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POSITIVE FORMS AND DILATIONS

WACLAW SZYMANSKI

ABSTRACT. By using the quadratic form and unbounded operator theory a new approach to the general dilation theory is presented. The boundedness condition is explained in terms of the Friedrichs extension of symmetric operators. Unbounded dilations are introduced and discussed. Applications are given to various problems involving positive definite functions.

1. Introduction. There are two principal conditions in general bounded dilation theory: positive definiteness and the boundedness condition. While the first one is naturally justified and generally accepted, the second one is rather complicated, usually not easy to verify, and several simplifications of this condition are known under some additional assumptions. This paper originated as an attempt to explain and understand this boundedness condition. It occurred to us that the positive quadratic form theory provides an appropriate framework for such an explanation. However, consequences of this theory are much deeper than just an explanation of the boundedness condition. They lead naturally to a new general dilation theory, which deals not only with bounded but also with unbounded dilations.

In §2 of this paper some known and some new results on positive quadratic forms and unbounded operators are discussed. In §3 a general dilation theory is presented, which contains the known bounded dilation theory as a special case. §4 deals with several applications of the previous results, in particular to $*$-semigroups, $*$-algebras, Gramians, operator moment problems, and quantum mechanics.

The general reference to quadratic forms and unbounded operators used here is [**8**]. There is an extensive literature on the bounded dilation theory. A general treatment can be found in [**6**, **7**, **1**].

All linear spaces are assumed to be complex. If X, Y are linear spaces, then $L(X,Y)$ ($B(X,Y)$, respectively) stands for the linear space of all linear (bounded linear, if X, Y are normed, respectively) mappings from X to Y. Moreover, $L(X) = L(X,X)$, $B(X) = B(X,X)$. The semigroup structure in $L(X)$ or $B(X)$ is always the multiplicative one, with the composition. I_X or I denotes the identity operator in X. A *subspace* of a Hilbert space H is a linear subset of H. An *operator* T in H is a linear mapping from a subspace $D(T)$ of H into H. $D(T)$, $\ker T$ denote the domain and the kernel of T, respectively, $C^\infty(T)$ denotes the intersection of all $D(T^n)$, $n = 1, 2, \ldots$. Let H, K be Hilbert spaces, let $D(T)$ be a subspace of

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H, and let $T\colon D(T) \to K$ be a linear mapping. T is *closable* if for each sequence $x_n \in D(T)$, $x_n \to 0$, $\|T(x_n - x_m)\| \to 0$ implies $Tx_n \to 0$. T is *closed* if for each sequence $x_n \in D(T)$, $x_n \to x$, $\|T(x_n - x_m)\| \to 0$ implies that x belongs to $D(T)$ and $Tx_n \to Tx$. N denotes the additive semigroup of all nonnegative integers. C denotes the complex plane.

2. Positive forms. Let X be a linear space. With no risk of confusion a mapping $p\colon X \times X \to C$ linear in the first variable, conjugate linear in the second one (such mappings are usually called sesquilinear or quadratic forms) and such that $p(x,x) \geq 0$ for $x \in X$ will be called a *positive form* on X. If p is a positive form on X, then the set

$$N_p = \{x \in X\colon p(x,x) = 0\}$$

is a linear subspace of X, by the Schwarz inequality. The quotient space X/N_p has a natural inner product

$$(x + N_p, y + N_p) = p(x,y), \qquad x, y \text{ in } X. \tag{2.1}$$

The Hilbert space defined as the completion of X/N_p under the norm given by this inner product will be denoted by X_p.

Throughout this paper the symbols N_p, X_p used in connection with a positive form on a linear space will have the fixed meaning just described.

(2.2) LEMMA. *Let X be a linear space. Let p, q be positive forms on X. If $N_q \subset N_p$, then the mapping $p^\sim\colon X/N_q \times X/N_q \to C$ defined by*

$$p^\sim(x + N_q, y + N_q) = p(x,y)$$

for x, y in X is a positive form on X/N_q and the Hilbert spaces X_p and $(X/N_q)_{p^\sim}$ are unitarily isomorphic.

PROOF. Since $N_q \subset N_p$, $p^\sim$ is well defined, and since p is positive, so is $p^\sim$. If $x, y \in X$, then

$$(x+N_p, y+N_p) = p(x,y) = p^\sim(x+N_q, y+N_q) = ((x+N_q)+N_{p^\sim}, (y+N_q)+N_{p^\sim}).$$

Hence the mapping $U\colon (X/N_q)/N_{p^\sim} \to X/N_p$ defined by $U((x+N_q)+N_{p^\sim}) = x+N_p$ for $x \in X$, extends to a unitary mapping from $(X/N_q)_{p^\sim}$ onto X_p. Q.E.D.

When dealing with two forms related as in this lemma, the two unitarily isomorphic Hilbert spaces mentioned above will be treated as identical.

Next, positive forms on subspaces of Hilbert spaces will be considered. Let M be a dense subspace of a Hilbert space H. A positive form p on M is called *closable* if for each sequence $x_n \in M$, $x_n \to 0$, $p(x_n - x_m, x_n - x_m) \to 0$ implies $p(x_n, x_n) \to 0$.

(2.3) PROPOSITION. *Let M be a dense subspace of a Hilbert space H. Let p be a positive form on M and let $M^\wedge$ be the completion of M under the norm $\| \|_+$ given by the inner product*

$$(x,y)_+ = (x,y) + p(x,y) \quad \textit{for } x, y \in M.$$

Then there is a closed positive form $p^\wedge$ on $M^\wedge$ that extends p, and M_p embeds isometrically into $M^\wedge_{p^\wedge}$. Moreover, p is closable if and only if $M^\wedge$ can be embedded injectively into H.

PROOF. If x belongs to $M^\wedge$, then there is a sequence $x_n \in M$ such that $\|x_n - x\|_+ \to 0$. By the Schwarz inequality for p,

$$|p(x_n, x_n) - p(x_m, x_m)| \leq p(x_n - x_m, x_n - x_m)^{1/2}[p(x_n, x_n)^{1/2} + p(x_m, x_m)^{1/2}].$$

Since x_n converges in the norm $\| \ \|_+$, $p(x_n, x_n) \leq \|x_n\|_+^2$, and $p(x_n - x_m, x_n - x_m) \leq \|x_n - x_m\|_+^2 \to 0$, it follows that the sequence $p(x_n, x_n)$ converges. The mapping $p^\wedge \colon M^\wedge \times M^\wedge \to C$ defined by $p^\wedge(x, x) = \lim p(x_n, x_n)$ for $x \in M^\wedge$ is a closed positive form on $M^\wedge$ which extends p. For each $x \in M$: $\|x + N_p\|^2 = p(x, x) = p^\wedge(x, x) = \|x + N_{p^\wedge}\|^2$. Hence the mapping $V \colon M/N_p \to M^\wedge/N_{p^\wedge}$ defined by $V(x + N_p) = x + N_{p^\wedge}$ for $x \in M$, extends to an isometry embedding M_p into $M^\wedge_{p^\wedge}$. Finally, let $i \colon M \to H$ be the inclusion mapping. Since $\|i(x)\| = \|x\| \leq \|x\|_+$ for $x \in M$, it follows that if $x \in M^\wedge$ and a sequence $x_n \in M$ is such that $\|x_n - x\|_+ \to 0$, then $i(x_n)$ converges in the norm $\| \ \|$. It is a routine matter to verify that the mapping $i^\wedge \colon M^\wedge \to H$ defined by $i^\wedge(x) = \lim i(x_n)$ for $x \in M^\wedge$ is injective if and only if p is a closable form. Q.E.D.

This proposition, in the case when p is defined by a positive operator on M, can be found in the proof of Theorem X.23 of [**8**]. It is stated and proved here, because for farther applications it is important to distinguish between properties coming just from positive forms, and properties, in which the presence of a positive operator is necessary.

(2.4) PROPOSITION. *If p is a positive form on a dense subspace M of a Hilbert space H and $x_n \to 0$ implies $p(x_n, x_m) \to 0$ for each sequence $x_n \in M$, then p is closable.*

PROOF. $p(x_n - x_m, x_n - x_m) = p(x_n, x_n) + p(x_m, x_m) - 2\operatorname{Re} p(x_n, x_m)$. Q.E.D.

The next proposition, whose proof is an application of the Riesz-Fischer theorem, is stated for the sake of completeness.

(2.5) PROPOSITION. *Let p be a positive form on a dense subspace M of a Hilbert space H. Then the following conditions are equivalent:*

(a) *For each x in M there is $m(x) \geq 0$ such that $|p(x, y)| \leq m(x)\|y\|$, $y \in M$.*

(b) *There is a unique positive operator $P \colon M \to H$ such that $p(x, y) = (Px, y)$ for all $x, y \in M$.*

The following theorem is the main result of this section. Some of its assertions are known. The known ones are stated here for two reasons: to formulate properly the new ones, and, more importantly, to gather in one place everything that is necessary to develop the dilation theory in the next sections.

