TAMING WILD EXTENSIONS WITH HOPF ALGEBRAS

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ABSTRACT. Let $K \subset L$ be a Galois extension of number fields with abelian Galois group G and rings of integers $R \subset S$, and let $\mathcal{A}$ be the order of S in KG . If $\mathcal{A}$ is a Hopf R -algebra with operations induced from KG , then S is locally isomorphic to $\mathcal{A}$ as $\mathcal{A}$ -module. Criteria are found for $\mathcal{A}$ to be a Hopf algebra when $K = \mathbb{Q}$ or when L/K is a Kummer extension of prime degree. In the latter case we also obtain a complete classification of orders over R in L which are tame or Galois H -extensions, H a Hopf order in KG , using a generalization of the discriminant.

Galois module theory seeks to describe the ring of integers S of a Galois extension $L \supset K$ of number fields with Galois group G as a G -module, either absolutely (i.e. over ZG) or relatively (i.e. over RG , R the ring of integers of K). In the relative case, the fundamental result is Noether's theorem: S is locally RG -isomorphic to RG , that is, S has a normal basis locally at each prime of R , if and only if L/K is tame, i.e. tamely ramified.

However, nontame extensions L/K abound (e.g. $K = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt{m})$, $m \equiv 2$ or $3 \pmod{4}$, or $L = \mathbb{Q}(\zeta)$, ζ a primitive m th root of unity, m not square-free). In attempting to extend the tame results to nontame extensions, one approach, introduced by H. W. Leopoldt [17] and studied by H. Jacobinski [15], F. Bertrandias, J.-P. Bertrandias and M. J. Ferton [2, 3, 4], A. M. Berge [1], and recently, M. Taylor [24, 28], is to replace RG by a larger order over R in KG , in particular, the order $\mathcal{A}$ of S in KG .

$$\mathcal{A} = \{ \alpha \in KG \mid \alpha S \subseteq S \},$$

and consider S as an $\mathcal{A}$ -module. For $K = \mathbb{Q}$ this approach was successful: $S \cong \mathcal{A}$ as $\mathcal{A}$ -module when $G = \text{Gal}(L/\mathbb{Q})$ is abelian. However, in [2 and 3] it is shown that $S \cong \mathcal{A}$ as $\mathcal{A}$ -module may fail, even locally, if G is dihedral or if L/K is a Kummer extension of prime degree.

This paper starts from the premise that it is of interest to know when $\mathcal{A}$ is a Hopf R -algebra with operations induced from those on the Hopf K -algebra KG (abusing language we henceforth call such an $\mathcal{A}$ a Hopf subalgebra of KG). There are several reasons for investigating such a premise.

In general, as Bergman [29] has eloquently explained, for an algebra A to act on another algebra S and to respect the algebra structure of S , it is natural for A to be

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at least a bialgebra. For to describe how A respects the unit map and the multiplication on S , it is necessary for A to act on R and on $S \otimes S$, and a natural way to define such actions is via maps $\epsilon: A \rightarrow R$ and $\Delta: A \rightarrow A \otimes A$ which make A into a bialgebra. In applying this general observation to $\mathcal{A}$, to require that $\mathcal{A}$ be a Hopf algebra, not just a bialgebra, is to require that $\mathcal{A}$ be closed under the inverse map, or antipode, of KG .

Asking when $\mathcal{A}$ is a Hopf algebra may be of intrinsic geometric interest. For if so, setting $Y = \text{Spec}(S)$, $X = \text{Spec}(R)$, then Y is acted upon by $\mathbf{A} = \text{Spec}(\mathcal{A}^*)$, the Cartier dual over X of the affine group scheme represented by $\mathcal{A}$, and $\mathbf{A}$ may be a more natural group scheme of operators on Y than is

$$\mathbf{G} = \text{Spec}((RG)^*).$$

Perhaps of most interest, however, is that fact that part of Noether's theorem may be recast as: if $\mathcal{A} = RG$, then locally $S \cong \mathcal{A}$ as $\mathcal{A}$ -module; and in this formulation the result can be generalized, at least for G abelian, to the case where $\mathcal{A}$ is an arbitrary Hopf subalgebra of KG . The proof of this, given in §4, is an almost immediate application of one of the main results in [9], which characterizes, for H cocommutative, the condition that there exist local normal bases for an object S of a Hopf R -algebra H in terms of a criterion for "tameness" which directly generalizes the criterion for tame ramification that the image of the trace map on S be all of R .

Thus in the wild case, when the order $\mathcal{A}$ of S in KG , G abelian, is a Hopf subalgebra of KG , the wild RG -extension S becomes a tame $\mathcal{A}$ -extension and has local normal basis at every prime of R . This result is a rare example of a general local normal basis criterion for wild extensions of arbitrary number fields K .

The main body of the paper is an investigation, for the simplest abelian extensions, cyclotomic extensions of Q and Kummer extensions of prime order, of conditions for which $\mathcal{A}$ is a Hopf subalgebra of KG .

When $K = Q$, the example of $L = Q(\sqrt[m]{m})$, $m \not\equiv 1 \pmod{4}$, for which $\mathcal{A} \cong (ZG)^*$, the dual of ZG , is almost the only possible example. In general, for $K = Q$, L an abelian extension of Q , $\mathcal{A}$ is a Hopf subalgebra of QG if and only if every odd prime is tamely ramified, and the first ramification group of L/Q at the prime 2 has order at most 2. The main obstacle is that the idempotents occurring in $\mathcal{A}$ which correspond to ramification groups of L/Q of order > 2 are not sent to $\mathcal{A} \otimes \mathcal{A}$ by the comultiplication on KG .

For Kummer extensions L/K of prime order l , we find several equivalent conditions for $\mathcal{A}$ to be a Hopf subalgebra of KG , involving a congruence condition on a Kummer generator of L , a condition on the ramification numbers of L/K at primes dividing l , and a trace condition. This latter condition is that $\mathcal{A}$ is Hopf if and only if $\text{tr}(S)$ is the $(l-1)$ th power of an ideal of R . The analysis of when $\mathcal{A}$ is a Hopf algebra utilizes Tate and Oort's classification of group schemes of order l over rings of integers; in particular, if $\mathcal{A}$ is a Hopf algebra, then $\mathcal{A} = H_{\mathcal{B}}$, the Hopf algebra corresponding to the ideal $\mathcal{B}$ with $\mathcal{B} \cdot \text{tr}(S) = lR$.

The determination of when $\mathcal{A}$ is a Hopf algebra is entirely a local question at completions of K , and is nontrivial only at primes p dividing l at which L/K is totally ramified and $\not\equiv (R_p G)^*$. Our approach in the prime order Kummer case is

to find all the Hopf subalgebras of $\mathcal{A}_p$, using the Tate-Oort theory, and then look for Galois extensions with respect to these Hopf algebras (an H -Galois extension S with $S^H = R$ is an H^* -Galois object in the sense of Chase and Sweedler [7]). We show that to each Hopf subalgebra of $\mathcal{A}$ there corresponds a unique Galois extension which is a suborder of $S \otimes_R R_p$ in $L \otimes K_p$, and then determine when $S \otimes R_p$ itself is such a Galois extension whose Hopf algebra is $\mathcal{A}$. To do this we develop a general codifferent criterion for an H -extension, H a Hopf algebra, to be Galois, based on the integral I of H , which yields a generalization of the classical discriminant criterion for $H = RG$, and also yields a Galois-theoretic proof of Larson and Sweedler's theorem that if H is a finite, unimodular Hopf algebra, then $H^* \cong H \otimes I$ as H -modules [16], and a proof of Pareigis' Frobenius criterion for Hopf algebras [19]. A by-product of the development is to give a complete local, then global classification of Hopf Galois extensions, and also tame H -extensions, which are orders over R in Kummer extensions of K of prime order. In particular, we show that there are orders over R in L which are Galois H -extensions for some H if and only if the Kummer order $\tilde{S}$ is a Galois $(RG)^*$ -extension, in which case the Galois H -extensions are in 1-1 correspondence with ideals of R which are $(l-1)$ th powers and contain $(lR)(\text{tr}(S))^{-1}$.

Throughout the paper, $L \supset K$ is a Galois extension of number fields, the Galois group $\text{Gal}(L/K) = G$, and O_K, O_L are the rings of integers of K, L , respectively.

1. Hopf algebras and their algebras. Hopf algebras (over a commutative ring R) as considered in this paper are in the sense of Sweedler [22], that is, H is a Hopf R -algebra if it is an R -bialgebra with antipode. A Hopf R -algebra H is finite if it is a finitely generated projective R -module [7, p. 55]. Throughout this paper, all Hopf algebras will be assumed finite. We denote the multiplication, unit, comultiplication, counit and antipode of H by μ, i, Δ, ϵ , and λ , respectively.

If H is a Hopf R -algebra, the space of (left) integrals I of H is the set

$$I = \{x \in H \mid yx = \epsilon(y)x \text{ for all } y \text{ in } H\}.$$

Let S be an R -algebra, finitely generated and projective as R -module, and H a Hopf algebra. Then S is an H -module algebra [22] if S is acted on by H via a measuring. If S is an H -module algebra, then the action $H \otimes S \rightarrow S$ induces a comodule map $\alpha: S \rightarrow S \otimes H^*$ which is an R -algebra homomorphism [7, p. 55]; S is then an H^* -object. Conversely, if S is an H -object, S is an H^* -module algebra.

If S is a H -module algebra, the fixed ring is

$$S^H = \{s \in S \mid \xi s = \epsilon(\xi)s \text{ for all } \xi \text{ in } H\}.$$

We have $IS \subseteq S^H$ for S any H -module algebra. We call S an H -extension of R if $S^H = R$ and S is an H -module algebra.

Let H, J be finite Hopf algebras which are dual: $H^* \cong J, J^* \cong H$, and S an R -algebra, finitely generated and projective as R -module, then S is a Galois H -extension of R if S is a Galois J -object in the sense of [7], and S is a tame H -extension of R if S is a tame J -object in the sense of [9]. We recall these definitions.

DEFINITION 1.1. The R -algebra S is a Galois J -object if S is a J -object via $\alpha: S \rightarrow S \otimes J$, and the map $\gamma: S \otimes S \rightarrow S \otimes J$ given by $\gamma(x \otimes y) = (x \otimes 1)\alpha(y)$, is an isomorphism.

S is a tame J -object if S is an H -module algebra, $H = J^*$, faithful as H -module, $\text{rank}_R(S) = \text{rank}_R(H)$ as projective R -modules, and for $I =$ the space of integrals of H , $IS = S^H = R$.

A Galois J -object is a tame J -object, by [9, (2.3)]. An H -extension S of R has normal basis if $S \cong H^*$ as H -module, and has local normal basis if for any prime ideal p of R , $S_p \cong H_p^*$ as H_p -module.

Let $L \supset K$ be a Galois extension of number fields with Galois group G , abelian. Let H be a Hopf O_K -algebra which is an order over O_K in KG . If O_L is an H -extension of O_K , then the criteria for O_L to be a tame H -extension reduce to the single condition $IO_L = O_K$, the analogue for H of the condition, for $H = O_K G$, that the trace map: $O_L \rightarrow O_K$ be surjective [9]. Thus for abelian extensions of number fields, O_L is a tame $O_K G$ -extension if and only if L/K is tamely ramified.

The main theorem of [9] is that the H -extension $O_L \supset O_K$ has local normal basis if and only if $IO_L = O_K$, I the space of integrals of H . Thus determining that O_L is a tame H -extension of O_K for some Hopf algebra H yields useful information on the local structure of O_L .

2. The order of O_L . Let $L \supset K$ be an abelian Galois extension of number fields with Galois group G . Following Leopoldt [17], let

$$\mathcal{A} = \{ \alpha \in KG \mid \alpha O_L \subseteq O_L \},$$

the order of O_L in KG , and set $\mathcal{A}^* = \text{Hom}_{O_K}(\mathcal{A}, O_K)$. In [17], Leopoldt proved that if $K = \mathbb{Q}$, O_L is always a free $\mathcal{A}$ -module; on the other hand, F. and J. P. Bertrandias and M. Ferton [3, 4] have shown that O_L need not be locally free over $\mathcal{A}$ for $L \supset K$ a Kummer extension of prime order.

