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HARDY SPACES OF HEAT FUNCTIONS

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ABSTRACT. We consider spaces of solutions of the one-dimensional heat equation on appropriate bounded domains in the (x,t)-plane. The domains we consider have the property that they are parabolically star-shaped at some point; i.e., each downward half-parabola from some center point intersects the boundary exactly once. We introduce parabolic coordinates (r,θ) in such a way that the curves $\theta =$ constant are the half-parabolas, and dilation by multiplying by r preserves heat functions. An integral kernel is introduced by specializing to this situation the very general kernel developed by Gleason and the author for abstract harmonic functions. The combination of parabolic coordinates and kernel function provides a close analogy with the Poisson kernel and polar coordinates for harmonic functions on the disc, and many of the Hardy space theorems for harmonic functions generalize to this setting. Moreover, because of the generality of the Bear-Gleason kernel, much of this theory extends nearly verbatim to other situations where there are polar-type coordinates (such that the given space of functions is preserved by the "radial" expansion) and the maximum principle holds. For example, most of these theorems hold unchanged for harmonic functions on a radial star in $\mathbf{R}^n$. As ancillary results we give a simple condition that a boundary point of a plane domain be regular, and give a new local Phragmén-Lindelöf theorem for heat functions.

1. Introduction. In this paper we study one dimensional heat functions $(u_{xx} = u_t)$ on an appropriate class of bounded domains. These domains, which we call parabolic stars, are characterized by the property that each downward half-parabola from some central point intersects the boundary exactly once. There are polar-type coordinates (r,θ) in a parabolic star, and these coordinates are natural for heat functions in that such functions are perserved by contraction and dilation; i.e., for fixed $\rho > 0$, $u(\rho r,\theta)$ is a heat function if $u(r,\theta)$ is.

We introduce a global kernel for heat functions in a parabolic star by specializing a very general result proved some years ago by A. M. Gleason and the author [**2**]. In this setting much of the Hardy-space theory for harmonic functions can be carried over to heat functions. Thus finite signed measures on the Shilov boundary Γ are in one-to-one correspondence with an appropriately defined space H_1 of heat functions, L_p functions on Γ correspond to a space H_p of heat functions, etc.

Although all our work here is specific to heat functions, part of our intent is to indicate how these Hardy-space theorems can be proved in many other settings. For example, most of our results of §4 apply, nearly verbatim, for harmonic functions on any bounded domain in $\mathbf{R}^n$ such that each ray from some point intersects the boundary once. The kernel function of [**2**] exists quite generally for any function

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space determined by local integral kernels. The only requirements are an appropriate polar-type coordinate system, and the maximum principle.

In §2 we introduce parabolic coordinates (r, θ) into a general parabolic star G. We also give a very simple condition that a boundary point of G be regular for the heat equation. This theorem, which is the result of joint work with G. N. Hile, extends for the plane earlier results of Effros and Kazdan for domains in $\mathbf{R}^n$ [**4**] and simplifies an earlier result of Petrowsky [**13**]. The theorem says that a point (x_0, t_0) of ∂G is regular provided it can be reached by an ascending curve $x = \gamma(t)$ in the complement of G which is nowhere flatter than a parabola. In particular this result shows that a regular domain can have such arcs removed and the result will still be regular.

In §3 we introduce the kernel from [**2**] and develop some necessary additional properties. In passing we prove a local Phragmén-Lindelöf theorem which may be of some interest in its own right. This result also represents joint work with Hile.

A global integral kernel for heat functions has also been given by Kemper [**10**, **11**], using techniques of Hunt and Wheeden [**8**, **9**] and Carleson [**3**] for harmonic functions. Rather than trying to adapt Kemper's kernel to our setting, we have used the Gleason-Bear kernel from [**2**] to emphasize its general applicability. We do, however, make use of Kemper's work in §4.

§4 is devoted to Hardy-space theorems about apppropriately defined H_p spaces of heat functions.

2. Parabolic coordinates. In this section we introduce "parabolic coordinates" for an appropriate class of domains in the (x, t)-plane. These coordinates are analogous to polar coordinates in that heat functions are preserved by contractions and dilations in the same way that harmonic functions are preserved by radial contractions and dilations.

Fix a point (a, T), and let P_θ be the half-parabola whose equation is

$$\theta = (x - a)/\sqrt{T - t}, \qquad t < T. \tag{2.1}$$

The vertex of P_θ is at (a, T), and the half-parabola points down. The union of the half-parabolas P_θ, for $-\infty < \theta < \infty$, fills the half-plane $t < T$. We will denote by $P_{-\infty}$ and $P_{+\infty}$ the left and right horizontal half-lines from (a, T).

For any point (x, t) with $t < T$ we define

$$x_r = rx + (1 - r)a; \qquad t_r = r^2 t + (1 - r^2)T, \tag{2.2}$$

and for any function $u(x, t)$ defined in the half-plane $t < T$ we let

$$u_r(x, t) = u(x_r, t_r). \tag{2.3}$$

As r runs from 0 to $+\infty$, the point (x_r, t_r) runs from (a, T) down along the half-parabola P_θ through (x, t), with $(x_1, t_1) = (x, t)$. Heat functions are preserved by the transformation $(x, t) \to (x_r, t_r)$, since $\partial^2 u_r/\partial x^2 = \partial u_r/\partial t$.

DEFINITION 2.1. Let G be a bounded open set in the half-plane $t < T$, such that ∂G intersects the line $t = T$ in a closed interval $a - h \le x \le a + k$, $t = T$. Assume further that each half-parabola P_θ, $-\infty < \theta < \infty$, from (a, T) intersects ∂G exactly once. Let $\hat{G}$ be the union of G and the open interval $(a - h, a + k) \times \{T\}$. Then either G or $\hat{G}$ will be called a *parabolic star* at (a, T). The hat on a star will be used systematically to denote the inclusion of the top open interval. The compact set $\Gamma = \Gamma(G) = \Gamma(\hat{G})$ is defined to be $\overline{G} - \hat{G}$, and is called the lower boundary of G or $\hat{G}$. $H(G)$ and $H(\hat{G})$ will respectively denote the space of heat functions on G, and the space of heat functions on $\hat{G}$, where the one-sided t-derivative is used on the top line. In order for a parabolic star to be regular for the Dirichlet problem, we will also assume that each of the top points $(a-h, T)$ and $(a+k, T)$ of Γ is the vertex of a downward pointing half-parabola which lies outside $\overline{G}$ (see Theorem 2.2 below).