(2.6) THEOREM. *Let H, K be Hilbert spaces, let M be a dense subspace of H, let p be a positive form on M, let $P \colon M \to H$, $T \colon M \to K$ be linear mappings.*

(a) *If $p(x, y) = (Tx, Ty)$ for $x, y \in M$, then p is closable (closed, resp.) if and only if T is a closable (closed, resp.) linear mapping.*

(b) *If $p(x, y) = (Tx, Ty)$ for $x, y \in M$, p is a closable form, and $p^\wedge$ is the closed positive form on $M^\wedge \subset H$ extending p (cf. Proposition (2.3)), then T extends to a closed linear mapping $T^\wedge \colon M^\wedge \to K$ such that $p^\wedge(x, y) = (T^\wedge x, T^\wedge y)$ for all $x, y \in M^\wedge$.*

(c) *If $p(x, y) = (Px, y)$ for $x, y \in M$, then p extends to a closed positive form $p^\wedge$ on $M^\wedge \subset H$, P extends to a selfadjoint, positive operator $P^\wedge \colon D(P^\wedge) \to H$ such*

that $M \subset D(P^\wedge) \subset M^\wedge$, $p^\wedge(x,y) = (P^\wedge x, y)$, *for* $x \in D(P^\wedge)$, $y \in M^\wedge$, *and* M_p *embeds isometrically into* $M^\wedge_{p^\wedge}$. *Moreover,*
(2.7)

$$p^\wedge(x,x) \leq \mu(x)\|x\|^2 \quad \text{for each } x \in C^\infty(P^\wedge), \text{ where } \mu(x) = \liminf \|P^{\wedge^{2^n}} x\|^{2^{-n}}.$$

(d) *If* $p(x,y) = (Tx,Ty) = (Px,y)$ *for all* $x,y \in M$, *then* $T^\wedge$ *defined in* (c) *is bounded and* $M^\wedge = H$ *if and only if there is a dense subset* G *of* $C^\infty(P^\wedge)$ *such that* $\sup\{\mu(x) \colon x \in G\}$ *is finite.*

PROOF. (a) is clear by the equality $p(x,x) = \|Tx\|^2$ for each $x \in M$.

(b) If $x \in M^\wedge$, then there is a sequence $x_n \in M$ that converges to x in the norm $\|\ \|_+$. Since $\|Tx\|^2 = p(x,x) \leq \|x\|_+^2$ for x in M, it follows that the sequence Tx_n converges in the norm $\|\ \|$. Put $T^\wedge x = \lim Tx_n$. It follows from the definition of $p^\wedge$ that

$$p^\wedge(x,x) = \lim P(x_n,x_n) = \lim(Tx_n,Tx_n) = (T^\wedge x, T^\wedge x).$$

Thus $p^\wedge(x,y) = (T^\wedge x, T^\wedge y)$ for all $x,y \in M^\wedge$. Since $p^\wedge$ is a closed form, $T^\wedge$ is a closed mapping, by (a). It should be mentioned that the inclusion $M^\wedge \subset H$ is understood in the sense that $M^\wedge$ is injectively embedded into H as in Proposition (2.3).

(c) The existence of $M^\wedge$, $p^\wedge$, and $P^\wedge$ such that $M \subset D(P^\wedge) \subset M^\wedge \subset H$, $p^\wedge$ is a closed, positive form on $M^\wedge$, extending p, $P^\wedge$ is a selfadjoint, positive extension of P and $p^\wedge(x,y) = (P^\wedge x, y)$ for all $x \in D(P^\wedge)$, $y \in M^\wedge$, follows from the Friedrichs extension theorem [**8**, Theorem X.23]. Since $p^\wedge$ extends p, M_p can be isometrically embedded into $M^\wedge_{p^\wedge}$, which is shown in the proof of Proposition (2.3). Let now $x \in C^\infty(P^\wedge) \subset D(P^\wedge)$. Then

$$p^\wedge(x,x) = (P^\wedge x, x) \leq \|P^\wedge x\|\|x\|,$$

and for each $n = 0, 1, 2, \ldots$

$$\|P^{\wedge^{2^n}} x\|^2 = (P^{\wedge^{2^{n+1}}} x, x) \leq \|P^{\wedge^{2^{n+1}}} x\|\|x\|.$$

These two inequalities are the first and the "$n \Rightarrow n+1$" step, respectively in the inductive proof of the inequality

$$p^\wedge(x,x) \leq \|P^{\wedge^{2^n}} x\|^{2^{-n}} \|x\|^{1+2^{-1}+\cdots+2^{-n}},$$

which holds for each $n = 0, 1, 2, \ldots$. Now (2.7) follows.

(d) Firstly notice that if $Q \colon M \to H$ is a selfadjoint operator, then $C^\infty(Q)$ is dense in H, which is remarked on p. 201 of [**8**, vol. II] after the definition of C^∞-vectors, and which is a consequence of Corollary 1, p. 203 of [**8**, vol. II]. A direct proof of this assertion can be given using the spectral theorem as follows: Let F be the operator-valued spectral measure of Q and let σ_n be the closed interval $[-n, n]$. Since $Q|F(\sigma_n)H$ is a bounded operator, for each n, it can be shown that $C^\infty(Q)$ contains the union of all $F(\sigma_n)H$, $n = 1, 2, \ldots$. This union is dense in H, because $F(\sigma_n)x \to x$ for each $x \in H$.

Now suppose that G is a dense subset of $C^\infty(P^\wedge)$ such that $c = \sup\{\mu(x) \colon x \in G\}$ is finite. Then, by (b) and (2.7),

$$\|T^\wedge x\|^2 \leq c\|x\|^2, \qquad x \in G.$$

Since $C^\infty(P^\wedge)$ is dense in H, as shown above, G is dense in H. Let now $x \in H$. Then there is a sequence $x_n \in G$ such that $x_n \to x$. The crucial property of the Friedrichs extension $P^\wedge$ of P is that the domain $D(P^\wedge)$ of $P^\wedge$ is contained in the domain $M^\wedge$ of the extended form $p^\wedge$. Therefore $x_n \in G \subset C^\infty(P^\wedge) \subset D(P^\wedge) \subset M^\wedge$. Since $\|T^\wedge x\|^2 \leq c\|x\|^2$ for $x \in G$, it follows that the sequence $T^\wedge x_n$ converges. Since $T^\wedge$ is a closed mapping, $x \in D(T^\wedge) = M^\wedge$ and $T^\wedge x_n \to T^\wedge x$. This proves that $M^\wedge = H$ and that $\|T^\wedge x\|^2 \leq c\|x\|^2$ for $x \in M^\wedge$. Hence $T^\wedge$ is bounded.

Conversely, if $M^\wedge = H$ and $T^\wedge : M \to K$ is a bounded mapping, then $(T^{\wedge *}T^\wedge x, y) = (T^\wedge x, T^\wedge y) = (P^\wedge x, y)$ for each $x \in D(P^\wedge)$, $y \in M^\wedge$. Hence $T^{\wedge *}T^\wedge x = P^\wedge x$ for $x \in D(P^\wedge)$. Since $D(P^\wedge)$ is dense in H, if $x \in H$, then there is a sequence $x_n \in D(P^\wedge)$ that converges to x. By the last equality, $P^\wedge x_n \to T^{\wedge *}T^\wedge x$. Since $P^\wedge$ is closed, $x \in D(P^\wedge)$ and $P^\wedge x_n \to P^\wedge x$. Hence $D(P^\wedge) = H$ and $P^\wedge = T^{\wedge *}T^\wedge$. Moreover, for each $x \in H$ and $n = 0, 1, 2, \ldots$:

$$\|P^{\wedge 2^n} x\|^{2^{-n}} \leq \|P^\wedge\| \|x\|^{2^{-n}}.$$

Thus $\mu(x) \leq \|P^\wedge\|$ for each $x \in H$. Q.E.D.

The arguments at the end of the proof of (c) in this theorem prove

(2.8) *If M is a dense subspace of a Hilbert space H and if $P\colon M \to H$ is a positive operator, then*

$$(Px, x) \leq \liminf \|P^{2^n} x\|^{2^{-n}} \|x\|^2$$

for each $x \in C^\infty(P)$.

However, positive operators may have no nonzero C^∞-vectors. The reasoning in the middle part of the proof of (d) justifies the following.

(2.9) COROLLARY. *Let M be a dense subspace of a Hilbert space H, let K be a Hilbert space and let $T\colon M \to K$ be a closed linear mapping. Then $M = H$ and T is bounded if and only if there is a dense subset G of M and $c \geq 0$ such that $\|Tx\| \leq c\|x\|$ for $x \in G$.*

Finally, notice that if p is a positive form on a linear space X, then there exist a Hilbert space K and a linear mapping $T\colon X \to K$ such that $p(x, y) = (Tx, Ty)$ for $x, y \in X$. Simply take $K = X_p$ and let T be the quotient map from X onto X/N_p.

3. Dilations. The purpose of this section is to show that a fairly general dilation theory, which contains the well-known bounded dilation theory, is governed by positive forms and can be completely derived from the main theorem of the preceding section. Besides, from the positive form standpoint it occurs to be natural to consider not only bounded dilations, but also closed, and even arbitrary ones, as long as the algebraic properties improve.

The basic setting of the dilation theory presented here is purely algebraic, i.e., no topology is involved. This has been done on purpose to exhibit the strength of positive forms. Whether the initial functions take values in the set of bounded operators of just in the set of linear mappings, is of secondary importance. Topological results can be obtained if an appropriate topology is introduced, when desired.

