TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 303, Number 2, October 1987

TOWERS AND INJECTIVE COHOMOLOGY ALGEBRAS

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ABSTRACT. Let Y be a space of finite type such that H^*Y is injective as an unstable algebra over the Steenrod algebra A and such that $\bar{H}^*Y$ is A-unbounded. Let X be a simply connected p-complete space. Then any map of A-algebras $f\colon H^*\Omega X \to H^*Y$ can be realized as a map of spaces.

Let Y be a space such that $H^*(Y, \mathbf{F}_p)$ is injective in the category of unstable modules over the mod p Steenrod algebra A. The purpose of this paper is to demonstrate that the functor $[Y,\ ]$ of homotopy classes of maps out of Y is very rigid.

Let **AU** be the category of unstable algebras over A and let **U** be the category of unstable modules over A. If X is a space, then $H^*(X, \mathbf{F}_p) = H^*X$ is an object of **UA** and $\bar{H}^*X$ is an object of **U**. $M \in \mathbf{U}$ is called *A-unbounded* if, for every $x \in M$, there exists an element $a \in A$ of positive degree so that $ax \neq 0$. We will prove the following result.

THEOREM A. *Let Y be space so that $\bar{H}^*Y$ is A-unbounded and injective in* **U**. *Let X be a simply connected space of finite type and X_p the Bousfield-Kan p-completion of X. Then the Hurewicz map $[Y, \Omega X_p] \to \mathrm{Hom}_{\mathbf{UA}}(H^*\Omega X_p, H^*Y)$ is onto.*

If, in addition, $\bar{H}^*(Y, \mathbf{Q}) = 0$, this may be strengthened: the Hurewicz map $[Y, \Omega X] \to \mathrm{Hom}_{\mathbf{UA}}(H^*\Omega X, H^*Y)$ is also onto.

Since Carlsson's paper [**5**] and the work of Lannes and Zarati [**9**], topologists have learned to recognize many spaces Y so $\bar{H}^*Y$ is injective in **U**. For example, if G is a finite group with p-Sylow subgroup $(\mathbf{Z}/p)^k$, the $\bar{H}^*BG$ is injective. Further infinite examples include $\Sigma B(\mathbf{Z}/p)^k$ and any of the many wedge summands of these spaces. A complete classification of **U**-injectives is given in [**18**] and, from this, it can be seen that the only such injectives which are not A-unbounded are the Brown-Gitler modules. These have been adequately discussed elsewhere (see [**6**], among many).

An immediate consequence of Theorem A will be the following result. Suppose Y satisfies the same hypotheses as in Theorem A.

Received by the editors March 1, 1986. Some of the contents of this paper were presented by the second author at a conference in honor of William Massey at Yale University in October 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55P99.

The first author enjoyed the hospitality of the SFB 170 in Göttingen, The Federal Republic of Germany.

THEOREM B. *Let X be a space of finite type and $f\colon H^*X \to H^*Y$ a map of unstable algebras over A. Then there exists a map $\Theta\colon \Sigma Y \to \Sigma X_p$ so that*

$$\Theta^* = \Sigma f\colon \overline{H}^*\Sigma X_p \to \overline{H}^*\Sigma Y.$$

The phenomenon of maps out of injectives being realized after suspension is a familiar one (cf. [**6**]).

Our results are not limited to loop spaces. For example, we have the following uniqueness result. Suppose $H^1Y = 0$ in the following.

THEOREM C. *Let Y and Z be two spaces so that $H^*Y \cong H^*Z$ in* **UA**, *$\overline{H}^*Y$ is injective in* **U** *and such that H^*Y is a free graded commutative algebra. Then there is a homotopy equivalence $Y_p \to Z_p$.*

Examples of such spaces include $B(G \ltimes (\mathbf{Z}/p)^k)$ where G is a subgroup of $\mathrm{Gl}_k(\mathbf{F}_p)$ generated by pseudo-reflections and such that p does not divide the order of G.

This is an active field of research. For a paper of similar nature, but different thrust, see [**12**]. And Jean Lannes has shown that the Hurewicz map $[B(\mathbf{Z}/p)^k, X_p] \to \mathrm{Hom}_{\mathbf{UA}}(H^*X_p, H^*B(\mathbf{Z}/p)^k)$ is a bijection for nilpotent spaces X with finite fundamental group [**19**]. One would not expect a similar result for most other examples of spaces with injective cohomology: in addition to being injective in **U**, $H^*B(\mathbf{Z}/p)^k$ is injective in **UA**.

Here is the plan of the paper. In §1 we state some preliminaries. In §2, we prove a more general theorem than Theorem A, showing essentially that the Hurewicz map $[Y, X] \to \mathrm{Hom}_{\mathbf{UA}}(H^*X, H^*Y)$ is onto for spaces X we call Massey-Peterson spaces, after the work of those authors in [**10** and **11**]. §3 applies the results of §2 to deduce Theorems A and B. Some of the technicalities of this third section recapitulate work done by Massey and Peterson in [**11**]. §§4 and 5 are devoted to applications: in §4 we demonstrate that Carlsson's seminal algebraic splitting is realized topologically and, in §5, we discuss the existence of complex bundles over some of our spaces Y. Theorem C is proved in §5.

1. Definitions and preliminaries. In this short section, we define our terms and state some preliminary results. Fix a prime p. All homology and cohomology will be with F_p coefficients.

We first define a category **U** of unstable modules over the Steenrod algebra A. This is a full subcategory of all graded modules over A subject to the following conditions. If $M \in \mathbf{U}$ and $x \in M$, then if $p > 2$

$$(1.1)\qquad \begin{aligned} P^i x &= 0 \quad \text{if } 2i > \deg(x),\\ \beta P^i x &= 0 \quad \text{if } 2i + 1 > \deg(x), \end{aligned}$$

and if $p = 2$,

$$\mathrm{Sq}^i x = 0 \quad \text{if } i > \deg(x).$$

Next let **AU** be the category of unstable algebras over A. Thus if $H^* \in \mathbf{AU}$, the H^* is a graded, commutative $\mathbf{F}_p$ algebra, H^* is an unstable A module, the multiplication map $H^* \otimes H^* \to H^*$ is a map of A modules where $H^* \otimes H^*$ is given the diagonal structure. Furthermore, if $p > 2$,

$$P^i x = x^p \quad \text{if } 2i = \deg(x), \tag{1.2}$$

or, if $p = 2$,

$$\mathrm{Sq}^i x = x^2 \quad \text{if } i = \deg(x).$$

The cohomology of a space is in **AU**.

The augmentation ideal functor $\mathbf{AU} \to \mathbf{U}$ has adjoint U: $\mathbf{U} \to \mathbf{AU}$.

Now, let V be a graded $\mathbf{F}_p$ vector space and $K(V)$ an Eilenberg-Mac Lane space with $\pi_* K(V) \simeq V$. Then $H^*K(V)$ is a Hopf algebra and module of primitives $PH^*K(V)$ is a projective in **U**. Furthermore, there is an isomorphism in **UA**: $U(PH^*K(V)) \simeq H^*K(V)$.

Because we need it in the next section, we turn now to a definition of Massey and Peterson [**8**]. Let $R \in \mathbf{AU}$ and **UR** the subcategory of **U** specified as follows. $M \in \mathbf{UR}$ if $M \in \mathbf{U}$ and M is an R module; furthermore, the action map $R \otimes M \to M$ should be in **U**. A morphism in **UR** is a morphism in **U** preserving the module structure.

