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ON THE MÖBIUS FUNCTION

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ABSTRACT. We investigate incomplete convolutions of the Möbius function of the form $\sum_{d|n;d\leq z}\mu(d)$. It is shown that for almost all integers n one can find z for which this sum is large.

1. Introduction. The function $M(n)=\sup_{z\leq n}|\sum_{d|n;d\leq z}\mu(d)|$ has been studied in various papers [**1**, **2**, **5**]. Erdös and Katai [**2**] proved that

$$M(n)\leq A^{\omega(n)} \qquad \text{(p.p.)}$$

if $A>\sqrt{2}$.

(We use (p.p.) to indicate that a property holds on a sequence of asymptotic density 1.)

This result was improved by Hall [**5**], who showed that $A>3/e$ is sufficient.

A recent result of G. Tenenbaum and the author [**9**] implies almost immediately

THEOREM 1.

$$M(n)\leq \psi(n)\log\log n, \qquad \text{(p.p.)}$$

where $\psi(n)$ is any function tending to ∞.

PROOF. Let $p_1(n)$ be the smallest prime factor of n. Then $\mu(d)+\mu(p_1(n)d)=0$ for all $d\not\equiv 0 \bmod p_1(n)$. Therefore

$$M(n)\leq \sup_{z\leq n}\left|\sum_{\substack{z<d\leq zp_1(n)\\ d|n}}1\right|.$$

In [**9**] it is shown that

$$\Delta(n)\leq\psi(n)\log\log n,$$

where Δ is Hooley's function [**7**], defined by

$$\Delta(n)=\sup_{z\leq n}\sum_{z\leq d<ez}1. \qquad \text{(p.p.)}$$

It follows by sieve methods that

$$p_1(n)\leq\psi(n). \qquad \text{(p.p.)}$$

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Thus

$$M(n) \leq \sup_{z\leq n}\left|\sum_{\substack{z<d\leq zp_1(n)\\ d|n}} 1\right| \leq \Delta(n)\log\psi(n)$$
$$\leq \psi(n)\log\psi(n)\log\log n. \qquad \text{(p.p.)}$$

Since $\psi(n)$ was arbitrary Theorem 1 follows.

In [**1** and **2**] the question for a lower bound for $M(n)$ was raised. The purpose of this paper is to establish such a lower bound.

We will prove

THEOREM 2. *Let*

$$\gamma < -\frac{\log 2}{\log(1-(\log 3)^{-1})} = 0.28754\ldots$$

Then

$$M(n) \geq (\log\log n)^{\gamma}. \qquad \text{(p.p.)}$$

Many of the techniques applied will be very similar to those applied in [**8**], where the same lower bound was obtained for $\Delta(n)$. However we need also some new devices which bear resemblance to those used in [**9**].

2. Notations and preliminary lemmas. We fix a function $v(x)$, to be specified later, with $v(x)\to\infty$ $(x\to\infty)$ and also a constant $\rho>1$.

Based on these two parameters we define $r_k=\rho^k v(x)$ and $r_{k,l}=\rho^k v(x)+l$ where k and l are any nonnegative integers. For any positive integer $n\leq x$ and a real number $z>0$ we set

$$n_z=\prod_{\substack{\log\log p<z\\ p|n}} p, \qquad n_z^*=\prod_{\substack{\log\log p<z\\ p^{\nu_p}\|n}} p^{\nu_p}.$$

We use $n_{(k)}$ for n_{r_k}; $n_{(k,l)}$ for $n_{r_{k,l}}$; $n^*_{(k)}$ for $n^*_{r_k}$; and $n^*_{(k,l)}$ for $n^*_{r_{k,l}}$. We define

$$\tilde{n}_{(k,l)}=\prod_{\substack{r_k\leq\log\log p<r_{k,l}\\ p|n}} p \quad\text{and}\quad \hat{n}_{(k,l)}=\prod_{\substack{r_k\leq\log\log p<r_{k,l}\\ p^{\nu_p}\|n}} p^{\nu_p}.$$

Assume that

$$n=n^*_{(k)}p_1^{(k)^{\nu_1}}\cdots p_{t(n)}^{(k)^{\nu_{t(n)}}}, \qquad p_1^{(k)}<\cdots<p_{t(n)}^{(k)}.$$

Then we set

$$\tilde{n}^{(s)}_{(k)}=\prod_{t\leq s}p_t^{(k)}, \qquad \hat{n}^{(s)}_{(k)}=\prod_{t\leq s}p_t^{(k)^{\nu_t}},$$
$$n^{(s)}_{(k)}=n_{(k)}\tilde{n}^{(s)}_{(k)}, \qquad n^{*(s)}_{(k)}=n^*_{(k)}\hat{n}^{(s)}_{(k)}.$$

For any quadruplet (n,k,l,η), where $n \le x$, k,l nonnegative integers, and $\eta > 0$ we denote by $\mu(n,k,l,\eta)$ the Lebesgue measure of the set

$$\mathcal{E}(n,k,l,\eta) = \bigcup_{\substack{dd' \mid \tilde{n}_{(k,l)} \\ \mu(dd')=1}} \left(\log \frac{d'}{d}\right) + [-\eta,\eta].$$

We now proceed with some auxiliary lemmas. They all are either identical or rather similar to the lemmas applied in [**8**].

LEMMA 1. *Let f be a nonnegative multiplicative function such that for all primes p*

$$0 \le f(p^\nu) \le \lambda_1 \lambda_2^\nu \qquad (\nu = 1,2,\dots),$$

where $0 < \lambda_1,\ 0 < \lambda_2 < 2$. Then for $x \ge 1$

$$\sum_{n \le x} f(n) \ll_{\lambda_1,\lambda_2} x \prod_{p \le x} (1 - p^{-1}) \sum_{\nu=0}^{\infty} f(p^\nu) p^{-\nu}.$$

This is a weakening of a theorem of Halberstam and Richert [**4**] generalizing a result of Hall.

LEMMA 2. *For $2 \le u \le v \le x$, we have*

$$\operatorname{card}\left\{ n \le x \colon \prod_{p \le u, p^\nu \| n} p^\nu \ge v \right\} \ll x \exp\left(-c_0 \frac{\log v}{\log u}\right)$$

where $c_0 > 0$ is an absolute constant.

This is established in [**3**] and, in a stronger version, in [**10**].

LEMMA 3. *Let $u(x)$ be any function tending to ∞ such that $u(x) \le \log\log x$. Let $\delta_0 > 0$ be a fixed constant. Then for each r with $u(x) \le r \le \log\log x$ we have uniformly in s, $u(x) \le s \le r$,*

$$|\omega(n_r/n_{r-s}) - s| \le \delta_0 s$$

for all $n \le x$ except a set of cardinality $\ll_{\delta_0} x \exp(-c(\delta_0)u(x))$, where $c(\delta_0) > 0$ depends only on δ_0.