Throughout this section two linear spaces E, E' are fixed. It will be assumed that they are related by a fixed "duality," i.e. a mapping $<\,,\,>\colon E \times E' \to C$ linear in the first variable, conjugate linear in the second one is defined. For instance, if E is a Hilbert space, then E' is usually taken equal E and the duality is the inner product in E. The linear space $L(E, E')$ will be denoted by L. If S is a set, then $F = F(S, E)$ will denote the linear space of all functions from S to E that equal zero off a finite subset of S. A function $A\colon S \times S \to L$ will be called *positive definite* (PD) if

$$\sum \langle f(t), A(s,t)f(s)\rangle \geq 0 \quad \text{for each } f \in F.$$

A positive definite function $A\colon S \times S \to L$ defines a positive form $q\colon F \times F \to C$ by

$$q(g, f) = \sum \langle g(t), A(s,t)f(s)\rangle, \qquad f, g \in F, \tag{3.1}$$

which will be called the form *associated with* A.

If $s \in S$, $x \in E$, then $f_{s,x}$ stands for the function from S to E whose only possible nonzero value is x attained at s. Clearly, $f_{s,x} \in F$. Let $A\colon S \times S \to L$ be a PD function and let q be the positive form associated with A. For $s \in S$ define a linear mapping $X(s)\colon E \to F/N_q$ by

$$X(s)x = f_{s,x} + N_q, \qquad x \in E. \tag{3.2}$$

Then for $s, t \in S$, $x, y \in E$

$$\langle y, A(s,t)x\rangle = q(f_{t,y}, f_{s,x}) = (X(t)y, X(s)x), \tag{3.3}$$

and for each $f \in F$

$$\sum X(s)f(s) = f + N_q. \tag{3.4}$$

This construction proves the so-called Kernel Theorem (cf. [**1**, §2]) in the bounded dilation theory. A similar theorem can be easily formulated in the general case discussed here. Also, the minimality problem can be formulated and solved analogously to the bounded dilation case.

The positive definiteness gives rise to a natural partial order in the set of all PD functions from $S \times S$ to L. Namely, if $A\colon S \times S \to L$, $B\colon S \times S \to L$ are PD, then $B \ll A$ if $A - B$ is PD. This partial order has been completely described in Theorem (2.2) of [**1**] for the case of bounded mappings and this description can be carried over to the present case without difficulty. Since positive forms are the main point of interest here, three more ways of comparing PD functions become available. This is described in the following theorem.

(3.5) THEOREM. *Let $A\colon S \times S \to L$, $B\colon S \times S \to L$ be PD functions. Let q, r be the positive forms associated with A, B, respectively. Let $X\colon S \to L(E, F/N_q)$ be as defined in* (3.2). *Then*

(a) *$N_q \subset N_r$ if and only if there is a Hilbert space K and a linear mapping $T\colon F/N_q \to K$ such that*

$$\langle y, B(s,t)x\rangle = (TX(t)y, TX(s)x), \qquad s, t \in S, \ x, y \in E.$$

(b) *For each sequence $f_n \in F$, $q(f_n, f_n) \to 0$, $r(f_n - f_m, f_n - f_m) \to 0$ implies $r(f_n, f_n) \to 0$ if and only if there is a Hilbert space K and a closed mapping $T^\wedge\colon D(T^\wedge) \to K$ such that $F/N_q \subset D(T^\wedge) \subset F_q$ and*

$$\langle y, B(s,t)x\rangle = (T^\wedge X(t)y, T^\wedge X(s)x), \qquad s, t \in S, \ x, y \in E.$$

(c) *There is a function* $m\colon F/N_q \to R$ *such that*

$$|r(g,f)| \leq m(g+N_q)q(f,f)^{1/2}, \qquad f,g \in F,$$

if and only if there is a positive, selfadjoint operator $Q\colon D(Q) \to F_q$ *such that* $F/N_q \subset D(Q) \subset F_q$ *and*

$$\langle y, B(s,t)x\rangle = (QX(t)y, X(s)x), \qquad s,t \in S,\ x,y \in E.$$

PROOF. (a) If $N_q \subset N_r$, define $K = F_r$ and $T\colon F/N_q \to F/N_r$ by $T(f+N_q) = f+N_r$, $f \in F$. By the assumption, T is well defined. It is plain that T is linear. If $s,t \in S$, $x,y \in E$, then, by (3.3),

$$\langle y, B(s,t)x\rangle = r(f_{t,y}, f_{s,x}) = (f_{t,y}+N_r, f_{s,x}+N_r) = (TX(t)y, TX(s)x).$$

Conversely, if such K, T exist, then for $f,g \in F$

$$\begin{aligned} r(g,f) &= \sum \langle g(t), B(s,t)f(s)\rangle = \sum (TX(t)g(t), TX(s)f(s)) \\ &= \Big(T\sum X(t)g(t), T\sum X(s)f(s)\Big) = (T(g+N_q), T(f+N_q)), \end{aligned}$$

by (3.4). Thus $q(f,f) = 0$, i.e. $f + N_q = 0$, implies $r(f,f) = \|T(f+N_q)\|^2 = 0$, $f \in F$.

(b) Suppose that $q(f_n, f_n) \to 0$, $r(f_n - f_m, f_n - f_m) \to 0$ implies $r(f_n, f_n) \to 0$ for each sequence $f_n \in F$. Taking arbitrary $f \in F$ and the constant sequence $f_n = f$ one sees immediately that $N_q \subset N_r$. Hence a positive form $p\colon F/N_q \times F/N_q \to C$ is well defined by the formula $p(g+N_q, f+N_q) = r(g,f)$, $f,g \in F$, and p is closable by the initial assumption. On the other hand, let $K = F_r$, $T\colon F/N_q \to K$ be as defined in (a). Then for $f,g \in F$

$$p(g+N_q, f+N_q) = r(g,f) = (g+N_r, f+N_r) = (T(g+N_q), T(f+N_q)).$$

By Theorem (2.6)(b), there is a closed linear mapping $T^\wedge\colon D(T^\wedge) \to K$ such that $F/N_q \subset D(T^\wedge)$ and $T^\wedge$ extends T. The last equality in (b) follows from the corresponding one for T in (a). Conversely, if such $K, T^\wedge$ exist, then, by a calculation similar to the one at the end of the proof of (a), $r(f,f) = \|T^\wedge(f+N_q)\|^2$ for $f \in F$. Hence (b) follows.

(c) If Q with the stated properties exists, then, again by a calculation as at the end of the proof of (a),

$$r(g,f) = (Q(g+N_q), f+N_q), \qquad f,g \in F,$$

which proves the "if" part of (c). For the "only if" part take m as stated. Then $N_q \subset N_r$. If p is the positive form on F/N_q defined in the proof on (b), then

$$|p(g+N_q, f+N_q)| \leq m(g+N_q)\|f+N_q\|, \qquad f,g \in F.$$

By Proposition (2.5), there is a positive operator $P\colon F/N_q \to F_q$ such that $p(g+N_q, f+N_q) = (P(g+N_q), f+N_q)$, $f,g \in F$. Taking $g = f_{t,y}$, $f = f_{s,x}$, $s,t \in S$, $x,y \in E$, one gets $\langle y, B(s,t)x\rangle = (PX(t)y, X(s)x)$. An application of Theorem (2.6)(c) finishes the proof. Q.E.D.

Until now S was an arbitrary set. From now on S *will be assumed to be a semigroup with unit.* Let $A\colon S \times S \to L$ be a function. A triple (K, π, R) will be called a *dilation* of A if K is a Hilbert space, $R\colon E \to K$ is a linear mapping and for each $s \in S$ an operator $\pi(s)\colon D(s) \to K$ is defined so that

(D1) $\pi(1) = I_K$.

(D2) RE is contained in each $D(s)$, $s \in S$.

(D3) The linear span M of $\{\pi(s)Rx\colon s \in S,\ x \in E\}$ is dense in K and it is contained in $D(s)$ for each $s \in S$.

(D4) $\pi(s)\pi(t)k = \pi(st)k$, $s, t \in S$, $k \in M$.

(D5) $\langle y, A(s,t)x\rangle = (\pi(t)Ry, \pi(s)Rx)$, $s, t \in S$, $x, y \in E$.

A dilation (K, π, R) of A will be called *closed* (*bounded*, respectively) if $\pi(s)$ is a closed (bounded, respectively) operator for each $s \in S$. If a dilation (K, π, R) is bounded, then, since $D(s)$ is dense in K, $\pi(s)$ can be extended to a bounded linear mapping on K, for each $s \in S$, π is a semigroup homomorphism, and the above definition coincides with the one commonly used in the bounded dilation theory. The density of M in K is assumed in (D3) for the sake of convenience. This assumption, however, is not a restrictive one, for it will become clear that if a dilation of a function can be found, then the attention can always be restricted to the closure of M. In the terminology known from bounded dilations (D3) is the minimality condition.

A calculation as at the end of the proof of Theorem (3.5)(a) shows that the linearity of R and of each $\pi(s)$, $s \in S$, together with (D5) implies that if A has a dilation, then A is PD. In the bounded dilation case it is known that there are PD functions with no bounded dilation (for a complete discussion of this problem see [**1**]). It will be shown that there are PD functions with no dilation at all. Necessary and sufficient conditions for the existence of various kinds of dilations will be given in terms of associated positive forms.

