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ABSTRACT. Denote by $\mathcal{M}_g^r$ the locus in the moduli space of curves of genus g of those curves which have a theta-characteristic of (projective) dimension at least r . We give an upper bound for the dimension of $\mathcal{M}_g^r$ and we determine this dimension completely for $r \leq 4$. For $r \leq 4$, we prove also that a generic point in every component of $\mathcal{M}_g^r$ has a single theta-characteristic of this dimension.

0. Introduction. Let $\mathcal{M}_g$ be the moduli space of smooth, complete curves of genus g over the complex field $\mathbb{C}$. We investigate the subloci $\mathcal{M}_g^r$ of $\mathcal{M}_g$. These are defined as the loci of curves having a theta-characteristic (i.e. a line bundle L such that $L \otimes L = K_C$) of (projective) dimension at least r and of the same parity as r .

By Clifford's Theorem, it is clear that $\mathcal{M}_g^r$ is empty if $r > \frac{1}{2}(g-1)$. On the other hand, one can easily see that hyperelliptic curves have theta-characteristics of all dimensions r with $0 \leq r \leq \frac{1}{2}(g-1)$.

In [H], Harris proves

(0.1) **THEOREM (CF. [H, THEOREM (1.10)]).** *Let $X \rightarrow S$ be a family of curves and L a line bundle on X such that the restriction $L(s)$ to every fiber $X(s)$ satisfies $L^2(s) \simeq K_{X(s)}$. Then the subset of S*

$$\{s \in S \mid h^0(X(s), L(s)) \geq r+1 \text{ and } h^0(X(s), L(s)) \equiv r+1 \pmod{2}\}$$

has codimension at most $\frac{1}{2}r(r+1)$ in S at all of its points.

Combining this with the above facts, one obtains

(0.2) **THEOREM (HARRIS).** *The locus $\mathcal{M}_g^r$ is empty if and only if $r > \frac{1}{2}(g-1)$. If $r \leq \frac{1}{2}(g-1)$, then any component of $\mathcal{M}_g^r$ has codimension at most $\frac{1}{2}r(r+1)$ in $\mathcal{M}_g$.*

One could ask if the lower bound of Harris for the dimension of the components of $\mathcal{M}_g^r$ is in fact an equality. This is not always the case, even for those components of $\mathcal{M}_g^r$ whose general point corresponds to a curve with a half-canonical series without fixed points, of dimension exactly r and giving a birational morphism of C in $\mathbf{P}^r$. A counterexample is provided, for instance, by the work of Accola [A] on curves of genus $3r$ which possess a (necessarily half-canonical) simple series of dimension r .

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In this case however g is very small compared with r , as these are Castelnuovo extremal curves. In his paper, Harris asks whether the situation becomes regular when g grows. Here we give an affirmative answer for $r \leq 4$.

In the first place, we find an upper bound for the dimension of the components of $\mathcal{M}_g^r$, namely $3g - 2r + 2$. This is sharp in the sense that for every r , there is one g ($g = 2r + 1$), for which it is attained. In this case $\mathcal{M}_g^r$ is the hyperelliptic locus.

For $r = 1$ and 2 , the upper bound coincides with the lower bound of Harris. It follows from this that $\mathcal{M}_g^2$ has pure codimension 3 in $\mathcal{M}_g$, as well as the classical result that $\mathcal{M}_g^1$ is a divisor in $\mathcal{M}_g$ (see [F, B]).

For $r \geq 3$, the upper bound is sharp only in the case mentioned above, namely $g = 2r + 1$. As a consequence, $\mathcal{M}_g^3$ has codimension 6 in $\mathcal{M}_g$ when $g \geq 8$. For $r \geq 4$ the upper bound may be refined when $g \gg r$ and from this refinement the solution in case $r = 4$ follows.

We show moreover that, for $r \leq 4$ and $g \gg r$, a generic point of a component of $\mathcal{M}_g^r$ has only one half-canonical series of this dimension which is simple and that for $r = 1$ and 2 it has no fixed points.

The proof uses the deformation theory developed by Arbarello and Cornalba in [A, C1, 2] combined with some ideas inspired by recent work of Díaz [D].

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1. Definitions and preliminaries. We recall first a few well-known facts and introduce notations that we are going to use throughout the paper.

In this work, C will always denote a projective, nonsingular curve of genus g defined over the complex field $\mathbf{C}$. If F is a sheaf on C , the cohomology groups $H^i(C, F)$ will often be written $H^i(F)$. If $f: X \rightarrow S$ is a morphism of schemes, $X(s)$ will denote the fiber over $s \in S$. For any scheme S , $T_S(s)$ will mean the tangent space to S at s , T_S the tangent sheaf on S .

(1.1) *With C as above, there exist irreducible, nonsingular varieties X, S , S quasi-projective and a flat projective morphism $p: X \rightarrow S$ such that*

- (a) *Any fiber of p is a nonsingular curve of genus g and one of them is C .*
- (b) *For every s in S , the Kodaira-Spencer map $T_S(s) \rightarrow H^1(X(s), T_{X(s)})$ is an isomorphism*
- (c) *p has a section.*

For such a family, there exists a Picard scheme $\text{Pic}^d(X/S)$ (that we shall write Pic^d for short), together with a Poincaré bundle on $X \times_S \text{Pic}^d$. This parametrizes line bundles of degree d on the fibers of p .

There is also a scheme G_d^r parametrizing linear series on the fibers of p (see [A, C1, §2]). If t is a point in G_d^r corresponding to a curve C , a line bundle L on C and a subspace of dimension $r + 1$ in $H^0(L)$, then there is an exact sequence ([A, C1, pp. 17–18])

$$(1.2) \quad 0 \rightarrow \text{Hom}(W, H^0(L)/W) \rightarrow T_{G_d^r}(t) \rightarrow H^1(\Sigma_L) \rightarrow \text{Hom}(W, H^1(L)).$$

Here Σ_L denotes the sheaf of differential operators of order at most one acting on L . The space $H^1(\Sigma_L)$ is naturally identified with $T_{\text{Pic}^d}(L)$ and the last morphism is given by cup-product.

(1.3) DEFINITION. We define a scheme P by means of the following pull-back diagram

$$\begin{array}{ccc} P & \rightarrow & \text{Pic}^{g-1} \\ \downarrow & & \downarrow \\ S & \rightarrow & \text{Pic}^{2g-2} \end{array}$$

where the morphisms from Pic^{g-1} and S to Pic^{2g-2} are obtained by means of the universal property of Pic^{2g-2} by using the square of the Poincaré bundle and the dualizing sheaf respectively.

The scheme P parametrizes curves of the family p and theta-characteristics on them so it projects onto S with degree 2^{2g} . It is known [M, Theorem, p. 184], that the parity of a theta-characteristic is locally constant. Therefore, P decomposes into two parts P^0 and P^1 corresponding to even and odd theta-characteristics respectively.