The set Γ is the Shilov boundary of the set of continuous functions on $\overline{G}$ which are in $H(\hat{G})$. This is well known, and follows easily by considering the functions Ψ_n of (3.6).

If $\hat{G}$ is a parabolic star at (a, T), we introduce parabolic coordinates (determined by $\hat{G}$ and (a, T)) into the half-plane $t \leq T$ as follows: $(1, \theta)$ will denote the point where P_θ intersects $\partial\hat{G}$ for $-\infty < \theta < \infty$, and $(1, -\infty)$, $(1, +\infty)$ will be the coordinates of the points $(a-h, T)$ and $(a+k, T)$ where Γ intersects the line $t = T$. If $(1, \theta)$, for $-\infty \leq \theta \leq \infty$, has cartesian coordinates (x, t), then (r, θ) will denote the point with cartesian coordinates (x_r, t_r) (see (2.2)). We topologize $[-\infty, \infty]$ in the usual way so the correspondence $\theta \to (1, \theta)$ is a homeomorphism onto Γ. The set of points (r, θ), $0 < r < 1$, $-\infty < \theta < \infty$, is the open set G whose closure is $\hat{G} \cup \Gamma$. The set $\hat{G}$ consists of the points (r, θ) with $0 \leq r < 1$, $-\infty \leq \theta \leq \infty$.

We show next that the Dirichlet problem for heat functions is solvable on parabolic stars. In fact, we show much more; namely, a boundary point of a plane domain is regular in the sense of the Perron method if it can be reached from outside G by an upward arc which is nowhere flatter than a parabola;

$$|x(t) - x(t')| \leq M|t - t'|^{1/2}.$$

This means, for example, that such slits can be removed from Dirichlet domains and the result will still be a Dirichlet domain. Note that some condition on such slits is necessary, since Pini has shown [**14**] that if ∂G is too flat at a boundary point (x_0, t_0), with G below (x_0, t_0), then (x_0, t_0) is definitely not regular. For other regularity conditions see Petrowsky [**13**] and Evans and Gariepy [**5**].

THEOREM 2.2. *Let (x_0, t_0) be a boundary point of a domain G. Let $x = \gamma(t)$, $t_1 \leq t \leq t_0$, be an arc in the complement of G, such that (x_0, t_0) is its topmost point, and*

$$|\gamma(t) - \gamma(t')| \leq M|t - t'|^{1/2} \tag{2.4}$$

for some M and all $t, t' \in [t_1, t_0]$. Then (x_0, t_0) is a regular point for G in the sense of the Perron method.

Condition (2.4) says that γ can be any downward arc from (x_0, t_0) which is never flatter than a parabola. In our application to a parabolic star, γ can be simply the arc $\theta = \theta_0$, $1 \leq r \leq 2$, where (x_0, t_0) has parabolic coordinates $(1, \theta_0)$. For $\theta_0 = \pm\infty$, the existence of γ is part of the definition of a star.

PROOF. We make essential use of a theorem of Effros and Kazdan [**4**], which says that a point of a domain in $\mathbf{R}^n \times \mathbf{R}$ is regular for the heat equation if it can be touched from outside G by a "parabolic tusk". For our case, $n = 1$, this says that (x_0, t_0) is regular if the region between two half-parabolas, P_a and P_b, with their vertices at (x_0, t_0), lies outside G.

Let $R = (a, b) \times (t_1, t_2)$ be a small rectangle with (x_0, t_0) in its interior, and γ extending through the bottom boundary line of R. We assume that $R \sim \gamma$ is contained in G, which is the worst case, and of course does not happen in the case of a parabolic star. To construct a barrier function v which is positive on $R \sim \gamma$, and zero at (x_0, t_0), we use the Effros-Kazdan result to construct first a heat function which is positive on the part of R below (x_0, t_0) and to the right of γ, and vanishes at (x_0, t_0). We do the same for the part of R to the left of γ and below (x_0, t_0). We then extend these two functions to the part of R above (x_0, t_0) by using their values on $t = t_0$ (which are positive except at (x_0, t_0)), and positive boundary values on the sides of R. The resulting function on $R - \gamma$ satisfies the heat equation except possibly on the line $t = t_0$. However, since the function is continuous across $t = t_0$ it is easy to show that it is a heat function on all of $R - \gamma$.

3. The parabolic kernel for a star domain. In this section we apply the main result of [**2**] to heat functions on a parabolic star, and prove some additional properties of the kernel.

We start with a fixed parabolic star $\hat{G}$ with its parabolic coordinates (r, θ) for the half-plane below the center. We apply Theorem 4.1 of [**2**] to this situation to obtain Theorem 3.1 below. The topological space X of [**2**] is $\hat{G}$, and the linear space of functions A of [**2**] is $H(\hat{G})$. The local kernel representation required by [**2**] also holds for top points of $\hat{G}$. The space F of [**2**] is assumed to be a linear subspace of $A = H(\hat{G})$ intersected with the bounded continuous functions on $X = \hat{G}$. Here we let F be the functions of $H(\hat{G})$ which extend continuously to $\overline{G}$. The Γ of [**2**] is our Γ, and restriction of functions in F to Γ is an isometrty onto $C(\Gamma)$ by Theorem 2.4. With these conventions, and parabolic coordinates for $\hat{G}$, Theorem 4.1 of [**2**] reads as follows.