Fix a semigroup S and a PD function $A\colon S \times S \to L$. Let q be the positive form on F associated with A. For each $u \in S$ define a PD function $A_u\colon S \times S \to L$ by

$$A_u(s,t) = A(us, ut), \qquad s, t \in S.$$

Let q_u be the positive form on F associated with A_u. For each $u \in S$ define a linear mapping $a(u)\colon F \to F$ by

$$(a(u)f)(t) = \sum_{s\colon us=t} f(s), \qquad t, u \in S,\ f \in F. \tag{3.6}$$

(3.7) PROPOSITION. (a) *The mapping $a\colon S \to L(F)$ defined in* (3.6) *is a semigroup homomorphism such that $a(1) = I_F$,*

(b) $q_u(g,f) = q(a(u)g, a(u)f)$, $u \in S$, $f, g \in F$,

(c) *For each $u \in S$ there is an isometry $i(u)$ from F_{q_u} in F_q.*

PROOF. (a) If $u, v, t \in S$, $f \in F$, then

$$\begin{aligned}(a(u)a(v)f)(t) &= \sum_{t'\colon ut'=t} (a(v)f)(t') = \sum_{t'\colon ut'=t} \left(\sum_{s\colon vs=t'} f(s)\right)\\ &= \sum_{s\colon uvs=t} f(s) = (a(uv)f)(t).\end{aligned}$$

(b) follows by a straightforward calculation.

(c) If $u \in S$, $f \in F$, then, by (b),

$$\|f + N_{q_u}\|^2 = q_u(f,f) = q(a(u)f, a(u)f) = \|a(u)f + N_q\|^2,$$

which shows that the mapping $f + N_{q_u} \to a(u)f + N_q$ extends to an isometry $i(u)$ from F_{q_u} into F_q. Q.E.D.

The next theorem is the principal result on the existence of dilations.

(3.8) THEOREM. *Let S be a semigroup, let $A\colon S \times S \to L$ be a PD function. Let q, q_u $(u \in S)$ be the positive forms on F associated with A, A_u, respectively. Let X be as defined in* (3.2).

(a) *A has a dilation if and only if N_q is contained in each N_{q_u}, $u \in S$.*

(b) *A has a closed dilation if and only if for each sequence $f_n \in F$, $q(f_n, f_n) \to 0$, $q_u(f_n - f_m, f_n - f_m) \to 0$ implies $q_u(f_n, f_n) \to 0$, for each $u \in S$.*

(c) *A has a bounded dilation if and only if*

(i) *for each $u \in S$ there is a positive operator $P(u)\colon F/N_q \to F_q$ such that $\langle y, A_u(s,t)x\rangle = (P(u)X(t)y, X(s)x)$, $s,t \in S$, $x, y \in E$, and*

(ii) *for each $u \in S$ there is a dense subset G_u of $C^\infty(P(u)^\wedge)$ such that $\sup\{\mu(u,x)\colon x \in G_u\}$ is finite, where $P(u)^\wedge$ is the Friedrichs extension of $P(u)$ and $\mu(u,x) = \liminf \|P(u)^{\wedge 2^n} x\|^{2^{-n}}$.*

The proof will be preceded by some comments on bounded dilations. The first necessary and sufficient condition for the existence of a bounded dilation was found in case of $*$-semigroups by Sz.-Nagy [**12**]. His condition was then generalized by several authors to an arbitrary semigroup case (see e.g. [**7**, **1**]). Apart from PD, the boundedness condition was introduced, which can be equivalently formulated as follows:

(BC) There is a real function c on S such that

$$q_u(f,f) \le c(u)q(f,f), \qquad f \in F,\ u \in S.$$

PD and (BC) together are known to be necessary and sufficient for the existence of a bounded dilation (see e.g. [**1**, Dilation Theorem]). The conditions (i) and (ii) of (c) in the above theorem are equivalent to (BC). There are two reasons why they are significant. The first one is that in many cases they are easier to check separately than (BC). The second one is that they explain what is happening behind (BC), in terms of properties of positive forms. In case of $*$-semigroups, which will be discussed in the next section, Szafraniec's result [**11**] is a major simplification of (BC).

PROOF OF THEOREM (3.8). To prove (a) suppose first that N_q is contained in each N_{q_u}, $u \in S$. Let a be the mapping defined by (3.6). If follows from the assumption and from (3.7)(b) that for each $u \in S$ the mapping $\pi(u)\colon F/N_q \to F/N_q$, $\pi(u)(f + N_q) = a(u)f + N_q$, $f \in F$, is well defined, and it is linear, because $a(u)$ is. By (3.4), $\pi(u)$ can be written as

$$\pi(u)\sum X(s)f(s) = \sum X(us)f(s), \qquad u \in S,\ f \in F.$$

This formula was first suggested by Masani for the proof of the existence of bounded dilations (cf. [**7**, p. 296]). It is clear that $\pi\colon S \to L(F/N_q)$ is a semigroup homomorphism. Notice that

(3.9) $\pi(u)$ equals the composition of the mapping T from Theorem (3.5)(a) (let $B = A_u$, $r = q_u$) and the isometry $i(u)$ from (3.7)(c).

Let $R = X(1)$. Then $\pi(s)R = X(s)$, $s \in S$, and (F_q, π, R) is a dilation of A. For the converse suppose that (K, π, R) is a dilation of A. Then for $f, g \in F$

$$q(g,f) = \left(\sum \pi(t)Rg(t), \sum \pi(s)Rf(s)\right), \tag{3.10}$$

and, by (3.6) and (3.7)(b),

$$\begin{aligned} q_u(g,f) = q(a(u)g, a(u)f) &= \left(\sum \pi(t)R(a(u)g)(t), \sum \pi(s)R(a(u)f)(s)\right) \\ &= \left(\sum \pi(ut')Rg(t'), \sum \pi(us')Rf(s')\right) \\ &= \left(\pi(u)\sum \pi(t)Rg(t), \pi(u)\sum \pi(s)Rf(s)\right). \end{aligned} \tag{3.11}$$

Hence N_q is contained in N_{q_u} for each $u \in S$.

It follows from (3.10) and (3.4) that the mapping $U\colon F/N_q \to K$ defined by $U(f + N_q) = \sum X(s)f(s) = \sum \pi(s)Rf(s)$, $f \in F$, extends to a unitary mapping from F_q onto K, by (D3). (This is the proof of the uniqueness of the minimal dilation.)

The proof of (b) is a consequence of Theorem (3.5)(b) and of (a) above. For a fixed $u \in S$ that theorem is applied to $B = A_u$, $r = q_u$. The closed operator $\pi(u)^\wedge$ resulting from there is identified with the closure of the operator $\pi(u)$ defined in the proof of (a), by (3.9). In the "only if" part of the proof of (b), (3.11) is used.

To prove (c) firstly notice that

(3.12) *The condition* (i) *implies that A has a closed dilation.*

For, it follows from Theorem (3.5)(c) that if (i) is satisfied, then the mapping $p_u(g + N_q, f + N_q) = q_u(g, f)$, $f, g \in F$, is a well-defined, positive form on F/N_q for each $u \in S$. It is plain that

$$p_u(g + N_q, f + N_q) = (P(u)(g + N_q), f + N_q), \qquad f, g \in F.$$

By (2.4), p_u is a closable form. Hence for each sequence $f_n \in F$ if $q(f_n, f_n) = \|f_n + N_q\|^2 \to 0$ and $q_u(f_n - f_m, f_n - f_m) = p_u(f_n - f_m + N_q, f_n - f_m + N_q) \to 0$, then $q_u(f_n, f_n) = p_u(f_n + N_q, f_n + N_q) \to 0$, $u \in S$. By (b), A has a closed dilation $(K, \pi^\wedge, R)$. Thus (3.12) is proved.

It follows from the definition of $\pi(u)$ in (a) and from the construction of $\pi(u)^\wedge$ that $p_u(g + N_q, f + N_q) = (\pi(u)^\wedge(g + N_q), \pi(u)^\wedge(f + N_q))$, $f, g \in F$. If (ii) is satisfied, then Theorem (2.6)(d) implies that $\pi(u)^\wedge$ is a bounded operator on F_q, for each $u \in S$.

Conversely, let (K, π, R) be a bounded dilation of A. Let U be the unitary mapping from F_q onto K as at the end of the proof of (a). Let $P(u) = U^*\pi(u)^*\pi(u)U$, $u \in S$. Then for $u, s, t \in S$, $x, y \in E$

$$(P(u)X(t)y, X(s)x) = (\pi(u)\pi(t)Ry, \pi(u)\pi(s)Rx) = \langle y, A_u(s,t)x\rangle,$$

which proves (i). The condition (ii) follows by the boundedness of $P(u)$, as in the proof of Theorem (2.6)(d). Q.E.D.

Notice that, by Theorem (3.5)(c), the condition (i) of (c) in the last theorem is equivalent to the following one:

(3.13) *There is a real function m on $S \times (F/N_q)$ such that*

$$|q_u(g,f)| \le m(u, g + N_q)q(f,f)^{1/2}, \qquad u \in S,\ f, g \in F.$$

The substantial difference between this condition and (BC) is that the constant $m(u, g + N_q)$ is allowed to depend on $g + N_q$, $g \in F$, whereas $c(u)$ in (BC) must not depend upon elements of F. This difference becomes even clearer when one realizes that PD and (BC) imply the existence of a bounded dilation, whereas PD and (3.13) imply merely the existence of a closed dilation, by (3.12).