PROOF. We first estimate the number of integers $n \le x$ for which $\omega(n_r/n_{r-s}) \le s(1-\delta_0)$ for any integer s. By Lemma 1 this number does not exceed

$$\sum_{u(x) \le s \le r} \sum_{n \le x} \alpha^{\omega(n_r/n_{r-s}) - \alpha s} \ll_\alpha x \sum_{s \ge u(x)} e^{-Q(\alpha)s},$$

where $\alpha = 1 - \delta_0$, $Q(\alpha) = \alpha \log \alpha - \alpha + 1 > 0$. The number of integers $n \le x$ for which $\omega(n_r/n_{r-s}) \ge s(1+\delta_0)$ for any s does not exceed

$$\sum_{u(x) \le s \le r} \sum_{n \le x} \beta^{\omega(n_r/n_{r-s}) - \beta s} \ll_\beta x \sum_{s \ge u(x)} e^{-Q(\beta)s},$$

where $\beta = 1 + \delta_0$, $Q(\beta) > 0$.

LEMMA 4. *Let $w(x)$ be any function defined for $x \geq 1$ such that $w(x) > 1$, $w(x) \to \infty$ for $x \to \infty$, and $v(x) \geq w(x)$. Moreover, we asume that*

$$l \geq \rho^k v(x)(\log\ 3 - 1)^{-1}(1 + \delta_1)$$
$$\text{for some } \delta_1 > 0, r_{k,l} \leq \log\log x \text{ and } 1 \geq \eta \geq 1/r_k.$$

Then there exists $c_1 = c_1(\delta_1) > 0$ such that $\mu(n, k, l, \eta) \geq \exp(r_{k,l}) w(x)^{-2}$ for all $n \leq x$ except a set of cardinality $\ll_{\delta_1} xw(x)^{-c_1}$.

PROOF. Set

$$F(z) = F(k, l, z) = \sum_{\substack{dd' | \tilde{n}_{(k,l)}; \log(d'/d) \leq z \\ \mu(dd')=1}} 1.$$

The $\mu(n, k, l, \eta)$ is the measure of the set of those z for which $F(z+\eta) - F(z-\eta) \neq 0$. We introduce the exponential sum

$$S_{k,l}(\theta, n) = \sum_{\substack{dd' | \tilde{n}_{(k,l)} \\ \mu(dd')=1}} (d'/d)^{i\theta}.$$

We have

$$\begin{aligned} F(z+\eta) - F(z-\eta) &\leq 2 \int_{-\infty}^{\infty} \left(\frac{\sin((u-z)/2\eta)}{(u-z)/2\eta} \right)^2 dF(u) \\ &= 2\eta \int_{-1/\eta}^{1/\eta} e^{i\theta z}(1 - |\theta\eta|) S_{k,l}(\theta, n) d\theta \end{aligned}$$

by Parseval's formula.

A second application of this formula implies

$$\int_{-\infty}^{\infty} (F(z+\eta) - F(z-\eta))^2\, dz \leq 8\pi\eta^2 \int_{-1/\eta}^{1/\eta} (1 - |\theta\eta|)^2 S_{k,l}(\theta, n)^2 d\theta.$$

This together with

$$\begin{aligned} (2\eta 3^{\omega(\tilde{n}_{(k,l)})-1})^2 &\leq \left(\int_{-\infty}^{\infty} (F(z+\eta) - F(z-\eta))\, dz \right)^2 \\ &\leq \mu(n, k, l, \eta) \int_{-\infty}^{\infty} (F(z+\eta) - F(z-\eta))^2\, dz \end{aligned}$$

gives

$$\mu(n, k, l, \eta) \geq 3^{2\omega(\tilde{n}_{(k,l)})-2} \left(2\pi \int_{-1/\eta}^{1/\eta} S_{k,l}(\theta, n)^2\, d\theta \right)^{-1}.$$

Thus to establish Lemma 4 it suffices to prove

$$\int_{-1/\eta}^{1/\eta} S_{k,l}(\theta, n)^2\, d\theta \leq 3^{2\omega(\tilde{n}_{(k,l)})-2} e^{-r_{k,l}} w(x)^2 (2\pi)^{-1}$$

for all $n \leq x$ except a set of cardinality $\ll_{\delta_1} xw(x)^{-c_1}$, where $c_1 = c_1(\delta_1)$ is a suitable constant. For this purpose we decompose

$$S_{k,l}(\theta, n) = \tfrac{1}{2}(S_{k,l}^{(1)}(\theta, n) + S_{k,l}^{(2)}(\theta, n))$$

where

$$S_{k,l}^{(1)}(\theta,n) = \sum_{dd'|\tilde{n}_{(k,l)}} (d'/d)^{i\theta} = \prod_{p|\tilde{n}_{(k,l)}} (1+2\cos(\theta\log p)),$$

$$S_{k,l}^{(2)}(\theta,n) = \sum_{dd'|\tilde{n}_{(k,l)}} \mu(dd')(d'/d)^{i\theta} = \prod_{p|\tilde{n}_{(k,l)}} (1-2\cos(\theta\log p)).$$

Since $S_{k,l}^2 \le \frac{1}{2}(S_{k,l}^{(1)2} + S_{k,l}^{(2)2})$ it is sufficient to show that

$$\int_{-1/\eta}^{1/\eta} S_{k,l}^{(i)}(\theta,n)^2\, d\theta \le (2\pi)^{-1} 3^{2\omega(\tilde{n}_{(k,l)})-2} e^{-r_{k,l}} w(x)^2, \tag{2.1}$$

for all $n \le x$ except a set of cardinality

$$\ll_{\delta_1} x w(x)^{-c_1} \qquad (i=1,2).$$

We show this only for $i=1$, the case $i=2$ being analogous.

For the range $|\theta| \le \exp(-r_{k,l})w(x)$ we take the trivial estimate

$$|S_{k,l}^{(1)}(\theta,n)| \le 3^{\omega(\tilde{n}_{(k,l)})}$$

and obtain

$$\int_{|\theta|\le\exp(-r_{k,l})w(x)} S_{k,l}^{(1)}(\theta,n)^2\, d\theta \le 2\cdot 3^{2\omega(\tilde{n}_{(k,l)})} \exp(-r_{k,l}) w(x). \tag{2.2}$$

For the estimate of the contribution from the remaining range we introduce

$$f_\theta(n) = S_{k,l}^{(1)}(\theta,n)^2 z^{\omega(\tilde{n}_{(k,l)})} y^{\omega_\theta(\tilde{n}_{(k,l)})}$$

with

$$\omega_\theta(r) = \sum_{\substack{\log p \le 1/|\theta| \\ p|r}} 1$$

and estimate $\sum_{n\le x} f_\theta(n)$ by Lemma 1.