We define T^r by means of the pull-back diagram

$$\begin{array}{ccc} T^r & \rightarrow & G_{g-1}^r \\ \downarrow & & \downarrow \\ P^{r+1} & \rightarrow & \text{Pic}^{g-1} \end{array}$$

where the superindex in P^{r+1} is understood modulo 2.

The scheme T^r is closed in G_{g-1}^r and parametrizes semicanonical series of dimension r on $X \rightarrow S$ whose corresponding bundle L satisfies $h^0 L \equiv r + 1 \pmod{2}$. It projects onto $\mathcal{M}_g^r \cap h(S)$, where $h: S \rightarrow \mathcal{M}_g$ is the classifying morphism induced by p .

(1.4) PROPOSITION. Let g and r satisfy $g > \frac{1}{2}(r^2 + r + 2)$. Let M be a component of $\mathcal{M}_g^r$. Then a generic point C of M cannot be a covering of a curve of genus $g \geq 1$. Moreover, if $g > \frac{1}{2}(r^2 + 3r + 2)$, $r \geq 2$, then C has only simple half-canonical series of dimension r .

PROOF. Assume the first statement false, i.e. C is a covering of degree $t \geq 2$ of a curve of genus $g' \geq 1$.

The curves of genus g which are coverings of degree t of some curve of genus g' depend on $2g - 2 - (2t - 3)(g' - 1)$ moduli (see [L, Satz 1]). Therefore, by using (0.2), one finds

$$2g - 2 \geq 2g - 2 - (2t - 3)(g' - 1) \geq 3g - 3 - \frac{1}{2}r(r + 1)$$

which contradicts the hypothesis on g .

Assume now that $g \geq \frac{1}{2}(r^2 + 3r + 2)$, $r \geq 2$, and C has a nonsimple semicanonical series of dimension r . So this series should give rise to a morphism in $\mathbf{P}^r$ which could be factored $C \rightarrow C' \rightarrow \mathbf{P}^r$, where the first morphism has degree $t \geq 2$ and C' is a rational curve contained in no hyperplane of $\mathbf{P}^r$. This latter condition implies

that the degree of C' is at least r , i.e. $\frac{1}{t}(g-1-k) \geq r$ where k is the number of fixed points of the semicanonical series. Hence

$$(1.4.a) \quad t \leq (1/r)(g-1-k) \leq (1/r)(g-1).$$

Moreover, as C is a covering of degree t of a rational curve, M is contained in the set of t -gonal curves and one has the inequality of dimensions (cf. (0.2)) $2g-2+2t-3 \geq 3g-3-\frac{1}{2}r(r+1)$. Therefore

$$(1.4b) \quad t \geq \frac{1}{2}(g+2) - \frac{1}{4}r(r+1).$$

From (1.4a) and (1.4b), in case $r > 2$ one finds

$$g \leq \frac{1}{2}(r^2 + 3r + 2)$$

which contradicts the hypothesis.

In the case $r = 2$, from (1.4.a) and (1.4.b) one finds $k = 0$, $t = \frac{1}{2}(g-1)$. Hence g is odd and for a generic point of the set of t -gonal curves of genus g (which is irreducible of dimension $3g-6$), the line bundle L giving rise to the unique linear series of degree t and dimension one satisfies $4L = K$. This cannot be true: for a hyperelliptic curve C , consider the line bundle $L_C = L_2 \otimes \mathcal{O}(\frac{1}{2}(g-5)P)$ where L_2 is the sheaf defining the g_2^1 and P is not a Weierstrass point in C . Then $4L_C \neq K$.

But by (3.1) and the irreducibility of the loci of special divisors on an hyperelliptic curve, one should be able to deform L to L_C . This would imply that $4L_C = K_C$ contradicting the choice of L_C .

The following lemma is implicit in [A,C2]. We include a proof here for the convenience of the reader.

(1.5) LEMMA (ARBARELLO-CORNALBA). *Let M be a subvariety of $\mathcal{M}_g$ of dimension at least g , $p: X \rightarrow U$ a family of curves such that the classifying map projects U onto an open dense subset of M and let*

$$(1.5.1) \quad \begin{array}{ccc} X & \xrightarrow{\quad} & S \times U \\ & \searrow & \downarrow \\ & & U \end{array}$$

be a family of birational morphisms from the fibers of p in a nonsingular algebraic surface S . Then, for a generic point u in U , the normal bundle N to the morphism $X(u) \rightarrow S$ satisfies $h^1(N) = 0$.

PROOF. The normal sheaf N to f is defined by means of the exact sequence

$$(1.5.2) \quad 0 \rightarrow T_C \rightarrow f^*T_S \rightarrow N \rightarrow 0.$$

Let D be the ramification divisor of f and N' the invertible rank one sheaf which fits in the exact sequence

$$0 \rightarrow T_C(D) \rightarrow f^*T_S \rightarrow N' \rightarrow 0.$$

There is a commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & 0 & & & H & \\
 & & \downarrow & & & \downarrow & \\
 0 & \rightarrow & T_C & \rightarrow & f^*T_S & \rightarrow & N \rightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \\
 0 & \rightarrow & T_C(D) & \rightarrow & f^*T_S & \rightarrow & N' \rightarrow 0 \\
 & & \downarrow & & & & \\
 & & T_C \otimes \mathcal{O}_D(D) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}
 \tag{1.5.3}$$

and clearly

$$H \simeq T_C \otimes \mathcal{O}_D(D) \simeq \mathcal{O}_D; \tag{1.5.4}$$

let t be a general point of U . Consider the Horikawa map $T_U(t) \rightarrow H^0(N)$ associated to the family of morphisms (1.5.1). By Lemma (1.4) in [A, C2], the image of this map intersects $H^0(H)$ in 0 and so it maps injectively in $H^0(N')$. Moreover, it is standard that the composition of the Horikawa morphism with the natural map $H^0(N) \rightarrow H^1(T_C)$ deduced from (1.5.2), is the Kodaira-Spencer map associated to $X \rightarrow U$ (see [Ho]). As t is general in U , the dimension of the image of this map is at least $\dim M$. So, because of the hypothesis on M , $h^0(CN') \geq \dim M > g$. As N' is a line bundle on a curve of genus g , this implies that it is nonspecial and so $h^1(N') = 0$. Then, from (1.5.3), (1.5.4) it follows that $h^1(N) = 0$ as stated.

(1.6) COROLLARY. *Let g be at least 6. For a generic point C of a component M of $\mathcal{M}_g^2$ any half-canonical linear series of dimension 2 on C is simple and gives rise to a morphism in $\mathbf{P}^2$ whose associated normal sheaf N satisfies $h^1(N) = 0$.*

PROOF. For $g = 6$, if L has fixed points, then C is hyperelliptic and (0.2) contradicts the genericity of C . If L has no fixed points, as $g - 1 = 5$ is prime, L is necessarily simple.