THEOREM 3.1. *There is a Borel probability measure ν on Γ and a measurable function $B(r, \theta; \varphi)$ on $G \times \Gamma$ such that $B(\cdot, \cdot, \varphi_0) \in H(\hat{G})$ for each $\varphi_0 \in \Gamma$, and*

$$g(r, \theta) = \int B(r, \theta; \varphi) \xi(\varphi)\, d\nu(\varphi) \tag{3.1}$$

defines a function $g \in H(\hat{G})$ for every bounded measurable function ξ on Γ, and

$$f(r, \theta) = \int B(r, \theta; \varphi) f(1, \varphi)\, d\nu(\varphi) \tag{3.2}$$

for every $f \in H(\hat{G}) \cap C(\overline{G})$ and every $(r, \theta) \in \hat{G}$.

The measure $B(r, \theta; \cdot)\, d\nu(\cdot)$ is of course the unique representing measure for the point (r, θ) and the continuous heat functions on $\overline{G}$. These measures are positive probability measures, so each $B(r, \theta; \cdot) \geq 0$ a.e. ν on Γ.

In the sequel we will sometimes deal with functions in $H(G)$ and sometimes with functions in $H(\hat{G})$. It will be important to know that the representation (3.2) holds for functions in $H(G)$ which are continuous on $\overline{G} \cap \{(x, t) \colon t < T\}$, and we state this property as a first lemma.

LEMMA 3.2. *If ξ is a measurable function on Γ which is bounded on $\Gamma \cap \{(x, t) \colon t \leq T - \varepsilon\}$ for each $\varepsilon > 0$, then* (3.1) *defines a function $g \in H(G)$. If*

$f \in H(G)$ and f is continuous on $\overline{G} \cap \{(x,t): t < T\}$, then (3.2) *holds for all* $(r,\theta) \in G$.

PROOF. It follows from the maximum principle that the representing measure $B(r,\theta;\cdot)\,d\nu(\cdot)$ is identically zero on the part of Γ above (r,θ). If ξ is measurable on Γ and bounded except possibly at $t = T$, then (3.1) exists for $(r,\theta) \in G$, and agrees on some neighborhood of each $(r,\theta) \in G$ with the function in $H(\hat{G})$ obtained by integrating a function $\bar{\xi}$, where $\bar{\xi}$ is bounded and measurable on Γ, and $\bar{\xi} = \xi$ up to a level above (r,θ). A similar argument applies to a function $f \in H(\hat{G})$ which is continuous on the part of $\overline{G}$ strictly below $t = T$. For any fixed $(r,\theta) \in G$, let $\bar{f}$ be a continuous function on Γ which agrees with f up to a level above (r,θ). The integral of $\bar{f}$ agrees with the integral of f on a neighborhood of (r,θ).

In the next few lemmas we draw out some of the properties of the representations (3.1) and (3.2). In particular we show that (3.1), like the function furnished by the Perron method, is continuous wherever the boundary function ξ is. We also show that ν can have no point masses.

LEMMA 3.3. *Let $m_{(r,\theta)}$ be the representing measure for (r,θ); i.e., $dm_{(r,\theta)} = B(r,\theta;\cdot)\,d\nu(\cdot)$. If U is any open subset of Γ containing $(1,\varphi_0)$, then $m_{(r,\theta)}(U) \to 1$ as $(r,\theta) \to (1,\varphi_0)$.*

PROOF. Let $f(\varphi)$ be a continuous nonnegative function on Γ, with $f(\varphi_0) = 1$ and $0 \le f \le 1$ and $f = 0$ off U. Let $f(r,\theta)$ be its extension to $H(\hat{G})$. Then

$$f(r,\theta) = \int f\,dm_{(r,\theta)} \le m_{(r,\theta)}(U) \le 1, \tag{3.3}$$

and $f(r,\theta) \to 1$ as $(r,\theta) \to (1,\varphi_0)$.

LEMMA 3.4. *Let ξ be a bounded measurable function on Γ and let $g(r,\theta)$ be its integral as in* (3.1). *Then $g(r,\theta) \to \xi(\varphi_0)$ as $(r,\theta) \to (1,\varphi_0)$ at every point φ_0 where ξ is continuous.*

PROOF. Let U be an open set about a continuity point φ_0, with $|\xi(\varphi)-\xi(\varphi_0)| < \varepsilon$ for $\varphi \in U$. Then

$$\begin{aligned} |g(r,\theta) - \xi(\varphi_0)| &\le \int |\xi(\varphi) - \xi(\varphi_0)|\,dm_{(r,\theta)}(\varphi) \\ &= \int_U + \int_{\Gamma - U}. \end{aligned} \tag{3.4}$$

The first integral is less than ε for all (r,θ), and the second integral is less than $2\|\xi\|_\infty m_{(r,\theta)}(\Gamma - U)$, which tends to zero as $(r,\theta) \to (1,\varphi_0)$ by Lemma 3.3.

The next result is a local Phragmén-Lindelöf theorem for heat functions. We will use this result to show the measure ν of Theorem 3.1 has no point masses, but the result is of some interest in its own right. It is more convenient for this proof to revert to cartesian coordinates since it is not necessary for this theorem that G be a parabolic star.

For convenience in stating the next theorem, we agree here that a heat function $u(x,t)$ in a domain G is $o(1/\sqrt{t-t_0})$ *over parabolas at the boundary point* (x_0,t_0)

provided there is some neighborhood U of (x_0, t_0) such that u is bounded in $G \cap U \cap \{(x,t)\colon t \le t_0\}$, and for every parabola $P = \{(x,t)\colon t - t_0 = A(x - x_0)^2\}$, with $A > 0$,

$$u(x,t)\sqrt{t - t_0} \to 0 \tag{3.5}$$

as $(x,t) \to (x_0, t_0)$, with (x,t) in G and above P. Notice that the condition above is formally weaker than simply requiring that (3.5) hold for $(x,t) \to (x_0, t_0)$ with $t > t_0$. Notice also that if u has zero boundary values on ∂G near (x_0, t_0), except possibly at (x_0, t_0), then u is automatically bounded on some set $U \cap G \cap \{(x,t)\colon t \le t_0\}$. (We assume here that the line $t = t_0$ is not the top of G to preclude the possibility that $u(x,t) \to \infty$ as $t \uparrow t_0$ for all x.)