One reason for interest in knowing if $\mathcal{A}$ is a Hopf algebra is:

THEOREM 2.1. *Let $L \supset K$ be an abelian extension of number fields with Galois group G . Suppose $\mathcal{A}$, the order of O_L in KG , is a Hopf subalgebra of KG . Then O_L is a tame $\mathcal{A}$ -extension of O_K and is locally isomorphic to $\mathcal{A}$ as $\mathcal{A}$ -module.*

PROOF. Suppose $\mathcal{A}$ is a Hopf algebra. Let $R = O_K$, $S = O_L$. By [9, Theorem 5.4], O_L is locally isomorphic to $\mathcal{A}^*$ as $\mathcal{A}$ -module if and only if $IS = R$ where I is the space of integrals of $\mathcal{A}$. Since $\mathcal{A}^*$ is locally isomorphic to $\mathcal{A}$ as $\mathcal{A}$ -module if $\mathcal{A}$ is a Hopf algebra (see e.g. [19] or Corollary 10.4 below), it suffices to show $IS = R$, a local question. So assume R is local. Let $\text{tr} = \sum_{\sigma \in G} \sigma$, then if $\text{tr}(S) = aR$ (R , being local, is a discrete valuation ring), $\theta = \text{tr}/a$ maps S onto R . Thus $\theta \in \mathcal{A}$. Since $\mathcal{A}$ is a Hopf subalgebra of KG and tr is an integral of KG , θ is an integral of $\mathcal{A}$. Thus $IS = R$.

3. Tate-Oort algebras. In most of this paper we study extensions of number fields $L \supset K$ which are cyclic of prime order l with Galois group G with generator σ , where K contains a primitive l th root of unity ζ . The Hopf algebras which arise are

finitely generated projective O_K -modules of rank l which are orders over O_K in KG . These have been classified by Tate and Oort [23], and are completely determined by their local structure at completions of O_K and at K [23, Lemma 4].

Let $K_{\mathfrak{p}}$ be the completion of K at a (finite) prime $\mathfrak{p}$, R the valuation ring. If $\mathfrak{p} \cap Z \neq (I)$, then the only Hopf R -algebra of interest is the group ring RG , which, since R contains $1/l$ and ζ , is isomorphic to $\text{Hom}_R(RG, R) = (RG)^*$.

The local structure of the Tate-Oort Hopf algebras when $\mathfrak{p} \cap Z = (I)$, involves certain constants $\omega_1, \dots, \omega_l$, obtained as follows.

Let $\chi: F_l \rightarrow Z_l \subseteq R$ be the unique multiplicative section of the residue class map $Z_l \rightarrow F_l$ [17, p. 44]. In RG , let

$$(3.1) \quad \begin{aligned} \theta_i &= - \sum_{m \in \mathbb{F}_l^*} \chi^i(m) \sigma^m, \quad i = 1, \dots, l-2, \\ \theta_{l-1} &= l - \sum_{j=0}^{l-1} \sigma^j \end{aligned}$$

[23, p. 9]. Then $\theta_1^i = \omega_i \theta_i$, $i = 1, \dots, l-1$, and $\theta_1^l = \omega_l \theta_1$, for some elements $\omega_1, \dots, \omega_l$, of R , where $\omega_1 = 1$, $\omega_2, \dots, \omega_{l-1}$ are units of R , and $\omega_l = l\omega_{l-1}$. (See [23, formula (17)] for an inductive definition of the ω_i .)

Let H be a Hopf R -algebra, free as an R -module of rank l . Then [23, p. 14] there exist a, b in R , $ab = \omega_l$, such that $H = R[\xi]$ where $\xi^l = b\xi$ as R -algebra, and the comultiplication $\Delta: H \rightarrow H \otimes H$ is given by

$$\begin{aligned} \Delta(\xi^i) &= 1 \otimes \xi^i + \xi^i \otimes 1 \\ &+ \frac{\omega_l}{1-l} \left[\sum_{j=1}^{i-1} \frac{\xi^j}{\omega_j} \otimes \frac{\xi^{i-j}}{\omega_{i-j}} + \sum_{j=i}^{l-1} a \frac{\xi^j}{\omega_j} \otimes \frac{\xi^{(l-1)+(i-j)}}{\omega_{(l-1)+(i-j)}} \right], \end{aligned}$$

the counit $\varepsilon: H \rightarrow R$ by $\varepsilon(\xi^i) = 0$ for $i > 0$, and the antipode $\lambda: H \rightarrow H$ by $\lambda(\xi) = -\xi$. The Hopf algebra H is thus defined by the constants a and b , which satisfy $ab = \omega_l$. If $p = 2$, $\lambda(\xi) = \xi$.

Denote the Hopf algebra $H = R[\xi]$, $\xi^l = b\xi$, by H_b .

With this notation, $RG = H_{\omega_l}$, $(RG)^* = H_1$.

The identification $RG = H_{\omega_l} = R[\theta]$, $\theta^l = \omega_l \theta$, is given by (3.1) and by

$$(3.2) \quad \sigma^m = 1 + \frac{1}{1-l} \left(\sum_{i=1}^{l-1} \frac{\chi^i(m)}{\omega_i} \theta^i \right) \quad \text{for } m = 1, \dots, l-1.$$

(cf. [23, p. 15, Remark 5]). There is an inclusion $H_b \subseteq H_{b'}$, if and only if there is an element u of R such that $u^{l-1}b' = b$, in which case the map is given as follows: if $H_b = R[\xi]$, $H_{b'} = R[\xi']$, then $\xi \mapsto u\xi'$: $(u\xi')^l = u^l b' \xi' = b(u\xi')$.

Whenever $\zeta \in R$, $RG \cong (RG)^*$, so ω_l has an $(l-1)$ th root $\tilde{\omega}_l$ in R . In general, the Hopf algebra H_b , $b \neq 0$ is a subalgebra of KG if and only if $H_b \otimes K \cong KG \cong (KG)^* \cong H_1 \otimes K$, if and only if b has an $(l-1)$ th root $\tilde{b}$ in R . In case $\tilde{b}$ exists, so does $\tilde{a} = \tilde{\omega}_l / \tilde{b}$, and so from $\tilde{a}^{l-1}b = \omega_l$ and $\tilde{b}^{l-1} \cdot 1 = b$ and (3.2) we obtain inclusions

$$RG = H_{\omega_l} \subseteq H_b \subseteq H_1 = (RG)^*.$$

PROPOSITION 3.3. *Let K be a local or global algebraic field containing a primitive l th root of unity ξ , and let R be the ring of integers of k . Then the set of isomorphism classes of Hopf R -algebras contained in KG , G cyclic of order l , is in 1-1 lattice-preserving correspondence with the set of ideals dividing lR which are $(l-1)$ th powers.*

PROOF. If H is a Hopf R -algebra of rank l , then H is uniquely determined by its images at completions of R . Let $\mathfrak{p}$ be a prime divisor of lR and $R_{\mathfrak{p}}$, $K_{\mathfrak{p}}$ be the completions of R , K at $\mathfrak{p}$, respectively. Then $H \otimes_R R_{\mathfrak{p}} = H_{\mathfrak{p}}$ for $\mathfrak{b}$ some divisor of $lR = (1 - \xi)^{l-1}R = \mathfrak{p}^{(l-1)e}$. Since $H_{\mathfrak{p}} \subseteq K_{\mathfrak{p}}G$, $\mathfrak{b}$ is an $(l-1)$ th power, so $\mathfrak{b}R = \mathfrak{p}^{(l-1)s}$ for some s , $0 \leq s \leq e$. If $l \notin \mathfrak{p}$ then $H \otimes_R R_{\mathfrak{p}} = R_{\mathfrak{p}}G \cong (R_{\mathfrak{p}}G)^* = H_1$, so $e = 0$. Thus to H corresponds the ideal $\mathcal{B} = \prod_{\mathfrak{p}|l} \mathfrak{p}^{(l-1)s}$. The lattice-preserving property follows from (3.2).

NOTATION. (3.4). Denote by $H_{\mathcal{B}}$ the Hopf subalgebra of KG corresponding to the ideal $\mathcal{B}$ of O_K . If $\mathcal{B} = \ell^{l-1}$, for each prime ideal $\mathfrak{p}$ of O_K , $H_{\mathcal{B}} \otimes_{O_K} \hat{O}_{K,\mathfrak{p}} = H_{\mathfrak{p}}$ where $\mathfrak{b}$ is the $(l-1)$ th power of a generator of $\ell \otimes_{O_K} \hat{O}_{K,\mathfrak{p}}$.

(3.5) For $H = H_{\mathfrak{b}} = R[\xi]$, the space of integrals I is the free R -module generated by $b - \xi^{l-1}$, as is easily checked. In KG , $b - \xi^{l-1}$ has a familiar look: if $RG = H_{\omega}$, $= R[\theta]$, $\theta = a\xi$, so

$$\begin{aligned} \xi^{l-1} &= \frac{1}{a} \theta^{l-1} = \frac{1}{a} \omega_{l-1} \theta_{l-1} \\ &= \frac{1}{a} \omega_{l-1} \left(l - \sum_{j=0}^{l-1} \sigma^j \right) \quad (\text{from (3.1)}) \end{aligned}$$

and so

$$b - \xi^{l-1} = \frac{\omega_{l-1}}{a} \sum_{j=0}^{l-1} \sigma^j = \frac{b}{l} \sum \sigma^j.$$

Of course $\sum_{j=0}^{l-1} \sigma^j$ generates the space of integrals of RG .

We will denote $\sum_{\sigma \in G} \sigma = \text{tr}$ since the action of tr on an RG -module gives the trace map.

4. The quadratic case. Every quadratic extension is tame. While this will follow as a special case of later results, we give here a short direct argument.

THEOREM 4.1. *Let R be a Dedekind domain with quotient field K , let L be a quadratic field extension of K with Galois group $G = \langle \sigma \rangle$ of order 2. Let S be the integral closure of R in L . Suppose $\text{tr}(S) = aR$, a principal ideal of R . Then S is a tame $H_{\mathfrak{b}}$ extension, $b = 2/a$, and $H_{\mathfrak{b}} = \{ \alpha \in KG \mid \alpha S \subseteq S \}$.*

PROOF. First we note that a divides 2, since $\text{tr}(1) = 2$ is in aR . So $RG = H_2 \subseteq H_{\mathfrak{b}} = R[\xi]$ by $\xi = (1 - \sigma)/a$, $\sigma = 1 - a\xi$.

First, $H_{\mathfrak{b}}$ acts on S . For suppose s is in S , $\text{tr}(s) = ar$. Then $(\sigma + 1)s = ar$ for some r in R , so $\sigma(s) = ar - s$, and $\xi s = ((1 - \sigma)/a)s = bs - r$, thus $H_{\mathfrak{b}}S \subseteq S$.

The space of integrals I of H is generated by $b - \xi = 2/a - ((1 - \sigma)/a) = (\sigma + 1)/a$, and if $\text{tr}(s) = a$, then $((\sigma + 1)/a)s = 1$. So $IS = R$, and S is a tame $H_{\mathfrak{b}}$ -module algebra.

Let $\mathcal{A} = \{\alpha \in KG \mid \alpha S \subseteq S\}$. Since $\mathcal{A}S \subseteq S$, $\mathcal{A}$ is integral over R , so $\mathcal{A} \subseteq (RG)^* = H_1 = R[y]$, $y^2 = y$. Further, $H_b \subseteq H_1$ by $\xi = by$. So any α in $\mathcal{A}$ has the form $\alpha = m + n(\xi/b)$, m, n in R . Let s be in S , not in R , with $\text{tr}(s) = a$. If α is in $\mathcal{A}$, then

$$\begin{aligned} (m + n(\xi/b))(s) &= ms + n\xi s/b \\ &= ms + n(bs - 1)/b \\ &= ms + ns - n/b \quad \text{is in } S, \end{aligned}$$

and so n/b is in R . But then $\alpha = m + n(\xi/b) = m + (n/b)\xi$ is in $R[\xi] = H_b$, and $\mathcal{A} \subseteq H_b$. That completes the proof.

COROLLARY 4.2. *If $L \supset K$ is any quadratic extension of number fields, then there is an O_K -Hopf algebra H such that O_L is a tame H -module algebra.*

If $\text{tr}(O_L)$ is a principal ideal of O_K this is immediate from (4.1). In general, this follows from the proof of Theorem 4.1, used as a local argument (see §17, below).