We have $f_\theta(n) = \prod_{p|n} f(p)$, where

$$f_\theta(p) = \begin{cases} (1+2\cos(\theta\log p))^2 yz, & \text{if } \exp(r_k) \le \log p \le \theta^{-1}, \\ (1+2\cos(\theta\log p))^2 z, & \text{if } \theta^{-1} < \log p < \exp(r_{k,l}), \\ 1, & \text{otherwise} \end{cases}$$

in the range $\exp(-r_{k,l})w(x) \le \theta \le \exp(-r_k)$ and

$$f_\theta(p) = \begin{cases} (1+2\cos(\theta\log p))^2 z, & \text{if } r_k \le \log\log p < r_{k,l}, \\ 1, & \text{otherwise} \end{cases}$$

in the range $\theta > \exp(-r_k)$.

We obtain for the first range

$$\begin{aligned} \sum_{n\le x} f_\theta(n) &\ll x\exp\left(\sum_{\exp(r_k)\le\log p\le\theta^{-1}} \frac{9yz-1}{p} \right. \\ &\qquad \left. + \sum_{\theta^{-1}<\log p<\exp(r_{k,l})} \frac{z(1+2\cos(\theta\log p))^2-1}{p} \right) \\ &\ll x\exp\left\{ (9yz-1)(\log^+(|\theta|^{-1}) - r_k + 1) \right. \\ &\qquad \left. + (3z-1)(r_{k,l} - \log^+(|\theta|^{-1})) + O(z) \right\}, \end{aligned}$$

the second sum over p being estimated, using the prime number theorem as explained in [**6**, Lemma 4].

For the second range we obtain

$$\sum_{n\le x} f_\theta(n) \ll x\exp\left(\sum_{r_k\le\log\log p<r_{k,l}} \frac{z(1+2\cos(\theta\log p))^2-1}{p}\right)$$
$$\ll x\exp\{(3z-1)l+O_z(\log\log(3+|\theta|))\}.$$

Now we choose $y=\frac{1}{3}$, $z=\frac{1}{3}$ and we obtain

$$\sum_{n\le x} f_\theta(n) \ll \begin{cases} x, & \text{if } \exp(-r_{k,l})w(x)\le|\theta|\le\exp(-r_k),\\ x(\log(3+|\theta|))^{c_2}, & \text{if } \exp(-r_k)<|\theta|\le 1/\eta,\end{cases} \tag{2.3}$$

where $c_2>0$ is an absolute constant.

To get estimates for $S_{k,l}^{(1)}(\theta,n)$ itself we need estimates for $\omega(\tilde n_{(k,l)})$ and $\omega_\theta(\tilde n_{(k,l)})$.

We set $\delta_2=\frac{1}{2}(1-(\log 3)^{-1})$ and obtain by

LEMMA 3. *$\omega(\tilde n_{(k,l)})-\omega_\theta(\tilde n_{(k,l)})\ge(1-\delta_2)(r_{k,l}-\log(|\theta|^{-1}))$ in the range $\exp(-r_{k,l})w(x)<|\theta|\le\exp(-r_k)$ for all $n\le x$ except a set of cardinality $\ll x\exp(-c_1\log w(x))$ for an appropriate $c_1=c_1(\delta_1)>0$.*

Together with (2.3) this yields

$$\sum_{n\le x}{}' S_{k,l}^{(1)}(\theta,n)^2 3^{-2\omega(\tilde n_{k,l})} \ll x3^{-(1-\delta_2)(r_{k,l}-\log(|\theta|^{-1}))}$$

for the range $\exp(-r_{k,l})w(x)<|\theta|\le\exp(-r_k)$, where the sum $\sum'$ is extended over all $n\le x$ except a set of cardinality $\ll xw(x)^{-c_1}$. Thus

$$\begin{aligned}\sum_{n\le x}{}' 3^{-2\omega(\tilde n_{(k,l)})}\int_{\exp(-r_{k,l})w(x)<|\theta|\le\exp(-r_k)} S_{k,l}^{(1)}(\theta,n)^2\,d\theta \\ \ll x\exp(-r_{k,l})w(x)^{-(1-\delta_2)\log 3+1}.\end{aligned} \tag{2.4}$$

For the estimate of the integral over the second range $\exp(-r_k)<|\theta|\le 1/\eta$ we observe that because of $l\ge r_k(\log 3-1)^{-1}(1+\delta_1)$ we can find $\delta_3=\delta_3(\delta_1)>0$ such that

$$l((1-\delta_3)\log 3-1)\ge r_k(1+\delta_3).$$

$|\omega(\tilde n_{(k,l)})-l|\le\delta_3 l$ for all $n\le x$ except a set of cardinality $\ll_{\delta_1} x\exp(-c_3(\delta_1)w(x))$, where $c_3>0$ depends only on δ_1. Thus

$$\begin{aligned}\sum_{n\le x}{}'\left(\int_{\exp(-r_k)<|\theta|\le 1/\eta} S_{(k,l)}^{(1)}(\theta,n)^2 d\theta\right)3^{-2\omega(\tilde n_{(k,l)})} \\ \ll \frac{x}{\eta}3^{-(1-\delta_3)l}\left(\log\left(3+\frac{1}{\eta}\right)\right)^{c_2}\end{aligned} \tag{2.5}$$

where $\sum'$ means that the sum is extended over all $n\le x$ except a set of cardinality $\ll_{\delta_1} x\exp(-c_3(\delta_1)w(x))$. But $3^{-(1-\delta_3)l}\ll\exp(-r_{k,l})\exp(-\delta_3 r_k)$. Now (2.2), (2.4), (2.5) give that for $x\ge x_0$

$$\sum_{n\le x}{}'\left(\int_{|\theta|\le 1/\eta} S_{(k,l)}^{(1)}(\theta,n)^2\,d\theta\right)3^{-2\omega(\tilde n_{(k,l)})}\le xe^{-r_{k,l}}w(x)^2,$$

where $\sum'$ is extended over all $n \le x$ except a set of cardinality $\ll_{\delta_1} xw(x)^{-c_1}$. This proves (2.1) and thus finishes the proof of Lemma 4.