If $g \geq 7$, (1.4) gives the first assertion.

Then use (0.2) and (1.5).

2. Infinitesimal study of T^r and applications.

I. *Some considerations about the tangent space to T^r .*

Let $p: X \rightarrow S$ be a family of curves satisfying the conditions of (1.1). Let t be a point in T^r corresponding to a curve C , a theta-characteristic L on C and a subspace W of dimension $r + 1$ (of $H^0(L)$). Because of the definition of T^r (cf. (1.3)), there is a pull-back diagram:

$$\begin{array}{ccc}
 T_{T^r}(t) & \rightarrow & T_{G_{g-1}^r}(t) \\
 \downarrow & & \downarrow \\
 T_S(C) & \rightarrow & T_{\text{Pic}^{2g-2}}(C, K)
 \end{array}
 \tag{2.1}$$

As the morphism of Pic^{g-1} in Pic^{2g-2} given by tensor square is étale, it induces an isomorphism of tangent spaces. So, one finds

$$H^1(\Sigma_L) \simeq T_{\text{Pic}^{g-1}}(C, L) \simeq T_{\text{Pic}^{2g-2}}(C, K) \simeq H^1(\Sigma_K).$$

We recall that, by hypothesis (cf. (1.1.b)), $T_S(C)$ is isomorphic to $H^1(T_C)$. The image of $T_S(C)$ in $H^1(\Sigma_K)$ are those first order infinitesimal deformations K_ε of K which give the canonical sheaf on the corresponding deformation C_ε of C , i.e. those deformations of K which maintain the g sections.

Therefore, by using (1.2), diagram (2.1) becomes

$$\begin{array}{c} 0 \\ \downarrow \\ \text{Hom}(W, H^0(L)/W) \\ \downarrow \\ 0 \rightarrow T_{T^r}(t) \rightarrow T_{G_{g-1}'}(t) \\ \downarrow \quad \downarrow \\ 0 \rightarrow H^1(T_C) \rightarrow H^1(\Sigma_L) \simeq H^1(\Sigma_K) \rightarrow \text{Hom}(H^0(K), H^1(K)) \rightarrow 0 \\ \downarrow \\ \text{Hom}(W, H^1(L)) \end{array}$$

where the square is a pull-back diagram.

It can be checked that the isomorphism $H^1(\Sigma_L) \simeq H^1(\Sigma_K)$ sends a cocycle $(s_{ij}) \in H^1(\Sigma_L)$ to $(\text{Id} \otimes s_{ij} + s_{ij} \otimes \text{Id}) \in H^1(\Sigma_K)$. So, the composed map $T_{G_{g-1}'}(t) \rightarrow \text{Hom}(H^0(K), H^1(K))$ factors through $\text{Hom}(H^0(K)/W \cdot W, H^1(K))$ where $W \cdot W$ denotes the image of $W \otimes W$ in $H^0(K)$ by means of the Petri morphism.

One finds a diagram:

$$\begin{array}{ccccc} (2.2) & & & & \\ & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ & \text{Hom}(W, H^0(L)/W) = \text{Hom}(W, H^0(L)/W) & & & 0 \\ & \downarrow & & \downarrow & \downarrow \\ 0 \rightarrow T_{T^r}(t) & \rightarrow & T_{G_{g-1}'}(t) & \rightarrow & \text{Hom}(H^0(K)/W \cdot W, H^1(K)) \\ & \downarrow & \downarrow & & \downarrow \\ 0 \rightarrow H^1(T_C) & \rightarrow & H^1(\Sigma_L) \simeq H^1(\Sigma_K) & \rightarrow & \text{Hom}(H^0(K), H^1(K)) \rightarrow 0 \\ & & \downarrow & & \downarrow \\ & & \text{Hom}(W, H^1(L)) & \rightarrow & \text{Hom}(W \cdot W, H^1(K)) \\ & & & & \downarrow \\ & & & & 0 \end{array}$$

This is exact, exactness in the upper row being deduced from the fact that the left lower square is a pull-back.

We shall repeatedly consider the following situation:

(2.3) Let t be a point of T^r corresponding to a curve C , a theta-characteristic L on C , and an $(r+1)$ -dimensional subspace W of $H^0(L)$. Let D be the fixed part of the series corresponding to W , k its degree, q an equation for D , $L' = L \otimes \mathcal{O}_C(-D)$, W' the

subspace of $H^0(L')$ whose image by the natural inclusion $\cdot q: H^0(L') \rightarrow H^0(L)$ is W . Let $t' = (C, L', W')$ be the corresponding point in G_{g-1-k}^r . Denote by f the morphism of C in $\mathbf{P}^r$ associated to W' .

Consider the following diagram (cf. [A, C1, (4.1)]):

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \uparrow \\
 & & & & H^0(K) \\
 & & & \uparrow & \\
 & & & H^0(K \otimes \Sigma_{L'}^*) & \\
 & & \uparrow & & \\
 & & \text{Ker } P & \xrightarrow{m} & H^0(2K) \\
 & & & & \uparrow \\
 & & & & 0
 \end{array}
 \quad
 \begin{array}{ccc}
 W' \otimes q^2 W' & \rightarrow & W' \otimes H^0(K - L') \xrightarrow{P} \\
 \parallel & & \parallel \\
 W' \otimes q^2 W' & \rightarrow & W' \otimes H^0(K - L') \xrightarrow{m_1}
 \end{array}$$

Here P is the Petri morphism, the vertical sequence in the right is exact, m_1 is the dual of the natural contraction map and m is obtained from m_1 by restriction.

Consider also the following diagram of exact sequences (cf. [A, C1, 5.1])

$$\begin{array}{ccccccc}
 0 & & 0 & & & & \\
 \downarrow & & \downarrow & & & & \\
 \mathcal{O}_C & = & \mathcal{O}_C & & & & \\
 \downarrow & & \downarrow & & & & \\
 0 \rightarrow \Sigma_{L'} & \rightarrow & W'^* \otimes L' & \rightarrow & N \rightarrow 0 & & \\
 \downarrow & & \downarrow & & \parallel & & \\
 0 \rightarrow T_C & \rightarrow & f^* T_{\mathbf{P}^r} & \rightarrow & N \rightarrow 0 & & \\
 \downarrow & & \downarrow & & & & \\
 0 & & 0 & & & &
 \end{array}$$

where N is the normal sheaf to f , the vertical sequence in the middle is obtained by pulling-back to C the Euler sequence in $\mathbf{P}^r$ and the morphism of $\Sigma_{L'}$ in $W'^* \otimes L' = \text{Hom}(W', L')$ is defined by contraction.