THEOREM 3.5 (LOCAL PHRAGMÉN-LINDELÖF). *Let G be a bounded domain with parabolic boundary Γ. Let (x_0, t_0) be a point of Γ which can be reached from below by a parabolic arc γ lying outside G (or any arc γ as in Theorem 2.2). Let U be a neighborhood of (x_0, t_0) and $u(x,t)$ a heat function in G such that $u(x,t) \to 0$ as (x,t) approaches any point of $\Gamma \cap U \sim \{(x_0, t_0)\}$. If u is $o(1/\sqrt{t - t_0})$ over parabolas at (x_0, t_0), then $u(x,t) \to 0$ as $(x,t) \to (x_0, t_0)$.*

COROLLARY. *If u is a heat function in G and u is locally bounded at (x_0, t_0) and $u(x,t) \to 0$ on Γ near (x_0, t_0), except possibly for (x_0, t_0), then $u(x,t) \to 0$ also as $(x,t) \to (x_0, t_0)$.*

PROOF. We will first prove the theorem in the case that u is bounded on U, and then show how the proof is modified to cover the case where u is possibly unbounded above $t = t_0$, but (3.5) holds over each parabola.

Since (x_0, t_0) is a regular point of G by Theorem 2.4, there is a heat function $\varphi \ge 0$ in $G \cap U$, with φ continuous on the closure of $G \cap U$, and $\varphi = 0$ only at (x_0, t_0). Hence there is a constant $c > 0$ such that $u + c\varphi \ge 0$ on $\partial U \cap \overline{G}$, and of course also on $\Gamma \cap U \sim \{(x_0, t_0)\}$. There is also a sequence $\{\Psi_n\}$ of heat functions which are nonnegative on G, with $\Psi_n(x_0, t_0) \to \infty$, and $\{\Psi_n(x_1, t_1)\}$ bounded for each $(x_1, t_1) \in G \cap U$; for example, let

$$\Psi_n(x,t) = k(x - x_n, t - t_n), \tag{3.6}$$

where k is the fundamental solution

$$k(x,t) = (4\pi t)^{-1/2} \exp(-x^2/4t), \tag{3.7}$$

and (x_n, t_n) is a sequence of points approaching (x_0, t_0) upward along γ.

Let (x_1, t_1) be any fixed point of $G \cap U$. We will show $u(x_1, t_1) + c\varphi(x_1, t_1) \ge 0$. Let $\varepsilon > 0$ and choose $n = n(\varepsilon)$ so that $u + \varepsilon\Psi_n \ge 0$ on $G \cap \partial V$ for some small neighborhood V of (x_0, t_0) such that (x_1, t_1) is outside V. Now we have

$$u + c\varphi + \varepsilon\Psi_n \ge 0 \tag{3.8}$$

on the boundary of a neighborhood $G \cap U \sim \overline{V}$ of (x_1, t_1), so

$$u(x_1, t_1) + c\varphi(x_1, t_1) + \varepsilon\Psi_n(x_1, t_1) \ge 0. \tag{3.9}$$

Now let $\varepsilon \to 0$ in (3.9). Although n depends on ε, $\{\Psi_n(x_1, t_1)\}$ is bounded, so

$$u(x_1, t_1) + c\varphi(x_1, t_1) \ge 0. \tag{3.10}$$

Letting $(x_1, t_1) \to (x_0, t_0)$, and recalling that $\varphi(x_0, t_0) = 0$, gives

$$\liminf_{(x_1,t_1)\to(x_0,t_0)} u(x_1, t_1) \geq 0. \tag{3.11}$$

The same argument applied to $-u$ completes the equality in the case where u is bounded on U.

Now assume only that u is bounded on $t \leq t_0$, and $o(1/\sqrt{t-t_0})$ above parabolas on $t > t_0$. Let $(x_1, t_1) \in U \cap G$. The same argument as before shows that (3.10) holds if $t_1 \leq t_0$, so assume $t_1 > t_0$. Let P be a parabola pointing upward from (x_0, t_0), with (x_1, t_1) above P. Let $\varepsilon > 0$. Notice that

$$\varepsilon\sqrt{t-t_0}\, k(x - x_0, t - t_0) \tag{3.12}$$

is bounded away from zero above every parabola $t - t_0 = A(x - x_0)^2$, $A > 0$. Hence (3.13)

$$\varepsilon k(x - x_0, t - t_0) \pm u(x,t) = \frac{1}{\sqrt{t-t_0}}\left[\varepsilon\sqrt{t-t_0}\, k(x - x_0, t - t_0) \pm \sqrt{t-t_0}\, u(x,t)\right]$$

approaches $+\infty$ as $(x,t) \to (x_0, t_0)$ above any parabola. Let V be a small neighborhood of (x_0, t_0) such that (x_1, t_1) is outside V, and

$$\pm u(x,t) + \varepsilon k(x - x_0, t - t_0) > 0 \tag{3.14}$$

on part of $\partial V \cap G$ above P.

Now we have

$$\pm u(x,t) + \varepsilon k(x - x_0, t - t_0) + c\varphi(x,t) > 0 \tag{3.15}$$

on the boundary of a neighborhood of (x_1, t_1); namely, on the part of $\partial U \cap G$ above P, the part of $P \cap G$ outside V (make c bigger if necessary for this part) and the upper part (possibly empty) of $\Gamma \cap U$ outside V. Therefore (3.15) holds with (x_1, t_1) for (x,t), and for every $\varepsilon > 0$. Hence we again have (3.10), and the desired conclusion.

The condition that $u(x,t)\sqrt{t-t_0} \to 0$ above parabolas cannot be relaxed, for example, to the condition that $u(x,t)\sqrt{t-t_0}$ be bounded above parabolas. The kernel function $k(x,t)$ on $t > 0$ near $(0,0)$ provides a counterexample. Moreover [**1**, Theorem 3.6], if $u(x,t)$ is *any* nonnegative heat function in any rectangle with (x_0, t_0) in its base, then $u(x,t)\sqrt{t-t_0}$ is bounded near (x_0, t_0).