(3.14) COROLLARY. *Let P_n be a sequence of positive, bounded operators on a Hilbert space H, let $P_0 = I$. Let $A: N \times N \to B(H)$ be defined by $A(m,m) = P_m$ for all $m \in N$, $A(m,n) = 0$ if $m, n \in N$, $m \neq n$. Then A is PD and*

(a) *A has a dilation if and only if $\ker P_n \subset \ker P_{n+1}$ for each $n = 1, 2, \ldots$.*

(b) *If either each P_n has a bounded inverse, or there is $m \in N$ such that $P_n = 0$, $n \geq m$, and each P_n has a bounded inverse for $n < m$, then A has a closed dilation.*

PROOF. If $f \in F = F(N, H)$, $k = 0, 1, 2, \ldots$, then

$$q_k(f,f) = \sum_n \|P_{n+k}^{1/2} f(n)\|^2.$$

Hence A is PD. Moreover, $q_k(f,f) = 0$ if and only if $f(n) \in \ker P_{n+k}^{1/2} = \ker P_{n+k}$ for each n. An application of Theorem (3.8)(a) proves (a). If Q is a bounded, boundedly invertible operator and $Qx_m \to 0$ for a sequence x_m of vectors, then $x_m \to 0$. Therefore (b) follows from the first equality in this proof and from Theorem (3.8)(b). Q.E.D.

Following Theorem (3.8)(b) a necessary and sufficient condition for the existence of a closed dilation of A defined in (3.14) can be formulated. Also, from this corollary it is easy to construct examples of PD functions without dilations. Here is one of them:

(3.15) EXAMPLE. Let H be at least a two-dimensional Hilbert space, let Q be an orthogonal projection in H, $Q \neq 0$, $Q \neq I$. The function $A: N \times N \to B(H)$ defined by $A(0,0) = I$, $A(1,1) = Q$, $A(n,n) = I$, $n = 2, 3, \ldots$, $A(m,n) = 0$ elsewhere, is PD and it has no dilation, by Corollary (3.14)(a).

In the next example a function is presented such that it has a dilation but it has no closed dilation.

(3.16) EXAMPLE. Let H be an infinite-dimensional Hilbert space, let P, Q be bounded, positive operators on H and let $A: N \times N \to B(H)$ be defined by $A(0,0) = I$, $A(1,1) = P$, $A(2,2) = Q$, $A(m,n) = 0$ elsewhere. Assume that $\ker P = \{0\}$. By Corollary (3.14)(a), A has a dilation. If $f \in F$, then

$$\begin{aligned}
q(f,f) &= \|f(0)\|^2 + \|P^{1/2} f(1)\|^2 + \|Q^{1/2} f(2)\|^2, \\
q_1(f,f) &= \|P^{1/2} f(0)\|^2 + \|Q^{1/2} f(1)\|^2, \\
q_2(f,f) &= \|Q^{1/2} f(0)\|^2, \\
q_k(f,f) &= 0 \quad \text{if } k > 2.
\end{aligned}$$

It follows from Theorem (3.8)(b) that A has a closed dilation if and only if $P^{1/2} x_n \to 0$, $\|Q^{1/2}(x_n - x_m)\| \to 0$ implies $Q^{1/2} x_n \to 0$ for each sequence $x_n \in H$ (in a sequence $f_n \in F$ in Theorem (3.8)(b) take $x_n = f_n(1)$). Now positive, bounded operators P, Q and a sequence x_n will be found so that $\ker P = \{0\}$, $P^{1/2} x_n \to 0$, and $Q^{1/2} x_n$ has a nonzero limit. The above defined function A with these P, Q will have a dilation without having a closed one. Assume that H is separable.

Let $e_1, e_2, \ldots$ be an orthonormal basis of H. Let $x_n = 2^n e_n$, let $P^{1/2}$ be given by the diagonal matrix with the diagonal entries 2^{-2n}, $n = 0, 1, 2, \ldots$, and let $Q^{1/2}$ be given by the exponential Hilbert matrix whose (i, j) entry is $2^{-(i+j+1)}$, $i, j = 0, 1, 2, \ldots$. Then $\ker P = \ker P^{1/2} = \{0\}$, and $P^{1/2} x_n = 2^{-n} e_n \to 0$. By [**4**, Problem 47], $Q^{1/2}$ is a bounded, positive operator on H. Finally, $Q^{1/2} x_n$ is a constant, nonzero sequence equal to $2^{-1} e_1 + 2^{-2} e_2 + \cdots$.

A straightforward weak convergence argument shows that the choice of P, Q and x_n with the above properties is possible only if x_n is an unbounded sequence.

(3.17) COROLLARY. *Let S be a semigroup, let H be a Hilbert space. Suppose a function $A\colon S \times S \to B(H)$ has a bounded dilation (K, π, R). Let M be the linear span of $\{\pi(s)Rx\colon s \in S,\ x \in H\}$. Let $P\colon M \to K$ be a linear mapping. Define $A_P\colon S \times S \to L(H)$ by $A_P(s,t) = R^*\pi(t)^* P\pi(s)R$, $s, t \in S$. Then*

(a) *A_P is PD if and only if P is positive.*

(b) *If P is positive, then A_P has a closed dilation.*

(c) *A_P has a bounded dilation if and only if P extends to a bounded, everywhere defined, positive operator $P^\wedge$ such that $\pi(s)^* P^{\wedge 1/2} K$ is contained in $P^{\wedge 1/2} K$ for all $s \in S$.*

(d) *If $B\colon S \times S \to B(H)$ is PD and $B \ll A$, then B has a closed dilation.*

PROOF. It follows from the definition of A_P that

$$\sum (f(t), A_P(s,t) f(s)) = \left(P \sum \pi(t) R f(t), \sum \pi(s) R f(s)\right), \qquad f \in F.$$

Hence (a) is proved.

If $u, s, t \in S$, $x, y \in H$, then

$$(y, A_P(us, ut)x) = (\pi(u)^* P \pi(u) \pi(t) R y, \pi(s) R x).$$

Hence (i) of Theorem (3.8)(c) is satisfied with $P(u) = \pi(u)^* P \pi(u)$, $u \in S$. By (3.12), A_P has a closed dilation, which proves (b).

The "if" part of (c) follows from Theorem (3.3) of [**1**]. For the "only if" part notice that, by Theorem (3.8)(c), and Theorem (2.6)(d), the Friedrichs extension $P(u)^\wedge$ of $P(u)$, $u \in S$, is a positive, everywhere defined, bounded operator. In particular, $P(1) = P$. Hence P extends to a bounded operator $P(1)^\wedge$ on K. The remainder of the assertion (c) follows from Theorem (3.3) of [**1**]. Finally, by Theorem (2.2) of [**1**], if $B \ll A$, then there is a bounded, positive operator $P \le I$ on K such that $B = A_P$. Now (d) follows from (b). Q.E.D.

In particular, this corollary allows one to give examples of functions which have a closed dilation without having a bounded one. One such example, with a two-dimensional H, is the function C_s of Example (3.11) in [**1**].

(3.18) COROLLARY. *Let S be a semigroup. Let $A\colon S \times S \to L$ be a PD function. Assume that there is a function $d\colon S \to S$ such that*

$$\langle y, A(us, ut)x \rangle = \langle y, A(s, d(u)t)x \rangle, \qquad u, s, t \in S,\ x, y \in E.$$

Then

(a) *A has a closed dilation.*

(b) *A has a bounded dilation if and only if there are real function b on $E \times E$ and c on S such that*

$$c(st) \le c(s)c(t), \qquad s, t \in S,$$

and

$$|\langle y, A(s,t)x\rangle| \leq b(x,y)c(s)c(t), \qquad s,t \in S,\ x,y \in E.$$

PROOF. (a) It follows from the assumption that

$$q_u(g,f) = q(a(d(u))g, f), \qquad u \in S,\ f,g \in F,$$

where a is defined by (3.6). By the Schwarz inequality for q,

$$|q(a(u)g, a(u)f)| = |q_u(g,f)| \leq q(a(d(u))g, a(d(u))g)^{1/2} q(f,f)^{1/2},$$

for each $u \in S$, $f,g \in F$, by (3.7)(b). This inequality applied to $d(u)$ gives for $f,g \in F$

$$|q(a(d(u))g, a(d(u))f)| \leq q(a(d(d(u)))g, a(d(d(u)))g)^{1/2} q(f,f)^{1/2}.$$

This proves that the mapping $P(u)\colon F/N_q \to F/N_q$, $P(u)(f+N_q) = a(d(u))f + N_q$, $f \in F$, is well defined. Clearly, $P(u)$ is linear and

$$q_u(g,f) = (P(u)(g+N_q), f+N_q), \qquad f,g \in F.$$

By (3.12), A has a closed dilation.

(b) Let $P(u)^\wedge$ be the Friedrichs extension of $P(u)$, $u \in S$. Since $P(u)$ maps F/N_q into itself, F/N_q is contained in $C^\infty(P(u)^\wedge)$.

Assume that the functions b, c exist. The existence of a bounded dilation will be proved by using Theorem (3.8)(c). For $u \in S$ let $G_u = F/N_q$. This set is dense in F_q, thus it is dense in $C^\infty(P(u)^\wedge)$. By the definition of $P(u)$ it follows from (3.7)(a) that

$$P(u)^{\wedge^n}(f+N_q) = a(d(u)^n)f + N_q, \qquad f \in F,\ n = 1,2,\ldots.$$

Therefore, if $f \in F$, $n = 1,2,\ldots$, then

$$\begin{aligned}
\|P(u)^{\wedge^{2^n}}(f+N_q)\|^2 &= (P(u)^{\wedge^{2^{n+1}}}(f+N_q), f+N_q) = q(a(d(u)^{2^{n+1}})f, f)\\
&= \sum \langle f(t), A(s,t)(a(d(u)^{2^{n+1}})f)(s)\rangle\\
&= \sum \langle f(t), A(d(u)^{2^{n+1}}s, t)f(s)\rangle\\
&\leq \sum b(f(s), f(t))c(d(u))^{2^{n+1}}c(s)c(t).
\end{aligned}$$

Hence $\liminf \|P(u)^{\wedge^{2^n}}(f+N_q)\|^{2^{-n}} \leq c(d(u))$, $f \in F$.