Taking homology one obtains:

$$\begin{array}{ccccccc}
 H^1(\mathcal{O}_C) & = & H^1(\mathcal{O}_C) & & & & \\
 \downarrow & & \downarrow p^* & & & & \\
 H^0(N) & \rightarrow & H^1(\Sigma_{L'}) & \rightarrow & \text{Hom}(W', H^1(L')) & \rightarrow & H^1(N) \rightarrow 0 \\
 \parallel & & \downarrow & & \downarrow & & \parallel \\
 H^0(N) & \rightarrow & H^1(T_C) & \rightarrow & H^1(f^* T_{\mathbf{P}^r}) & \rightarrow & H^1(N) \rightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & & 0 & & 0 & &
 \end{array}$$

Therefore $H^1(f^*T_{P^r})$ is identified with the dual of $\text{Ker } P$. Moreover, because of (1.2), the image of $T_{G_{g-1-k}^r}(t)$ in $H^1(\Sigma_{L'}) = T_{\text{Pic}^{g-1-k}}(L')$ is the image of the morphism above from $H^0(N)$ to $H^1(\Sigma_{L'})$. So one obtains the following diagram [A, C1, p. 35]:

$$(2.5) \quad \begin{array}{ccccccc} T_{G_{g-1-k}^r}(t') & \xrightarrow{h} & H^1(\Sigma_{L'}) & \rightarrow & \text{Hom}(W', H^1(L')) & \rightarrow & H^1(N) \rightarrow 0 \\ \parallel & & \downarrow g_{L'} & & \downarrow & & \parallel \\ T_{G_{g-1-k}^r}(t) & \rightarrow & H^1(T_C) & \xrightarrow{m^*} & (\text{Ker } P)^* & \rightarrow & (\text{Ker } m)^* \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Suppose now that the point t corresponds to a complete theta-characteristic of dimension r , i.e. $W = H^0(L)$ and hence $W' = H^0(L')$.

Assume that t is a generic point of a component of T^r . Then, up to a finite base-change, there are k sections $s_1 \cdots s_k$ of $p: X \rightarrow S$ defined in a neighborhood of C in S such that, when restricted to the image of T^r in S they give rise to the fixed points of the theta-characteristic in the fibers of p .

One obtains then a commutative diagram:

$$\begin{array}{ccccc} T^r & \rightarrow & G_{g-1-k}^r & \rightarrow & G_{g-1}^r \\ \downarrow & & & & \downarrow \\ S & \rightarrow & & & \text{Pic}^{g-1} \end{array}$$

Taking tangent spaces, one has a factorization of one of the morphisms in (2.2)

$$(2.6) \quad \begin{array}{ccccccc} & 0 & & 0 & & 0 & & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & T_{T^r}(t) & \rightarrow & T_{G_{g-1-k}^r}(t') & \rightarrow & T_{G_{g-1}^r}(t) & \rightarrow & \text{Hom}(H^0(K)/W \cdot W, H^1(K)) \\ & j \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \rightarrow & H^1(T_C) & \rightarrow & H^1(\Sigma_{L'}) & \simeq & H^1(\Sigma_L) \simeq H^1(\Sigma_K) & \rightarrow & \text{Hom}(H^0(K), H^1(K)) \rightarrow 0 \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & \text{Hom}(W', H^1(L')) & \rightarrow & \text{Hom}(W, H^1(L)) & \rightarrow & \text{Hom}(W \cdot W, H^1(K)) \\ & & & & \downarrow & & & & \downarrow \\ & & & & H^1(N) & & & & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

where the isomorphism $H^1(\Sigma_{L'}) \simeq H^1(\Sigma_L)$ is the differential of the isomorphism from Pic^{g-1-k} to Pic^{g-1} given by tensor product with the sheaf

$$\mathcal{O}_X(s_1(S) + \cdots + s_k(S)).$$

As the upper left rectangle is a pull-back, so is the upper left square.

From the commutative diagram

$$\begin{array}{ccccc} \text{Pic}^{g-1-k} & \rightarrow & \text{Pic}^{g-1} & \rightarrow & \text{Pic}^{2g-2} \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

one obtains, by taking tangent spaces:

$$\begin{array}{ccccc}
 & & & H^1(T_C) & \\
 & & & \downarrow i & \ddots \\
 H^1(\Sigma_{L'}) & \cong & H^1(\Sigma_L) & \cong & H^1(\Sigma_K) & \ddots & \text{Id} \\
 g_{L'} \downarrow & & g_L \downarrow & & g_K \downarrow & \ddots & \\
 H^1(T_C) & = & H^1(T_C) & = & H^1(T_C) & \ddots &
 \end{array}$$

Therefore i (interpreted through the isomorphisms in the upper row) is a section of $g_{L'}$. Hence, with the notations of (2.5), (2.6)

$$(2.7) \quad j(T_{T^r}(t)) \subset g_{L'}(h(T_{G'_{g-1-k}}(t))) = \text{Ker } m^*$$

where the last equality follows from (2.5).

Then, one finds

$$\begin{aligned}
 (2.8) \quad \dim T^r &\leq \dim T_{T^r}(t) \leq \dim j(T_{T^r}(t)) \leq \dim \text{Ker } m^* \\
 &= 3g - 3 - \dim \text{Ker } P + \dim \text{Ker } m.
 \end{aligned}$$

We recall now that, by hypothesis, $2L = K$, i.e. $K - L = L$. Therefore, if s and s' are elements in W' , $s \otimes q^2 s' - s' \otimes q^2 s$ belongs to $\text{Ker } P$. In particular, $\text{Ker } P$ has dimension at least $\frac{1}{2}r(r+1)$, as it contains the independent elements $s_i \otimes q^2 s_j - s_j \otimes q^2 s_i$, $0 \leq i < j \leq r$, for a basis s_i of W' .

(2.9) DEFINITION. *The elements of $\text{Ker } P$ of the form $s \otimes q^2 s' - s' \otimes q^2 s$ will be called decomposable. The set of decomposable elements will be denoted by G and its projectivization by G' .*

We point out that the point in G' coming from $s \otimes q^2 s' - s' \otimes q^2 s$ depends only on the one-dimensional linear subseries of (L', W') generated by s and s' . In fact G' is isomorphic to the Grassmannian of lines in $\mathbf{P}^r$ canonically immersed in a linear subspace of dimension $\frac{1}{2}r(r+1) - 1$ of the projectivization of $\text{Ker } P$.