All of our results so far hold, usually with the same proofs, for harmonic functions on a regular radial star in $\mathbf{R}^n$. In particular, the analogous Phragmén-Lindelöf theorem holds, and a sufficient hypothesis is that $u(P) = o(d(P, Q_0)^{2-n})$, where $d(P, Q_0)$ is the distance from P to the boundary point Q_0. Local boundedness of u is of course sufficient.

Now we return to a star domain $\hat{G}$ with its coordinates (r, θ).

LEMMA 3.6. *The (unique) representing measures $m_{(r,\theta)}$ for points (r, θ) of $\hat{G}$ have no point masses.*

PROOF. Let m be the representing measure for a fixed point $(r_0, \theta_0) \in \hat{G}$, and let ξ be the characteristic function of $\{\varphi_0\}$, for any $\varphi_0 \in \Gamma$. Let $g(r, \theta)$ be the (3.1) integral of ξ. Then $g(r, \theta) = m_{(r,\theta)}(\{\varphi_0\})$ for all (r, θ). By Lemma 3.3, $g(r, \theta) \to 0$ as $(r, \theta) \to (1, \varphi)$ for all $\varphi \neq \varphi_0$, and g is bounded. By Theorem 3.5, $g(r, \theta) \to 0$ as $(r, \theta) \to (1, \varphi_0)$ also. Hence $g \equiv 0$, and $m_{(r,\theta)}(\{\varphi_0\}) = 0$ for all (r, θ).

Now we rearrange the kernel-measure combination in a way that singles out the center (a,T) of the star $\hat{G}$, and makes the Harnack properties of the functions more apparent.

DEFINITION 3.7. *A domain $\hat{G}$ is Harnack ordered provided for every two cartesian points (x_1,t_1), (x_2,t_2) in $\hat{G}$ with $t_2 < t_1$ there is a strictly descending curve in $\hat{G}$ from (x_1,t_1) to (x_2,t_2).*

A bounded domain is Harnack ordered iff its lower boundary consists of two curves $x=\eta_i(t)$, $a \le t \le b$, with $\eta_1(t) < \eta_2(t)$ for $a < t < b$. The functions η_i can have jump discontinuities (η_1 can jump down and η_2 can jump up) in which case the horizontal segments which constitute the jumps will be part of Γ.

If $\hat{G}$ is Harnack ordered and $t_2 < t_1$, then there is a constant M depending on (x_1,t_1) and (x_2,t_2) such that for every nonnegative function $u \in H(\hat{G})$

$$u(x_2,t_2) \le Mu(x_1,t_1). \tag{3.16}$$

If $\hat{G}$ is a parabolic star, so all continuous functions on Γ extend to $H(\hat{G})$, then

$$m_2 \le M m_1 \tag{3.17}$$

where m_i is the unique representing measure for (x_i,t_i). The constant M of (3.16) is bounded as (x_2,t_2) ranges over any compact subset of G strictly below (x_1,t_1). The facts above follow from [**6**, **12**, **14**].

Let $E = \{\varphi\colon B(0,0;\varphi)=0\}$, and let ξ be the characteristic function of E, and $f(r,\theta)$ its integral. Then $f \ge 0$ and $f(0,0)=0$. It follows from Harnack's theorem that $f \equiv 0$ on $\hat{G}$, so $B(r,\theta;\varphi)=0$ for ν-almost all φ on E, for each $(r,\theta)\in\hat{G}$. In this case the behavior of ν on E is irrelevant for the integral representation, and we can assume without loss of generality that $\nu(E)=0$. (I.e., replace ν if necessary by $\tilde{\nu}$ where $\tilde{\nu}(A)=\tilde{\nu}(A\sim E)$. Instead of relabeling we simply assume $\nu(E)=0$.) Now we redefine $B(r,\theta;\varphi)$ for $\varphi\in E$ so that $B(r,\theta;\varphi)\equiv 1$ for $(r,\theta)\in\hat{G}$ and $\varphi\in E$. The new B and the new ν satisfy all the conditions of Theorem 3.1, and in addition we have

$$B(0,0;\varphi) > 0 \quad \text{for all } \varphi \in \Gamma. \tag{3.18}$$

Now we rewrite the representing measures in terms of the representing measure $B(0,0;\cdot)\,d\nu(\cdot)$ for $(0,0)$.

LEMMA 3.8. *Let μ be the representing measure for $(0,0)$:*

$$d\mu(\cdot) = B(0,0;\cdot)\,d\nu(\cdot). \tag{3.19}$$

There is a nonnegative measurable function $K(r,\theta;\varphi)$ on $\hat{G}\times\Gamma$ such that $K(r,\theta;\cdot)\in L_1(\mu)$ for each $(r,\theta)\in\hat{G}$, $K(r,\theta;\varphi)$ is bounded on $F\times\Gamma$ for each closed set $F\subset G$, $K(\cdot,\cdot;\varphi)\in H(\hat{G})$ for each $\varphi\in\Gamma$, and

$$B(r,\theta;\varphi)\,d\nu(\varphi) = K(r,\theta;\varphi)\,d\mu(\varphi). \tag{3.20}$$

PROOF. For $(r,\theta)\in G$ we know that

$$B(r,\theta;\varphi) \le M(r,\theta)B(0,0;\varphi) \tag{3.21}$$

for ν-almost all φ, by Harnack's theorem. Hence if we let

$$K(r,\theta;\varphi) = B(r,\theta;\varphi)/B(0,0;\varphi) \tag{3.22}$$

then we have (3.20) with $K(r, \theta; \cdot) \leq M(r, \theta)$ for μ-almost all φ. For (r, θ) on the top line of $\hat{G}$ (i.e., $\theta = \pm\infty$), Harnack's theorem still implies that the representing measure for (r, θ) is absolutely continuous with respect to μ, and vice versa. Hence $K(r, \theta; \cdot) \in L_1(\mu)$ for all $(r, \theta) \in \hat{G}$, although $K(r, \theta; \cdot)$ is not bounded for $\theta = \pm\infty$. Clearly K is measurable and $K(\cdot, \cdot; \varphi) \in H(\hat{G})$ for all φ.