3. Proof of Theorem 2. Given now

$$\gamma < -\frac{\log 2}{\log(1-(\log 3)^{-1})}$$

then we fix $\varepsilon_1 > 0$ such that

$$(1-10\varepsilon_1)\frac{\log 2}{\gamma} > -\log(1-(\log 3)^{-1}). \tag{3.1}$$

Then we set

$$\begin{aligned} &\rho = \min\left(\exp\left((1-8\varepsilon_1)\frac{\log 2}{\gamma}\right), \frac{\log 3}{\log 3 - 1} + \frac{1}{2}\frac{1-\log 2}{\log 3 - 1}\right),\\ &v(x) = (\log\log x)^{6\varepsilon_1}, \quad w(x) = (\log\log x)^{\varepsilon_1},\\ &K = \left[\frac{1+\varepsilon_1}{\log 2}\gamma\log\log\log x\right]. \end{aligned} \tag{3.2}$$

These choices imply that $\rho^K v(x) \le (\log\log x)^{1-\varepsilon_1}$ and $2^K > 2(\log\log x)^{\gamma}$.

In the following considerations we always assume that x is sufficiently large: $x \ge x_0(\gamma)$. We are now asking for blocks of divisors $d_1 < d_2 < \cdots < d_s$ such that $\mu(d_1) = \mu(d_2) = \cdots = \mu(d_s) \ne 0$, which are not interrupted by other divisors.

To make our demands more precise we introduce the two sequences:

$$\xi_k = \frac{1}{100}\sum_{l=1}^{k}\frac{1}{l^2} \quad \text{and} \quad \varsigma_k = \log 2 - \frac{1}{100}\sum_{l=1}^{k}\frac{1}{l^2}, \qquad k \ge 0.$$

Later we will still need

$$\eta_k = 1/100k^2.$$

We now introduce the property $B(k)$. We say that an integer $n \le x$ has *property* $(B(k))$, if the following is true:

There are 2^k divisors of $n_{(k)}$ having the following property $(P(k))$.

$$\begin{gathered} d_1 < \cdots < d_{2^k}, \quad \mu(d_1) = \cdots = \mu(d_{2^k}) \ne 0,\\ |\log d_{2^k} - \log d_1| \le \xi_k \quad \text{and} \quad d \,|\, n,\ \mu(d) \ne 0,\\ d \notin \{d_1, \dots, d_{2^k}\} \Rightarrow \log d < \log d_1 - \varsigma_k \ \textit{or}\ \log d > \log d_2 + \varsigma_k. \end{gathered}$$

We will prove by induction in k for $0 \le k \le K$ the *statement* $S(k)$:

All integers $n \le x$ have property $(B(k))$ except those of a set of cardinality $\le c_4(\gamma)xw(x)^{-c_5(\gamma)}(k+1)$. If $k = K$ this means that all integers $n \le x$ except a set of cardinality $\le c_4(\gamma)xw(x)^{-c_5(\gamma)}(K+1)$ have property $(B(K))$, which proves Theorem 2, since $2^K > 2(\log\log x)^{\gamma}$.

Proof of $S(0)$. $S(0)$ means that there is a *single divisor* $d_1 \,|\, n_{(0)} = n_{v(x)}$ with property

$$\mu(d_1) \ne 0, \quad |\log d_1 - \log d| \ge \log 2 \quad \text{for all } d \,|\, n,\ d \ne d_1,\ \mu(d) \ne 0. \tag{$P(0)$}$$

We set $z_0 = \frac{1}{2}v(x)$ and write

$$n = n^*_{z_0} p_1^{(z_0)\alpha_1} \cdots p_{r(n)}^{(z_0)\alpha_{r(n)}}.$$

We claim that for all $n \le x$ except a set of cardinality $\le xw(x)^{-2c_5(\gamma)}$ the divisor $p_1^{(z_0)}$ has property $P(0)$. We denote the exceptional set by $\mathcal{E}(x)$.

$n \in \mathcal{E}(x)$ implies that there is a $d|n_{z_0}$ such that $|\log d - \log p_1^{(z_0)}| < \log 2$ or that $|\log p_2^{(z_0)} - \log p_1^{(z_0)}| < \log 2$. There are $\ll xw(x)^{-A}$ integers $n \le x$ for which $n_{z_0}^* \ge x^{1/6}$ or $p_1^{(z_0)} \ge x^{1/6}$ or $\omega(n_{z_0}^*) \ge ((\log 5)/(\log 2) - 1)z_0$, by Lemmas 2 and 3, where A is arbitrarily large. Denote by $m_{z_0}^*$ any integer equal to $n_{z_0}^*$ for some $n \le x$ and by $h(r)$ an integer all of whose prime factors are $> r$.

Then we have

$$\begin{aligned}\operatorname{card}\mathcal{E}(x) \ll &\sum_{\substack{m_{z_0}^*\,:\, m_{z_0}^* < x^{1/6} \\ \omega(m_{z_0}) < ((\log 5)/(\log 2) - 1)z_0}} \sum_{p_1}{}' \sum_{h(p_1-1) \le x/m_{z_0}^* p_1} \\ &+ \sum_{m_{z_0}^* < x^{1/6}} \sum_{p_1 \ge \exp\exp z_0} \sum_{p_2\,:\, p_1 \le p_2 \le 2p_1} \frac{x}{m_{z_0}^* p_1 p_2} + xw(x)^{-A} \\ = &\sum\nolimits_1 + \sum\nolimits_2 + xw(x)^{-A}, \quad \text{say}.\end{aligned}$$

In $\sum'_{p_1}$ the sum is extended over all p_1 with $\exp\exp z_0 \le p_1 < x^{1/6}$ for which there exists a $d \,|\, m_{z_0}$ with $|\log d - \log p_1| < \log 2$.

Since now $m_{z_0}^* p_1 \le x^{1/3}$, the last sum $\sum'_{h(p_1-1) \le x/m_{z_0}^* p_1}$ is $\ll x/m_{z_0}^* p_1 \log p_1$ by the sieve. Thus we obtain

$$\sum\nolimits_1 \le x \sum_{m_{z_0}^* < x^{1/6}} \frac{1}{m_{z_0}^*} \sum_{d \,|\, m_{z_0}} \sum_{\substack{p_1\,:\, |\log p_1 - \log d| < \log 2 \\ p_1 \ge \exp\exp z_0}} \frac{1}{p_1 \log p_1}.$$

In the inner sum $\log\log p_1$ is contained in an interval of length $\ll e^{-z_0}$. Thus the p_1-sum is $\ll e^{-2z_0}$.