(2.10) LEMMA. *Let t be a generic point of a component of T^r corresponding to a complete theta-characteristic of dimension r . Assume $\text{Ker } m$ intersects the linear span $\langle G \rangle$ of G in 0 (cf. (2.9)). Then $\dim \text{Ker } P = \frac{1}{2}r(r+1) + \dim \text{Ker } m$ and the image $T_{T^r}(t)$ in $H^1(T_C)$ is the kernel of m^* .*

PROOF. As $\langle G \rangle \cap \text{Ker } m = 0$, $\dim \text{Ker } P \geq \dim \langle G \rangle + \dim \text{Ker } m = \frac{1}{2}r(r+1) + \dim \text{Ker } m$. Then, (2.8) gives

$$\dim T^r \leq \dim \text{Ker } m^* \leq 3g - 3 - \dim \text{Ker } P + \dim \text{Ker } m \leq 3g - 3 - \frac{1}{2}r(r+1).$$

On the other hand, by (0.1), $\dim T^r \geq 3g - 3 - \frac{1}{2}r(r+1)$. Therefore all the inequalities are equalities and the result follows using (2.7).

II. Nonexistence of fixed points.

(2.11) PROPOSITION. *If the generic point of a component of T^r is a complete half-canonical series of dimension r such that the associated morphism m (cf. (2.4)) satisfies $\text{Ker } m = 0$, then the half-canonical series has no fixed points.*

PROOF. The hypothesis in (2.10) is satisfied. Therefore, taking into account that $(\text{Ker } m)^* = H^1(N)$ (cf. (2.5)), diagram (2.6) may be completed to:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow T_{T'}(t) & \rightarrow & T_{G'_{g-1-k}}(t') & \rightarrow & \text{Hom}(H^0(K)/W \cdot W, H^1(K)) & & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow H^1(T_C) & \rightarrow & H^1(\Sigma_{L'}) & \rightarrow & \text{Hom}(H^0(K), H^1(K)) \rightarrow 0 & & \\
 m^* \downarrow & & \downarrow & & \downarrow & & \\
 0 \rightarrow (\text{Ker } P)^* & \rightarrow & \text{Hom}(W, H^1(L')) & \rightarrow & \text{Hom}(W \cdot W, H^1(K)) & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & & 0 & & 0 & &
 \end{array}$$

Here the 0 in the lower row is obtained by diagram chasing, using the fact that the upper left square is a pull-back.

By the Snake lemma and the exactness of the left column, $\text{coker } 1 = 0$. By (2.10), $\dim \text{Ker } P = \frac{1}{2}r(r+1)$. Then, a computation using the first row and column gives

$$\dim T_{G'_{g-1-k}}(t') = 4g - 3 - (r+1)^2.$$

From the central column

$$\dim T_{G'_{g-1-k}}(t') = 4g - 3 + (r+1)h^1(L') = 4g - 3 - (r+1)^2 - k(r+1)$$

where the last equality follows by Riemann-Roch because of the hypothesis of the series being complete. Therefore $k = 0$.

III. An upper bound in the dimension of the components of $\mathcal{M}_g^r$.

(2.12) LEMMA. *Completing notations in (2.3), let V' be a two-dimensional linear subspace of W' , F the fixed part of the corresponding one-dimensional series and R the ramification divisor of the morphism $C \rightarrow \mathbf{P}^1$ it induces. Then, the image by m (cf. (2.4)) of the one-dimensional subspace of $\text{Ker } P$ associated to this series (cf. (2.9)) is the one-dimensional linear space in $H^0(2K)$ corresponding to the divisor $R + 2D + 2F$.*

PROOF. Choose a basis a, b of V' so that b has no multiple zeros outside F . Write $a = fb$ with f a meromorphic function. Then f , as a morphism of C in $\mathbf{P}^1$, is unramified at infinity.

By definition (cf. (2.4)), m is obtained from m_1 by restriction and m_1 is the dual of the natural contraction

$$H^1(\Sigma_{L'}) \rightarrow \text{Hom}(W', H^1(L')) = \text{Hom}(W' \otimes H^0(K - L'), H^1(K)).$$

More explicitly, once an affine covering U_i of C has been chosen, any element in $H^1(\Sigma_{L'})$ is represented by a cocycle (s_{ij}) . Then $m_1^*((s_{ij}))(w) = (s_{ij}(w)) \in H^1(L')$ if $w \in W'$ or $m_1^*((s_{ij}))(w \otimes \omega) = (s_{ij}(w)\omega) \in H^1(K)$ if $w \in W'$, $\omega \in H^0(K - L')$.

In particular, if v_{ij} is the derivation associated to s_{ij} , then

$$m_1^*(s_{ij})(a \otimes q^2b - b \otimes q^2a) = (s_{ij}(a) \cdot q^2b - s_{ij}(b)q^2a) = v_{ij}(f)q^2b^2.$$

Therefore, the dual of the restriction of m to the subspace $a \otimes q^2b - b \otimes q^2a$ operates as contraction with f

$$\begin{aligned} H^1(T_C) &\rightarrow \langle a \otimes q^2b - b \otimes q^2a \rangle^* \simeq H^1(K), \\ (v_{ij}) &\rightarrow v_{ij}(f)q^2b^2. \end{aligned}$$

Hence, $m(a \otimes q^2b - b \otimes q^2a) = (df)q^2b^2$. Denote by D_a and D_b the divisors of zeros of a and b respectively.

By the choice of a and b , df is a meromorphic differential whose divisor of zeros is R and whose divisor of poles is twice the divisor of poles of f , i.e. $2(D_b - F)$. Therefore the divisor of $df \cdot q^2b^2$ is

$$R - 2(D_b - F) + 2D + 2D_b = R + 2F + 2D$$

as asserted.

(2.13) THEOREM. Any component M of $\mathcal{M}_g^r$ has dimension at most $3g - 2r - 2$. For $r \geq 3$ equality holds only for $g = 2r + 1$ and in this case M is the hyperelliptic locus. For $r \geq 4$ and $g \geq \max(12r - 22, \frac{1}{2}(r^2 + 3r + 2))$, one has $\dim M \leq 3g - 4r + 3$.

(2.14) COROLLARY. For $r = 3$ and $g \geq 8$ and for $r = 4$ and $g \geq 26$, at a generic point of a component of T^r projecting onto M , $\text{Ker } m \cap \langle G \rangle = 0$ (cf. (2.9), (2.4) for the notations).

PROOF OF (2.13), (2.14). A general point of M is a curve which has a complete semicanonical series of dimension $r + 2k$, $k \geq 0$. If $k > 0$, M is a component of $\mathcal{M}_g^{r+2k}$. As the upper bound we try to prove is a decreasing function of r , we may assume $k = 0$.

Let T be a component of T^r projecting onto M and t a generic point of T . By (2.12), the set G defined in (2.9) cuts $\text{Ker } m$ in 0. Hence G' does not intersect the linear subspace $L = \mathbf{P}(\text{Ker } m)$. Therefore

$$\dim \text{Ker } m + \dim G' \leq \dim \text{Ker } P - 1,$$

As G' is a Grassmannian of lines in $\mathbf{P}^r$, it has dimension $(2r - 1)$. So,

$$\dim \text{Ker } m \leq \dim \text{Ker } P - 2(r - 1) - 1$$

and (2.8) gives

$$(2.13.1) \quad \dim j(T_{T^r}(t)) \leq 3g - 3 - \dim \text{Ker } P + \dim \text{Ker } m \leq 3g - 2r - 2$$

which proves the first assertion in (2.13).