The measure μ of (3.19) will be referred to as the *standard measure* of $\hat{G}$ with its specified center.

THEOREM 3.9. *If $u \in H(\hat{G})$, then for all $r \in (0, 1)$,*

$$u(0,0) = \int u(r, \theta)\, d\mu(\theta). \tag{3.23}$$

If $0 \leq r < 1$, then

$$u(r\rho, \theta) = \int u(r, \varphi) K(\rho, \theta; \varphi)\, d\mu(\varphi) \tag{3.24}$$

$$= \int u(\rho, \varphi) K(r, \theta; \varphi)\, d\mu(\varphi). \tag{3.25}$$

As special cases, we have for all r, $\rho \in [0, 1)$ and all $\varphi_0 \in [-\infty, \infty]$,

$$\int K(r, \theta; \varphi_0)\, d\mu(\theta) = 1, \tag{3.26}$$

$$K(r\rho, \theta; \varphi_0) = \int K(r, \varphi; \varphi_0) K(\rho, \theta; \varphi)\, d\mu(\varphi). \tag{3.27}$$

PROOF. If $r < 1$, then $u_r(0,0) = u(0,0)$ and u_r is continuous on $\overline{G}$. Hence

$$u_r(0,0) = \int u_r(1, \varphi)\, d\mu(\varphi),$$

which is the same as (3.23). Similarly, if $r, \rho \in [0, 1)$, then u_r is continuous on $\overline{G}$, so

$$\begin{aligned} u_r(\rho, \theta) &= u(r\rho, \theta) \\ &= \int u_r(1, \varphi) K(\rho, \theta; \varphi)\, d\mu(\varphi) \\ &= \int u(r, \varphi) K(\rho, \theta; \varphi)\, d\mu(\varphi); \end{aligned} \tag{3.28}$$

(3.25) results from (3.24) by interchanging r and ρ.

It follows immediately from (3.24) that if u is a bounded function in $H(\hat{G})$ and u has "parabolic" limits zero a.e. μ (i.e. as $r \to 1$, $u(r, \varphi) \to 0$ for each fixed φ in some set of μ-measure one), then $u \equiv 0$. We will improve on this result in the next section.

The mean value property of (3.23) characterizes heat functions. This is hardly surprising, but the result does not seem to be explicit in the literature even for mean values on rectangles. We formalize what we mean by mean value property in the following definition.

DEFINITION 3.10. *A function u, continuous on some parabolic star $\hat{G}$, centered at a point P, has the mean value property (MVP) at P iff u satisfies (3.23) for arbitrarily small values of r. The coordinates and standard measure are those of $\hat{G}$.*

THEOREM 3.11. *If u is continuous on an open set S and has the MVP at each point of S, then u is a heat function on S.*

PROOF. The local stars can all have different shapes, or can all be rectangles in the closest analogy to the usual theorem for harmonic functions.

We assume that u has the MVP in S and show that each $P_1 \in S$ is interior to a rectangle on which u is a heat function. Fix $P_1 \in S$, with P_1 interior to a rectangle R, with $\overline{R} \subset S$. Let v be the heat function in $\hat{R}$ which agrees with u on $\Gamma(R)$. We let $w = u - v$ and show that $w \equiv 0$ in R. Since v is a heat function in R and u has the MVP in R by hypothesis, w has the MVP in R. Assume $w \not\equiv 0$ in $\hat{R}$ and to be specific assume the maximum value of w in $\overline{R}$ is $m > 0$. Let P_2 be a lowest point in $\overline{R}$ such that $w(P_2) = m$. Let $\hat{G}(P_2)$ be the hypothesized star at P_2, and assume $\hat{G}(P_2) \subset \hat{R}$. Since $w < m$ on the lower boundary of $\hat{G}(P_2)$, except possibly at the two top points, we have a contradiction from (3.23).

Clearly the same argument holds for sets S which contain some horizontal top boundary segments.

The argument above yields a slightly different mean value characterization for harmonic functions in $\mathbf{R}^n$. If G is any regular radial star at 0, and μ is the representing measure for 0 on $\Gamma = \partial G$, then the local MVP with μ on homothetic copies of G characterizes the harmonic functions.

4. Some boundary properties of $H(G)$. In this section we deal with a fixed star $\hat{G}$, with coordinates (r, θ), standard measure μ, and representing kernel $K(r, \theta; \varphi)$.

For the work of this section it is convenient and apposite to use the terminology of Hardy theory. Thus we let $H_\infty(G)$ (resp. $H_\infty(\hat{G})$) denote the bounded functions in $H(G)$ (resp. $H(\hat{G})$). Similarly, for $p \geq 1$ we let $H_p(G)$ denote the functions $f \in H(G)$ such that $\|f_r\|_p$ is bounded for $0 \leq r < 1$, where

$$\|f_r\|_p = \left\{ \int |f(r, \theta)|^p \, d\mu(\theta) \right\}^{1/p}. \tag{4.1}$$

Our terminological pilfering is unabashed, since our thesis here is that Hardy space theorems are not peculiar to harmonic functions or to discs and balls, but depend rather on the existence of local kernels, the maximum principle, and the appropriate coordinate system.

We start with the fact that $\|f_r\|_p$ is an increasing function of r.

THEOREM 4.1. *If $f \in H(G)$, then $\|f_r\|_p$ is an increasing function of r for each p, $1 \leq p \leq \infty$.*

PROOF. For $1 \leq p \leq \infty$ the result follows from Jensen's theorem and (3.19), and for $p = \infty$ from the maximum principle.

We already know from Theorem 3.1 that each bounded measurable function ξ on Γ gives rise to some $f \in H_\infty(\hat{G})$. We show next that each $f \in H_\infty(G)$ is obtained this way, and consequently that $H_\infty(G) = H_\infty(\hat{G})$.