We get

$$\sum\nolimits_1 \ll x \sum_{m_{z_0}^* < x^{1/6}} \frac{1}{m_{z_0}^*} e^{-z_0} \left(\frac{5}{2}\right)^{z_0} e^{-z_0}.$$

By the sieve

$$\operatorname{card}\left\{n \le x \colon n_{z_0}^* = m_{z_0}^*\right\} \sim xe^{-z_0}/m_{z_0}^*$$

such that

$$\sum_{m_{z_0}^* \le x^{1/6}} \frac{x}{m_{z_0}^*} e^{-z_0} \ll x.$$

Thus $\sum_1 \ll x(5/2e)^{z_0}$.

For $\sum_2$ we get

$$\sum\nolimits_2 \ll \sum_{m_{z_0}^* \le x^{1/6}} \sum_{p_1 \ge \exp\exp z_0} \frac{x}{m_{z_0}^* p_1 \log p_1} \ll xe^{-z_0}.$$

This concludes the proof of $S(0)$.

Induction step $S(k) \Rightarrow S(k+1)$. The induction step is similar to the proof of Theorem 2 in [**8**] but there are additional difficulties. Since the induction step is rather complicated, we start by giving an outline.

Outline of the induction step. Assume that n has property $(B(k))$ and let the block of 2^k divisors of $n_{(k)}: d_1, \ldots, d_{2^k}$ be contained in the interval $I_k = [\log d_1 - \varsigma_k, \log d_{2^k} + \varsigma_k]$. We then consider the *translates* $I_k + \log d$, where d consists of larger prime factors of n. Our aim is to show that almost always *two such translates merge into a block of the double size* 2^{k+1}.

That would conclude the induction step $S(k) \Rightarrow S(k+1)$ if the d's are not too large. The aim, to establish the merger of two translates, is roughly achieved as follows:

We denote by $\mathcal{B}(k,l)$ the exceptional set of integers for which no two translates $I_k + \log d$, $I_k + \log d'$, d, $d' | n_{(k,l)}$ have merged. We then will show that card $\mathcal{B}(k,l)$ is exponentially decreasing for increasing l. We have already shown in Lemma 4 that the measure of

$$\bigcup_{\substack{dd' \mid \tilde{n}_{(k,l)} \\ \mu(dd')=1}} \log(d'/d) + [-\eta, \eta]$$

is fairly large for most η.

This leaves many possibilities for the subsequent prime divisors $p_1^{(k,l)}$ and $p_2^{(k,l)}$ that the difference $\log p_2^{(k,l)} - \log p_1^{(k,l)}$ is close to a logarithm $\log(d'/d)$. But then $\log d + \log p_1^{(k,l)} + I_k$ contains the block of 2^{k+1} divisors:

$$\log d_j + \log d + \log p_1^{(k,l)}, \quad \log d_j + \log d' + \log p_2^{(k,l)} \qquad (j = 1, \ldots, 2^k).$$

Thus, if $n_{(k,l)}$ does not have property $(B(k))$ and therefore by definition $n_{(k,l)} \in \mathcal{B}(k,l)$, the *conditional probability* that for small j, $n_{(k,l+j)}$ still does not have property $(B(k))$ and thus $n_{(k,l+j)} \in \mathcal{B}(k, l+j)$ is not too close to 1. This fact accounts for the shrinking of $\mathcal{B}(k,l)$ with increasing l.

There is one additional difficulty to overcome. We have to guarantee that the new larger block of 2^{k+1} divisors is not interrupted by other divisors with different μ-values. This is accomplished by only considering translates $I_k + \log d$, which do *not contain* $\log \tilde{d}$-values *other than* the translates of the $\log d_j$. We will call such divisors d *pure.*

Thus instead of the measure of

$$\mathcal{E}(n,k,l,\eta) = \bigcup_{\substack{dd' \mid \tilde{n}_{(k,l)} \\ \mu(dd')=1}} \log(d'/d) + [-\eta, \eta]$$

we have to consider the measure of

$$\mathcal{D}(n,k,l,\eta) = \bigcup_{\substack{dd' \mid \tilde{n}_{(k,l)} \\ \mu(dd')=1 \\ d,d' \text{ pure}}} \log(d'/d) + [-\eta, \eta].$$

In Lemma 5 we will show that the contribution in $\mathcal{E}(n,k,l,\eta)$ of d, d' that are not pure is very small. Thus meas $\mathcal{D}(n,k,l,\eta)$ is approximately meas $\mathcal{E}(n,k,l,\eta)$.

After this outline we now give the details of the induction step.

DEFINITIONS. We denote by $\mathcal{B}(k)$ the set of all $n \le x$ that possess property $(B(k))$ and by $\mathcal{B}(k,l)$ the set of all integers $n \le x$ that possess the property $(B(k))$, but for which there exists no block of 2^{k+1} divisors d_j, $1 \le j \le 2^{k+1}$, $d_j | n_{(k,l)}$ with property $(P(k+1))$.

Given $n \in \mathcal{B}(k,l)$ and a block of 2^k divisors $d_j | n_{(k)}$, $1 \le j \le 2^k$, with property $(P(k))$. We set $I_k(n_{(k)}) = [\log d_1 - \varsigma_k, \log d_{2^k} + \varsigma_k]$. If there are several blocks we arbitrarily choose one of them to define $I_k(n_k)$. Many of the following definitions will depend on this choice of $I_k(n_{(k)})$.

Given any positive integer r, we call $d | n/n^*_{(k)}$ *r-pure* if $I_k(n_{(k)}) + \log d$ contains *no* $\log d'$, $d' \mid (n,r)$ *other than* the translates $\log d' := \log d_j + \log d$ $(1 \le j \le 2^k)$. For $\eta > 0$ we denote by $\lambda(n,k,l,\eta)$ the Lebesgue measure of the set

$$\mathcal{D}(n,k,l,\eta) = \bigcup_{\substack{dd' \mid \tilde{n}_{(k,l)} \\ \mu(dd')=1 \\ d,d' \ n_{(k,l)}\text{-pure}}} \log(d'/d) + [-\eta, \eta].$$

Let now $\varepsilon_2 > 0$ be a constant to be determined later. Then we define

$$L_k = \rho^k(\rho - 1 - 2\varepsilon_2)v(x) \quad \text{and} \quad M_k = \rho^k(\rho - 1 - \varepsilon_2)v(x).$$

We will prove

LEMMA 5. *For all $n \in \mathcal{B}(k)$ except a set of cardinality $\ll_\gamma x\exp(-c_6(\gamma)w(x))$ we have $\mu(n,k,l,\eta) - \lambda(n,k,l,\eta) \le \exp(r_{k,l}(1 - c_7(\gamma)))$, for $L_k < l \le M_k$, where $c_6(\gamma) > 0$, $c_7(\gamma) > 0$ depend only on γ.*

In preparation for the proof of Lemma 5 we first give some more definitions and prove some auxiliary lemmas.