Assume now

(a) $r \geq 3$, $g > 2r + 1$ and $\dim M = 3g - 2r - 2$ or

(b) $r \geq 4$, $g \geq \max(\frac{1}{2}(r^2 + 3r + 2), 12r - 22)$, and $\dim M > 3g - 4r + 3$.

Condition (a) implies that L has the maximal dimension of a linear subspace in $\mathbf{P}$ not intersecting G' . Therefore the linear space generated by L and a generic point in G' intersects G' in other points.

Condition (b) implies (cf. (2.13.1)) that $\dim \text{Ker } m \geq \dim \text{Ker } P - 4r + 7$. Therefore L meets the variety of chords of G' (which has dimension $4r - 7$).

In both cases there is a pair of points in G having the same image by m . In case (a) one of the points in the pair (and hence also the other) may be assumed to be generic in G .

Using (2.12) we find two one-dimensional linear subseries g_1, g_2 of (L, W) with fixed parts F_1, F_2 and ramification divisors R_1, R_2 such that

$$(2.13.2) \quad R_1 + 2F_1 = R_2 + 2F_2.$$

Let R be the greatest effective divisor contained in R_1 and R_2 and write $R_i = R + A_i$. Then $A_1 + 2F_1 = A_2 + 2F_2$ and A_1, A_2 have no points in common, so $A_2 \leq 2F_1$ (2.13.3).

Consider the morphism $(f_1, f_2): C \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$ obtained as the product of the two morphisms associated to the two one-dimensional linear series considered above.

This morphism is ramified at the points shared by the two ramification divisors of f_1 and f_2 , i.e. at R . Hence, one finds a diagram defining N' (cf. (1.5.3)):

$$\begin{array}{ccccccc} 0 & \rightarrow & T_C & \rightarrow & f_1^*T_{\mathbf{P}^1} \oplus f_2^*T_{\mathbf{P}^1} & \rightarrow & N \rightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \rightarrow & T_C(R) & \rightarrow & f_1^*T_{\mathbf{P}^1} \oplus f_2^*T_{\mathbf{P}^1} & \rightarrow & N' \rightarrow 0 \\ & & \downarrow & & & & \downarrow \\ & & T_C \otimes \mathcal{O}_R(R) & & & & 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

By Hurwitz's formula, $f_i^*T_{\mathbf{P}^1} \simeq T_C(R_i)$. Therefore, computing with the lower row, one finds

$$N' = T_C(R_1) \otimes T_C(R_2) \otimes (T_C(R))^v = T_C(R_1 + A_2).$$

Hence

$$h^1(N') = h^0(K - N') = h^0(2K - R_1 - A_2) \geq h^0(2K - R_1 - 2F_1) \geq 1$$

where the first inequality comes from (2.13.3) and the last from the fact (see (2.12)) $R_1 + 2F_1 + 2D \equiv 2K$.

As $\dim M > g$ and $h^1(N') = h^1(N)$, this means by (1.5) that the morphism (f_1, f_2) is composed with an involution.

In case (b) (1.4) asserts that C is not a covering of a curve of genus $g \geq 1$ and that the morphism $C \rightarrow \mathbf{P}^r$ induced by the half-canonical series, is simple.

In case (a) a proof as in (1.4) using the conditions $\dim M = 3g - 2r - 2$ and $g > 2r + 1$ gives the same result.

The morphism f_i is obtained from a one-dimensional linear subseries of $(L, H^0(L))$. Equivalently, f_i is obtained by composing f with a projection from a codimension 2 subspace X_i in $\mathbf{P}^r$. The condition for (f_1, f_2) to be composed with an involution of degree k is that f_1 and f_2 should be composed with the same involution. This means that the intersection of the hyperplanes generated by X_i and a generic point of C contains $k - 1$ further points.

In case (a), by the genericity of X_1 and the principle of general position, this implies $k \leq r - 1$. As the Hurwitz scheme of coverings of degree $r - 1$ and genus g of $\mathbf{P}^1$ has dimension $2g + 2(r - 1) - 2$, one obtains

$$3g - 2r - 2 + 2(r - 1) = \dim M + \dim Gr(r - 2, \mathbf{P}^r) \leq 2g + 2(r - 1) - 2$$

so $r \geq g/2$ which contradicts (a).

Consider now case (b). We are assuming that the morphism (f_1, f_2) given by the two linear subseries g_1, g_2 of $(L, H^0(L))$ considered above may be factored $C \xrightarrow{1} \mathbf{P}^1 \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$ where the first morphism has degree k and the second bidegree (n_1, n_2) with $kn_i \leq g - 1$. Hence the dimension of M must be at most the dimension of the set of k -gonal curves, namely $2g + 2k - 5$.

Assume $n_i \geq 3$, then $k \leq \frac{1}{3}(g - 1)$ and $\dim M \leq 2g + \frac{2}{3}(g - 1) - 5$ which contradicts (b).

If $n_1 = n_2 = 1$, then $R_1 = R_2$ and by (2.13.2) $F_1 = F_2$. So, $g_1 = g_2$ which is not the case.

If $n_1 = n_2 = 2$, then g_1, g_2 are one-dimensional subseries of $1^*H^0(\mathcal{O}_{\mathbf{P}^1}(2)) + F_1$ and $1^*H^0(\mathcal{O}_{\mathbf{P}^1}(2)) + F_2$ respectively. As, by hypothesis, the line bundle for both series is the same, one obtains $F_1 \equiv F_2$. If $F_1 = F_2$, because $1^*H^0(\mathcal{O}_{\mathbf{P}^1}(2))$ is 3-dimensional, g_1 and g_2 would share a section. But this implies that the line in $\langle G' \rangle$ joining the points corresponding to g_1 and g_2 is entirely contained in G' . By assumption, the image by m of the 2-plane of G corresponding to this line is a line in $H^0(2K)$, so G would cut $\text{Ker } m$ and this contradicts (2.11).

If $F_1 \neq F_2$, then $h^0(F_1) \geq 2$ and C is $(g - 1 - 2k)$ -gonal. Therefore, by (b)

$$2g + 2 + 2(g - 1 - 2k) - 5 > 3g - 4r + 3, \quad 2g + 2 + 2k - 5 > 3g - 4r + 3.$$

But these inequalities are incompatible with (b).