Let **UAR** be the obvious subcategory of **UA** consisting of R algebras. The forgetful functor $\mathbf{UAR} \to \mathbf{UR}$ has a left adjoint $U_R(\ )$. The existence of $U_R(\ )$ is discussed in [**10**].

The other tool we need is the algebraic loops functor. Let Σ: $\mathbf{U} \to \mathbf{U}$ be the obvious degree shifting functor, it has adjoint Ω: $\mathbf{U} \to \mathbf{U}$. Ω is right exact and has left derived functors Ω_s, $s \geqslant 0$. However, $\Omega_s = 0$ if $s > 1$; see, for instance, [**13**]. We will repeatedly use the following observations.

LEMMA 1.3. *If $P \in \mathbf{U}$ is projective and $M \subseteq P$ is a submodule, then $\Omega_1 M = 0$.*

PROOF. Since $\Omega_2 = 0$, there is an injection $\Omega_1 M \to \Omega_1 P$. However, $\Omega_1 P = 0$, because P is projective. Q.E.D.

$M \in U$ is of finite type if M^n is finite for each n.

LEMMA 1.4. *If $M \in \mathbf{U}$ is a trivial A module of finite type, then $\Omega_1 M$ is a trivial A module.*

PROOF. Ω_1 commutes with sums, so it is sufficient to show the result for $M = \Sigma^n \mathbf{F}_p \cong \overline{H}^*S^n$. Let $P_n = PH^*K(\mathbf{Z}/p, n)$ and define K_n by the exact sequence

$$0 \to K_n \to P_n \xrightarrow{f} \Sigma^n \mathbf{F}_p \to 0$$

where f is nonzero. Direct calculation shows that there is an exact sequence

$$0 \to \Sigma^j \mathbf{F}_p \to K_n \to \Omega P_n \to \Omega \Sigma^n \mathbf{F}_p \to 0$$

where $j = np - 1$ if n is even or $j = np - p + 1$ if n is odd. Q.E.D.

2. Maps of spaces with injective cohomology into towers of principal fibrations. In this section, we describe and prove the principal technical lemma of this paper: roughly, that if Y is a space whose cohomology is injective in the category **U** and X is a space of a special sort, then the evaluation map $[Y, X] \to \mathrm{Hom}_{\mathbf{AU}}(H^*X, H^*Y)$ is onto.

All spaces will be of finite type; that is, H^*X will be of finite type.

The first task is to describe explicitly the properties the space X must have.

DEFINITION 2.1. A space X is a Massey-Peterson space if there is a tower of principal fibrations.

$$\begin{array}{ccccccccc} \cdots \to & X_s & \xrightarrow{p_s} & X_{s-1} & \to & \cdots \to & X_1 & \to & X_0 = K_0 \\ & \uparrow i_s & & & & & \uparrow i_1 & & \\ & \Omega K_s & & & & & \Omega K_1 & & \end{array}$$

and maps $f_s\colon X \to X_s$ so that $p_s f_s = f_{s-1}$, the induced map $X \to \varprojlim X_s$ is a homotopy equivalence and

1. K_s, $s \geqslant 0$, is an Eilenberg-Mac Lane space with $\pi_* K_s$ a graded $\mathbf{F}_p$ vector space,

2. $\Omega K_s \xrightarrow{i_s} X_s \xrightarrow{p_s} X_{s-1}$ is the principal fibration classified by a k-invariant $k_s\colon X_{s-1} \to K_s$ and f_{s-1}^* induces an isomorphism $H^*X_{s-1}//k_s^* \cong H^*X$,

3. for $s \geqslant 1$, there is a module $M_s \in \mathbf{U}$ and an isomorphism in **UAR** with $R = H^*X_{s-1}//k_s^*$

$$\psi_s\colon H^*X_{s-1}//k_s^* \otimes U(M_s) \to H^*X_s$$

so that $0 = f_s^*\psi_s\colon M_s \to H^*X$, and

4. the inclusion of the fiber $i_s\colon \Omega K_s \to X_s$ induces an exact sequence

$$0 \to N_s \to M_s \xrightarrow{i_s^*} PH^*\Omega K_s$$

where N_s is a trivial module.

We call such spaces Massey-Peterson spaces because these authors delineated their advantages in [**10**]; interesting examples appear there and in [**11**].

We elucidate the strength of the Definition 2.1. Let us examine the fibration

$$\Omega K_s \xrightarrow{i_s} X_s \xrightarrow{p_s} X_{s-1}.$$

Because $p_s i_s$ is null-homotopic, $H^*\Omega K_s$ is a trivial $H^*X_{s-1}//k_s^*$ algebra. Therefore, i_s^* is completely determined by 2.1.3 and 2.1.4. Indeed, i_s^* is the composition

$$H^*X_s \cong H^*X_{s-1}//k_s^* \otimes U(M_s) \xrightarrow{\varepsilon\otimes 1} U(M_s) \to U(PH_*\Omega K_s) \cong H^*\Omega K_s.$$

But there is more structure in this fibration. Because it is a principal fibration, H^*X_s is an $H^*\Omega K_s$ comodule algebra. Furthermore, i_s^* is a coalgebra map, where $H^*\Omega K_s$ is given a coalgebra structure over itself by its Hopf algebra structure. Therefore, there is a commutative diagram

$$\begin{array}{ccc} H^*X_s & \xrightarrow{\psi} & H^*\Omega K_s \otimes H^*X_s \\ \downarrow i_s^* & & \downarrow 1 \otimes i_s^* \\ H^*\Omega K_s & \xrightarrow{\psi} & H^*\Omega K_s \otimes H^*\Omega K_s. \end{array}$$

We let ψ be our name for coactions of any sort. The map $M_s \to PH^*\Omega K_s$ also determines ψ: $H^*X_s \to H^*\Omega K_s \otimes H^*X_s$. $H^*\Omega K_s \otimes H^*X_s$ is given the structure of an $H^*X_{s-1}//k_s^*$ algebra by concentrating the action on the right-hand factor. Thus $H^*\Omega K_s \otimes H^*X_s$ is an object in **UAR** and the coaction of H^*X_s is completely determined by the commutative diagram

$$\begin{array}{ccc} M_s & \xrightarrow{i \oplus 1} & PH^*\Omega K_s \oplus M_s \\ \downarrow & & \downarrow \\ H^*X_s & \xrightarrow{\psi} & PH^*\Omega K_s \otimes H^*X_s \end{array} \tag{2.2}$$

With these remarks in hand, we come to our main result. We say that $M \in \mathbf{U}$ is **A**-*unbounded* if, for every $x \in M$, there exists an element $a \in A$ of positive degree so that $ax \neq 0$. All examples of **U**-injectives given in the introduction are A-unbounded.

THEOREM 2.3. *Let X be a Massey-Peterson space and Y a space so that H^*Y is A-unbounded and an injective object in* **U**. *Then the evaluation map* $[Y, X] \to \operatorname{Hom}_{\mathbf{AU}}(H^*X, H^*Y)$ *is onto.*

PROOF. We identify X with $\varprojlim X_s$ via the given homotopy equivalence. Let θ: $H^*X \to H^*Y$ be a map of unstable algebras over the Steenrod algebra. Our task is to show that there exists a map g: $Y \to X$ so that $g^* = \theta$. Define θ_s: $H^*X_s \to H^*Y$ to be the obvious composition $H^*X_s \to H^*X \xrightarrow{\theta} H^*Y$. Because of 2.1, it will be sufficient to prove that there exist maps g_s: $Y \to X_s$ so that $p_s g_s = g_{s-1}$ and $\theta_s = g_s^*$. This will be accomplished by induction.