We set $q_k = r_{k+1} - r_k$ and $s_k = [q_k(1+\varepsilon_3)]$, where $\varepsilon_3 > 0$ will be determined later.

We denote by $\mathcal{R}(k)$ the set of all $n \in \mathcal{B}(k)$ with the following properties:

(i) $n_{(k,q_k)} \mid n^{(s_k)}_{(k)}$,

(ii) $\omega(n_{(k)}) \le r_k(1+\varepsilon_4)$,

(iii) $\log p^{(k)}_s \ge \exp(r_k(1-\varepsilon_5) + s)$ for $1 \le s \le s_k$,

(iv) $n^{*(s_k)}_{(k)} \le x^{1/6}$,

LEMMA 6. $\operatorname{card}(\mathcal{B}(k)\backslash\mathcal{R}(k)) \le C(\gamma,\varepsilon_3,\varepsilon_4,\varepsilon_5)x\exp(-c(\gamma,\varepsilon_3,\varepsilon_4,\varepsilon_5)v(x))$ *where the constants $c > 0$ and $C > 0$ depend only on the indicated parameters.*

PROOF. For any of the properties (i)–(iii) we estimate the set of $n \le x$ not possessing this property by Lemma 3. For the estimate of the set exceptional with respect to (iv) we observe that $\log\log p^{(k)}_{s_k} \le r_k + s_k(1+\varepsilon_3)$ for most n and then apply Lemma 2 for the estimate of $n^*_{(k,s_k(1+\varepsilon_3))}$; observing that $k \le K$ and thus $r_k \le (\log\log x)^{1-\varepsilon_1}$.

DEFINITIONS. We introduce the set

$$\mathcal{F}(n,k,l) = \{d | \tilde{n}_{(k,l)} : d \text{ not } n_{(k,l)}\text{-pure}\}.$$

For $d | \tilde{n}_{(k,l)}$ we define $c_{k,l}(d,n) = \operatorname{card}\{d' | \tilde{n}_{(k,l)} : (d,d') = 1\}$ and obtain

$$\mu(n,k,l,\eta) - \lambda(n,k,l,\eta) \le 2\eta \sum_{d \in \mathcal{F}(n,k,l)} c_{k,l}(d,n) = 2\eta C(n,k,l), \quad \text{say.}$$

Thus

$$\sum_{n\in\mathcal{R}(k)}(\mu(n,k,l,\eta)-\lambda(n,k,l,\eta))\le 2\eta\sum_{n\in\mathcal{R}(k)}C(n,k,l). \tag{3.3}$$

We introduce the set

$$\mathcal{G}(n,k)=\left\{d|\tilde{n}_k^{(s_k)}:\, d \text{ not } n_k^{(s_k)}\text{-pure}\right\}$$

and define

$$b_k(d,n)=\operatorname{card}\left\{d'|\tilde{n}_k^{(s_k)}:\,(d,d')=1\right\}.$$

Since $L_k<l\le M_k$, we have for $n\in\mathcal{R}(k)$: $\tilde{n}_{(k,l)}|\tilde{n}^{(s_k)}$ and therefore $c_{k,l}(d,n)\le b_{k,l}(d,n)$ and $\mathcal{F}(n,k,l)\subseteq\mathcal{G}(n,k)$. Therefore we have the majorization

$$C(n,k,l)\le\sum_{d\in\mathcal{G}(n,k)}b(d,n).$$

We introduce the sequence of sets

$$\mathcal{H}(n,k,s)=\left\{d|\tilde{n}_k^{(s)},d \text{ not } n_k^{(s)}\text{-pure}\right\},\qquad 1\le s\le s_k,$$

such that

$$\mathcal{H}(n,k,s_k)\supseteq\mathcal{G}(n,k)\quad\text{ for all } n\in\mathcal{R}(k).$$

We set

$$B(k,s)=\sum_{n\in\mathcal{R}(k)}\sum_{d\in\mathcal{H}(n,k,s)}b_k(d,n)$$

such that

$$B(k,s_k)\ge\sum_{n\in\mathcal{R}(k)}C(n,k,l)\quad\text{ for } L_k\le l\le M_k. \tag{3.4}$$

We now prove

LEMMA 7. *For $1\le s\le s_k$, $\varepsilon_6>0$ we have*

$$B(k,s)\le C(\gamma,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6)x\exp(-r_k(1-\varepsilon_5)2^{r_k(1+\varepsilon_4)s_k}(3/2+\varepsilon_6)^s.$$

PROOF. If $d\in\mathcal{H}(n,k,s)$ we have $d=d^*$ or $d=d^*p_s^{(k)}$, where $d^*|n_{(k)}^{(s-1)}$. We have

$$\begin{aligned}B(k,s)=\sum_{n\in\mathcal{R}(k)}\Bigg\{&\sum_{d^*\in\mathcal{H}(n,k,s-1)}b_k(d^*,n)+b_k(d^*p_s^{(k)},n)\\&+\sum_{\substack{d^*\,|\,n_k^{(s-1)},d^*\notin\mathcal{H}(n,k,s-1)\\ d^*p_s^{(k)}\in\mathcal{H}(n,k,s)}}b_k(d^*p_s^{(k)},n)\\&+\sum_{\substack{d^*\,|\,n_{(k)}^{(s-1)}:\ d^*\notin\mathcal{H}(n,k,s-1)\\ d^*\in\mathcal{H}(n,k,s)}}b_k(d^*,n)\Bigg\}.\end{aligned}$$

Since $b_k(d^* p_s^{(k)}, n) = \frac{1}{2} b_k(d^*, n)$, we have
(3.5)

$$B(k,s) = \frac{3}{2} B(k,s-1) + \sum_{n \in \mathcal{R}(k)} \left(\sum_{\substack{d \mid \tilde{n}_{(k)}^{(s-1)} :\ d \notin \mathcal{H}(n,k,s-1) \\ d p_s^{(k)} \in \mathcal{H}(n,k,s)}} b_k(d p_s^{(k)}, n) + \sum_{\substack{d \mid \tilde{n}_{(k)}^{(s-1)} \\ d \notin \mathcal{H}(n,k,s-1) \\ d \in \mathcal{H}(n,k,s)}} b_k(d,n) \right)$$

$$= \frac{3}{2} B(k,s-1) + E(k,s), \qquad \text{say}.$$

Estimate of $E(k,s)$. We have

$$E(k,s) \le \sum_{n \in \mathcal{R}(k)} \sum_{d \mid \tilde{n}_{(k)}^{(s-1)}}{}' s^{2_k - \omega(d)},$$