EXAMPLES. Let $K = Q$, $L = Q(\sqrt{d})$, d square-free. Then O_L is a tame H_b -module algebra, where

$$\begin{cases} b = 2 \ (H_b = RG) & \text{if } d \equiv 1 \pmod{4}, \\ b = 1 \ (H_b = (RG)^*) & \text{if } d \equiv 2, 3 \pmod{4}. \end{cases}$$

For $K \neq Q$, numerous examples of quadratic extensions $L \supset K$ for which O_L is a Galois, hence tame H_b -module algebra, for $H_b \neq O_K G$ or its dual, are described in [8].

5. Absolutely abelian extensions. In this section we show that unless the abelian extension $L \supset Q$ is tamely ramified except possibly at the prime 2, and then only with ramification group cyclic of order ≤ 2 , the ring of integers O_L of L is not tame for any Z -Hopf subalgebra of QG , $G = \text{Gal}(L/Q)$.

THEOREM 5.1. *Let G be a finite abelian group, and let $\mathcal{A}$ be an order over Z in QG generated by ZG and, for each prime p dividing the order of G , an idempotent*

$$e_p = \frac{1}{|G_p|} \sum_{\sigma \in G_p} \sigma \quad (|G_p| = \text{order of } G_p)$$

corresponding to some (possibly trivial) p -subgroup G_p of G . Then $\mathcal{A}$ is a Hopf subalgebra of QG if and only if $|G_2| \leq 2$ and G_p is trivial for all odd p .

PROOF. Suppose $m = |G_p| > 2$ for some p , and fix $\pi, \rho \neq 1$ in G_p . Now in QG ,

$$\Delta e_p = \frac{1}{m} \sum_{\sigma \in G_p} \sigma \otimes \sigma.$$

Since $\mathcal{A}$ is generated over Z by elements of G and idempotents e_q for q dividing the order of G , Δe_p is in $\mathcal{A} \otimes \mathcal{A}$ if and only if Δe_p is a Z -linear combination of the generators $\sigma \otimes \tau$, $\sigma \otimes e_q e_q \otimes \tau$, and $e_q \otimes e_{q'}$ in $\mathcal{A} \otimes \mathcal{A}$, for σ, τ in G and q, q' running through prime divisors of the order of G .

We suppose we can write Δe_p as such a $\mathbb{Z}$ -linear combination, and (in $QG \otimes QG$) collect the coefficients of $\pi \otimes \pi$, $\pi \otimes \rho$, $\rho \otimes \pi$, $\rho \otimes \rho$. Since $\pi, \rho \neq 1$ in G , the only generators of $\mathcal{A} \otimes \mathcal{A}$ which contribute nonzero coefficients are the generators $\pi \otimes \pi$, $\pi \otimes \rho$, $\rho \otimes \pi$ and $\rho \otimes \rho$ themselves, together with $e_p \otimes \pi$, $e_p \otimes \rho$, $\pi \otimes e_p$, $\rho \otimes e_p$, and $e_p \otimes e_p$. (The nonidentity terms in e_q , $q \neq p$, lie in G_q , and $G_q \cap G_p = (1)$.)

We write

$$\begin{aligned} \Delta e_p &= \frac{1}{m} \sum_{\sigma \in G_p} \sigma \otimes \sigma = a_{0,0} e_p \otimes e_p + \sum_{\sigma \in G_p} a_{\sigma,0} \sigma \otimes e_p \\ &\quad + \sum_{\tau \in G_p} a_{0,\tau} e_p \otimes \tau + \sum_{\sigma, \tau \in G_p} a_{\sigma,\tau} \sigma \otimes \tau \\ &\quad + (\text{other terms not containing } \sigma \otimes \tau \text{ for } \sigma, \tau \neq 1 \text{ in } G_p), \end{aligned}$$

with all coefficients in $\mathbb{Z}$. Then, collecting coefficients of $\pi \otimes \pi$:

$$(5.2) \quad \frac{1}{m} = \frac{1}{m^2} a_{0,0} + \frac{1}{m} a_{\pi,0} + \frac{1}{m} a_{0,\pi} + a_{\pi,\pi}$$

of $\pi \otimes \rho$:

$$(5.3) \quad 0 = \frac{1}{m^2} a_{0,0} + \frac{1}{m} a_{\pi,0} + \frac{1}{m} a_{0,\rho} + a_{\pi,\rho}$$

of $\rho \otimes \pi$:

$$(5.4) \quad 0 = \frac{1}{m^2} a_{0,0} + \frac{1}{m} a_{\rho,0} + \frac{1}{m} a_{0,\pi} + a_{\rho,\pi}$$

of $\rho \otimes \rho$:

$$(5.5) \quad \frac{1}{m} = \frac{1}{m^2} a_{0,0} + \frac{1}{m} a_{\rho,0} + \frac{1}{m} a_{0,\rho} + a_{\rho,\rho}.$$

Multiplying the four equations by m and taking the differences (5.2)–(5.3) and (5.4)–(5.5), yields

$$\begin{aligned} 1 &= (a_{0,\pi} - a_{0,\rho}) + m(a_{\pi,\pi} - a_{\pi,\rho}), \\ -1 &= (a_{0,\pi} - a_{0,\rho}) + m(a_{\rho,\pi} - a_{\rho,\rho}), \end{aligned}$$

impossible if $m > 2$. Thus Δe_p is not in $\mathcal{A} \otimes \mathcal{A}$ if $|G_p| > 2$. Since G_p is a p -group, if $\mathcal{A}$ is a Hopf subalgebra of QG we must have $G_p = (1)$ if p is odd, and $|G_2| \leq 2$.

Conversely, if $\mathcal{A} = ZG + ZGe_2$ where $G_2 = \langle \sigma \rangle$ has order 2, then $e_2 = (1 + \sigma)/2$, $\mathcal{A}$ contains $\bar{e}_2 = e_2 - \sigma = (1 - \sigma)/2$, and $\Delta e_2 = e_2 \otimes e_2 + \bar{e}_2 \otimes \bar{e}_2$. So $\mathcal{A}$ is a $\mathbb{Z}$ -Hopf subalgebra of QG .

NOTE. For G of odd order, Theorem 5.1 follows from a theorem of R. Larson [30].

COROLLARY 5.6. *If $L \supset Q$ is an abelian extension with Galois group G , then the order $\mathcal{A}$ of O_L in QG is a Hopf subalgebra of QG if and only if either $L \supset Q$ is tamely ramified (i.e. $\mathcal{A} = \mathbb{Z}G$), or the only prime which ramifies wildly in L is 2 and the first ramification group of G corresponding to 2 has order 2.*

This follows from the description of the order $\mathcal{A}$ given in [17], cf. [1].

PROPOSITION 5.7. *Let $L \supset Q$ be an abelian extension with Galois group G . Then O_L is tame with respect to some Hopf subalgebra of KG if and only if L is tamely ramified except possibly at 2, and the first ramification group of G for the prime 2 has order dividing 2.*

PROOF. If L satisfies the ramification conditions, then the order $\mathcal{A}$ of O_L in KG is a Hopf algebra by Corollary 5.6. That O_L is tame then follows from Theorem 2.1.

Conversely, if O_L does not satisfy the ramification hypothesis, then the order $\mathcal{A}$ of O_L in QG is not a Hopf algebra. But then, as Bergé notes [1, p. 17], O_L cannot be locally free for any order in KG other than $\mathcal{A}$, so, in particular, O_L cannot be a tame J -module for any order J which is a Hopf subalgebra of KG . This completes the proof of Proposition 5.7.

6. Orders of Kummer extensions. We now proceed to the case of Kummer extensions of prime order.

Let $L \supset K$ be a Kummer extension of number fields of prime order l . If the order $\mathcal{A}$ of O_L in KG is a Hopf algebra, it is a Hopf algebra of the kind described by Tate and Oort [23], so by (3.3) $\mathcal{A} = H_{\mathcal{B}}$ for some ideal $\mathcal{B} = \ell^{l-1}$ dividing lO_K . Using this fact, we obtain a necessary condition for $\mathcal{A}$ to be a Hopf algebra.

THEOREM 6.1. *Let $L \supset K$ be a Kummer extension of prime order l . If $\mathcal{A}$, the order of O_L in KG , is a Hopf algebra isomorphic to $H_{\mathcal{B}}$ and $\mathcal{B}\mathcal{W} = lO_K$, then $\text{tr}(O_L) = \mathcal{W}$. Hence $\text{tr}(O_L)$ is the $(l-1)$ th power of an ideal of O_K .*

PROOF. If $\mathcal{A}$ is a Hopf algebra, then O_L is locally isomorphic to $\mathcal{A}^*$ as $\mathcal{A}$ -module, $\mathcal{A}^* = \text{Hom}_{O_K}(\mathcal{A}, O_K)$. Now $\mathcal{A}^*$ is the trivial Galois $\mathcal{A}^*$ -object, so $I\mathcal{A}^* = O_K$, I the space of integrals of $\mathcal{A}$, by [9, Proposition 2.3]. Since O_L is locally isomorphic to $\mathcal{A}^*$ as $\mathcal{A}$ -module, $lO_L = O_K$.

Locally at $\mathfrak{p}$, $\mathcal{A} = H_{\mathfrak{p}^{(l-1)s}}$ for some s , $0 \leq s \leq e$, where $lO_{K,\mathfrak{p}} = \mathfrak{p}^{(l-1)e}$ and p is a uniformizing parameter for $\mathfrak{p}$. So $\mathcal{A}$ corresponds to the ideal $\mathcal{B} = \prod \mathfrak{p}^{(l-1)s}$. But if θ generates the space of integrals of $H_{\mathfrak{p}^{(l-1)s}}$, then since $O_{K,\mathfrak{p}}G = H_{\mathfrak{p}^{(l-1)e}}$, $\text{tr} = p^{(l-1)(e-s)}\theta$. Thus

$$\text{tr } O_L = \prod p^{(l-1)(e-s)} = (lO_K) \left(\prod \mathfrak{p}^{(l-1)s} \right)^{-1} = (lO_K)(\mathcal{B})^{-1}.$$

Since $lO_K = (1 - \zeta)^{l-1}O_K$ and $\mathcal{B}$ is an $(l-1)$ th power by Theorem 3.3, $\mathcal{W} = \text{tr } O_L$ is an $(l-1)$ th power of an ideal of O_K .

One objective of the remainder of this paper is to prove the converse of this result, Theorem 17.3 below.

7. The local case. I: Which Hopf algebra can occur? In the following sections we will focus on the situation where K is a local field containing a primitive l th root of unity ζ , l prime, and $L \supset K$ is a Kummer extension, $L = K[z]$, $z^l = w \in K$, with Galois group $G = \langle \sigma \rangle$ acting on L by $\sigma(z) = \zeta z$. Let R be the valuation ring of K . We shall determine the tame and the Galois H_b -extensions of R contained in S , the integral closure of R in L , and, in particular, find criteria for S itself to be a tame H_b -extension of R for some b , where H_b is the Tate-Oort Hopf algebra $H_b = R[\xi]$, $\xi^l = b\xi$.

In this section we show that if T is an order over R in L which is a faithful J -extension of R for some Hopf algebra J of rank l , then J must be a sub-Hopf algebra of KG , hence, since J must be of the form H_b for some b in R , b must have an $(l-1)$ th root in R . We first look at L itself.

PROPOSITION 7.1. *Let K be a local or global number field containing $1/l$ and a primitive l th root of unity, l prime. Let L be a Galois field extension of K with Galois group $G = \langle \sigma \rangle$, cyclic of order l . For $b \neq 0$ in K , H_b acts faithfully on L if and only if b has a $(l-1)$ th root $\tilde{b}$ in K , if and only if $H_b \cong KG$.*

PROOF. Given b , let $K' = K[\tilde{b}]$, $L' = L \otimes K'$, $H'_b = H_b \otimes K'$. We have $H_{\omega_l} = KG \cong (KG)^* = H_1$, so by (3.2), ω_l has an $(l-1)$ th root $\tilde{\omega}_l$ in K .

Let $\varphi: H_b \otimes L \rightarrow L$ be a measuring. Then φ induces $\varphi': H'_b \otimes L' \rightarrow L'$, a measuring. But $H'_b = H_{\tilde{b}^{l-1}} \cong H_{\omega_l}$, so the action φ yields an action of $H_{\omega_l} = K'G$ on L' , that is, a map $G \rightarrow \text{Aut}_{K'}(L')$.