To begin, X_0 is an Eilenberg-Mac Lane space; hence g_0 exists.

Suppose g_q, $q < s$, exist and satisfy the specified properties. Because the composition $X \to X_{s-1} \xrightarrow{k_s} K_s$ is null-homotopic, $g_{s-1}^* k_s^* = \theta_{s-1} k_s^* = 0$. So there exists a lifting g of g_{s-1}:

$$\begin{array}{ccc} & & X_s \\ & g \nearrow & \downarrow \\ Y & \xrightarrow[g_{s-1}]{} & X_{s-1}. \end{array}$$

Before continuing we note that because $g_{s-1}^* k_s^* = 0$, g_{s-1}^* gives H^*Y the structure of an $H^*X_{s-1}//k_s^*$ module.

We seek now to suitably modify g. Define $\bar{g}$ to be the composition

$$M_s \to H^*X_s \xrightarrow{g^*} H^*Y.$$

The first map is given by 3 of 2.1. Define $\bar{\theta}_s$ to be the composition

$$M_s \to H^*X_s \xrightarrow{\theta_s} H^*Y.$$

Since **U** is an abelian category, we may form the map

$$\bar{\theta}_s - \bar{g}: M_s \to H^*Y.$$

Consider the following diagram where the top row is the sequence given by 2.1.4:

$$\begin{tikzcd} 0 \arrow[r] & N_s \arrow[r, "j"] & M_s \arrow[r] \arrow[d, "\bar{\theta}_s - \bar{g}"] & PH^*\Omega K_s \arrow[dl, dashed, "\bar{\alpha}"] \\ & & H^*Y & \end{tikzcd}$$

Because N_s is a trivial A module and H^*Y is A-unbounded $(\bar{\theta}_s - \bar{g})j = 0$. Because H^*Y is injective, $\bar{\alpha}$ exists. And, because ΩK_s is an Eilenberg-Mac Lane space, there exists a map α: $Y \to \Omega K_s$ so that the following composition is $\bar{\alpha}$:

$$PH^*\Omega K_s \subseteq H^*\Omega K_s \xrightarrow{\alpha^*} H^*Y.$$

Define g_s to be the composition

$$Y \xrightarrow{\alpha \times g} \Omega K_s \times X_s \to X_s$$

where the second map is the action of the fiber on the total space. Certainly $p_s g_s = g_{s-1}$. Furthermore, we have a commutative diagram where the bottom row is f_s^*

$$\begin{tikzcd} M_s \arrow[r, "i \oplus 1"] \arrow[d] & PH^*\Omega K_s \oplus M_s \arrow[r, "\bar{\alpha} + \bar{g}"] \arrow[d] & H^*Y \arrow[d, "="] \\ H^*X_s \arrow[r, "\psi"] & H^*\Omega K_s \otimes H^*X_s \arrow[r, "\alpha^* \otimes g^*"] & H^*Y. \end{tikzcd}$$

The composition across the top is $(\bar{\theta}_s - \bar{g}) + \bar{g} = \bar{\theta}_s$. Since the composition across the bottom is a morphism in **UAR**, where $H^*X_{s-1}//k_s^*$, 3 of 2.1 implies that this map must be θ_s. Hence $g_s^* = \theta_s$. This completes the inductive step. Q.E.D.

The argument given to prove 2.4 is adapted from one given by the second author in [**15**] and by Haynes Miller [**12**].

3. $\mathbf{F}_p$-complete nice spaces are Massey-Peterson spaces. It is the purpose of this section to demonstrate that we can apply the results of the previous section to a whole class of interesting spaces, including the Bousfield-Kan $\mathbf{F}_p$-completions of loop spaces, $BU(n)$, $BO(2n+1)$, and the classifying spaces of some finite groups. These spaces, among others, are Massey-Peterson spaces.

First we define the type of spaces we are interested in. Let $S(\ )$ be the left adjoint of the augmentation ideal functor from graded supplemented $\mathbf{F}_p$ algebras to graded $\mathbf{F}_p$ vector spaces. $S(V)$ is a polynomial algebra if $p = 2$ and an exterior algebra tensor a polynomial algebra if $p > 2$.

DEFINITION 3.1. Let T^* be a graded supplemented $\mathbf{F}_p$ algebra. Then T^* is a graded complete intersection (GCI) algebra if there exist graded vector spaces V_0 and V_1 and a coexact sequence of algebras

(3.1a) $$\mathbf{F}_p \to S(V_1) \xrightarrow{\partial} S(V_0) \xrightarrow{\varepsilon} T^* \to \mathbf{F}_p$$

such that $S(V_0)$ is a projective $S(V_1)$ module.

The sequence (3.1a) is called a *presentation* for T^*.

A space X will called *nice* if H^*X is a GCI algebra. We adopt this terminology because our notion of a GCI algebra is dual to Bousfield's notion of a nice coalgebra.

Examples of nice spaces include any space whose cohomology algebra is the algebra underlying a Hopf algebra, any space whose cohomology is free, or even such examples as $\mathbf{C}P^n$ or $\mathbf{R}P^n$, if $p = 2$. The spaces studied by Massey and Peterson [**11**] are, of course, nice spaces.

The first point is that if H^*X is nice, the algebra resolution guaranteed by 3.1 can be extended to a resolution in **UA**.

PROPOSITION 3.2. *If H^*X is a GCI algebra, then there exists a coexact sequence in* **UA**

$$H^*K(W_1) \xrightarrow{\partial^*} H^*K(W_0) \xrightarrow{\varepsilon^*} H^*X \to \mathbf{F}_p$$

so that

(1) *the composition map of supplemented $\mathbf{F}_p$ algebras*

$$S(W_1) \xrightarrow{i} H^*K(W_1) \to H^*K(W_0)$$

is an injection,

(2) *$H^*K(W_0)$ is a projective $S(W_1)$ module, and*

(3) *$H^*K(W_0)//\partial^*i \cong H^*K(W_0)//\partial^* \cong H^*X$.*

PROOF. Let

$$\mathbf{F}_p \to S(V_1) \xrightarrow{\partial} S(V_0) \xrightarrow{\varepsilon} H^*X \to \mathbf{F}_p$$

be the presentation guaranteed by the hypothesis. If we set $W_0 = V_0$, then ε can be extended to a surjective map ε^*: $H^*K(W_0) \to H^*X$.

