M_2 such that $f^{M_2}(I) \supset H$. Since f is nomadic, $O(I)$ has at most two components [**BCR**, Theorem 10]. But f^2 is also nomadic, so there must exist a point $z \neq x_0$, $z \in (0,1)$, such that $f(z) = x_0$. It follows that some iterate of f maps I onto an interval containing x_0 and that some later iterate f^{M_2} satisfies $f^{M_2}(I) \supset H$. Thus $f^k(I) \supset [\delta, 1-\delta]$ for all $k \geq N = M_1 + M_2$.

PROPOSITION 4.5. *Let f be continuous on I_0 with f^2 nomadic. Define a function g on $I_0 \times I_0$ to $I_0 \times I_0$ by $g(x,y) = (f(x), f(y))$. Then there exists a set A that is dense and of type G_δ in $I_0 \times I_0$ such that each point in A has a dense orbit relative to g.*

PROOF. Let $U = U_1 \times U_2$, $V = V_1 \times V_2$ be open rectangles whose closures are contained in $(0,1) \times (0,1)$. By Proposition 4.4, there exists a positive integer N such that $f^N(U_1) \supset V_1$ and $f^N(U_2) \supset V_2$. Thus $g^N(U) \supset V$. Hence the inverse orbit of V under g is dense. Since V is open and g is continuous, this set is also open. Now let $V_1, V_2, \ldots$ be a countable base for the topology of $(0,1) \times (0,1)$ with $\overline{V}_i \subseteq (0,1) \times (0,1)$. Let $A = \bigcap_{k=1}^{\infty} O^{-1}(V_k)$, where $O^{-1}(V_k)$ is the inverse orbit of V_k under g. Then A is a dense G_δ and each point of A has a dense orbit relative to g.

PROPOSITION 4.6. *Let f be continuous on I_0 with f^2 nomadic. Then there exists a residual set B such that if $x \in B$, there exists a residual set $B_x \subset B$ such that if $y \in B_x$ with $y \neq x$, then the set $\{f^n(y) - f^n(x)\}$ is dense in $[-1,1]$.*

PROOF. Let A be the set of Proposition 4.5. By the Kuratowski-Ulam Theorem [**O**], there exists a residual set $B \subset I_0$ such that if $x \in B$, then $B_x = \{y \colon (x,y) \in A\}$ is a dense G_δ in I_0. But, if $(x,y) \in A$, then the set $g^n(x,y)$ is dense in $I_0 \times I_0$ by Proposition 4.5. This implies that $\{f^n(x) - f^n(y)\}$ is dense in $(-1,1)$. Since B and B_x are residual sets, so is $B_x \cap B$, so there is no loss of generality in assuming $B_x \subset B$.

We are now ready for the proof of Theorem 4.3.

PROOF OF THEOREM 4.3. Let B and B_x have the meanings indicated in Proposition 4.6. We assume the Continuum Hypothesis. Let Ω be the first uncountable ordinal. Well-order the second category sets of type G_δ in I_0 by Ω:

$$G_0, G_1, \ldots, G_\alpha, \ldots, \qquad \alpha < \Omega.$$

Let $x_0 \in G_0 \cap B$, and choose $x_1 \neq x_0$, $x_1 \in G_1 \cap B_{x_0}$. Suppose we have chosen x_β for all $\beta < \alpha$ such that $x_\beta \neq x_\gamma$ for $\gamma < \beta$, $x_\beta \in G_\beta \cap B_{x_\gamma}$ for all $\gamma < \beta$. Consider the set $S_\alpha = (G_\alpha \cap \bigcap_{\gamma<\alpha} B_{x_\gamma}) - \{x_\beta \colon \beta < \alpha\}$. This set is a countable intersection of dense sets of type G_δ, thus S_α is also a dense G_δ. Let $x_\alpha \in S_\alpha$ and let $S = \{x_\alpha \colon \alpha < \Omega\}$. We show that S has the desired properties.