Now since L/K is a Galois field extension of prime degree l and $[K':K]$ divides $l-1$, L' is a field. For if $L = K[z]$, $f(x) = \text{Irr}(z, K)$, the minimal polynomial of z over K , and $g(x) = \text{Irr}(z, K')$, then, since L/K is Galois, $f(x) = \prod \sigma(g(x))$ where σ runs through a transversal of the stabilizer of $g(x)$ in G . Since $\deg(f(x)) = l$, prime, $\deg(g(x)) = 1$ or l . If $\deg(g(x)) = l$, L' is a field; if $\deg(g(x)) = 1$, then z is in K' , so $\text{Irr}(z, K)$ has degree $\leq \deg[K':K] < l$, impossible.

Thus if $L = K[z]$, $z^l = w$, then $L' = K'[z]$ is a field, and the only actions of G on L' are those given by $\sigma(z) = \zeta z$ for ζ some primitive l th root of unity.

Let $H'_{\omega_l} = K'G = K'[\theta]$, $\theta^l = \omega_l \theta$; $H_b = K[\xi]$, $\xi^l = b\xi$; then we have an isomorphism of Hopf algebras $H'_{\omega_l} \rightarrow H'_b$ by $\theta \mapsto \tilde{a}\xi$, $\tilde{a} = \tilde{\omega}_l/\tilde{b}$. Here

$$\theta = - \sum_{m=1}^{l-1} \chi^{-1}(m) \sigma^m.$$

Thus any action of G on L' extends uniquely to an action of H'_b on L' , and so given the action of G , $\sigma(z) = \zeta z$, for some root of unity ζ , we have

$$\xi z^i = \left(-\frac{1}{\tilde{a}} \sum_m \chi^{-1}(m) \xi^{im} \right) z^i.$$

Since $H_b = K[\xi]$ acts on L and $z^i \in L$, then ξz^i is in L for all i , that is, for all i , $-(1/\tilde{a}) \sum_m \chi^{-1}(m) \xi^{im}$ is in L . But since $\sum_m \chi^{-1}(m) \xi^{im} \in L$, that is the case if and only if $\tilde{a}$ is in L or $\sum_m \chi^{-1}(m) \xi^{im} = 0$ for all i ; and the latter possibility cannot occur, since otherwise ξ would act trivially on L and the action of H_b on L would be unfaithful.

Thus if H_b acts on L , then $\tilde{b} = \tilde{\omega}_l/\tilde{a}$ is in K , and $H_b \cong H_{\omega_l}$. That completes the proof of Proposition 7.1.

COROLLARY 7.2. *Let $L \supset K$ be a Galois extension of local fields, cyclic of order l , prime, where K contains $1/l$ and a primitive l th root of unity. Let R be the valuation ring of K , T an integral R -subalgebra of L with $TK = L$, which is an H_b -extension of R for b in R . Then H_b is an order over R in KG containing RG , and b is an $(l-1)$ th power in R .*

PROOF. If H_b acts on S , $H_b \otimes K$ acts on L , so since R is integrally closed, Proposition 7.1 implies that $\tilde{b}$, an $(l-1)$ th root of b , is in R , and the inclusion $H_b \subseteq KG$ then follows from (3.2). Since b divides l by definition of H_b , we have $RG \subseteq H_b$.

The uniqueness in Theorem 7.1 may also be obtained as a special case of Theorem 3.1 of [27].

8. Kummer orders. Now we begin the classification of H_b -extensions S as in (7.2) which are tame. First we consider the case $b = 1$, $H_b = (RG)^*$.

PROPOSITION 8.1. *Let $L \supset K$ be a Galois extension of local fields with Galois group G , cyclic of order l , and suppose K contains a primitive l th root of unity. Let R be the valuation ring of K , and let S be the integral closure of R in L . Let $\tilde{S}$ be the Kummer order of S [12], $\tilde{S} = \sum_{\chi \in \hat{G}} S_{\chi}$, where*

$$S_{\chi} = \{s \in S \mid \sigma(s) = \chi(\sigma)s \text{ for all } \sigma \text{ in } G\}.$$

Then $\tilde{S}$ is a tame $(RG)^$ -extension of R contained in S , $\tilde{S} = R[z]$, z^l in R , and the tame $(RG)^*$ -extensions of R contained in S are the G -graded subalgebras T of $\tilde{S}$,*

$$T = \sum_{i=0}^{l-1} R c_i z^i, \quad c_i \text{ in } R, c_i \neq 0, \text{ with } c_0 = 1.$$

PROOF. Let p be a uniformizing parameter for the maximal ideal $\mathfrak{p}$ of R .

First we identify $\tilde{S}$. Let $L = K[y]$, y^l in K . By altering y by an l th power, we can choose $y^l = w$ in R with $v_{\mathfrak{p}}(w)$, the $\mathfrak{p}$ -adic valuation of w , satisfying $0 \leq v_{\mathfrak{p}}(w) < l$, so y is in S .

If $v_{\mathfrak{p}}(w) = 0$, i.e. w is a unit, then $\tilde{S} = R[y]$. For since y^l is in R , the map $\chi: G \rightarrow K$ by $\chi(\sigma) = \sigma(y)/y$ is a character of G which generates the character group $\hat{G}$, and

$$S_{\chi^i} = S \cap L_{\chi^i} = S \cap Ky^i \supseteq Ry^i;$$

since y^i is a unit of S , $S_{\chi^i} = Ry^i$.

If $v_{\mathfrak{p}}(w) = r > 0$, let $rs - kl = 1$, and let $z = y^s/p^k$. Then $z^l = y^{sl}/p^{kl} = w^s/p^{kl}$ and $v_{\mathfrak{p}}(z^l) = rs - kl = 1$. In that case, z is the root of the Eisenstein polynomial $Z^l - z^l$, so L/K is totally ramified and $S = R[z]$. In that case, if $\chi(\sigma) = \sigma(z)/z$, then $S_{\chi^i} = Rz^i$, and $S = \tilde{S}$.

Set $(RG)^* = \sum Re_{\chi}$, $e_{\chi} = (\sum_{\sigma} \chi(\sigma^{-1})\sigma)/l$; the integral I of $(RG)^* = Re_{\chi_0}$. Then $e_{\chi}S = S_{\chi}$ and in particular, $IS = S_{\chi_0} = S^G = R$. Since $\tilde{S}$ is a faithful $(RG)^*$ -module of rank l , $\tilde{S}$ is a tame $(RG)^*$ -extension of R . $\tilde{S}$ is a Galois $(RG)^*$ -extension of R if and only if $\tilde{S} = R[z]$ with z a unit of R , if and only if $v_{\mathfrak{p}}(w) = 0$ (cf. Example 11.6 below).

Write $\tilde{S} = R[z]$ with $\sigma(z) = \chi(\sigma)z$.

Let T be an $(RG)^*$ -module subalgebra of S . Then

$$T = (RG)^*T = \sum Re_{\chi}T = \sum e_{\chi}T = \sum T_{\chi'}$$

and $T_{\chi'} \subseteq S_{\chi'}$. Thus T is a G -graded subalgebra of $\tilde{S}$. Tameness means simply that $T_{\chi_0} = R$ and each $T_{\chi'} \neq (0)$, from which the description of T given in the statement of the theorem is clear.

COROLLARY 8.2. *With L, K, S, R, G as in Proposition 8.1, there exists a Galois $(RG)^*$ -module subalgebra of S if and only if $S = R[z]$ with z^l a unit of R , in which case $\tilde{S}$ is the unique such Galois $(RG)^*$ -module algebra.*

PROOF. T is a Galois $(RG)^*$ -module algebra if and only if $T = \sum T_\chi$ with $T_\chi = Rz_\chi$ and $T_\chi T_\psi = T_{\chi\psi}$ for all χ, ψ in $\hat{G}$, in particular, $T_\chi^l = R$. Thus T is Galois if and only if each z_χ is a unit of S , in which case $T_\chi = S_\chi$, $T = \tilde{S}$ and $\tilde{S} = R[z]$ with z^l a unit of R .

9. Galois extensions. In contrast to the situation for $H_b = H_1 = (RG)^*$, we have

THEOREM 9.1. *Let R be a local ring containing a primitive l th root of unity, l prime, with l contained in the maximal ideal $\mathfrak{p}$ of R ; let G be cyclic of order l . Let H_b be a Tate-Oort Hopf R -algebra, $RG \subseteq H_b \subseteq (RG)^*$. Suppose $b \in \mathfrak{p}$. Then any tame H_b -extension of R is Galois.*

PROOF (from [13, Theorem 4.4]). Let $H_b = R[\xi]$, $\xi^l = b\xi$; then $\phi = \xi^{l-1} - b$ generates the space of integrals of H_b . If S is tame, then $\phi s = 1$ for some s in S .

We claim that $s, \xi s, \xi^2 s, \dots, \xi^{l-1} s$ is an R -basis of S . To see this, it suffices to show it mod p . But mod p , $\xi^{l-1} s \equiv 1$ and $\xi^l s \equiv 0$. Suppose

$$\sum_{i=0}^{l-1} r_i \xi^i s \equiv 0 \pmod{p}.$$

If k is the least index with $r_k \neq 0$, then since $\xi^l s \equiv 0 \pmod{p}$,

$$0 \equiv \xi^{l-1-k} \left(\sum_{i=1}^{l-1} r_i \xi^i s \right) \equiv r_k \xi^{l-1} s \equiv r_k.$$

So, mod p , $s, \xi s, \xi^2 s, \dots, \xi^{l-1} s$ are linearly independent, so are a basis. Thus, $s, \xi s, \dots, \xi^{l-1} s$ span S over R [5, II, §3, No. 2, Corollaire 2]. But since S is tame S is free over R of rank l . Hence $s, \xi s, \dots, \xi^{l-1} s$ form a basis of S .

Let $h_0, \dots, h_{l-1}$ be a dual basis in H_b^* for $1, \xi, \xi^2, \dots, \xi^{l-1}$ in H_b . Now S is an H_b^* -object via the map $\alpha: S \rightarrow S \otimes H_b^*$ given by

$$\alpha(s) = \sum_i \xi^i s \otimes h_i.$$

Define $\gamma: S \otimes S \rightarrow S \otimes H$ by

$$\gamma(s \otimes t) = \sum_i s \xi^i t \otimes h_i.$$

Then S is a Galois H_b -extension of R if and only if γ is an isomorphism. Since $S \otimes S$ and $S \otimes H_b^*$ are of equal ranks as free R -modules, it suffices to show that γ is surjective modulo p [5, II, §3, No. 2, Corollaire 1]. So for the rest of the proof, assume R is a field with $b = l = 0$.

We show that γ is surjective by finding, for each i , elements a_k and b_k in S such that $\gamma(\sum a_k \otimes b_k) = 1 \otimes h_i$, as follows: we set $b_k = \xi^{l-1-k} s$ for all k , and

$$a_k = \begin{cases} 0 & \text{for } k > i, \\ 1 & \text{for } k = i, \\ -\sum_{m>k} a_m (\xi^k b) & \text{for } k < i. \end{cases}$$

Then

$$\begin{aligned}
 \gamma\left(\sum_k a_k \otimes b_k\right) &= \sum_j \left(\sum_k a_k \xi^j b_k\right) \otimes h_j \\
 &= \sum_j \left(\sum_k a_k \xi^j \xi^{l-1-k} s\right) \otimes h_j \\
 &= \sum_j \left(\sum_{k \geq j} a_k \xi^{l-1+j-k} s\right) \otimes h_j
 \end{aligned}$$

since $\xi^l = b\xi = 0$,

$$= \sum_j \left(\sum_{k \geq j} a_k \xi^j b_k\right) \otimes h_j.$$

For $j > i$, $k \geq j$, $a_k = 0$, so

$$\sum_{k \geq j} a_k \xi^j b_k = 0.$$

For $j = i$,

$$\sum_{k \geq i} a_k \xi^i b_k = a_i \xi^i b_i + \sum_{k > i} a_k \xi^i b_k = a_i \xi^i b_i;$$

since $a_i = 1$ and $\xi^i b_i = \xi^{l-1} s = 1$, $\xi^i b_i = 1$.