To proceed further let IA be the augmentation ideal of the Steenrod algebra. Then IA has an $\mathbf{F}_p$ basis of admissible Steenrod operations P^I or Sq^I if $p = 2$, $I \neq 0$. Define the $e(I)$ to be the usual excess function. Then we may define $\overline{W}_0 \subseteq H^*K(W_0)$ by

$$\overline{W}_0 = \mathrm{Span}\{P^I(w) \mid P^I \in I(A), e(I) < \deg(w), w \in W_0\}.$$

If $\{w_\alpha\}$ is a basis for W_0, $\{P^I(w_\alpha) \mid e(I) < \deg(w_\alpha)\}$ is a basis for $\overline{W}_0$. Define an isomorphism of algebras

$$\varphi\colon S(\overline{W}_0) \otimes S(W_0) \to H^*K(W_0)$$

by $\varphi(w) = w$ if $w \in W_0$ and $\varphi(P^I(w_\alpha)) = P^I(w_\alpha) - v$ where $v \in S(W_0) \subseteq H^*K(W_0)$ is an element such that $\varepsilon^*(v) = P^I\varepsilon^*(w_\alpha)$. That φ is an isomorphism follows from the usual calculations of the cohomology of Eilenberg-Mac Lane spaces.

There is an evident diagram of algebras, with the top row coexact

$$\begin{array}{ccccccccc}
\mathbf{F}_p & \to & S(\overline{W}_0)\otimes S(V_1) & \overset{1\otimes\partial}{\to} & S(\overline{W}_0)\otimes S(W_0) & \to & H^*X & \to & \mathbf{F}_p \\
 & & & & \downarrow\varphi & & \| & & \\
 & & & & H^*K(W_0) & \to & H^*X & \to & \mathbf{F}_p
\end{array}$$

V_1 is as in the presentation of H^*X. Define $W_1 = \overline{W}_0 \oplus V_1$ and $\partial^*\colon H^*K(W_1) \to H^*K(W_0)$ to be the evident extension of $\varphi(1 \otimes \partial)$. Then we may complete the diagram to

$$\begin{array}{ccccccccc}
\mathbf{F}_p & \to & S(W_1) & \overset{1\otimes\partial}{\to} & S(\overline{W}_0)\otimes S(W_0) & \to & H^*X & \to & \mathbf{F}_p \\
 & & \downarrow i & & \downarrow\varphi & & \| & & \\
 & & H^*K(W_1) & \overset{\partial^*}{\to} & H^*K(W_0) & \overset{\varepsilon^*}{\to} & H^*X & \to & \mathbf{F}_p .
\end{array}$$

Both rows are coexact. $S(\overline{W}_0) \otimes S(W_0)$ is a projective $S(W_1)$ module, by hypothesis. Hence $H^*K(W_0)$ is a projective $S(W_1)$ module. Q.E.D.

Now we can begin to build an unstable Adams type tower for X and this will be the tower needed for 2.1. Let H^*X be a GCI algebra and let

$$H^*K(W_1) \overset{\partial^*}{\to} H^*K(W_0) \overset{\varepsilon^*}{\to} H^*X \to \mathbf{F}_p$$

be the presentation given by the previous proposition. Define $K_1\colon K(W_0) \to K(W_1)$ and $f_0\colon X \to K(W_0)$ such that $k_1^* = \partial^*$ and $f_0^* = \varepsilon^*$. Define X_1 to be the pull-back, via k_1, of the trivial path fibration over $K(W_1)$. Then there is a diagram

$$(3.3)\qquad \begin{array}{ccccc}
 & & X_1 & \overset{p}{\to} & \mathrm{pt} \\
 & f_1 \nearrow & \downarrow p_1 & & \downarrow \\
X & \overset{f_0}{\to} & K(W_0) & \overset{k_1}{\to} & K(W_1)
\end{array}$$

with $p_1 f_1 = f_0$.

Define a module $M_1 \in \mathbf{U}$ be the short exact sequence

$$0 \to M_1 \to PH^*K(W_1) \to W_1 \to 0$$

where W_1 is given trivial A module structure.

The following two propositions demonstrate that X_1 is the first stage in a Massey-Peterson type tower (2.1). Let $R = H^*K(W_0)//k_1^*$. R is isomorphic, via f_0^*, to H^*X.

Proposition 3.4. *There is an isomorphism in* **UAR** $\psi_1\colon H^*K(W_0)//k_1^* \otimes U(\Omega M_1) \to H^*X_1$ *so that* $f_1^* \circ \psi_1\colon \Omega M_1 \to H^*X$ *is zero.*

Proof. Consider the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{H^*K(W_1)}(\mathbf{F}_p, H^*K(W_0)) \Rightarrow H^*X_1.$$

Using the conclusion of the previous proposition, we see that there is a split sequence of algebras

$$\mathbf{F}_p \to S(W_1) \to H^*K(W_1) \to S(\overline{W}_1) \to \mathbf{F}_p$$

where $\overline{W}_1 = \mathrm{Span}\{P^I(w) \mid P^I \in IA,\ e(I) > \deg(w), w \in W_1\}$. Therefore, there is a spectral sequence

$$\mathrm{Tor}_{S(\overline{W}_1)}\big(\mathbf{F}_p, \mathrm{Tor}_{S(W_1)}\big(\mathbf{F}_p, H^*K(W_2)\big)\big) \Rightarrow \mathrm{Tor}_{H^*K(W_1)}\big(\mathbf{F}_p, H^*K(W_0)\big)$$

By 3.2(2), (3),

$$\mathrm{Tor}_{S(W_1)}(\mathbf{F}_p, H^*K(W_0)) \cong H^*K(W_0)//k_1^*$$

and $\mathrm{Tor}_{S(W_1)}(\mathbf{F}_p, H^*K(W_0))$ is a trivial $S(\overline{W}_1)$ module. Hence

$$\mathrm{Tor}_{H^*K(W_1)}\big(\mathbf{F}_p, H^*K(W_0)\big) \cong H^*K(W_0)//k_1^* \otimes \mathrm{Tor}_{S(\overline{W}_1)}\big(\mathbf{F}_p, \mathbf{F}_p\big).$$

The computation of $\mathrm{Tor}_{S(W_1)}(\mathbf{F}_p, \mathbf{F}_p)$ and the differentials in the spectral sequence are handled in the usual way (see [**14**, §5], for example). Thus we may conclude that there is a coexact sequence in **UA**

$$\mathbf{F}_p \to H^*K(W_0)//k_1^* \xrightarrow{P_1^*} H^*X_1 \xrightarrow{i^*} U(\Omega M_1) \to \mathbf{F}_p.$$

We now demonstrate that f_1^* splits this sequence. As in Theorem 6.4 of [**14**], we may conclude the following. Let $F^{-1} = F^{-1}H^*X_1 \subseteq H^*X_1$ be the sub-**UR**-module of H^*X_1 given by the first Eilenberg-Moore filtration. Then there is an isomorphism $U_R(F^{-1}) \to H^*X_1$.

There is an obvious quotient map in **UR** q: $F^{-1} \to \Omega M_1$, where ΩM_1 is given the trivial R module structure. Define φ: $F^{-1} \to H^*X_1 \otimes U(\Omega M_1)$ by $\varphi(x) = x \otimes 1 + 1 \otimes q(x)$. Then φ induces a map

$$U_R(\varphi)\colon H^*X_1 \cong U_R(F^{-1}) \to H^*X_1 \otimes U(\Omega M_1).$$

Define θ to be the composition

$$H^*X_1 \xrightarrow{U_R(\varphi)} H^*X_1 \otimes U(\Omega M_1) \xrightarrow{f_1^* \otimes 1} H^*X \otimes U(\Omega M_1).$$

θ is an isomorphism and the following diagram commutes:

$$\begin{array}{ccc} H^*X_1 & \xrightarrow{\theta} & H^*X \otimes U(\Omega M_1) \\ \downarrow f_1^* & & \downarrow 1 \otimes \varepsilon \\ H^*X & \xrightarrow{=} & H^*X \end{array}$$

where ε is the augmentation. Let $\psi_1 = \theta^{-1}$. Q.E.D.