where the $\sum'$-sum is extended over all $d \mid \tilde{n}_k^{(s-1)}$ for which there exists a $\tilde{d} \mid n_{(k)}^{(s-1)}$ with $\log \tilde{d} \in \log(d p_s^{(k)}) + I_k(n_{(k)})$ *or* a $\tilde{d} \mid n_{(k)}^{(s-1)}$ with $\log d \in \log(\tilde{d} p_s^{(k)}) + I_k(n_{(k)})$. Denoting the interval $I_k(n_{(k)})$ by $[a_k(n_{(k)}), b_k(n_{(k)})]$ we have for $s \ge 2$

$$E(k,s) \ll \sum_{\substack{l \in \mathcal{B}(k):\ l \le x^{1/6} \\ \omega(l/l^*_{(k)}) = s-1,\ \omega(l_{(k)}) \le r_k(1+\varepsilon_4) \\ \log p_{s-1}^{(k)}(l) \ge \exp(r_k(1-\varepsilon_5) + (s-1))}} \sum_{\tilde{d} \mid l, \mu(\tilde{d}) \ne 0} \sum_{\substack{d \mid (l/l^*_{(k)}) \\ \mu(d) \ne 0}} 2^{s_k - \omega(d)} \cdot \sum_p{}'' \sum_{l \cdot p \cdot h(p-1) \le x} 1$$

where the $\sum''$-sum is extended over all $p \ge p_{s-1}^{(k)}$ for which

$$|\log p + \log d + a_k(l_{(k)}) - \log \tilde{d}| < \log 2 \quad \textit{or}$$
$$|\log d + a_k(l_{(k)}) - \log \tilde{d} - \log p| < \log 2.$$

We recall that $h(r)$ denotes an integer all of whose prime factors are $> r$. Since $l \cdot p \le 2x^{1/3}$ the inner sum is $\ll x/(lp \log p)$ by the sieve.

The interval for $\log\log p$ in $\sum_p''$ has length $\ll 1/\log p_{s-1}^{(k)}$ such that

$$\sum{}'' \frac{x}{lp \log p} \ll \frac{x}{l(\log p_{(s-1)}^{(k)})^2}.$$

Moreover,

$$\sum_{\substack{\tilde{d} \mid l \\ \mu(\tilde{d}) \ne 0}} \sum_{\substack{d \mid (l/l^*_{(k)}) \\ \mu(d) \ne 0}} 2^{s_k - \omega(d)} \ll 2^{s_k - s} 3^s 2^{\omega(l)}.$$

Thus,

$$E(k,s) \ll \left(\sum_{\substack{l \le x^{1/6}:\ \omega(l/l^*_{(k)})=s-1 \\ \omega(l_{(k)}) \le r_k(1+\varepsilon_4) \\ \log p^{(k)}_{s-1}(l) \ge \exp(r_k(1-\varepsilon_5)+(s-1))}} \frac{x}{l(\log p^{(k)}_{s-1})^2} 2^{\omega(l)} \right) 2^{s_k - s} 3^s.$$

Since

$$\operatorname{card}\left\{ n \le x\colon n^{*(s-1)}_{(k)} = l \right\} \gg \frac{x}{l \log p^{(k)}_{s-1}} \qquad \text{for } l \le x^{1/6}$$

we obtain

$$\sum_{l \le x^{1/6}:\ \omega(l/l^*_{(k)})=s-1} \ll x.$$

Therefore,

$$E(k,s) \ll x 2^{r_k(1+\varepsilon_4)+s_k} \exp(-r_k(1-\varepsilon_5) - (s-1)) 3^s \tag{3.6}$$

for $s \ge 2$.

The estimate of $E(k,1)$ is accomplished in a similar manner. We omit the condition $\log p^{(k)}_{s-1} \ge \exp(r_k(1-\varepsilon_5)+(s-1))$ and observe instead, that $\log p^{(k)}_1 \ge \exp r_k$. This leads to the estimate (3.6) also for $s = 1$.

Now we prove Lemma 7 by induction in s. We choose the integer $s_0 = s_0(\varepsilon_6) > 0$ such that

$$\varepsilon_6 (3/2)^{s_0} \ge (3/e)^{s_0}. \tag{3.7}$$

First it is easily proven by induction, using (3.5) and (3.6), that

$$B(k,s) \le x C_0 (5/2)^s,$$

where

$$C_0 = C'(\gamma, \varepsilon_3, \varepsilon_4, \varepsilon_5) \exp(-r_k(1-\varepsilon_5)) 2^{r_k(1+\varepsilon_4)+s_k}$$

for $s \le s_0$. This gives

$$B(k,s) \le x C_0 (\tfrac{5}{3})^{s_0} (\tfrac{3}{2})^{s_0} \le x C_0 (\tfrac{5}{3})^{s_0} (\tfrac{3}{2} + \varepsilon_6)^{s_0}.$$

For $s \ge s_0$ we continue the induction, observing (3.5), (3.6), and (3.7). This concludes the proof of Lemma 7.

PROOF OF LEMMA 5. From (3.3), (3.4), and Lemma 7 we obtain that

$$\begin{aligned} &\sum_{n \in \mathcal{R}(k)} (\mu(n,k,l,\eta) - \lambda(n,k,l,\eta)) \\ &\qquad \le 2\eta C(\gamma, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) x \exp(r_k(1-\varepsilon_5)) 2^{r_k(1+\varepsilon_4)+s_k} (\tfrac{3}{2} + \varepsilon_6)^{s_k}. \end{aligned} \tag{3.8}$$

We now fix the constants $c_7(\gamma), \varepsilon_2, \ldots, \varepsilon_6$ in a manner such that

$$(\rho - 1 - 2\varepsilon_2) > (\log 3 - 1)^{-1}(1 + \varepsilon_2) \tag{3.9}$$

and

$$(3.10)\qquad \begin{aligned} &-(1-\varepsilon_5)+\{1+\varepsilon_4+(\rho-1)(1+\varepsilon_3)\}\log 2\\ &\qquad+(\rho-1)(1+\varepsilon_3)\log(\tfrac{3}{2}+\varepsilon_6)\\ &\qquad\le[1+(\rho-1-2\varepsilon_2)(1-2c_7(\gamma))], \end{aligned}$$

which is possible because of (3.1) and (3.2). Then (3.8) gives, that

$$\sum_{n\in\mathcal{R}(k)}(\mu(n,k,l,\eta)-\lambda(n,k,l,\eta))\ll_\gamma \exp(r_{k,l}(1-2c_7(\gamma)))$$

for $L_k < l \le M_k$. This implies that

$$\mu(n,k,l,\eta)-\lambda(n,k,l,\eta)\le \exp(r_{k,l}(1-c_7(\gamma)))$$

for all $n\in\mathcal{R}(k)$ except a set of cardinality $\ll_\gamma x\exp(-r_{k,l}c_7(\gamma))$. This together with Lemma 6 implies Lemma 5. Because of (3.9) Lemma 4 is applicable. As an immediate corollary of Lemmas 4 and 5 we obtain