For $j < i$,

$$\sum_{k \geq j} a_k \xi^j b_k = \sum_{k > j} a_k \xi^j b_k + a_j \xi^j b_j.$$

Now $\xi^j b_j = 1$, and, substituting for a_j , we get

$$= \sum_{k > j} a_k \xi^j b_k - \sum_{m > j} a_m \xi^j b_m = 0.$$

Thus $\gamma(\sum a_k \otimes b_k) = 1 \otimes h_i$, completing the proof.

10. Frobenius conditions on Galois H -extensions. We develop some general theory for H -extensions which may be of independent interest.

Let R be a commutative ring with unity, and H a finite (i.e. finitely generated and projective as R -module) R -Hopf algebra. Finiteness implies that the space of left integrals of H ,

$$I = \{ \theta \in H \mid h\theta = \varepsilon(h)\theta, \text{ for all } h \text{ in } H \}$$

is a rank one projective R -module, as is the space of right integrals. Following Larson and Sweedler [16], H is called unimodular if the space of left integrals equals the space of right integrals.

Let S be an R -algebra, finitely generated and projective as R -module ("finite"), and an H -extension.

If $S^H = R$, then S is a Galois H -extension of R if and only if the map

$$j: S \# H \rightarrow \text{End}_R(S), \quad j(s \# h)(t) = sh(t)$$

is an isomorphism [7, Theorem 9.3]. Denote the image of S in $\text{End}_R(S)$ under j by S_l , the set of left multiplications by elements of S .

THEOREM 10.1. *Let H be a finite unimodular Hopf algebra with space of integrals I , and S a finite R -algebra and an H -module algebra with $S^H = R$. Then S is a Galois H -extension of R if and only if the map $\varphi: I \otimes S \rightarrow S^* (= \text{Hom}_R(S, R))$, $\varphi(\theta, s)(t) = \theta(st)$ for θ in I , s, t in S , is an isomorphism.*

PROOF. For M an R -submodule of $\text{End}_R(S)$ denote by $I \cdot M$ the set $\{\theta m \mid m \text{ in } M, \theta \text{ in } I\}$. Then since $IS \subseteq S^H = R$, $I \cdot S_l \subseteq S^* \subseteq \text{End}_R(S)$. The image of φ is then $I \cdot S_l$. Since $I \otimes S$ and S^* are both finitely generated projective R -modules of equal ranks, φ is an isomorphism if and only if φ is an epimorphism, if and only if $I \cdot S_l = S^*$. So we shall show that S is Galois if and only if $I \cdot S_l = S^*$.

LEMMA 10.2. $I \cdot (S \# H)_l = I \cdot S_l$.

Assuming the lemma, the proof of the theorem proceeds as follows.

Suppose $I \cdot S_l = S^*$. Then we have the diagram

$$\begin{array}{ccc} S \otimes S^* & = & S \otimes I \cdot S_l = S \otimes I \cdot (S \# H)_l \\ \wr \mu & & \downarrow m \\ \text{End}(S) & \supseteq & j(S \# H) \end{array}$$

where

$$\begin{aligned} m(s \otimes \theta(s \# h))(t) &= \left(\sum s(\theta_{(1)} s') \# \theta_{(2)} h \right)(t) \\ &= \sum s(\theta_{(1)} s') (\theta_{(2)} h)(t) \\ &= j \left(\sum s(\theta_{(1)} s') \# \theta_{(2)} h \right)(t) \end{aligned}$$

and $\mu(s \# f)(t) = sf(t)$.

The diagram commutes: for given s, s' in S , θ in I , we have $\mu(s \otimes \theta \cdot s')(t) = s\theta(s't)$, while

$$m(s \otimes \theta \cdot s')(t) = \sum s(\theta_{(1)} s') (\theta_{(2)} t) = s\theta(s't) \quad (\text{by measuring}).$$

Thus $j(S \# H) = \text{End}_R(S)$, and S is Galois.

Conversely, suppose S is a Galois H^* -object. Then $\text{End}_R(S) \cong S \# H$, and by Morita theory

$$\text{End}_R(S) \cong S \otimes I \cdot (S \# H)_l = S \otimes I \cdot S_l$$

by Lemma 10.2, where the map $S \otimes I \cdot S_l$ to $\text{End}_R(S)$ is μ . Thus the diagram

$$\begin{array}{ccc} S \otimes S^* & \xleftarrow{\quad} & S \otimes I \cdot S_l \\ \cong & & \cong \\ & \text{End}_R(S) & \end{array}$$

commutes, and so the inclusion $I \cdot S_l \subset S^*$ induces an isomorphism $S \otimes I \cdot S_l \cong S \otimes S^*$. Since S is R -faithfully flat, $I \cdot S_l = S^*$.

We are left only with proving the lemma: $I \cdot S_l = I \cdot (S\sharp H)_l$.

PROOF OF LEMMA 10.2. For $x, y \in S$, $h \in H$, $\phi \in I$, we have

$$\begin{aligned}
 (\phi(y\sharp h))(x) &= \sum \phi_{(1)}(y)\phi_{(2)}h(x) \\
 &= \sum \phi_{(1)}(y)\phi_{(2)}\varepsilon(h_{(1)})h_{(2)}(x) \quad \text{since } (1 \otimes \varepsilon)\Delta = \text{id}, \\
 &= \sum \phi_{(1)}(\varepsilon(h_{(1)}^\lambda))(y)\phi_{(2)}h_{(2)}(x) \\
 &= \left(\sum \phi_{(1)}\varepsilon(h_{(1)}^\lambda) \otimes \phi_{(2)}h_{(2)}\right)(y \otimes x) \\
 &= \Delta(\phi\varepsilon(h_{(1)}^\lambda))(1 \otimes h_{(2)})(y \otimes x) \\
 &= \Delta(\phi h_{(1)}^\lambda)(1 \otimes h_{(2)})(y \otimes x) \quad \text{since } \phi \text{ is a right integral,} \\
 &= \left(\sum \phi_{(1)}h_{(1)}^\lambda \otimes \phi_{(2)}h_{(2)}^\lambda h_{(3)}\right)(y \otimes x) \\
 &= \left(\sum \phi_{(1)}h_{(1)}^\lambda \otimes \phi_{(2)}\varepsilon(h_{(2)}^\lambda)\right)(y \otimes x) \\
 &= \left(\sum \phi_{(1)}h_{(1)}^\lambda \varepsilon(h_{(2)}^\lambda) \otimes \phi_{(2)}\right)(y \otimes x) \\
 &= \left(\sum \phi_{(1)}h^\lambda \otimes \phi_{(2)}\right)(y \otimes x) = \sum \phi_{(1)}h^\lambda(y)\phi_{(2)}(x) \\
 &= \phi(h^\lambda y \cdot x) = \phi(h^\lambda y)_l(x).
 \end{aligned}$$

So $I \cdot (S\sharp H)_l \subseteq I \cdot S_l$. The opposite inclusion is clear.

EXAMPLE 10.3. Let K be a domain of characteristic p . Let $H = K[f_1, \dots, f_n]$ with $f_i^p = 0$, f_i commuting and primitive, $\varepsilon(f_i) = 0$ for all i .

Let $L = K[x_1, \dots, x_n]$ with $x_i^p = a_i$ in K , acted on by H with f_i acting by $\partial/\partial x_i$.

For $R = (r_1, r_2, \dots, r_n)$, set

$$x^R = x_1^{r_1} \cdots x_n^{r_n}, \quad f^R = f_1^{r_1} \cdots f_n^{r_n},$$

and

$$y^R = x^R / (r_1)! \cdots (r_n)!.$$

Setting $P - 1 = (p - 1, p - 1, \dots, p - 1)$, the space of integrals of H is generated by $\theta = f^{P-1}$. Then $\{y^R \mid 0 \leq r \leq p - 1\}$ is a K -basis of L , and if $\{\varphi_R\}$ is a dual basis, we have $\varphi_{P-1} = \theta$. Then $\varphi_R(y^S) = \theta(y^{P-1-R}y^S)$, and $\theta \cdot L_l = L^*$. By Theorem 10.1 L is a Galois H -extension of K .

Using Theorem 10.1 we may give a Galois-theoretic proof of a well-known result of Larson and Sweedler [16]. The usual proof (cf. [16, 18, 19, 22]), uses a Hopf module approach, which we avoid.

COROLLARY 10.4. $H \cong I \otimes H^*$ as left H -modules, where I is the space of integrals of H and $I \otimes H^*$ is a left H -module via the action of H on H^* given by $(x \cdot f)(y) = f(yx)$.

PROOF. We can assume that the H -action on H^* is given by $(x \cdot f)(y) = f(x^\lambda \cdot y)$, for the antipode $\lambda: H^* \rightarrow H^*$ induces an isomorphism between $H_1^* = H^*$ with action $(xf)(y) = f(x^\lambda \cdot y)$ and $H_2^* = H^*$ with action $(xf)(y) = f(yx)$.

If θ is an integral of H , then

$$\sum_{(\theta)} x^\lambda \theta_{(1)} \otimes \theta_{(2)} = \sum_{(\theta)} \theta_{(1)} \otimes x \theta_{(2)}$$

(cf. [22, p. 104]).

Define $\varphi: I \times H^* \rightarrow H$ by

$$\langle \varphi(\theta, f), g \rangle = \langle \theta, fg \rangle$$

for f, g in H^* , θ in I . Since H^* is a Galois H^* -object, φ is an isomorphism by Theorem 10.1. Then φ is an H -module isomorphism. For

$$\begin{aligned} \langle x\varphi(\theta, f), g \rangle &= \langle x, g_{(1)} \rangle \langle \varphi(\theta, f), g_{(2)} \rangle = \langle x, g_{(1)} \rangle \langle \theta, fg_{(2)} \rangle \\ &= \langle x, g_{(1)} \rangle \langle \theta_{(1)}, f \rangle \langle \theta_{(2)}, g_{(2)} \rangle = \langle \theta_{(1)}, f \rangle \langle x\theta_{(2)}, g \rangle \\ &= \langle x^\lambda \theta_{(1)}, f \rangle \langle \theta_{(2)}, g \rangle = \langle x^\lambda, f_{(1)} \rangle \langle \theta_{(1)}, f_{(2)} \rangle \langle \theta_{(2)}, g \rangle \\ &= \langle x^\lambda, f_{(1)} \rangle \langle \theta, f_{(2)}g \rangle = \langle \theta, \langle f_{(1)}, x^\lambda \rangle f_{(2)}g \rangle \\ &= \langle \theta, (x \cdot f)g \rangle = \langle \varphi(\theta, x \cdot f), g \rangle. \end{aligned}$$

So $x\varphi(\theta, f) = \varphi(\theta, x \cdot f)$, completing the proof.

COROLLARY 10.5 (PAREIGIS [19]). *As left H -modules, $H^* \cong H$, i.e. H is a Frobenius R -algebra, if and only if I is R -free.*

REMARK 10.6. The condition that H is unimodular, i.e. that the spaces of left integrals and right integrals are equal, is obvious if H is commutative. Unimodularity has been studied by Larson and Sweedler [16], who showed that a finite Hopf algebra over a field is unimodular if H has a left integral θ with $\varepsilon(\theta) \neq 0$, which is equivalent to H being semisimple; or if H has an antipode of order 2 and H^* is separable. They give an example of a finite cocommutative Hopf algebra H with H^* connected over a field of characteristic 2 which is not unimodular.

The trivial Galois H^* -object is H^* itself, which is acted upon by H . Theorem 10.1 then specializes, for unimodular H , to the result of Larson and Sweedler [16] that for a finite bialgebra with antipode, the bilinear form $\beta: H^* \times H^* \rightarrow R$, $\beta(p, q) = (pq)\theta$, associated to a generator θ of the space of integrals of H , is nonsingular.

11. Discriminants. We may define a codifferent using the integrals of H .

PROPOSITION 11.1. *Let R be a domain with quotient field K , H a finite unimodular Hopf R -algebra with space of integrals I . Let S be a finite R -algebra and an H -extension of R such that $L = S \otimes_R K$ is a Galois $H \otimes_R K$ -extension of K . Let $C = \{x \in L \mid \theta x \in R \text{ for all } \theta \text{ in } I\} \supseteq S$. Then $I \cdot C_I = S^*$. Hence S is a Galois H -extension of S^* if and only if $C = S$.*

PROOF. Both conditions are true if and only if they are true locally, so we may assume R is a local ring and $I = R\theta$ for some θ . Since L is Galois, $I \cdot L_I = L^*$ by Theorem 10.1, and so, viewing θ as in $\text{Hom}_R(S, R) \subseteq \text{Hom}_K(L, K)$, $S^* \subseteq \theta \cdot L_I$, and $S^* = \theta \cdot C_I$, where $C = \{x \text{ in } L \mid \theta x \in S^*\}$. But

$$\begin{aligned} S &\subseteq C \\ \beta|_S &\searrow \swarrow \beta \\ S^* & \end{aligned}$$

commutes, where $\beta(x)(y) = \theta(xy)$. Since $\beta: C \rightarrow S^*$ is an isomorphism, $S = C$ if and only if $\beta|_S$ is an isomorphism, if and only if $I \cdot S_I = S^*$. Theorem 10.1 applies to complete the proof.