PROPOSITION 3.5. *There is an exact sequence of unstable A modules*

$$0 \to N_1 \to \Omega M_1 \to PH^*\Omega K(W_1)$$

where N is a trivial A module.

PROOF. There is a short exact sequence with W_1 a trivial A module

$$0 \to M_1 \to PH^*K(W_1) \to W_1 \to 0$$

and hence, the exact sequence

$$0 \to \Omega_1 W_1 \to \Omega M_1 \to PH^*\Omega K(W_1).$$

Now apply Lemma 1.4. Q.E.D.

We now supply the proposition that will allow us to calculate the homology of successive stages in the tower.

Suppose we have a finite tower of principal fibrations

$$(3.5)\qquad \begin{tikzcd} & K_s & & & & \\ X_s \arrow[r, "p_s"] & X_{s-1} \arrow[u, "k_s"'] \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] & X_0 = K_0 \\ \Omega K_s \arrow[u, "i_s"] & & & \Omega K_1 \arrow[u] & \end{tikzcd}$$

and maps $f_q\colon X \to X_q$ so that $p_q f_q = f_{q-1}$ and

(3.6.1) There exists $M_s \in \mathbf{U}$ and an isomorphism in **UAR** with $R = H^*X_{s-1}//k_s^*$

$$\psi_s\colon H^*X_{s-1}//k_s^* \otimes U(M_s) \to H^*X_s$$

so that $0 = f_s^*\psi_s\colon M_s \to H^*X$.

(3.6.2) There is an exact sequence in **U**, induced by i_s^*

$$0 \to N_s \to M_s \to PH^*\Omega K_s$$

where N_s has trivial A module structure.

Then we may proceed as follows. Define $\varepsilon\colon P_{s+1} \to M_s$ to be a projective cover of M_s and K_{s+1} to be an Eilenberg-Mac Lane space so that $H^*K_{s+1} = U(P_{s+1})$. Let $k_{s+1}\colon X_s \to K_{s+1}$ be the extension of ε guaranteed by (3.6.1). Define X_{s+1} and f_{s+1} by the following homotopy pull-back diagram:

$$\begin{tikzcd} & X_{s+1} \arrow[r] \arrow[d, "p_{s+1}"] & * \arrow[d] \\ X \arrow[ur, "f_{s+1}"] \arrow[r, "f_s"] & X_s \arrow[r, "k_{s+1}"] & K_{s+1} \end{tikzcd}$$

Notice that f_s^* induces an isomorphism $H^*X_s//k_{s+1}^* \cong H^*X$.

Define M_{s+1} to be the kernel of ε. Then we have the following result.

THEOREM 3.7. *There is an isomorphism in* **UAR**, $R = H^*X_s//k_{s+1}^*$,

$$\psi_{s+1}\colon H^*X_s//k_{s+1}^* \otimes U(\Omega M_{s+1}) \to H^*X_{s+1},$$

so that $0 = f_{s+1}^*\psi_{s+1}\colon \Omega M_{s+1} \to H^*X$. *Furthermore, there is an exact sequence*

$$0 \to N_{s+1} \to \Omega M_{s+1} \to PH^*\Omega K_{s+1}$$

such that N_{s+1} *is a trivial* A *module.*

PROOF. Consider the Eilenberg-Moore spectral sequence $\mathrm{Tor}_{H^*K_{s+1}}(\mathbf{F}_p, H^*X_s) \Rightarrow H^*X_{s+1}$.

Because H^*X_s is a free module over the image of k_{s+1}^*, the usual change of rings theorems (cf. 5.2 of [**14**]) imply that

$$\mathrm{Tor}_{H^*K_{s+1}}(\mathbf{F}_p, H^*X_s) \cong H^*X_s//k_{s+1}^* \otimes \mathrm{Tor}_{U(M_{s+1})}(\mathbf{F}_p, \mathbf{F}_p).$$

Then, again arguing as in §5 of [**14**], we get a coexact sequence in **UA**

$$\mathbf{F}_p \to H^*X_s//k^*_{s+1} \to H^*X_{s+1} \to U(\Omega M_{s+1}) \to \mathbf{F}_p.$$

To produce ψ_{s+1} we argue as at the end of the proof of 3.4 that f^*_{s+1} splits this sequence. Finally, the exact sequence

$$0 \to M_{s+1} \to PH^*K_{s+1} \xrightarrow{\varepsilon} M_s \to 0$$

shows that there is an exact sequence

$$0 \to \Omega_1 M_s \to \Omega M_{s+1} \to \Omega PH^*K_{s+1} \cong PH^*\Omega K_{s+1}.$$

On the other hand, (3.6.2) implies that there is a short exact sequence

$$0 \to N_s \to M_s \to K \to 0$$

where $K \subseteq PH^*\Omega K_s$. The latter is projective; hence, 1.3 implies $\Omega_1 K = 0$. Therefore $\Omega_1 M_s \cong \Omega_1 N_s$, which is a trivial A module by 1.4 and (3.6.2). Q.E.D.

The following is now immediate using induction, 3.4, and 3.7.

THEOREM 3.8. *Let X be a space such that H^*X is a GCI algebra. Then there exists a tower of principal fibrations*

$$\begin{array}{ccccccccccc} & & & & K_s & & & & & & \\ & & & & {\scriptstyle k_s}\uparrow & & & & & & \\ \cdots & \to & X_s & \xrightarrow{p_s} & X_{s-1} & \to & \cdots & \to & X_1 & \to & X_0 = K_0 \\ & & {\scriptstyle i_s}\uparrow & & & & & & \uparrow & & \\ & & \Omega K_s & & & & & & \Omega K_1 & & \end{array}$$

and maps $f_s\colon X \to X_s$ so that $p_s f_s \simeq f_{s-1}$ and

(1) *K_s, $s \geqslant 0$, is an Eilenberg-Mac Lane space and $\pi_* K_s$ is a graded $\mathbf{F}_p$ vector space,*

(2) *There exist A modules $M_s \in$ **U** and isomorphisms in **UAR**, $R = H^*X_{s-1}//k^*_s$*

$$\psi_s\colon H^*X_{s-1}//k^*_s \otimes U(M_s) \to H^*X_s$$

so that $0 = f^*_s\psi_s\colon M_s \to H^*X$,

(3) *i^*_s induces a short exact sequence in* **U**

$$0 \to N_s \to M_s \xrightarrow{i^*_s} PH^*\Omega K_s$$

where N_s is a trivial A module.

(4) *f^*_s induces an isomorphism $H^*X_s//k^*_{s+1} \to H^*X$.*

In the next two corollaries let X be a space such that H^*X is a GCI algebra, let X_s be as in 3.8 and $X_\infty = \varprojlim X_s$. There is an induced map $f\colon X \to X_\infty$.

COROLLARY 3.9. *X_∞ is the Bousfield-Kan $\mathbf{F}_p$-completion of X.*

PROOF. Because X_s is p-complete for each s, it follows from III.6.2 of [**4**] that we need only check that $\varinjlim H^*X_s \cong H^*X_s$. But this follows from 3.8(4). Q.E.D.