By Theorem (3.8)(c), A has a bounded dilation. Conversely, if (K, π, R) is a bounded dilation of A, then for $x, y \in E$, $s,t \in S$

$$|\langle y, A(s,t)x\rangle| = |(\pi(t)Ry, \pi(s)Rx)| \leq \|\pi(t)\|\|\pi(s)\|\|Ry\|\|Rx\|.$$

Now the functions: $b(x,y) = \|Rx\|\|Ry\|$, $x,y \in E$, and $c(s) = \|\pi(s)\|$, $s \in S$, satisfy the suitable conditions. Q.E.D.

4. Applications.

4.1. *$*$-Semigroups.* A $*$-semigroup is a semigroup with a mapping $*$ from S into itself such that $(st)^* = t^*s^*$, $(s^*)^* = s$, $s,t \in S$ and $1^* = 1$.

Let S be a $*$-semigroup, let L be as defined in the previous section and let $A\colon S \times S \to L$ be a function. A dilation (K, π, R) of A will be called a *$*$-dilation* if $\pi(s^*) \subset \pi(s)^*$, $s \in S$.

(4.1.1) PROPOSITION. *Let S be a $*$-semigroup, let $A\colon S \times S \to L$ be a function.*

(a) *A has a $*$-dilation if and only if A is PD and*

$$\langle y, A(us,t)x\rangle = \langle y, A(s,u^*t)x\rangle, \qquad u,s,t \in S,\ x,y \in E. \tag{4.1.2}$$

(b) *If A has a $*$-dilation (K,π,R), then $\pi(u)$ is a closable operator in K for each $u \in S$. If $\pi(u)^\wedge$ denotes the closure of $\pi(u)$, $u \in S$, then $(K,\pi^\wedge,R)$ is a closed $*$-dilation of A.*

(c) *A has a bounded $*$-dilation if and only if A has a $*$-dilation and there are functions b,c that satisfy the conditions of Corollary* (3.18)(b).

PROOF. (a) Suppose A is PD and (4.2) holds. Then

$$\langle y, A(us,ut)x\rangle = \langle y, A(s,u^*ut)x\rangle, \qquad u,s,t \in S,\ x,y \in E.$$

By Corollary (3.18)(a) with $d(u) = u^*u$, $u \in S$, A has a closed dilation. By the construction of this dilation, the linear span M of $\{\pi(s)Rx\colon s \in S,\ x \in E\}$ is dense in K, and $\pi(u)$ is the closure of $\pi(u)|M$, $u \in S$. Now fix $u \in S$. By (4.1.2), for $s,t \in S$, $x,y \in E$:

$$(\pi(t)Ry, \pi(u)\pi(s)Rx) = (\pi(u^*)\pi(t)Ry, \pi(s)Rx).$$

Hence $\pi(u^*)|M \subset (\pi(u)|M)^*$. By Theorem VIII.1, (a), (c) of [**8**], $(\pi(u)|M)^*$ is a closed operator, and $(\pi(u)|M)^* = \pi(u)^*$. Therefore $\pi(u^*) \subset \pi(u)^*$.

The converse implication is immediate.

(b) If (K,π,R) is a $*$-dilation of A, then $\pi(u^*) \subset \pi(u)^*$, $u \in S$. Since $D(u^*)$ is dense in K, so is $D(\pi(u)^*)$, hence $\pi(u)$ is closable, by Theorem VIII.1, (b) of [**8**]. Since $\pi(u)^*$ is a closed extension of $\pi(u^*)$, and $\pi(u^*)^\wedge$ is the least closed extension of $\pi(u^*)$, it follows from Theorem VIII.1, (c) of [**8**], that $\pi(u^*)^\wedge \subset \pi(u)^* = \pi(u)^{\wedge^*}$. Hence $(K,\pi^\wedge,R)$ is a closed $*$-dilation of A.

(c) If A has a $*$-dilation (K,π,R), and b,c exist, as stated in Corollary (3.18) (b), then, by (4.1.2) and that corollary, A has a bounded dilation $(K,\pi^\wedge,R)$. From its construction it follows that $\pi(u)^\wedge$ is the closure of $\pi(u)$, and, by Theorem (3.8) (c), $\pi(u)^\wedge$ is a bounded, everywhere defined operator, for each $u \in S$. By (b), $\pi(u^*)^\wedge \subset \pi(u)^{\wedge^*}$. Thus $\pi(u^*)^\wedge = \pi(u)^{\wedge^*}$, $u \in S$. The converse implication is clear. Q.E.D.

The last part of the proof shows that the above definition of a $*$-dilation coincides with the common one in case of bounded dilations.

It may happen that if S is a $*$-semigroup, then a function $A\colon S \times A \to L$ has a bounded dilation without having a $*$-dilation. The following example is a consequence of Proposition (4.3) of [**1**].

(4.1.3) EXAMPLE. Let S be a group. Then S is a $*$-semigroup with $u^* = u^{-1}$, $u \in S$. Suppose that there is a $*$-semigroup homomorphism $\pi\colon S \to B(H)$, i.e., a unitary representation in a Hilbert space H, such that the double commutant of $\pi(S)$ is strictly larger than the set of all scalar multiples of the identity in H. Then there is an orthogonal projection Q that does not commute with $\pi(S)$. If $P = \frac{1}{2}(I+Q)$, then the function $B(s,t) = \pi(t)^*P\pi(s)$, $s,t \in S$, has a bounded dilation, and it has no bounded $*$-dilation, as shown in Proposition (4.3) of [**1**]. On the other hand, if $x,y \in H$, $s,t \in S$, then

$$|(y, B(s,t)x)| \le \|\pi(s)\|\|\pi(t)\|\|P\|\|x\|\|y\|.$$

It follows from Proposition (4.1.1)(c) that B has no $*$-dilation.

(4.1.4) COROLLARY. *Let S be a $*$-semigroup, let H be a Hilbert space. Suppose $A\colon S\times S \to B(H)$ has a bounded $*$-dilation (K,π,R). Let M, P, and $A_P\colon S\times S \to L(H)$ be as defined in Corollary* (3.17). *Then*

(a) *A_P has a $*$-dilation if and only if P is positive and $\pi(u)P = P\pi(u)|M$, $u \in S$.*

(b) *A_P has a bounded $*$-dilation if and only if P extends to a bounded, everywhere defined operator $P^\wedge$ such that $\pi(u)P^\wedge = P^\wedge\pi(u)$, $u\in S$.*

PROOF. If $x,y\in H$, $u,s,t\in S$, then

$$(y, A_P(us,t)x) = (\pi(t)Ry, P\pi(u)\pi(s)Rx),$$
$$(y, A_P(s,u^*t)x) = (\pi(t)Ry, \pi(u)P\pi(s)Rx).$$

Hence (a) follows from Corollary (3.17)(a) and Proposition (4.1.1)(a). The part (b) is a consequence of (a) and Theorem (4.1) of [**1**]. Q.E.D.

From this corollary examples of functions which have $*$-dilation and have no bounded $*$-dilations can be obtained. Here is one.

(4.1.5) EXAMPLE. Let H be an infinite-dimensional Hilbert space and let M be its dense subspace. Let $P\colon M \to H$ be a positive, selfadjoint operator which cannot be extended to a bounded operator on H. Let F_P be its operator-valued spectral measure. Choose a real number k so that the orthogonal projection $Q = F_P([-k,k])$ is proper (not 0 or I). Let $R\colon QH \to H$ be the inclusion mapping. The semigroup N is a $*$-semigroup with $n^* = n$, $n\in N$. Let $\pi\colon N\to B(H)$ be defined by $\pi(n) = Q^n = Q$, $n\in N$. The function $A\colon N\times N\to B(H)$, $A(m,n) = R^*\pi(n)^*\pi(m)R$, $m,n\in N$, has a bounded $*$-dilation (H,π,R). Since QH is a subspace of M and $PQ|M = QP$, the function $A_P\colon N\times N \to L(QH)$, $A_P(m,n) = R^*\pi(n)^*P\pi(m)R$, $m,n\in N$, has a $*$-dilation, and it has no bounded $*$-dilation, by Corollary (4.1.4).

Finally, the previous results will be interpreted for the case originally studied by Sz.-Nagy [**12**].

(4.1.6) COROLLARY. *Let S be a $*$-semigroup. Let $\Phi\colon S\to L$ be a function. Define $A\colon S\times S\to L$ by $A(s,t) = \Phi(t^*s)$, $s,t\in S$.*

(a) *If A is PD, then A has a $*$-dilation.*

(b) *A has a bounded $*$-dilation if and only if A is PD and there are real functions b on $E\times E$, c on S, such that $c(st)\le c(s)c(t)$, $s,t\in S$, and $|\langle y,\Phi(s)x\rangle| \le b(x,y)c(s)$, $s\in S$, $x,y\in E$.*

PROOF. By the definition of A, $A(us,t) = A(s,u^*t)$, $u,s,t\in S$. Thus (4.1.2) holds, and, by (4.1.1)(a), A has a $*$-dilation. Hence (a) is proved. Concerning (b), this is a rephrasing of (4.1.1)(c) by using Φ. One should notice that the inequalities assumed in (b) imply

$$|\langle y, A(s,t)x\rangle| \le b(x,y)c(s)c(t^*), \qquad s,t\in S,\ x,y\in E,$$

which is not exactly the second inequality in (3.18)(b). However, a look at the proof of (3.18)(b) makes it clear that it does not matter, as far as that proof is concerned. Q.E.D.