THEOREM 4.2. *$f \in H_\infty(G)$ iff there is some bounded measurable function F on Γ such that*

$$f(r, \theta) = \int F(\varphi) K(r, \theta; \varphi) \, d\mu(\varphi) \tag{4.2}$$

for all $(r, \theta) \in G$.

PROOF. We need only prove the one implication, so assume $f \in H_\infty(G)$, and $\|F_\infty\| = M$. Let

$$f_r(\theta) = f(r,\theta) = f_r(1,\theta), \qquad -\infty < \theta < \infty.$$

The functions $\{f_r\}$ form a net, as $r \to 1$, in the closed M-ball of $L_1(\mu)^*$. Hence some subnet, $\{f_{r_\alpha}\}$, $r_\alpha \to 1$, converges w* (pointwise on $L_1(\mu)$) to a bounded measurable function F with $\|f\|_\infty \le M$. For $(\rho,\theta) \in G$, $K(\rho,\theta;\cdot) \in L_1(\mu)$, so

$$\begin{aligned} f(\rho,\theta) &= \lim_{r_\alpha \to 1} f(r_\alpha \rho, \theta) \\ &= \lim_{r_\alpha \to 1} \int f_{r_\alpha}(1,\varphi) K(\rho,\theta;\varphi)\, d\mu(\varphi) \\ &= \int F(\varphi) K(\rho,\theta;\varphi)\, d\mu(\varphi). \end{aligned} \tag{4.3}$$

Since the right side of (4.3) defines a heat function on all of $\hat{G}$, (4.3) gives the automatic extension of each $f \in H_\infty(G)$ to a function in $H_\infty(\hat{G})$.

An exactly similar proof shows that each $f \in H_p(G)$, $p > 1$, is the integral of a function $F \in L_p(\mu)$, with $\|F\|_p \le \sup_r \|f_r\|_p$ (cf. [**7**] for similar results on a rectangle). The fact that the integral (4.2) for $F \in L_p(\mu)$, does yield a function $f \in H(G)$ will follow from Theorem 4.4, and the fact that $\|f_r\|_p \le \|F\|_p$ follows from Jensen's inequality. Hence we have the following result.

THEOREM 4.3. *$f \in H_p(G)$ and $\|f_r\|_p \le M$ for $0 \le r < 1$, with $p > 1$, iff there is $F \in L_p(\mu)$ such that $\|F\|_p \le M$ and (4.2) holds for all $(r,\theta) \in G$.*

Next we show there is a one-to-one correspondence between finite signed measures on Γ and functions in $H_1(G)$.

THEOREM 4.4. *If α is a finite signed measure on Γ, and*

$$g(r,\theta) = \int K(r,\theta;\varphi)\, d\alpha(\varphi), \tag{4.4}$$

for $(r,\theta) \in G$, then $g \in H(G)$.

PROOF. For $(r,\theta) \in G$, $K(r,\theta;\cdot)$ is zero for φ above (r,θ), so we may as well assume that α is a measure on $(-\infty,\infty)$ (i.e., has no mass at $(1,\pm\infty)$).

We use the mean value property characterization of Theorem 3.10. Let $(r_0,\theta_0) \in G$ and let R be a rectangle in G which is top-centered at (r_0,θ_0). Let μ_0 be the standard measure for R, so μ_0 represents (r_0,θ_0) for all heat functions on $\overline{R}$, and in particular for the functions $K(r,\theta;\varphi_0)$. We show that the integral of g over $\gamma = \Gamma(R)$ gives $g(r_0,\theta_0)$, which suffices by Theorem 3.11. In the following, (r,θ) are $\hat{G}$ coordinates, and $\Gamma = \Gamma(\hat{G})$.

$$\begin{aligned} \int_\gamma g(r,\theta)\, d\mu_0(r,\theta) &= \int_\gamma \int_\Gamma K(r,\theta;\varphi)\, d\alpha(\varphi)\, d\mu_0(r,\theta) \\ &= \int_\Gamma \int_\gamma K(r,\theta;\varphi)\, d\mu_0(r,\theta)\, d\alpha(\varphi) \\ &= \int_\Gamma K(r_0,\theta_0;\varphi)\, d\alpha(\varphi) \\ &= g(r_0,\theta_0). \end{aligned} \tag{4.5}$$

The Fubini interchange is justified since the kernel is bounded on $\gamma \times \Gamma$.

Alternatively, we could write the integral (4.4) as a pointwise limit of Riemann sums, which are in $H(\hat{G})$. These sums form an equicontinuous family on G, and hence the pointwise limit is uniform on compact subsets of G. The mean value proof appears to be simpler.

THEOREM 4.5. *If g is defined on G by* (4.4) *for a finite signed measure α on $(-\infty,\infty)$, then for all $r \in (0,1)$,*

$$\|g_r\|_1 \leq \|\alpha\| \tag{4.6}$$

PROOF.

$$\begin{aligned} \int |g_r(\theta)|\,d\mu(\theta) &= \int |g(r,\theta)|\,d\mu(\theta) \\ &= \int \left| \int K(r,\theta;\varphi)\,d\alpha(\varphi) \right| d\mu(\theta) \\ &\leq \iint K(r,\theta;\varphi)\,d|\alpha|(\varphi)\,d\mu(\theta) \\ &= \iint K(r,\theta;\varphi)\,d\mu(\theta)\,d|\alpha|(\varphi) \\ &= \|\alpha\|. \end{aligned} \tag{4.7}$$

The last equality is (3.26).