Deciding when $f^*\colon H^*X_\infty \to H^*X$ is an isomorphism is more problematical. From VI.6.3 of [**4**] we have the following.

COROLLARY 3.10. *If X is a nilpotent space then $f^*\colon H^*X_\infty \to H^*X$ is an isomorphism.*

The hypothesis of 3.10 is satisfied if X is a loop space or if X is simply connected. It is possible to extend the results of 3.10 to any space with finite fundamental group.

The following is the main technical result of this section.

THEOREM 3.11. *Let X be an $\mathbf{F}_p$-complete nilpotent space such that H^*X is a GCI algebra. Then X is a Massey-Peterson space.*

PROOF. $f\colon X \to X_\infty$ is a homotopy equivalence. Q.E.D.

The following result subsumes Theorem A of the introduction, because $(\Omega X)_p = \Omega(X_p)$ if X is simply connected.

THEOREM 3.12. *If X is an $\mathbf{F}_p$-complete nilpotent space such that H^*X is a GCI algebra and Y is a space so that H^*Y is injective in $\mathbf{U}$ and A-unbounded, then the evaluation map $[Y, X] \to \mathrm{Hom}_{\mathbf{AU}}(H^*X, H^*Y)$ is surjective.*

PROOF. Combine 2.4 and 3.10. Q.E.D.

COROLLARY 3.13. *If X is a nilpotent space such that H^*X is a GCI algebra and Y is a space so that H^*Y is injective in $\mathbf{U}$ and A-unbounded, and Y has the property that $\bar{H}^*(Y, \mathbf{Q}) = 0$, then the evaluation map $[Y, X] \to \mathrm{Hom}_{\mathbf{AU}}(H^*X, H^*Y)$ is surjective.*

PROOF. Under the hypotheses on Y, standard fracture lemmas (e.g. 1.5 of [**13**]) imply that $[Y, X] \to [Y, X_p]$ is a surjection. Q.E.D.

To end this section, we supply a proof of Theorem B of the introduction.

COROLLARY 3.14. *Let Y be a space such that H^*Y is injective and A-unbounded. Let X be a connected space and $\theta\colon H^*X \to H^*Y$ a map of unstable A algebras. Then there exists a map $f\colon \Sigma Y \to \Sigma X_p$ so that $f^* = \Sigma\theta$.*

PROOF. Define $\bar{\theta}$ to be the composition

$$H^*\Omega\Sigma X_p \xrightarrow{\eta^*} H^*X_p \xrightarrow{\theta} H^*Y$$

where η is the unit of the adjunction. By 3.12, there exists a map $g\colon Y \to \Omega\Sigma X_p$ so that $g^* = \bar{\theta}$. Let f be the adjoint of g. Q.E.D.

4. First application: A topological realization of an algebraic splitting of Carlsson. In this section we provide an application of the results of the previous two sections. We prove that an algebraic splitting given by G. Carlsson [**5**], is in fact given topologically; this splitting was used to prove a weak form of the Segal conjecture and showed that $\bar{H}^*B(\mathbf{Z}/2)^k$ was an unstable injective.

We must begin by defining the (much-studied) dual Brown-Gitler modules $J(n)$. We use the notation of [9], which draws on [13]. Define a functor $(\ )_n^*$ from $\mathbf{U}$ to $\mathbf{F}_p$ vector spaces by

$$(M)_n^* = \operatorname{Hom}_{\mathbf{F}_p}(M^n, \mathbf{F}_p), \qquad n \geqslant 0.$$

M^n is the vector space of elements of degree n in M. This functor is corepresentable: there exists an unstable A module $J(n) \in \mathbf{U}$ and a natural isomorphism

$$\operatorname{Hom}_{\mathbf{U}}(M, J(n)) \xrightarrow{\cong} (M)_n^*. \tag{4.1}$$

Because $(\ )_n^*$ is an exact functor, $J(n)$ is injective in $\mathbf{U}$. Because of (4.1) they could be considered to be "free" or "cofree" injectives; indeed, we have the following result of Lannes and Zarati [9].

PROPOSITION 4.2. *If $K \in \mathbf{U}$ is injective, there is a set C of integers and a split monomorphism $K \to \prod_{n \in C} J(n)$.*

We specialize, for the remainder of the section, to the prime 2.

Let us recall Carlsson's splitting. If $M \in \mathbf{U}$, then Sq^n defines a map Sq^n: $M^n \to M^{2n}$. By dualizing, we obtain a map which we call ρ_n:

$$\rho_n = (\mathrm{Sq}^n)^*: (M)_{2n}^* \to (M)_n^*. \tag{4.4}$$

This is a natural transformation of functors and, thus, must be induced by a morphism in $\mathbf{U}$: ρ_n: $J(2n) \to J(n)$.

We record the following result from [13]. Let $\alpha(k)$ be the number of ones in the binary expansion of the positive integer k.

LEMMA 4.5. *ρ_n is an isomorphism in degrees less than $\alpha(2n-1)+1$.*

Carlsson's idea was to consider the following limits.

DEFINITION 4.6. Let $n > 0$ be an integer. Define

$$K(n) = \varprojlim \left\{ J(2^k(2n-1)), \rho_{2^k(2n-1)} \right\}.$$

Because this sort of inverse limit of a system of injectives is injective, $K(n)$ is injective in $\mathbf{U}$. See [9].

By the formulas (4.1) and (4.4), we see that there are maps f_k: $\overline{H}^*B\mathbf{Z}/2 \to J(2^k)$ so that the following diagram commutes:

$$\begin{array}{ccc} \overline{H}^*B\mathbf{Z}/2 & \xrightarrow{=} & \overline{H}^*B\mathbf{Z}/2 \\ \downarrow f_{k+1} & & \downarrow f_k \\ J(2^{k+1}) & \xrightarrow{\rho_{2^k}} & J(2^k) \end{array}$$

Therefore there is a map f: $\overline{H}^*B\mathbf{Z}/2 \to K(1)$; the following result is seminal.

THEOREM 4.7 [5]. *f is a split monomorphism in* $\mathbf{U}$.

As a consequence $\overline{H}^*B\mathbf{Z}/2$ is injective in $\overline{U}$. Carlsson extends this result by showing that there is an isomorphism in $\mathbf{U}K(1) \otimes \cdots \otimes K(1) \to K(n)$, and, thus, that there is a split monomorphism $\otimes_n \overline{H}^*B\mathbf{Z}/2 \to K(n)$. In the same vein, our results below can be extended; we invite the reader familiar with, in the words of Jean Lannes, "Brown-Gitler technology" to perform this extension.