LEMMA 8. *We have $\lambda(n,k,l,\eta)\ge \exp(r_{k,l})w(x)^{-3}$ for all $n\in\mathcal{B}(k)$ except a set of cardinality $\ll_\gamma xw(x)^{-c_5(\gamma)}$.*

Conclusion of the Proof of Theorem 2. To complete the induction step and thus the proof of Theorem 2 we want to show that

$$\operatorname{card}\mathcal{B}(k,l)\le c_4(\gamma)xw(x)^{-c_5(\gamma)}\quad\text{for some } l\in[L_k,M_k].$$

We denote by $\mathcal{C}(k,l)$ the subset of $\mathcal{B}(k,l)$ of those integers which satisfy the three extra conditions:

(a) $\log n^*_{(k,l)}\le \exp(r_{k,l})w(x)$,
(b) $\omega(n_{(k,l)})\le 2r_{k,l}$,
(c) $\lambda(n,k,l,\eta)\ge\exp(r_{k,l})w(x)^{-3}$.

By Lemma 2, 3, and 8 we have

$$\operatorname{card}(\mathcal{B}(k,l)/\mathcal{C}(k,l))\ll_\gamma xw(x)^{-c_5(\gamma)}.$$

Thus to complete the proof of Theorem 2 it suffices to show that

$$(3.11)\qquad \operatorname{card}\mathcal{C}(k,l)\le xw(x)^{-2c_5(\gamma)}\quad\text{for some } l\in[L_k,M_k].$$

Assume that

$$n=n^*_{(k,l)}p_1^{(k,l)}\cdots p_r^{(k,l)},\qquad p_1^{(k,l)}\le\cdots\le p_r^{(k,l)}.$$

We consider the set $\mathcal{A}(k,l)$ of $n\in\mathcal{C}(k,l)$, whose prime factors $p_1^{(k,l)},p_2^{(k,l)},p_3^{(k,l)}$ satisfy the following conditions:

$$\text{(i)}\qquad \exp(r_{k,l})w(x)\le\log p_1^{(k,l)}\le 2\exp(r_{k,l})w(x),$$

$$\text{(ii)}\qquad \log p_2^{(k,l)}-\log p_1^{(k,l)}\in\bigcup_{\substack{dd'\mid\tilde{n}_{(k,l)}\\ dd'\ n_{(k,l)}\text{-pure}}}\log(d'/d)+[-\eta_{k+1},\eta_{k+1}],$$

$$\text{(iii)}\qquad \log p_3^{(k,l)}\ge\log(n_{(k,l)}p_1^{(k,l)}p_2^{(k,l)}).$$

These conditions ensure that there exists a block of 2^{k+1} divisors of $n_{(k,l+j)}$, $j\le 2\log w(x)$, satisfying $(P(k+1))$, namely the divisors $p_1^{(k,l)}d'd_i$, $p_2^{(k,l)}dd_i$ ($1\le i\le 2^k$). Condition (iii) ensures that this block is not destroyed by larger prime factors.

Thus $\mathcal{C}(k,l+j) \subseteq \mathcal{C}(k,l)/\mathcal{A}(k,l)$ such that

$$\operatorname{card}\mathcal{C}(k,l+j) \le \operatorname{card}\mathcal{C}(k,l) - \operatorname{card}\mathcal{A}(k,l). \tag{3.12}$$

We now give a lower bound for $\operatorname{card}\mathcal{A}(k,l)$. Denote by $m^*_{(k,l)}$ an integer equal to $n^*_{(k,l)}$ for some $n \in \mathcal{C}(k,l)$.

We have

$$\operatorname{card}\mathcal{A}(k,l) \gg \sideset{}{^*}\sum_{m^*_{(k,l)}p_1^{(k,l)}p_2^{(k,l)} h(m^*_{(k,l)}p_1^{(k,l)}p_2^{(k,l)}) \le x} 1$$

where $*$ means that $n^*_{(k,l)} \in \mathcal{B}(k)$ and that the $n^*_{(k,l)}, p_i^{(k,l)}$ satisfy (i)–(iii).

By the sieve we have

$$\operatorname{card}\mathcal{A}(k,l) \gg \sideset{}{^*}\sum_{m^*_{(k,l)}, p_1^{(k,l)}, p_2^{(k,l)}} \frac{x}{m^*_{(k,l)}p_1^{(k,l)}p_2^{(k,l)} \log p_2^{(k,l)}}.$$

For a fixed pair $(m^*_{(k,l)}, p_1^{(k,l)})$ the $p_2^{(k,l)}$ cover a union of at most $3^{\omega(\tilde{m}_{(k,l)})} \le 3^{2r_{k,l}}$ disjoint intervals with total logarithmic length $\ge \frac{1}{2}\exp(r_{k,l})w(x)^{-3}$. Moreover all the limit points have logarithm of order $\exp(r_{k,l})w(x)$. This implies that the $p_2^{(k,l)}$-sum is $\gg \exp(-r_{k,l})w(x)^{-5}$. The $p_1^{(k,l)}$-sum is $\gg 1$. Finally,

$$\operatorname{card}\mathcal{A}(k,l) \gg \left(\sum_{m^*_{(k,l)}:\ \log m^*_{(k,l)} \le \exp(r_{k,l})w(x)} \frac{x}{m^*_{k,l}} \right) \exp(-r_{k,l})w(x)^{-5},$$

$$\begin{aligned} \operatorname{card}\mathcal{C}(k,l) &\le \sum_{m^*_{(k,l)}} \sum_{h(\exp\exp r_{k,l}) \le x/m^*_{(k,l)}} 1 \\ &\ll \left(\sum_{m^*_{(k,l)}:\ \log m^*_{(k,l)} \le \exp(r_{k,l})w(x)} \frac{x}{m^*_{(k,l)}} \right) \exp(-r_{k,l}). \end{aligned}$$

Thus

$$\operatorname{card}\mathcal{A}(k,l) \gg \operatorname{card}\mathcal{C}(k,l)w(x)^{-5}.$$

Together with (3.11) this gives

$$\operatorname{card}\mathcal{C}(k,M_k) \le \operatorname{card}\mathcal{C}(k,l)(1-w(x)^{-5})^{(M_k-L_k)/2j} \ll x\exp(-w(x)^{1/2}),$$

which is sufficient.

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