To see S is second category in every interval, we need only note that S intersects every second category set of type G_δ. (If S were first category in some interval I, then $I - S$ would be residual in I and would therefore contain a G_δ set dense in I. This set would be a second category G_δ missing S.) Now, let $x, y \in S$, say $x = x_\alpha$, $y = x_\beta$ with $\alpha < \beta$. Then $y \in B_x$ so $\{f^n(y) - f^n(x)\}$ is dense in $[-1,1]$ as required.

REMARK. Let $D = \{x\colon O(x) \text{ is dense}\}$. Then D is a dense G_δ. Let $T = D \cap S$. Then T is second category in every interval and is scrambled for f. If p is a periodic point for f, then $\limsup_{n\to\infty} |f^n(p) - f^n(x)| > 0$ for every $x \in T$. This condition is sometimes part of the requirement of a scrambled set [**K**, **S1**].

REMARK. Propositions 4.5 and 4.6 and Theorem 4.3 are not valid in general without the assumption that f^2 be nomadic. One need only consider a nomadic function with a cut-point [**BCR**], that is a point z such that $f([0,z]) = [z,1]$ and $f([z,1]) = [0,z]$ and $O^{-1}(z) = z$. It is clear that if $x < y < z$ for such a function, then $\lim_{n\to\infty} \sup |f^n(y) - f^n(x)| < 1$.

Nonetheless, in that case, weaker conclusions are possible since the function f^2 is nomadic on $[0,z]$ and on $[z,1]$ and has infinitely many fixed points. (See Theorem 11 of [**BCR**] for some results concerning behavior of nomadic functions with cut points.) Thus f^2 exhibits chaotic-like behavior on each of the two intervals. More specifically, f^2 exhibits the behavior of Theorem 4.3 on each of these intervals. In particular, if x and y are in $[0,z]$, then $\{f^n(x) - f^n(y)\}$ are dense in $[-z,z]$ and have similar behavior for $x, y \in [z,1]$.

Theorem 4.3 showed that continuous functions with f^2 nomadic are chaotic. We now obtain a converse.

THEOREM 4.7. *Let $f\colon I_0 \to I_0$ be continuous. If there exists a set S, dense in I_0 such that* $\limsup_{n\to\infty} |f^n(x) - f^n(y)| = 1$ *for all $x \neq y$ in S, then f^2 is nomadic.*

PROOF. Let K be a compact set with interior points such that $f^2(K) \subset K$. We show $K = I_0$. This implies that f^2 is nomadic [**BCR**]. Let U be an interval contained in K, let $x, y \in S \cap U$ and let $\varepsilon > 0$. Choose δ such that $0 < \delta < \varepsilon$ and $f([\delta, 1-\delta]) \supset [\varepsilon, 1-\varepsilon]$. By the definition of S, there is a positive integer N such that $|f^N(x) - f^N(y)| > 1 - \delta$. Thus $f^N(U) \supset [\delta, 1-\delta] \supset [\varepsilon, 1-\varepsilon]$ and $f^{N+1}(U) \supset f([\delta, 1-\delta]) \supset [\varepsilon, 1-\varepsilon]$ because of the definition of δ. One of the numbers N or $N+1$ is even, say N. Then since $U \subset K$, $K \supset f^N(U) \supset [\varepsilon, 1-\varepsilon]$.

This inclusion $K \supset [\varepsilon, 1-\varepsilon]$ is valid for every $\varepsilon > 0$, thus $K = I_0$. Thus the only compact invariant set for f^2 with interior is I_0, as was to be shown.

Combining Theorems 4.3 and 4.7 we obtain Theorem 4.8.

THEOREM 4.8. *Assume the Continuum Hypothesis. Let f be continuous on I_0. Then there exists a dense set S such that for $x, y \in S$ $(x \neq y)$,*

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| = 1$$

if and only if f^2 is nomadic. In that case there is a set T of second category in every interval such that for $x, y \in T$ $(x \neq y)$, the set $\{f^n(x) - f^n(y)\}$ is dense in $[-1,1]$.

In particular, we obtain the following corollary that relates nomadicity to chaos.

COROLLARY 4.9. *Under the Continuum Hypothesis, a continuous function f defined on I_0 is chaotic if and only if f^2 is nomadic.*

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