We seek to realize the splitting 4.7 topologically. We first produce a spectrum whose cohomology is $K(1)$. In [**2**] Brown and Gitler produced 2-complete spectra $T(n)$ so that

$$H^*T(n) \cong J(n) \tag{4.8a}$$

and if $\Sigma^\infty X$ is the suspension spectrum of a space of finite type, then the evaluation map

$$[T(n), \Sigma^\infty X] \to \mathrm{Hom}_{\mathrm{U}}(\overline{H}^*X, J(n)) \cong \overline{H}_n X \tag{4.8b}$$

is onto. In [**3**], Brown and Peterson, following Mahowald, produced maps

$$\psi_n \colon T(n) \to T(2n) \tag{4.9}$$

so that $\psi_n^* = \rho_n$; in particular, the following diagram commutes:

$$\begin{array}{ccc} [T(2n), \Sigma^\infty X] & \to & \mathrm{Hom}_{\mathrm{U}}(\overline{H}^*X, J(2n)) \\ \downarrow \psi_n^* & & \downarrow \rho_n \\ [T(n), \Sigma^\infty X] & \to & \mathrm{Hom}_{\mathrm{U}}(\overline{H}^*X, J(n)) \end{array}$$

PROPOSITION 4.9. *If X is a CW complex of finite type so that* $\mathrm{Sq}^k\colon \overline{H}^kX \to \overline{H}^{2k}X$ *is injective for every k, then for every even number n,* $\psi_k^*\colon [T(2n), \Sigma^\infty X] \to [T(n), \Sigma^\infty X]$ *is onto.*

PROOF. It has been known since Brown and Gitler's original work [**2**] that the Adams spectral sequence

$$\mathrm{Ext}_A^{s,t}(\overline{H}^*X, H^*T(n)) \Rightarrow [\Sigma^{t-s}T(n), \Sigma^\infty X]$$

collapsed at E_2 for n even and $t - s \leqslant 0$. That the induced map of E_2 terms

$$\psi_k^*\colon \mathrm{Ext}_A^{s,t}(\overline{H}^*X, H^*T(n)) \to \mathrm{Ext}_A^{s,t}(\overline{H}^*X, H^*T(n))$$

is onto, under our hypotheses, follows from the calculations of Lannes and Zarati [**8**], the first author [**7**], or from lambda algebra calculations using Brown and Gitler's original techniques. There is an explicit calculation in [**7**], using ideas of W. Singer.

The result follows. Q.E.D.

Now, by (4.8b), there exists a map $\theta_2\colon T(2) \to \Sigma^\infty B(\mathbf{Z}/2)$ so that $\theta_2^* = f_2$. Then using (4.9) and induction, there exist maps $\theta_k\colon T(2^k) \to \Sigma^\infty B\mathbf{Z}/2$ so that $\theta_k^* = f_k$ and the following diagram commutes:

$$\begin{array}{ccc} T(2^k) & \xrightarrow{\psi_{2^k}} & T(2^{k+1}) \\ \downarrow \theta_k & & \downarrow \theta_{k+1} \\ \Sigma^\infty B\mathbf{Z}/2 & \xrightarrow{=} & \Sigma^\infty B\mathbf{Z}/2 \end{array} \tag{4.10}$$

Define $\mathbf{T}(1) = \varinjlim \{T(2^k), \psi_{2^k}\}$. Then 4.5 implies that $H^*\mathbf{T}(1) \cong K(1)$. Because of (4.10) we have constructed a map $\theta\colon \mathbf{T}(1) \to \Sigma^\infty B\mathbf{Z}/2$ so that

$$\theta^* = f\colon \overline{H}^*B\mathbf{Z}/2 \to H^*\mathbf{T}(1) \cong K(1).$$

THEOREM 4.11. *θ is split: there exists a map γ: $\Sigma^\infty B\mathbf{Z}/2 \to \mathbf{T}(1)$ so that $\theta\gamma$ is a homotopy equivalence.*

Before proving this result, we state some facts from [**6**]. For any spectrum Y, let σ: $\Sigma^\infty\Omega^\infty Y \to Y$ be the counit of the adjunction.

LEMMA 4.12. (1) *σ: $\Sigma^\infty\Omega^\infty\Sigma T(2k) \to \Sigma T(2k)$ has a section: there exists a map τ: $\Sigma T(2k) \to \Sigma^\infty\Omega^\infty\Sigma T(2k)$ so that $\sigma\tau \cong \mathrm{id}$.*

(2) *$H^*\Omega^\infty\Sigma T(2k)$ is an exterior algebra.*

PROOF. In [**6**] it was shown that σ: $\Sigma^\infty\Omega^\infty T(2k) \to T(2k)$ has a section τ_0. (1) follows by setting τ to be the composition

$$\Sigma T(2k) \xrightarrow{\tau_0} \Sigma\Sigma^\infty\Omega^\infty T(2k) \to \Omega^\infty\Sigma^\infty\Sigma T(2k).$$

Because of the existence of τ_0, the fibration

$$\Omega^\infty\Sigma^\infty\Sigma\Omega^\infty T(2k) \xrightarrow{\Omega^\infty\Sigma\sigma} \Omega^\infty\Sigma T(2k)$$

has a section. $H^*\Omega^\infty\Sigma^\infty\Sigma\Omega^\infty T(2k)$ is an exterior algebra, by Proposition 3.5 of [**17**]. (2) follows. Q.E.D.

PROOF OF 4.11. $\Sigma^\infty B\mathbf{Z}/2$ and $\mathbf{T}(1)$ are 2-complete; therefore it is only necessary to produce a map γ: $\Sigma^\infty B\mathbf{Z}/2 \to \mathbf{T}(1)$ so that $(\theta\gamma)^*$ is the identity in cohomology. We will produce the adjoint of the suspension of γ; that is, a map γ_1: $\Sigma B\mathbf{Z}/2 \to \Omega^\infty\Sigma\mathbf{T}(1)$.

4.12(1) implies that σ^*: $H^*\Sigma T(2^k) \to H^*\Omega^\infty\Sigma T(2^k)$ is an injection; because ψ_{2^k}: $T(2^k) \to T(2^{k+1})$ is an isomorphism in cohomology in degrees less than k, σ^*: $H^*\Sigma\mathbf{T}(1) \to H^*\Omega^\infty\Sigma\mathbf{T}(1)$ is an injection. Because the image of σ^* lies in the primitives of $H^*\Omega^\infty\Sigma\mathbf{T}(1)$, there is an injection σ^*: $H^*\Sigma\mathbf{T}(1) \to PH^*\Omega^\infty\Sigma\mathbf{T}(1)$.

Now 4.12(2) and the fact that ψ_{2^k} is a k-equivalence imply that $H^*\Omega^\infty\Sigma\mathbf{T}(1)$ is an exterior algebra; hence the composition

$$PH^*\Omega^\infty\Sigma\mathbf{T}(1) \xrightarrow{i} H^*\Omega^\infty\Sigma\mathbf{T}(1) \xrightarrow{\pi} QH^*\Omega^\infty\Sigma\mathbf{T}(1)$$

is an injection. Therefore we have constructed an injection

$$j = \pi i\sigma^*\colon H^*\Sigma\mathbf{T}(1) \to QH^*\Omega^\infty\Sigma\mathbf{T}(1).$$

From [**13** or **9**] we know that $\Sigma\overline{H}^*B\mathbf{Z}/2$ is a $\mathbf{U}$-injective; therefore if

$$g\colon \Sigma K(1) \cong H^*\Sigma\mathbf{T}(1) \to \overline{H}^*\Sigma B\mathbf{Z}/2$$

is any map splitting f (from 4.7) there exists a map g_0 making the following diagram commute:

$$(4.13)\qquad \begin{array}{ccc} H^*\Sigma\mathbf{T}(1) & \xrightarrow{j} & QH^*\Omega^\infty\Sigma\mathbf{T}(1) \\ \downarrow g & \swarrow g_0 & \\ \overline{H}^*\Sigma B\mathbf{Z}/2 & & \end{array}$$

where Q denotes the indecomposable functor.