If $n_1 = 1, n_2 = 2$, then $g_1 = 1^*H^0\mathcal{O}_{\mathbf{P}^1}(1) + F_1, g_2 \subset 1^*H^0\mathcal{O}_{\mathbf{P}^1}(2) + F_2$; so $R_2 = R_1 + A_1 + A_2$ where $A_i \in 1^*H^0\mathcal{O}_{\mathbf{P}^1}(1)$ are the pull-back of the ramification points of the double covering $\mathbf{P}^1 \rightarrow \mathbf{P}^1$. From (2.13.2) $A_1 + A_2 + 2F_2 = 2F_1$. As A_1, A_2 have disjoint supports, this implies that all points in the support of A_i have even multiplicity in A_i and are in the support of R_1 and 1 is ramified over at most $2g + k - 2$ distinct points. Hence C depends on at most $2g + k - 5$ moduli. As $k \leq \frac{1}{2}(g - 1)$, this contradicts (b) and ends the proof of (2.13).

We point out that, in case $r = 3$ and $g \geq 8$ and in case $r = 4$ and $g \geq 26$, we have proved that $\text{Ker } m$ intersects the variety of chords of G in 0 and in both cases this chordal variety coincides with the span $\langle G \rangle$ of G in $\text{Ker } P$. Hence (2.14) is also proved.

(2.15) COROLLARY. *Let M be a component of $\mathcal{M}_g^r$, C a generic point in M . If $r \leq 3$ or $r = 4$ and $g \geq 38$, then C has no half-canonical linear series of dimension greater than r .*

PROOF. Assume that C has a semicanonical series of dimension greater than r . Then M would be contained in a component of $\mathcal{M}_g^{r+1}$ or $\mathcal{M}_g^{r+2}$. So, by (2.13), $\dim M \leq 3g - 2r - 4$, and $\dim M \leq 3g - 17$ if $r = 4$ and $g \geq 38$. But this contradicts (0.2).

IV. Uniqueness of the half-canonical series.

(2.16) THEOREM. For $r = 1$, $r = 2$ and $g \geq 6$, $r = 3$ and $g \geq 9$ or $r = 4$ and $g \geq 38$, a generic point of any component M of $\mathcal{M}_g^r$ has only one half-canonical series of dimension r .

PROOF. Let M be a component of $\mathcal{M}_g^r$ with g and r satisfying the hypothesis. Let C be a generic point in M . From (2.15), a half-canonical linear series of dimension r on C is complete. Assume C had two of them, then they would correspond to two different line bundles L_1 and L_2 on C . Let t_1 and t_2 be the corresponding points in T^r . By the genericity of C in M and the fact that the image in $H^1(T_C)$ of the tangent spaces to T^r at both points has dimension equal to the dimension of M (cf. (2.8), (2.10), (2.14)), these images must be the same. Moreover they are the kernels of the corresponding morphisms $m = m_{L_i}$ (cf. (2.10)). By duality, the images of the morphisms m_{L_i} are also the same.

From (2.5) $(\text{Ker } m)^*$ is identified with $H^1(N)$ and this is zero when $r = 1$ and also when $r = 2$ (cf. (1.6)). When $r = 3$ or 4 , $\text{Ker } m$ intersects the space $\langle G_{L_i} \rangle$ in 0 (cf. (2.14)).

For $r = 1$ or 2 , $\langle G_{L_i} \rangle = G_{L_i}$. For $r = 3$ (resp. 4), G_{L_i} has dimension 4 (resp. 6) and $\langle G_{L_i} \rangle$ has dimension 6 (resp. 10). As $\langle G_{L_i} \rangle \cap \text{Ker } m = 0$, these are also the dimensions of the images of G_{L_i} and $\langle G_{L_i} \rangle$ by m . Therefore G_{L_1} and G_{L_2} must intersect. By (2.12), this means that there are one-dimensional linear subseries of $(L_i, H^0(L_i))$ such that the corresponding fixed parts F_i and ramification divisors R_i satisfy

$$(2.16.1) \quad R_1 + 2F_1 + 2D_1 = R_2 + 2F_2 + 2D_2,$$

where D_i denotes the fixed part of $(L_i, H^0(L_i))$. Moreover for $r = 1$ or 2 the one-dimensional series may be assumed to be generic in $H^0(L_i)$, so $F_i = 0$.

This pair of linear series gives rise to the morphism $(f_1, f_2): C \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$ ramified over the divisor R of points shared by R_1 and R_2 . All we need to prove is that this morphism is birational. Then the proof is finished as in (2.13) by application of (1.5).

By (1.4), (f_1, f_2) is not composed with a nonrational involution.

In case $r = 1$ and 2 , $D_1 = D_2 = 0$ by (2.11) and we found already $F_1 = F_2 = 0$. If the morphism were composed with a rational involution, then $L_1 = L_2$ contradicting the hypothesis.

We study now the case $r = 3$, the case $r = 4$ being similar will be left to the reader. Assume (f_1, f_2) could be factored as $C \xrightarrow{1} \mathbf{P}^1 \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$ where 1 has degree k and the second morphism has bidegree (n_1, n_2) .

If $n_1 = n_2 = 1$, then $R_1 = R$ and from (2.16.1) $F_1 + D_1 = F_2 + D_2$. As

$$L_i = 1^* \mathcal{O}_{\mathbf{P}^1}(1) + F_i + D_i,$$

this gives $L_1 = L_2$ and contradicts the hypothesis.

If $n_1 = 1$, $n_2 \geq 2$, then $R_2 = R_1 + \sum A_i$ where each A_i is the divisor of a fiber of L and there are at least two different A_i in the summation. Now (2.16.1) is

$$2F_1 + 2D_1 = \sum A_i + 2F_2 + 2D_2,$$

therefore all points in A_i are counted with multiplicity at least 2 and so they appear in the ramification divisor of 1. It follows that the set of coverings of $\mathbf{P}^1$ such as 1 depend on at most $2g + k - 5$ moduli. As $k \leq \frac{1}{2}(g - 1)$ and $\dim M \geq 3g - 9$, this implies $g \leq 9$ and in this case $k = 4$. Now for $g = 9$, one would have $L_2 = 1^* \mathcal{O}_{\mathbf{P}^1}(2)$. Therefore C would be contained in a quadric cone in $\mathbf{P}^3$. But this is impossible by a moduli count (cf. (0.2) and [A, C3, Lemma (3.13)]).

If $n_1, n_2 \geq 2$, then $L_i = \pi^* \mathcal{O}_{\mathbf{P}^1}(2) + T_i$ where T_i is effective. As $L_1 \neq L_2$ and $2L_1 = K = 2L_2$, $T_1 \neq T_2$ and $2T_1 \equiv 2T_2$. So C is $2(g - 1 - 2k)$ -gonal. This implies $2g - 5 + 4(g - 1 - k) \geq 3g - 9$. As C is also k -gonal, one finds $2g - 5 + 2k \geq 3g + 9$. Hence $k \geq (g - 4)/2$. This inequality together with the preceding one gives $g \leq 10$ and if $g = 10$ for all trigonal curves satisfy $6g_3^1 = K$ which is not the case (see the proof of (1.4)). The case $g = 9$ may be discarded as above.