To define a discriminant of an H -extension S of R , first assume R is local, so that $I = R\theta$ and S is a free R -module. Let $\{x_1, \dots, x_n\}$ be a basis of S as a free R -module.

Define $\delta_H(x_1, \dots, x_n) = \det(\theta(x_i x_j))$.

Let $\{f_1, \dots, f_n\}$ be a dual basis in S^* to $\{x_1, \dots, x_n\}$. Let $\{y_1, \dots, y_n\}$ in C be such that $f_i = \theta \cdot y_i$. Write $y_i = \sum_j a_{ij} x_j$, a_{ij} in K . Then

$$\delta_{ij} = f_i(x_j) = \theta(y_i x_j) = \theta\left(\sum_k a_{ik} x_k x_j\right) = \sum_k a_{ik} \theta(x_k x_j)$$

so $(a_{ik})(\theta(x_k x_j)) =$ the $n \times n$ identity matrix.

Thus $(\theta(x_k x_j))$ is invertible if and only if all a_{ik} are in R , if and only if all y_i are in S , if and only if $S^* = \theta \cdot S_I$, if and only if S is a Galois H -object.

Globalizing, we get the following:

DEFINITION 11.2. $\delta_H(S/R)$, the discriminant of S with respect to H , is the ideal of R generated by $\{\det \theta(x_i x_j)\}$ for θ in I and $\{x_1, \dots, x_n\}$ running through K -bases of L contained in S .

PROPOSITION 11.3. *Under the same assumptions as in Proposition 11.1, $\delta_H(S/R) = R$ if and only if S is a Galois H^* -object.*

PROOF. Both conditions are true if and only if they are true locally. So we can assume S is a free R -module with basis $\{x_1, \dots, x_n\}$, and $I = R\theta$, in which case the above argument applies.

REMARKS. 11.4. When $H = RG$, S is a Galois H -extension of R if and only if S is a Galois extension of R with group G , in the sense of Chase, Harrison, Rosenberg [6]. In that case, $H = RG$, which is unimodular with space of integrals generated by $\theta = \sum_{\sigma \in G} \sigma = \text{tr}$; $\delta_H(S/R)$ is the classical discriminant. The above results then specialize to the results on pages 92–93 of DeMeyer and Ingraham [10].

EXAMPLE 11.5 (CLASSICAL). Suppose R is a domain with quotient field K , and R contains a primitive n th root of unity ζ ; let $S = R[z]$ with $z^n = b$, and $H = RG$, G cyclic of order n with generator σ acting on S by $\sigma(z) = \zeta z$. Then $\delta_H(S/R) = \det(\text{tr}(z^i z^j))$. Since

$$\text{tr}(z^r) = \begin{cases} 0, & r \not\equiv 0 \pmod{n}, \\ nz^r, & n \mid r, \end{cases}$$

we have

$$\det(\text{tr}(z^i z^j)) = \det \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & & & nb \\ \vdots & & \ddots & \\ 0 & nb & & 0 \end{pmatrix} = \pm n^n b^{n-1}.$$

Hence S is a Galois H -extension of R , $H = RG$ if and only if n and b are units of R . Of course, $\delta_H(S/R)$ is the classical discriminant.

Note here that if n is a unit of R , then $RG = (RG)^*$. Consider, then,

EXAMPLE 11.6. Same S , but do not assume R contains a primitive n th root of unity. Let $H = (RG)^* = \sum_{k=0}^{n-1} Re_k$, $e_k(\sigma^j) = \delta_{k,j}$. Then $I = Re_0$.

Define H on S by $e_k(z^j) = \delta_{k,j}z^j$. Then

$$\begin{aligned}\delta_H(S/R) &= \det(e_0(z^i z^j)) \\ &= \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & b \\ \vdots & & \ddots & \\ 0 & b & & 0 \end{pmatrix} \\ &= \pm b^{n-1}.\end{aligned}$$

Hence S is a Galois H -extension of R for $H = (RG)^*$ if and only if b is a unit of R . See also Chase and Sweedler [7, Example 4.16].

12. The local case. II: A chain of Galois module algebras. The discriminant permits us to construct a chain of Galois module algebras inside the ring of integers of a Kummer extensions of local fields.

EXAMPLE 12.1. Let R be the completion at some prime lying over l of a finite extension of $Z[\xi]$, ξ a primitive l th root of unity, l an odd prime. Let p generate the maximal ideal of R , and let $lR = (pR)^{e(l-1)}$. Let H_b be the Tate-Oort Hopf algebra $R[\xi]$, with $\xi^p = b\xi$, and comultiplication

$$\begin{aligned}\Delta(\xi^i) &= (1 \otimes \xi^i) + (\xi^i \otimes 1) \\ &+ \frac{w_i}{1-l} \left[\sum_{j=1}^{i-1} \frac{\xi^j}{\omega_j} \otimes \frac{\xi^{i-j}}{\omega_{i-j}} + \sum_{j=i}^{l-1} \frac{a^{\xi^j}}{\omega_j} \otimes \frac{\xi^{l-1+i-j}}{\omega_{l-1+i-j}} \right]\end{aligned}$$

where $ab = \omega_l$.

Let K be the quotient field of R , and $L = K[z]$, $z^l = w$, w in R . For $0 \leq s \leq e$, let $S = R[x]$, $x = (z-1)/p^s$. Then x satisfies $(1+p^s z)^l = w$, or

$$x^l + \binom{l}{l-1} \frac{x^{l-1}}{p^s} + \cdots + \binom{l}{r} \frac{x^r}{p^{(l-r)s}} + \cdots + \binom{l}{1} \frac{x}{p^{(l-1)s}} = \frac{w-1}{p^{sl}}.$$

Since $s \leq e$, all coefficients of x^r , $r > 0$, are in R , and x is integral over R if and only if $w \equiv 1 \pmod{p^{sl}}$.

Suppose $w = 1 + p^{sl}c$, c in R .

Let H_b , $b = p^{s(l-1)}$ act on S by $\xi x = 1 + p^s x = z$, $\xi z = p^s z$. Then H_b sends Rx into S . But then, using the measuring property:

$$\begin{aligned}\xi^i(x^r x^s) &= x^r(\xi^i x^s) + (\xi^i x^r)x^s \\ &+ \frac{\omega_i}{1-l} \left[\sum_{j=1}^{i-1} \left(\frac{\xi^j x^r}{\omega_j} \right) \left(\frac{\xi^{i-j} x^s}{\omega_{i-j}} \right) + \sum_{j=i}^{l-1} a \left(\frac{\xi^j x^r}{\omega_j} \right) \left(\frac{\xi^{l-1+i-j} x^s}{\omega_{l-1+i-j}} \right) \right],\end{aligned}$$

one sees easily by induction on k that H sends Rx^k into S for all $k > 0$. Thus H_b acts on S .

Now $RG = H_{p^{e(l-1)}} \subseteq H_b = H_{p^{s(l-1)}} \subseteq H_1 = (RG)^*$; if $l = up^{e(l-1)}$ for some unit u of R , then $\theta = (\sum \sigma)/p^{(e-s)(l-1)}u$ generates the space of integrals of H .

Thus

$$\theta(z^r) = \begin{cases} 0, & l \nmid r, \\ z^r p^{s(l-1)}, & l \mid r. \end{cases}$$

and so

$$\begin{aligned} \delta_H(1, z, z^2, \dots, z^{l-1}) &= \det(\theta(z^{i+j})) \\ &= \det \begin{pmatrix} p^{s(l-1)} & 0 & \dots & 0 \\ 0 & & & p^{s(l-1)}w \\ \vdots & & \ddots & \\ 0 & p^{s(l-1)}w & & 0 \end{pmatrix} \\ &= w^{l-1} p^{s(l(l-1))}. \end{aligned}$$

Now $z = 1 + p^s x$, so

$$z^r = \sum_{k=0}^r p^{sk} \binom{r}{k} x^k.$$

So

$$\delta_H(1, x, x^2, \dots, x^{l-1}) \det(A)^2 = p^{s(l(l-1))} w^{l-1},$$

where

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & p^s & & \cdot \\ 1 & \binom{2}{1} p^s p^{2s} & & \cdot \\ & \dots & & 0 \\ \cdot & \cdot & \dots & p^{(l-1)s} \end{pmatrix}.$$

Thus $(\det A)^2 = p^{l(l-1)s}$ and $\delta_H(1, x, x^2, \dots, x^{l-1}) = w^{l-1}$, a unit of R . Thus $S = R[x]$, $x = (z-1)/p^s$, is a Galois $H_{p^{s(l-1)}}$ -extension of R .

If $L = K[z]$, $z^l = w$, $w = 1 + p^{ql+r}u$, $0 \leq r < l$, $0 \leq q \leq e$, $ql+r \geq 1$, u a unit of R , then we get a chain of Galois extensions of R contained in the integral closure S of R in L : $S_0 \subsetneq S_1 \subsetneq S_2 \subsetneq \dots \subsetneq S_q$ where $S_s = R[(z-1)/p^s]$ is a Galois $(H_{p^{s(l-1)}})$ -extension of R . In particular, S_0 is as in Example 11.6, and is the Kummer order $\tilde{S}$ of S arising in Theorem 8.1.

Summarizing, we have shown

THEOREM 12.2. *Let K be a local field with valuation ring R , maximal ideal $\mathfrak{p} = pR$ and $l \in \mathfrak{p}$. Let $L \supset K$ be a Kummer extension of degree l , $L = K[z]$, $z^l = w = 1 + up^k$, u a unit of R , k maximal, $k > 0$, $k = ql + r$, where $0 \leq r < l$ and $0 < r$ if $q < e$. Let S be the integral closure of R in L . Then there is a chain of Galois extensions of R , $S_0 \subsetneq S_1 \subsetneq \dots \subsetneq S_h$, all contained in S , where S_k is a Galois $H_{p^{k(l-1)}}$ -extension of R , $0 \leq k \leq h$, and $h = \min\{q, e\}$.*

13. Some lemmas on Kummer extensions of prime order. Throughout this section, let K be a local field, a finite extension of $\mathcal{Q}_l$, with valuation ring R , maximal ideal $\mathfrak{p} = pR$, $lR = \mathfrak{p}^{e(l-1)}$, and L a Kummer extension of K of prime order l .

We wish to show that the chain of Galois extensions described in Theorem 12.2 contains all the Galois extensions of R contained in L . We need a preliminary lemma.

LEMMA 13.1. *Let $z^l = 1 + up^{lq+r}$, u a unit of R , where $q < e$ and $0 < r < l$. Set $x = (z - 1)/p^q$. Then $v_L(x) = r$.*

PROOF. Since $z^l = 1 + up^{lq+r}$, x satisfies

$$0 = ((1 + p^q x)^l - 1 - up^{lq+r})/p^{ql}$$

or

$$(13.2) \quad 0 = x^l + \frac{lp^{q(l-1)}}{p^{ql}} x^{l-1} + \cdots + \binom{l}{k} \frac{p^{qk} x^k}{p^{ql}} + \cdots + \frac{lp^q}{p^{ql}} - up^r.$$

Since $q < e$, $L \supset K$ is totally ramified, and $v_L(p) = l$. In order that equation (13.2) hold, (13.2) must contain two terms whose valuations are equal and minimal. Now

$$v_L(x^l) = lv_L(x), \quad v_L(up^r) = lr$$

and for $1 \leq k \leq l-1$,

$$\begin{aligned} v_L\left(\binom{l}{k} \frac{x^k}{p^{q(l-k)}}\right) &= el(l-1) + kv_L(x) - lq(l-k) \\ &\geq el(l-1) - l(e-1)(l-1) + kv_L(x) \\ &\geq l(l-1) + kv_L(x). \end{aligned}$$

Thus

$$v_L(up^r) < v_L\left(\binom{l}{k} \frac{x^k}{p^{q(l-k)}}\right)$$

unless $r = l-1$ and $v_L(x) = 0$. But then $0 = v_L(x^l)$ and x^l is the unique term in minimal valuation, impossible. So we must have $lv_L(x) = lr$, and $v_L(x) = r$, as claimed.