If $E = E'$ is a Hilbert space, $L = B(E)$, and $b(x,y) = \alpha\|x\|\|y\|$, with some $\alpha\ge 0$, $x,y\in E$, then part (b) of this corollary is Szafraniec's result [**11**].

4.2. *$*$-algebras.* The Stinespring theorem says that each completely positive function on a C^*-algebra D has a bounded $*$-dilation (K,π,R) such that π is a

$*$-representation of D (see e.g. [**2**] for this theorem and appropriate definitions). This is one of the cases in which (BC) is a consequence of PD (see [**6**, §9, Theorem 3] for a straightforward proof). The basic reason why it happens here is that each positive, linear functional on a C^*-algebra is bounded. This, as well as a certain converse of this, is explained in the following corollary. Let S be a $*$-semigroup. Call a function $h\colon S \to C$ *positive* if $h(s^*s) \geq 0$ for each $s \in S$.

(4.2.1) COROLLARY. *Let S be a $*$-semigroup. Consider the following conditions:*

(a) *For each positive function $h\colon S \to C$ there is a real function c on S and a constant $\alpha(h)$ such that $c(st) \leq c(s)c(t)$, $s, t \in S$, and $|h(s)| \leq \alpha(h)c(s)$, $s \in S$.*

(b) *For each Hilbert space H, each dense subspace M of H, and each function $\Phi\colon S \to L(M, H)$ if the function $A\colon S \times S \to L(M, H)$, $A(s,t) = \Phi(t^*s)$, $s, t \in S$, is PD, then A has a bounded $*$-dilation.*

Then (a) *implies* (b). *If S is a $*$-algebra, then* (b) *is equivalent to* (a) *with $h\colon S \to C$ being a positive, linear functional.*

PROOF. Suppose (a) holds. Let Φ be as in (b). Fix $x \in M$. Then $h_x\colon S \to C$, $h_x(s) = (\Phi(s)x, x)$, $s \in S$, is positive. By the assumption, $|(\Phi(s)x, x)| \leq \alpha(x)c(s)$, $s \in S$. By the polarization, if $x, y \in M$, $s \in S$, then

$$|(\Phi(s)x, y)| \leq \tfrac{1}{4}(\alpha(x+y) + \alpha(x-y) + \alpha(x+iy) + \alpha(x-iy))c(s).$$

By Corollary (4.1.6)(b), A has a bounded $*$-dilation.

Suppose now that S is a $*$-algebra and that (b) holds. Let $h\colon S \to C$ be a positive, linear functional. Then $p\colon S \times S \to C$ defined by $p(s,t) = h(t^*s)$, $s, t \in S$, is a positive form on S. Let $M = S/N_p$, $H = S_p$, and $\Phi(u)(s + N_p) = us + N_p$, $u, s \in S$. So far, this is the GNS construction. Now $A(s,t) = \Phi(t^*s)$, $s, t \in S$, is PD. By (b), A has a bounded $*$-dilation. By Corollary (4.6)(b), there are real functions b on $M \times M$, c on S, such that $c(st) \leq c(s)c(t)$, $s, t \in S$, and $|(t + N_p, \Phi(u)(s + N_p))| \leq c(u)b(t + N_p, s + N_p)$, $u, s, t \in S$. If $s = t = 1$, then

$$\begin{aligned} |h(u)| = |p(u,1)| &= |(1 + N_p, \Phi(u)(1 + N_p))| \\ &\leq c(u)b(1 + N_p, 1 + N_p), \qquad u \in S. \end{aligned}$$

Thus (a) follows. Q.E.D.

If S is a C^*-algebra, then (a) of this corollary is satisfied, because each positive, linear functional on a C^*-algebra is bounded. In this case the implication (a)$\Rightarrow$(b) is the Stinespring theorem.

4.3. *Gramians.* Let H, K be Hilbert spaces, let S be a subset of H, let M be the linear subspace of H spanned by S, and let $h\colon S \to K$ be a function. The results of the previous sections will be applied to answer the following

QUESTION. When does there exist a linear (closed, bounded) mapping $T\colon M \to K$ such that $T|S = h$?

If H is finite dimensional, then an obvious answer to this question is: if all elements of S are linearly independent.

Let now H be arbitrary. Without loss of generality it can be assumed that M is dense in H. The inner product $(\ ,\ )$ in H is a PD, scalar-valued function on $H \times H$. Its restriction to $S \times S$, which is also PD, will be denoted by A. Let $F = F(S, C)$.

The positive form q associated with A has the form $q(g,f) = \sum(t,s)g(t)\overline{f(s)}$, $f,g \in F$. Since for each $f \in F$

$$(4.3.1) \qquad \|f + N_q\|^2 = q(f,f) = \left\|\sum f(s)s\right\|^2,$$

it follows that the mapping $U(f+N_q) = \sum f(s)s$, $f \in F$, is an isometry form F/N_q onto M and it extends to a unitary isomorphism of F_q onto H. Let $B\colon S \times S \to C$ be defined by $B(s,t) = (h(s), h(t))$, $s,t \in S$ (the inner product here is in K). Let r denote the positive form on F associated with B, i.e.,

$$(4.3.2) \qquad r(g,f) = \sum(h(t), h(s))g(t)\overline{f(s)}, \qquad f,g \in F.$$

The same way as U was defined above, one finds a unitary V from F_r onto the closed linear span of $h(S)$, such that $V(f + N_r) = \sum f(s)h(s)$, $f \in F$.

(4.3.3) COROLLARY. (a) *There is a linear mapping* $T\colon M \to K$ *such that* $T|S = h$ *if and only if* $\sum f(s)s = 0$ *implies* $\sum f(s)h(s) = 0$ *for each* $f \in F$.

(b) *There is a closed linear mapping* $T\colon D(T) \to K$ *such that* $M \subset D(T)$ *and* $T|S = h$ *if and only if* $q(f_n, f_n) \to 0$, $r(f_n - f_m, f_n - f_m) \to 0$ *implies* $r(f_n, f_n) \to 0$ *for each sequence* $f_n \in F$.

(c) *The following conditions are equivalent:*

(i) *There is a bounded linear mapping* $T\colon H \to K$ *such that* $T|S = h$.

(ii) *There is* $c \geq 0$ *such that* $r(f,f) \leq cq(f,f)$, $f \in F$.

(iii) *There is a positive operator* $P\colon F/N_q \to K$ *such that* $r(g,f) = (Pg, f)$, $f,g \in F$, *and there is a dense subset* G *of* $C^\infty(P^\wedge)$ *such that* $\sup\{\mu(x)\colon x \in G\}$ *is finite, where* $P^\wedge$ *is the Friedrichs extension of* P, *and* $\mu(x)$ *is defined as in* (2.7).

PROOF. The "only if" part of (a) is clear. For the "if" part, it follows from the assumption, (4.3.1), and (4.3.2), as in the proof of Theorem (3.5)(a), that one can define a linear mapping $T'\colon F/N_q \to F/N_r$ by $T'(f + N_q) = f + N_r$, $f \in F$. If U, V are the unitary operators introduced before this corollary, and if $T = VT'U^*$, then $Ts = VT'(f_{s,1} + N_q) = V(f_{s,1} + N_r) = h(s)$, $s \in S$.

The proof of (b) is similar to the proof of Theorem (3.5)(b), with the above unitary identification. The equivalence of (i) and (ii) in (c) is clear. The equivalence of (ii) and (iii) is a consequence of (a), (b), and Theorem (2.6)(d). Q.E.D.

It follows from (a) in this corollary that, like in the case of a finite-dimensional H, an answer to the question raised at the beginning for arbitrary H with linear T is: if each finite number of vectors in S is linearly independent.

Finally, the title of this subsection will be explained. Following Halmos [**4**, Solution of Problem 48], if $s_1, \ldots, s_n$ are vectors in a Hilbert space H, then the $n \times n$ matrix whose (i,j) entry is (s_i, s_j) is called the *Gramian* of $s_1, \ldots, s_n$, and it will be denoted by $G(s_1, \ldots, s_n)$. The condition (ii) in (c) in the last corollary is equivalent to

(4.3.4) There is $c \geq 0$ such that $G(h(s_1), \ldots, h(s_n)) \leq cG(s_1, \ldots, s_n)$, for each $n = 1, 2, \ldots,$ $s_1, \ldots, s_n \in S$.

Recently Halmos has asked [**5**] whether a weaker condition

(4.3.5) For each $n = 1, 2, \ldots,$ $s_1, \ldots, s_n \in S$ there is $c(s_1, \ldots, s_n) \geq 0$ such that

$$G(h(s_1), \ldots, h(s_n)) \leq c(s_1, \ldots, s_n)G(s_1, \ldots, s_n)$$

is equivalent to the existence of a bounded operator $T\colon H \to K$ such that $T|S = h$. After a look at (c) of Corollary (4.3.3) and at Proposition (2.5), one may suspect that the answer is no. To justify this answer, suppose that $T\colon H \to K$ is a linear mapping. Then for each $n = 1, 2, \ldots, s_1, \ldots, s_n \in H$, $x = (x_1, \ldots, x_n) \in C^n$,

$$(G(Ts_1, \ldots, Ts_n)x, x) = \left\| T \sum x_i s_i \right\|^2, \quad (G(s_1, \ldots, s_n)x, x) = \left\| \sum x_i s_i \right\|^2.$$

Hence (4.3.5) means that T is bounded on each finite-dimensional subspace of M. This holds for any, not necessarily bounded linear mapping.