V. Conclusions for $r \leq 4$.

(2.17) THEOREM. *The locus $\mathcal{M}_g^1$ (resp. $\mathcal{M}_g^2$) has pure codimension 1 (resp. 3) in $\mathcal{M}_g$ if $g \geq 3$ (resp. $g \geq 5$) and a generic point of any of its components is a curve which has only one half-canonical series of dimension 1 (resp. 2 if $g \geq 6$). Moreover this half-canonical series is not composed with an involution (resp. if $g \geq 6$) and has no fixed points.*

PROOF. The dimensionality statement follows from (0.2) and (2.13). The uniqueness of the half-canonical series from (2.16).

For $r = 1$, (1.4) says that the series cannot be composed with a nonrational involution. If it were composed with a rational involution, then the dimension of the series would be at least 2, contradicting (2.15). For $r = 2$ the simplicity of the series is contained in (1.4).

From (2.5), the condition $\text{Ker } m = 0$ is equivalent to $h^1(N) = 0$. From (1.6) this is satisfied for $r = 2$ and it is obviously satisfied for $r = 1$. Then (2.11) gives the nonexistence of fixed points.

We have obtained similar results for $r = 3$ and 4 that we sum up in the following theorem. We point out however that the bounds given on the genus for $r = 4$ are not the best possible and could be improved by ad hoc methods.

(2.18) THEOREM. *The locus $\mathcal{M}_g^3$ (resp. $\mathcal{M}_g^4$) has pure codimension 6 (resp. 10) if $g \geq 8$ (resp. 26). If $g \geq 9$ (resp. $g \geq 38$), a generic point in a component of this locus has only one half-canonical series of dimension 3 (resp. 4) and this gives rise to a birational morphism in $\mathbf{P}^3$ (respectively $\mathbf{P}^4$).*

3. Appendix. Irreducibility of G_d^1 . We include a proof of this fact here because we have not been able to find a proper reference in the literature. Denote by $\mathcal{M}_{g,d}^1$ the locus of d -gonal curves in $\mathcal{M}_g$.

(3.1) Let $p: X \rightarrow S$ be a family as in (1.1). Choose a d such that $\rho(g, d, 1) = 2d - g - 2 < 0$. Then there is a single component of G_d^1 over every component of the pull-back of $\mathcal{M}_{g,d}^1$ to S .

PROOF. We shall assume that this is not the case for a certain component M of the pull-back of $\mathcal{M}_{g,d}^1$ to S and reach a contradiction.

(3.2) *It is known that G_d^1 is nonsingular and has dimension $2d + 2g - 5$ if $g \geq 2$ (cf. [A, C1, p. 35]).*

As the set of d -gonal curves $\mathcal{M}_{g,d}^1$ is irreducible of dimension $2g + 2d - 5$ if $g \geq 2$ and a generic d -gonal curve has only one linear series g_d^1 [A, C2, Theorem 2.6], there is exactly one component of G_d^1 projecting onto M .

Consider a component G of G_d^1 projecting to but not onto M . We claim that a generic point of G is a linear series without fixed points. Otherwise if k were the number of fixed points of a generic series in G , then $\dim G = \dim G_{d-k}^1 + k$ and this contradicts (3.1).

Replace S by the image of G in the S above by means of the natural map and denote by q the morphism $G \xrightarrow{q} S$. Now $\dim S \leq \dim M - 1 \leq 2g + 2d - 6$ and therefore, by (3.2), the dimension of the fibers of q is $a \geq 1$.

Write $G \times_S G = \bigcup T_i$, the irreducible components of the product. The fibers of the surjective morphism $G \times_S G \rightarrow S$ are the product of the fibers of q , hence their generic dimension is $2a$. Therefore there is one T_i , say T , projecting onto S with generic fiber of dimension $2a$. Consider the pull-back diagram:

$$\begin{array}{ccc} & q_1 & \\ T & \rightarrow & G \\ q_2 \downarrow & & \downarrow q \\ G & \rightarrow & S \\ & q & \end{array}$$

As T projects onto S , so does the image of T by q_2 . Therefore the dimension of the generic fiber of q_1 is at most the dimension of the fibers of q . Hence,

$$\begin{aligned} \dim S + 2a &= \dim T \leq \dim q_1(T) + \dim \text{fiber } q_1 \leq \dim q_1(T) + \dim \text{fiber } q \\ &= \dim q_1(T) + a \leq \dim G + a = \dim S + 2a. \end{aligned}$$

Hence $q_1(T) = G$ and similarly $q_2(T) = G$ and a generic point of T corresponds to a pair of linear series which have no fixed points (as this happens for the generic point in G). Moreover, as $\dim T = G + 2a > \dim G$, T is not contained in the diagonal of $G \times G$ and the two linear series in the pair are different.

Consider the morphism $f: C \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$ associated to this pair of linear series.

Assume f were birational. Then, by [A, C2, Proposition (2.4)], $\dim T = g + 4d - 7$. Hence $\dim G = g + 4d - 7 - a$. This, together with (3.2), gives $1 \leq a = 2d - g - 2$ contradicting the hypothesis $\rho < 0$.

Therefore f is composed with an involution, i.e., f may be factored $C \xrightarrow{1} C' \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$ where 1 has degree $m \geq 2$, C' has genus g' and the two rulings of $\mathbf{P}^1 \times \mathbf{P}^1$ cut linear series on C' of degree d/m whose pull-back to C are the two series on C considered above. Hence

$$\dim T \leq \dim T' + [2g - 2 - m(2g' - 2)]$$

where T' is a component of $G_{d/m}^1 \times G_{d/m}^1$ for curves of genus g' whose general point gives rise to a birational morphism of C' in $\mathbf{P}^1 \times \mathbf{P}^1$ and the second summand is the number of moduli of an m -cover of C' of degree g .

If $g' \geq 2$, by [A, C2, Proposition (2.4)] $\dim T' = g' + 4d/m - 7$.

Hence $\dim G = \dim T - a \leq 2g + (1 - 2m)(g' - 1) + 4d/m - 9 < 2g + 2d - 5$ and this contradicts (3.2).

If $g' = 0$, $C' = \mathbf{P}^1$. Then $m < d$ because, otherwise $g_d^1 = \pi^* \mathcal{O}_{\mathbf{P}^1}(1) = h_d^1$.

Hence

$$\dim G \leq \dim G_m^1 + \dim G_{d/m}^1(\mathbf{P}^1) = 2g + 2m - 5 + 2(d/m - 1) < 2g + 2d - 5.$$

Similarly, if $g' = 1$, C' is elliptic and,

$$\dim G \leq \dim m_1 + \dim G_{d/m}^1(C') + 2g - 2 = 2g + 2d/m < 2g + 2d - 5.$$

In both cases this contradicts (3.2).

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