The following result will help to identify when the ring of integers of L is a Galois extension.

PROPOSITION 13.3. *Suppose $L = K[z]$ is totally ramified. Suppose $z^l = 1 + up^{lq+r}$, u a unit, $q < e$, $2 \leq r \leq l-1$. Let $T = R[x]$ where $x = (z - 1)/p^q$. Then T is not integrally closed.*

PROOF. We have

$$\frac{z^l - 1}{p^{ql}} = \left(\prod_{\xi \neq 1} \frac{\xi z - 1}{p^q} \right) \frac{z - 1}{p^q} = up^r.$$

Let

$$y = \frac{1}{p^{r-1}} \prod_{\xi \neq 1} \frac{\xi z - 1}{p^q} = \frac{1}{p^{r-1}} \prod_{i=1}^{l-1} \sigma^i(x).$$

Then $yx = up$, and, using Lemma 13.1, $v_L(y) = l - r \geq 0$, and y is in S , the integral closure of R in L .

However,

$$\begin{aligned} p^{r-1}y &= \frac{z^l - 1}{p^{ql}} \Big/ \frac{z - 1}{p^q} = \frac{1 + z + z^2 + \cdots + z^{l-1}}{p^{q(l-1)}} \\ &= \frac{1}{p^{q(l-1)}} \sum_{k=0}^{l-1} (1 + p^q x)^k = \frac{1}{p^{q(l-1)}} \sum_{k=0}^{l-1} \sum_{m=0}^k \binom{k}{m} p^{mq} x^m \\ &= \frac{1}{p^{q(l-1)}} \sum_{m=0}^{l-1} \left(\sum_{k=m}^{l-1} \binom{k}{m} \right) p^{mq} x^m \\ &= \frac{1}{p^{q(l-1)}} \sum_{m=0}^{l-1} \binom{l}{m+1} p^{mq} x^m. \end{aligned}$$

So

$$y = \frac{1}{p^{q(l-1)+(r-1)}} \sum_{m=0}^{l-1} \binom{l}{m+1} p^{mq} x^m.$$

Let c_m be the coefficient of x^m , $m = 0, \dots, l-1$. Since $q(l-1) + r - 1 < e(l-1) = v_K(l)$, c_m is in R for all $m = 0, 1, 2, \dots, l-2$. But

$$c_{l-1} = \frac{p^{(l-1)q}}{p^{(l-1)q+(r-1)}}$$

is not in R if $r > 1$. So y is not in $T = R[x]$, and T is not integrally closed.

Finally we need to know how we can adjust a generator z of a Kummer extension $L = K[z]$. We retain the hypothesis of this section.

Recall that $lR = \mathfrak{p}^{e(l-1)}$.

PROPOSITION 13.4. *Let $L \supset K$ be a Kummer extension, then $z \in L$ may be chosen so that $L = K[z]$, $z^l = w \in R$ and*

- (i) w generates $\mathfrak{p}$, or
- (ii) $w = 1 + up^k$, u a unit of R , $k = lq + r \geq 1$, $0 \leq r < l$, and
 - (a) $r \neq 0$ or
 - (b) $q \geq e$.

If $w = 1 + up^k$ with $k = lq + r \geq 1$, $k < le$ and $r \neq 0$, then k is maximal for all possible z with $L = K[z]$, $z^l \in R$.

PROOF. Let $L = K[y]$, $y^l = v$. If $vR = \mathfrak{p}^t$ and $l \nmid t$, find s, m with $ts = 1 + lm$, then $(yp^{-m})^s = z$ satisfies

$$z^l = w = (yp^{-m})^{sl} = v^s p^{1-ts} = (p^t u')^s p p^{-ts} = up \quad \text{for some units } u, u' \text{ of } R.$$

If $l \nmid t$, $t = lq$, then $(y/p^q)^l$ is a unit of R , so we can assume $y^l = v \notin \mathfrak{p}$. Suppose $v = 1 + up^k$, u a unit of R , for some $k > 0$. If $l \nmid k$, $k = lq$, let $-u \equiv v_1^l \pmod{\mathfrak{p}}$ (possible since $R/\mathfrak{p}$ is a finite field of characteristic l) and set $c = 1 + v_1 p^q$, and $z = cy$, then

$$\begin{aligned} z^l &= (yc)^l = (1 + up^k)(1 + v_1 p^q)^l \\ &= 1 + up^k + v_1^l p^{ql} + v_1 p^{k+ql} + lv_2, \end{aligned}$$

some $v_2 \in \mathfrak{p}^q$,

$$\equiv \begin{cases} 1 \pmod{\mathfrak{p}^{k+1}} & \text{if } k < le, \\ 1 \pmod{\mathfrak{p}^{el}} & \text{if } k \geq le. \end{cases}$$

Repeating this construction as needed, we may eventually find z with $z^l = w = 1 + up^k$ with $l \nmid k$ or $k \geq le$. If $k = 0$ the argument is similar.

To show maximality, first note that given z with $L = K[z]$, $z^l \in R$, all other elements of L with $y^l \in R$ have the form $y = cz^s$, $c \in R$, $1 \leq s \leq l-1$.

Suppose $z^l = w = 1 + up^k$, $k \not\equiv 0 \pmod{l}$, u a unit of R , $k < le$. For any $c = 1 + vp^d$, v a unit of R , and any s , $1 \leq s \leq l-1$, we have

$$\begin{aligned} (cz^s)^l &= (1 + vp^d)^l (1 + up^k)^s \\ &= (1 + v^l p^{dl} + lp^d u_0)(1 + u_1 p^k), \quad u_0, u_1 \text{ units.} \end{aligned}$$

If $k < le$ then

$$(cz^s)^l = 1 + u_2 p^n, \quad u_2 \text{ a unit,}$$

where $n = \min\{k, dl\}$. Hence if $k < le$, $k \not\equiv 0 \pmod{l}$, then k is maximal.

14. The local case. III: Galois orders in L . Let $L = K[z]$, $z^l = w \in R$, be a Kummer extension of local fields, with $l \in \mathfrak{p} = pR$, the maximal ideal of R . In this section we will classify the Galois and tame extensions of R which are orders over R in L .

The case where $l \nmid v_{\mathfrak{p}}(w)$ was done in Proposition 8.1. So throughout this section assume w is a unit of R . In view of Proposition 13.4 we may assume $w = 1 + p^k u$, u a unit of R , where $k = ql + r$, and $1 \leq r < l$ or $q \geq e$.

Recall that inside S , the integral closure of R in L , is the chain $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_h$ of Galois extensions of R , where $h = \min\{q, e\}$ (Proposition 12.2).

THEOREM 14.1. *Let S be a Galois extension of R which is an order over R in L . Then $S = S_m$ for some $m \leq h$.*

PROOF. The results of §7 imply that if S is a Galois extension of R such that $SK = L$, then S is a Galois $H_{p^{m(l-1)}}$ -extension for some m , $0 \leq m \leq e$, with induced action from that of KG on L . By the results of [14], $S = R[t]$ with $1 + p^m t = y$ satisfying $y^l = v$ in R , $v \equiv 1 \pmod{p^{ml}}$ a unit of R . Now $S_m = R[x]$ where $1 + p^m x = z$, $z^l = w \in R$. Since $K[y] = K[z]$, $z = cy^r$ some $c \in K$, and $y = dz^s$, some d in K . Since z, y are both units of S , c, d are units of R . Substituting, we have

$$(14.2) \quad 1 + p^m x = c(1 + p^m t)^r \quad \text{and} \quad 1 + p^m t = d(1 + p^m x)^s;$$

thus $c, d \equiv 1 \pmod{p^m}$. Writing $c = 1 + p^m e$, $d = 1 + p^m f$, $e, f \in R$, and substituting into (14.2) yields easily that $x \in R[t]$, $t \in R[x]$, so $S = S_m$.

Now we ask when the integral closure of R in L is a Galois extension. Again assume $l \mid v_k(w)$.

THEOREM 14.3. *Let $L = K[z]$, $z^l = w = 1 + p^k u$, $k = ql + r$, is a unit of R , and $q \geq e$ or $1 \leq r < l$. Let S be the integral closure of R in L . If $q \geq e$ then $S = S_e$ is a Galois extension. If $q < e$, then S is a Galois extension if and only if $r = 1$, in which case $S = S_q$, a Galois $H_{p^{q(l-1)}}$ -extension.*

PROOF. Since $S_h \subset S$ where $h = \min\{e, q\}$, and S_h is the largest Galois extension contained in S , S is a Galois extension if and only if $S = S_h$.

If $h = e$, S_e is a Galois RG -extension so is integrally closed, and $S_e = S$.

If $h = q < e$, set $S_q = R[x]$, $1 + p^q x = z$, then $v_L(x) = r$ by Lemma 13.1. If $r > 1$ then S_q is not integrally closed, hence $S_q \neq S$, by Proposition 13.3. If $r = 1$, then x satisfies the Eisenstein equation

$$\frac{(1 + p^q X)^l - (1 + p^{q(l+1)} u)}{p^{ql}} = 0$$

which is in $R[X]$ since $lR = \mathfrak{p}^{e(l-1)}$ and $q \leq e$. Thus [21, Chapitre I, Corollaire to Proposition 17], $S_q = R[x] = S$.

15. The local case. IV: The order of S in KG . Assume R is a local ring. Let T be a Galois H_b -extension of R , where $\tilde{b}$, an $(l-1)$ th root of b , is in R . Then the description of Galois H_b -extensions of R with normal basis given by Hurley [14] applies. Namely, let $H_b = R[\xi]$, $\xi^l = b\xi$, then $T = R[y]$ with $\xi y = 1 + \tilde{b}y = z$, $\xi z = \tilde{b}z$, $z^l = w$ is a unit of R congruent to 1 modulo $\tilde{b}^l$, and

$$v = \frac{1}{b} \left(1 - \sum_{i=1}^{l-1} z^i \right)$$

generates a normal basis for T over R , in the sense that $\{v, \xi v, \xi^2 v, \dots, \xi^{l-1} v\}$ is a basis for T as a free R -module.

EXAMPLE 15.1. Let $l = 2$, $H = H_{p^q}$, T a Galois H -extension of R . Then $T = R[x]$ where $x = (z - 1)/p^q$ satisfies $x^2 + (2/b)x = 2u$, u in R and $v = x = (1 - z)/p^q$ generates a normal basis $\{x, \xi x = z\}$.

Using the existence of the normal basis, we have

PROPOSITION 15.2. *Suppose $L \supset K$ is a Kummer extension of local fields of order l with Galois group G , R is the valuation ring of K , l is in $\mathfrak{p}$, the maximal ideal of R , and S is the integral closure of R in L . Suppose T is a tame H_b -extension contained in S . If $\mathcal{A} = \{\alpha \in KG \mid \alpha T \subseteq T\}$, the order of T in KG , then $H_b = \mathcal{A}$.*

PROOF. Since $H_b \subseteq KG$, $H_b \subseteq \mathcal{A}$.

First assume b is in $\mathfrak{p}$, the maximal ideal of R . Suppose α is in $\mathcal{A}$, $\alpha T \subseteq T$. Since T is a tame H_b -module algebra, where $H_b = R[\xi]$, T is Galois, so T has a basis over

R consisting of $v, \xi v, \xi^2 v, \dots, \xi^{l-1} v$ for some v in T . Let $\alpha = \sum_{i=0}^{l-1} d_i \xi^i$ in $H_b \otimes K = KG$, $d_i \in K$. Then

$$\alpha v = \sum_{i=0}^{l-1} d_i \xi^i v.$$

If αv is in T , then d_i must be in R for all i , and so α is in $R[\xi] = H_b$. Thus in this case, $\mathcal{A} = H_b$.

Now suppose b is not in p . Then $H_b = H_1$ is the integral closure of RG in KG . Since $\mathcal{A}$ is an order over R in KG , $\mathcal{A} \subseteq H_1$.

That completes the proof.