Now we will invoke some results of Kemper on heat kernels [**10**, **11**]. In the one dimensional case (treated specifically in [**10**]), Kemper assumes a domain G of the form $\eta_1(t) < x < \eta_2(t)$, $0 < t < T$, where η_1 and η_2 satisfy a Lipschitz condition of order $\frac{1}{2}$. Following the ideas of Hunt and Wheeden [**8**, **9**] and ultimately of Carleson [**3**] for harmonic functions, Kemper obtains the kernel $K(r,\theta;\varphi)$ as a limit of functions $h_I(r,\theta)/\mu(I)$, where h_I is the parabolic measure of an interval I, and thelimit is taken as I decreases to $\{\varphi\}$. Thus

$$\begin{aligned} K(r,\theta;\varphi) &= \lim_{I\to\{\varphi\}} \frac{1}{\mu(I)} \int_I dm_{(r,\theta)}(\varphi) \\ &= \lim_{I\to\{\varphi\}} \frac{m_{(r,\theta)}(I)}{\mu(I)} \quad \text{(a.e. } \mu). \end{aligned}$$

The critical part of all the works cited above is showing that the function $K(\cdot,\cdot;\varphi)$ has zero boundary values on $\Gamma \sim \{\varphi\}$. The kernel $K(\cdot,\cdot;\varphi)$ can be characterized as the unique nonnegative heat function which is one at $(0,0)$ and has zero boundary values on $\Gamma \sim \{\varphi\}$. From this it follows that $K(r,\theta;\varphi)$ is a continuous function of φ for fixed $(r,\theta) \in G$. Kemper's methods appear to extend to any Harnack ordered parabolic star. However, we will emphasize the specific properties we need in the following definition.

DEFINITION 4.6. *A parabolic star G is a standard star provided the kernel K of* (3.15) *has the properties:*

(i) *$K(r,\theta;\varphi_0)$ approaches zero as $(r,\theta) \to (1,\varphi)$ for all $\varphi \neq \varphi_0$;*

(ii) *$K(r_0,\theta_0;\varphi)$ is continuous for $\varphi \in \Gamma$ for each fixed $(r_0,\theta_0) \in G$.*

THEOREM 4.7. *If G is a standard star, then the mapping* (4.4) *of finite signed measures on Γ is onto $H_1(G)$.*

PROOF. Let $g \in H_1(G)$, with $\|g_r\| \leq M$ for all $r < 1$. The measures $g_r(\theta)\, d\mu(\theta)$ are all in the M-ball of $C(\Gamma)^*$, which is w*-compact. Hence there is a w*-convergent subnet $\{g_{r_j}(\theta)\, d\mu(\theta)\}$, $r_j \to 1$, which converges pointwise on $C(\Gamma)$ to some measure α, with $\|\alpha\| \leq M$. Since $K(\rho, \theta; \cdot) \in C(\Gamma)$ for $(\rho, \theta) \in G$, we have

$$\begin{aligned} g(r_j\rho, \theta) &= \int g(r_j, \varphi) K(\rho, \theta; \varphi)\, d\mu(\varphi) \\ &\to \int K(\rho, \theta; \varphi)\, d\alpha(\varphi). \end{aligned} \tag{4.8}$$

We also have $g(r_j\rho, \theta) \to g(\rho, \theta)$ as $r_j \to 1$, which gives (4.4).

COROLLARY 4.8 (HERGLOTZ THEOREM; cf. [**1**]). *If G is a standard star and g is a nonnegative function in $H(\hat{G})$, then*

$$g(r, \theta) = \int K(r, \theta; \varphi)\, d\alpha(\varphi) \tag{4.9}$$

for some nonnegative measure α with $\|\alpha\| = g(0, 0)$.

PROOF. If $g \geq 0$, then by the mean value property (3.18), $\|g_r\|_1 = g(0, 0)$ for all r.

We will show finally that the mapping (4.4) of measures α onto $H_1(G)$ is one-to-one, for which we need the following lemma.

LEMMA 4.9. *If I is an open interval in $(-\infty, \infty)$, and G is a standard star, then*

$$\lim_{r \to 1} \int_I K(r, \theta; \varphi)\, d\mu(\theta) = \begin{cases} 1 & \text{if } \varphi \in I, \\ 0 & \text{if } \varphi \notin \bar{I}. \end{cases} \tag{4.10}$$

PROOF. If $\varphi \notin \bar{I}$, then $K(r, \theta; \varphi) < \varepsilon$ for all $\theta \in I$ and all $r \geq r_0$ by condition (i) of Definition 4.6. Hence the limit is zero for $\varphi \notin \bar{I}$. The other limit follows by replacing I by $\Gamma \sim I$.

Now we show that the mapping of measures onto $H_1(G)$ is one-to-one (cf. [**7**, pp. 375, 387–389] for the case where G is a rectangle).

THEOREM 4.10. *If G is a standard star then the mapping* (4.4) *of signed measures onto $H_1(G)$ is one-to-one.*

PROOF. Let α be a finite signed measure on $(-\infty, \infty)$, and assume that for each $(r, \theta) \in G$,

$$f(r, \theta) = \int K(r, \theta; \varphi)\, d\alpha(\varphi) = 0. \tag{4.11}$$

Let I be any interval $(a, b) \subset \Gamma$ such that $\alpha(\{a\}) = \alpha(\{b\}) = 0$. By Lemma 4.9

$$\lim_{r \to 1} \int_I K(r, \theta; \varphi)\, d\mu(\theta) = \chi_I(\varphi) \quad \text{a.e. } [\alpha]. \tag{4.12}$$

Since K is bounded for $\theta \in I$, $\varphi \in \Gamma$ for every fixed $r < 1$,

$$
\begin{aligned}
0 &= \int_I f(r,\theta)\, d\mu(\theta) \\
&= \int_I \int_\Gamma K(r,\theta;\varphi)\, d\alpha(\varphi)\, d\mu(\theta) \qquad (4.13)\\
&= \int_\Gamma \int_I K(r,\theta;\varphi)\, d\mu(\theta)\, d\alpha(\varphi).
\end{aligned}
$$

The limit in (4.12) is bounded convergence in φ, so taking the limit in (4.13) as $r \to 1$ gives

$$0 = \int_\Gamma \chi_I(\varphi)\, d\alpha(\varphi) = \alpha(I). \tag{4.14}$$

There are at most a countable number of points where α has mass, so $\alpha(I) = 0$ for all I with endpoints off this countable set. Hence $\alpha \equiv 0$.

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