Define g_1 to be the composition

$$H^*\Omega^\infty\Sigma\mathbf{T}(1) \xrightarrow{\pi} QH^*\Omega^\infty\Sigma\mathbf{T}(1) \xrightarrow{g_0} H^*\Sigma B\mathbf{Z}/2.$$

g_1 is a map of cohomology algebras; hence 3.4 implies there exists a map γ_1: $\Sigma B\mathbf{Z}/2^\infty \to \Omega^\infty\Sigma\mathbf{T}(1)$ so that $\gamma_1^* = g_1$. Define γ to be the desuspension of the composition

$$\Sigma^\infty\Sigma B\mathbf{Z}/2 \xrightarrow{\Sigma^\infty\gamma_1} \Sigma^\infty\Omega^\infty\Sigma\mathbf{T}(1) \xrightarrow{\sigma} \Sigma\mathbf{T}(1).$$

(4.13) and the definition of j imply that $\gamma^* = g$; the choice of g (from 4.7) implies that $(\theta\gamma)^*$ is the identity. Q.E.D.

5. Second application: Maps out of a space whose cohomology is a ring of invariants. Suppose G is a group of order prime to p and r: $G \to \mathrm{Gl}_n(\mathbf{F}_p)$ a representation of G. Let $P_n = H^*B(\mathbf{Z}/p)^n$. Then there is a natural action of G on P_n; let P_n^G denote the ring of invariants. In this section we discuss two subjects. First, if G is among those groups such that P_n^G is a free graded $\mathbf{F}_p$ algebra, then we show that any space Y such that $H^*Y \cong P_n^G$ is essentially unique. Second, for any such G, we show that if Y is a space so that $H^*Y \cong P_n^G$ then the algebraic Chern classes of Smith and Stong **[16]**, which are invariants of the representation, can be realized as Chern classes of a complex bundle.

To begin, we will completely clarify the notation. If r is a representation of G, then r defines an action of G on $H^1B(\mathbf{Z}/p)^n \cong (\mathbf{F}_p)^n$; we extend this action to all of P_n by requiring that it commute with Steenrod operations and be an action through algebra automorphisms.

We are interested in the ring of invariant P_n^G because of the following lemma.

LEMMA 5.1. *If G is finite and of order prime to p, then P_n^G is an injective in* $\mathbf{U}$.

PROOF. We show that the inclusion $P_n^G \to P_n$ is split over the Steenrod algebra. Since P_n is injective, this suffices. Define a map a: $P_n \to P_n^G$ by

$$a(x) = |G|^{-1} \sum_{g\in G} gx.$$

$|G| \in \mathbf{F}_p$ is a unit; a is a multiple of the usual transfer and is often called the averaging map. a is an A module map and splits the inclusion. Q.E.D.

Next we notice that there exists a space whose cohomology is P_n^G. The representation r determines an action of G on $(\mathbf{Z}/p)^n$ and, hence, a semidirect product $G \ltimes (\mathbf{Z}/p)^n$. The Serre spectral sequence and the assumption that $p \nmid |G|$ immediately imply that $H^*B(G \ltimes (\mathbf{Z}/p)^n) \cong P_n^G$. In what follows, however, we may choose any space with cohomology P_n^G.

For our first application, suppose that r: $G \to \mathrm{Gl}_n(\mathbf{F}_p)$ is faithful and reduced and that the image of r is generated by pseudo-reflections; that is, elements $g \in \mathrm{Gl}_n(\mathbf{F}_p)$ such that $1 - g$: $(\mathbf{F}_p)^n \to (\mathbf{F}_p)^n$ has rank one.

Then from [1] we have that

$$P_n^G \cong E(w_1, \ldots, w_n) \otimes \mathbf{F}_p[z_1, \ldots, z_n]$$

so that $\beta w_i = z_i$ and $\deg(z_i) \geq 2$.

Fix such a G with these stipulations. Let Y_0 be any space such that $H^*Y_0 \cong P_n^G$. Let $Z(G,n)$ be its Bousfield $\mathbf{F}_p$ *localization*. Then we have the following uniqueness statement, which is Theorem C of the introduction.

THEOREM 5.2. *Let Y be any space so that $H^*Y \cong P_n^G$. Then there is a map g: $Y \to Z(G,n)$ so that g^* is an isomorphism. g is unique up to homotopy equivalences of $Z(G,n)$ and is the Bousfield localization of Y.*

PROOF. The second claim follows from the first and properties of localization. For the first, because $Z(G,n)$ is local and $H_1Z(G,n) = 0$, $Z(G,n)$ is simply connected. Thus, again because it is local, it is $\mathbf{F}_p$-complete. 3.12 now implies the existence of g. Q.E.D.

Let us return to general G.

We now define the Smith-Stong Chern classes of an orbit [16]. Let $V \subseteq H^2B(\mathbf{Z}/p)^n$ be the kernel of the Bockstein. The action of G on P_n determines an action of G on V. Let $B \subseteq V$ be an orbit of this action. Adjoin to P_n a variable X and form the polynomial

$$g_B(X) = \prod_{b \in B} (X + b) \in P_n[X].$$

If k is the number of elements in B, define the ith Chern class $c_i(B)$ to be the coefficient of X^{k-i} in $f_B(X)$.

Notice that $c_i(B) \in P_n^G$ for all i.

To justify calling these elements Chern classes, we make the following argument. Let $\mathbf{F}_p[y_1, \ldots, y_k]$ be a polynomial algebra on generators of degree 2, given the unique structure as an unstable algebra over A. Enumerate the elements of B; that is, write $B = \{b_1, \ldots, b_k\}$. Then there is a map of A algebras

$$f\colon \mathbf{F}_p[y_1, \ldots, y_k] \to P_n \tag{5.3}$$

given by $f(y_j) = b_j$. The standard Chern class $c_i \in \mathbf{F}_p[y_1, \ldots, y_k]$ is the coefficient of X^{k-i} in the polynomial

$$g(x) = \prod_i (X + y_i) \in \mathbf{F}_p[y_1, \ldots, y_k].$$

Clearly $f(c_i) = c_i(B)$.

For any space X, let i_*: $H^*(X; \mathbf{Z}) \to H^*(X, \mathbf{Z}/p)$ be the reduction of coefficients. Recall that k is the number of elements in the orbit B.

THEOREM 5.4. *Suppose p does not divide the order of G. If Y is a space so that $H^*Y = P_n^G$ and $\overline{H}^*(Y, \mathbf{Q}) = 0$, then there exists a complex bundle of dimension k, $\zeta_B \downarrow Y$, so that $i_*c_i(\zeta_B) = c_i(B)$.*

PROOF. We wish to combine 5.1 with 3.13 to produce a map ζ_B: $Y \to BU(k)$. To do this we need only supply an appropriate morphism of cohomology algebras. But $H^*BU(k) \cong \mathbf{F}_p[c_1, \ldots, c_k] \subseteq \mathbf{F}_p[y_1, \ldots, y_k]$ as the subalgebra generated by the Chern classes.

Consider the composition

$$H^*BU(k) \to \mathbf{F}_p[y_1, \ldots, y_k] \xrightarrow{f} P_n$$

where f is as in (5.3). By the remarks following (5.3) it is clear that this composition factors as map $\hat{f}$: $H^*BU(k) \to P_n^G$ followed by inclusion and that $\hat{f}(c_i) = c_i(B)$. Q.E.D.

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