4.4. *Operator moment problems.* As the next application of results of §3 the following problem will be solved: Let S be a semigroup. Let H, K be Hilbert spaces, let $X\colon S \to L(H, K)$ be a function and let M be the linear subspace of K spanned by vectors $X(s)x$, $s \in S$, $x \in H$.

QUESTION. When can X be represented in the form $X(s) = \pi(s)R$, $s \in S$, where $R\colon H \to M$ is a linear mapping, $\pi(s)$ is a linear mapping from a linear subspace $D(s)$ of K containing M into K for each $s \in S$, and $\pi(st)|M = \pi(s)\pi(t)|M$, $s, t \in S$?

A pair (π, R) satisfying the above conditions will be called a *solution* of the moment problem for X. A solution (π, R) will be called *closed* (*bounded, contractive*, respectively) if $\pi(s)$ is closed (bounded, contractive, respectively) for each $s \in S$. This is a general formulation of several operator moment-like problems whose particular cases ($S = N$, $S =$ all nonnegative real numbers, $H = K$, and all operators in question are bounded) were considered in [**9** and **10**].

For simplicity assume that M is dense in K. Let $F = F(S, H)$. Let $q\colon F \times F \to C$ be defined by $q(g, f) = \sum (X(t)g(t), X(s)f(s))$, $f, g \in F$, and for $u \in S$ define $q_u(g, f) = \sum (X(ut)g(t), X(us)f(s))$, $f, g \in F$. Since

$$(4.4.1) \qquad \begin{aligned} q(f, f) &= \left\| \sum X(s)f(s) \right\|^2 \quad \text{and} \\ q_u(f, f) &= \left\| \sum X(us)f(s) \right\|^2, \qquad f \in F,\ u \in S, \end{aligned}$$

it is clear that q and q_u, $u \in S$, are positive forms on F.

It follows from (4.4.1) that the mapping $f + N_q \to \sum X(s)f(s)$, $f \in F$, is an isometry from F/N_q onto M. Since M is assumed to be dense in K, this mapping extends to a unitary isomorphism of F_q onto K. Therefore in what follows F_q will be identified with K, and F/N_q will be identified with M.

A comparison of the definition of q with (3.1) and (3.3) makes it clear that the function X is the starting point here instead of a PD function A in §3. The reason for which A is not explicitly stated in the form $A(s, t) = X(t)^*X(s)$, $s, t \in S$, is to avoid the use of the adjoint in the present case. As an immediate consequence of Theorem (3.8) one gets the following solution of the moment problem.

(4.4.2) COROLLARY. *Let S, H, K, X, M be as above.*

(a) *The moment problem for X has a solution if and only if for each $f \in F$ $\sum X(s)f(s) = 0$ implies $\sum X(us)f(s) = 0$ for each $u \in S$.*

(b) *The moment problem for X has a closed solution if and only if for each sequence $f_n \in F$ $q(f_n, f_n) \to 0$, and $q_u(f_n - f_m, f_n - f_m) \to 0$ implies $q_u(f_n, f_n) \to 0$, for each $u \in S$.*

(c) *The following conditions are equivalent:*

(i) *The moment problem for X has a bounded (contractive, respectively) solution.*

(ii) *For each $u \in S$ there is $c(u) \geq 0$ such that $q_u(f,f) \leq c(u)q(f,f)$ for each $f \in F$ ($q_u(f,f) \leq q(f,f)$, $f \in F$, respectively).*

(iii) *For each $u \in S$ there is a positive operator $P(u)\colon M \to K$ such that*

$$(X(ut)y, X(us)x) = (P(u)X(t)y, X(s)x), \qquad u,s,t \in S,\ x,y \in H,$$

and (ii) *of Theorem* (3.8)(c) *holds (with $\sup\{\mu(x,u)\colon x \in G_u\} \leq 1$ for each $u \in S$, respectively).*

Recall that such $P(u)$ can be obtained by checking an appropriate inequality—cf. Proposition (2.5).

A formal adjustment of Example (3.15) ((3.16), respectively) provides an example of a function X for which the moment problem has no solution (has a solution, but has no closed one, respectively).

4.5. *Reconstruction of quantum mechanics.*

(4.5.1) THEOREM. *Let S be a semigroup, let H be a Hilbert space, let $\tau\colon S \to B(H)$ be a semigroup homomorphism, and let M be a subspace of H which is invariant for $\tau(u)$ and $\tau(u)^*\tau(u)$, $u \in S$. Let q be a positive form on M such that $q(\tau(u)x, \tau(u)y) = q(x, \tau(u)^*\tau(u)y)$, $x,y \in M$, $u \in S$, and $q(x,x) \leq c\|x\|^2$ for all $x \in M$, with some $c \geq 0$.*

Then there is a semigroup homomorphism $\pi\colon S \to B(M_q)$ such that

(i) *$\pi(u)(x + N_q) = \tau(u)x + N_q$, $u \in S$, $x \in M$, and*

(ii) *$\|\pi(u)\| \leq \|\tau(u)\|$, $u \in S$.*

PROOF. To simplify the notation in the proof fix $u \in S$ and let $Z = \tau(u)$. By the assumed properties of q and τ, and by the Schwarz inequality for q,

$$q(Zx, Zy) = q(x, Z^*Zy) \leq q(x,x)^{1/2} q(Z^*Zy, Z^*Zy)^{1/2}, \qquad x,y \in M.$$

Therefore the formula $T(x + N_q) = Zx + N_q$, $x \in M$, properly defines a linear mapping $T\colon M/N_q \to M/N_q$. If $x,y \in M$, then

$$(T(x+N_q), T(y+N_q)) = q(Zx, Zy) = q(x, Z^*Zy) = (x + N_q, Z^*Zy + N_q).$$

Hence the range of T is contained in $D(T^*)$, and $T^*T(x + N_q) = Z^*Zx + N_q$, $x \in M$. Thus $P = T^*T$ is a positive, densely defined operator in M_q, and

$$(T(x+N_q), T(y+N_q)) = (x + N_q, P(y + N_q)), \qquad x,y \in M.$$

If $P^\wedge$ denotes the Friedrichs extension of P, then M/N_q is dense in $C^\infty(P^\wedge)$. Moreover, if $x \in M$, then

$$\begin{aligned}\|P^{\wedge^{2^n}}(x+N_q)\|^{2^{-n}} &= (P^{\wedge^{2^n}}(x+N_q), P^{\wedge^{2^n}}(x+N_q))^{2^{-n-1}} \\ &= q((Z^*Z)^{2^n}x, (Z^*Z)^{2^n}x)^{2^{-n-1}} \\ &\leq c^{2^{-n-1}}\|(Z^*Z)^{2^n}x\|^{2^{-n}} \leq (c^{1/2}\|x\|)^{2^{-n}}\|Z\|^2.\end{aligned}$$

It follows from Theorem (2.6)(d) that $\|T(x+N_q)\| \leq \|Z\|\|x+N_q\|$, $x \in M$. Hence T extends to a bounded linear operator $\pi(u)$ on M_q, which satisfies (i) and (ii). Since τ is a semigroup homomorphism, it is easy to check that the mapping $u \to \pi(u)$ is also a semigroup homomorphism. Q.E.D.

Notice that without the assumption $q(x,x) \leq c\|x\|^2$, $x \in M$, the definition of the operator T in the above proof still makes sense. In this case, by Theorem (2.6)(b), (c), T is a closable operator. Letting $\pi(u)$ be the closure of T, $u \in S$, one obtains a mapping $u \to \pi(u)$ from S to the family of densely defined, closed operators in M_q such that $M/N_q \subset D(\pi(u))$, $u \in S$, (i) of Theorem (4.5.1) is satisfied, and $\pi(s)\pi(t)|M/N_q = \pi(st)|M/N_q$, $s,t \in S$. The reconstruction of quantum mechanics from the field theory axioms—Theorem 6.1.3 of [**3**]—is a particular case of the above theorem. Using the notation and terminology of Chapter 6 of [**3**] to obtain this reconstruction from Theorem (4.5.1) one takes

$S =$ the additive semigroup of all nonnegative real numbers—positive time,

$H = \mathcal{E} = L_2(\mathcal{D}'(R^d), d\mu)$—the "path space" for quantum operators,

$\tau =$ the restriction to S of the time translation subgroup $T(\ \ )$,

$M = \mathcal{E}_+$—the positive time subspace of H,

$q =$ the positive form on M defined by using the time reflection θ (cf. [**3**, p. 91, (6.1.11)]).

Clearly, M is invariant under the positive time translation $\tau(u)$, $u \geq 0$, as well as under $\tau(u)^*\tau(u) = I_H$, because $\tau(u)$ is a unitary operator, for each $u \in S$. Proposition 6.1.2 of [**3**] shows that $q(x,x) \leq \|x\|^2$, $x \in M$. The other assumption made on q in Theorem (4.5.1) follows from the unitarity of $\tau(u)$, $u \in S$. Finally, $M_q = \mathcal{H}$—the quantum mechanical Hilbert space, and $\pi(\ ) = T(\ )^\wedge$ is a contraction semigroup, because $\|\tau(u)\| = 1$.

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DEPARTMENT OF MATHEMATICAL SCIENCES, WEST CHESTER UNIVERSITY, WEST CHESTER, PENNSYLVANIA 19383