COROLLARY 15.3. *Suppose K, R, L are as in (15.2) and S is the integral closure of R in L . Suppose $L = K[z]$, $z^l = w = 1 + p^k u$, $k \geq 1$ maximal, $k = ql + r$, u a unit of R . If $r = 1$ or $q \geq e$ then the order $\mathcal{A}$ of S is a Hopf algebra.*

16. The local case. V: Trace, ramification number. Again assume $L \supset K$ is a Kummer extension of local fields of prime order l , let R be the valuation ring of K with maximal ideal $\mathfrak{p} = pR$, S the integral closure of R in L . Assume $L = K[z]$, $z^l = w$ a unit of R , $w = 1 + p^k u = 1 + p^{ql+r} u$, u a unit of R , $q \geq e$ or $1 \leq r < l$. If $q \geq e$ then S is a Galois extension of R with group G , $\text{tr}(S) = R$ and $\mathfrak{p}$ is unramified in S .

Suppose $q < e$, then $\mathfrak{p}$ is totally ramified in S . Let $\mathcal{P}$ be the maximal ideal of S . Set $G_i = \{\sigma \in G \mid \sigma x \equiv x \pmod{\mathcal{P}^{i+1}} \text{ for all } x \text{ in } S\}$, the i th ramification group. The ramification number t of L/K is the number t so that $G_t = G$, $G_{t+1} = (1)$.

THEOREM 16.1. *With the above notation, suppose $L = K[z]$, z^l a unit of R , and suppose L/K is totally ramified. Then the following are equivalent:*

- (i) S is a Galois $H_{p^{q(l-1)}}$ -extension,
- (ii) z may be chosen with $z^l = 1 + up^{ql+1}$, u a unit of R ,
- (iii) $t = (e - q)l - 1$,
- (iv) $\text{tr}(S) = p^{(e-q)(l-1)} R$.

Note that (i), (iii), (iv) all hold when L/K is unramified (in which case $q = e$).

PROOF. (iii) $\Rightarrow$ (iv). Let t be the ramification number, then

$$v_p(\text{tr}(S)) = [(t+1)(l-1)/l]$$

by [22, Lemma 4, p. 91]. Then (iii) $\Rightarrow$ (iv) is obvious.

(i) $\Leftrightarrow$ (ii) is Theorem 14.3.

(i) $\Rightarrow$ (iii). If S is a Galois $H_{p^{q(l-1)}}$ -extension then

$$S = S_q = R[x], \quad x = (z - 1)/p^q.$$

We have

$$\sigma(x) = \sigma\left(\frac{z-1}{p^q}\right) = \frac{\xi z - 1}{p^q} = \frac{z-1}{p^q} + \frac{\xi z - z}{p^q} = x + \frac{x(\xi - 1)}{p^q}$$

so $\sigma(x) - x$ is in $\mathfrak{p}^{e-q} = \mathcal{P}^{l(e-q)}$, and is not in $\mathcal{P}^{l(e-q)+1}$. So $t = (e - q)l - 1$.

(iv) $\Rightarrow$ (i). Suppose $v_k(\text{tr}(S)) = q(l-1)$, some q . Let t be the ramification number, $h = t + 1$. Then $q(l-1) = [h(l-1)/l]$. Write $h = cl + r$, $0 \leq r < l$. Then

$$\begin{aligned} q(l-1) &= [(cl+r)(l-1)/l] \quad [22, \text{p. 91}] \\ &= c(l-1) + [r(l-1)/l], \quad \text{so } c = q \text{ and } r = 0 \text{ or } 1. \end{aligned}$$

CLAIM. S is an $H_{p^{(e-q)(l-1)}}$ -module algebra.

PROOF OF CLAIM. Let π be a uniformizing parameter for $\mathcal{P}$, the maximal ideal of S .

If $G = \langle \sigma \rangle$, for any x in S , $\sigma(x) \equiv x \pmod{\pi^h}$, so for each i ,

$$\sigma^i(x) = x + u_i \pi^h \quad \text{for some } u_i \text{ in } S.$$

Recall that $RG = R[\theta]$, $\theta^l = \omega_l \theta$ where

$$\theta = - \sum_{m \in \mathbb{F}_l^*} \chi^{-1}(m) \sigma^m.$$

Thus

$$\begin{aligned} \theta(x) &= - \sum_{m=1}^{l-1} \chi^{-1}(m) \sigma^m(x) \\ &= - \sum_{m=1}^{l-1} \chi^{-1}(m) x - \sum_{m=1}^{l-1} \chi^{-1}(m) u_m \pi^h \\ &= \left(- \sum_{m=1}^{l-1} \chi^{-1}(m) u_m \right) \pi^h. \end{aligned}$$

If $h = lq + r$, $0 \leq r < l$, let $\xi = \theta/p^q$ in KG ; then $R[\xi] = H_b$ for $b = \omega_l/p^{q(l-1)}$. For any x in S ,

$$\xi(x) = \left(- \sum_{m=1}^{l-1} \chi^{-1}(m) u_m \right) \pi^h / p^q,$$

and $\pi^h/p^q = u_1 \pi^r$, an element of S (where u_1 is a unit of S). Thus $H_b = R[\xi]$ maps S to S . Since $H_b = R[\xi] \subseteq KG$ and the action of H_b on S is the restriction of that of KG on L , S is an H_b -module algebra, completing the proof of the claim.

Now the space of integrals of $H_{p^{(e-q)(l-1)}}$ is generated by

$$\phi = b - \xi^{l-1} = \left(\sum_{i=0}^{l-1} \sigma^i \right) / p^{q(l-1)}.$$

So $IS = \phi S = \text{tr}(S)/p^{q(l-1)} = R$. It follows that S is a tame, hence Galois $H_{p^{(e-q)(l-1)}}$ -extension. Thus (iv) $\Rightarrow$ (i), completing the proof of Theorem 16.1.

COROLLARY 16.2 (cf. [3]). Let $L \supset K$ be a Kummer extension of local fields of prime order l . Then l does not divide the ramification number t of L/K .

PROOF. The conclusion is true if S is a Galois H_b -extension for some H_b , by Theorem 16.1. So assume S is not Galois. In that case, $v_k(\text{tr}(S))$ is not a multiple of $l-1$. So if $h = t + 1 = cl + r$, $0 \leq r < l$, again by [22, p. 91],

$$v_k(\text{tr}(S)) = [(cl+r)(l-1)/l] = c(l-1) + [r(l-1)/l]$$

is not a multiple of $l - 1$. So $r \geq 2$, and $t \not\equiv -1$ or $0 \pmod{l}$. That completes the proof.

17. Globalization. In this section we obtain global versions of the local results of the previous sections.

Let $L \supset K$ be a Kummer extension of number fields of prime order l with Galois group G and with rings of integers $R = O_K$, $S = O_L$. For each (finite) prime $\mathfrak{p}$ of K , let $\hat{R}_{\mathfrak{p}}$, $\hat{K}_{\mathfrak{p}}$ be the completions at $\mathfrak{p}$, and let $\hat{S}_{\mathfrak{p}} = S \otimes_R \hat{R}_{\mathfrak{p}}$, $L_{\mathfrak{p}} = L \otimes_K \hat{K}_{\mathfrak{p}}$. Then $\hat{L}_{\mathfrak{p}} \supset \hat{K}_{\mathfrak{p}}$ is a Kummer extension, $\hat{L}_{\mathfrak{p}} = \hat{K}_{\mathfrak{p}}[z]$, $z^l \in \hat{K}_{\mathfrak{p}}$ (even though if p splits completely in S , $\hat{L}_{\mathfrak{p}}$ will be a direct sum of fields, rather than a field).

PROPOSITION 17.1. *Suppose $L \supset K$ are as above, and at each prime $\mathfrak{p}$ of R , we are given a Tate-Oort Hopf algebra $\hat{H}_{\mathfrak{p}} \subseteq \hat{K}_{\mathfrak{p}}G$ and a tame $\hat{H}_{\mathfrak{p}}$ -extension $\hat{T}_{\mathfrak{p}}$ of $\hat{R}_{\mathfrak{p}}$ contained in $\hat{S}_{\mathfrak{p}}$ such that at all but a finite number of primes $\mathfrak{p}$, $\hat{T}_{\mathfrak{p}} = \hat{S}_{\mathfrak{p}}$. Then there exists a unique Hopf algebra H contained in KG and a tame H -extension T of R contained in S such that $T \otimes_R \hat{R}_{\mathfrak{p}} = \hat{T}_{\mathfrak{p}}$ and $H \otimes \hat{R}_{\mathfrak{p}} = \hat{H}_{\mathfrak{p}}$.*

If we can find unique $H_{\mathfrak{p}}$ and $T_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$, the localization of R at $\mathfrak{p}$, so that $T_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}} = \hat{T}_{\mathfrak{p}}$, $H_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}} = \hat{H}_{\mathfrak{p}}$, such that $T_{\mathfrak{p}} = S_{\mathfrak{p}}$ for all but a finite number of primes, then, since $H_{\mathfrak{p}} = R_{\mathfrak{p}}G$ for all $\mathfrak{p} \nmid l$, $H = \bigcap H_{\mathfrak{p}}$, and $T = \bigcap T_{\mathfrak{p}}$ by standard module theory over Dedekind domains. So (17.1) follows from

PROPOSITION 17.2. *Let R be a discrete valuation ring with quotient field K , L a finite extension of K , S the integral closure of R in L . Let $\hat{K}$ be the completion of K with respect to the valuation on R , $\hat{R}$ = valuation ring, $\hat{L} = L \otimes_K \hat{K}$, $\hat{S} = S \otimes_R \hat{R}$. Let T_1 be an order over $\hat{R}$ in $\hat{L}$. Then there exists a unique order T over R in L with $\hat{T} = T \otimes_R \hat{R} = T_1$.*

PROOF. Since T_1 is an order over $\hat{R}$ in $\hat{L}$, there exists some m so that $p^m \hat{S} \subseteq T_1 \subseteq \hat{S}$. Let $i: L \rightarrow \hat{L}$ be the canonical inclusion. Let $T = \{x \in S \mid i(s) \in T_1\}$. Then $p^m S \subseteq T$. We claim $T_1 = \hat{T}$. We have the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \rightarrow & T & \rightarrow & S & \rightarrow & S/T \\ & & \downarrow i & & \downarrow i & & \downarrow \\ 0 & \rightarrow & T_1 & \rightarrow & \hat{S} & \rightarrow & \hat{S}/T_1 \rightarrow 0 \end{array}$$

Since $\hat{R}$ is R -faithfully flat, $\hat{R} \otimes S/T = \hat{S}/\hat{T}$. But then

$$\begin{aligned} \hat{S}/\hat{T} &= \hat{R} \otimes_R (S/T) = (\hat{R}/p^m \hat{R}) \otimes_R (S/T) = (R/p^m R) \otimes_R (S/T) \\ &= R \otimes_R (S/T) = S/T. \end{aligned}$$

Now $\hat{T} \subseteq T_1$, so $S/T = \hat{S}/\hat{T} \rightarrow \hat{S}/T_1$ is surjective, but by definition of T_1 $S/T \rightarrow \hat{S}/T_1$ is injective. Thus $\hat{S}/T_1 \cong S/T \cong \hat{S}/\hat{T}$, and $T_1 = \hat{T}$.

To show T is unique with $\hat{T} = T_1$, suppose T' has $\hat{T}' = T_1$. Then $T' \subseteq T$, and $\hat{R} \otimes_R T' = \hat{T}' = T_1 = \hat{T} = \hat{R} \otimes_R T$. By faithful flatness of $\hat{R}$, $T' = T$.

Using 17.1, the globalizations of the local results for Kummer extensions of the previous three sections are immediate. The globalization of the trace criterion of Theorem 16.1 for $\mathcal{O}_L$ is the converse of Theorem 6.1:

THEOREM 17.3. *Let $L \supset K$ be a Kummer extension of prime order l of number fields. Let $\mathcal{W} = \text{Tr}(O_L)$. Then the order $\mathcal{A}$ of O_L is a Hopf algebra and O_L is a tame $\mathcal{A}$ -extension if and only if $\mathcal{W}$ is the $(l-1)$ th power of an ideal of O_K . If so, $\mathcal{A} = H_{\mathcal{B}}$ where $\mathcal{B}\mathcal{W} = lO_K$.*

Here is the global version of the congruence criterion 14.2:

THEOREM 17.4. *Let $L = K[z]$, $z^l = w \in K$ be a Kummer extension of prime order of number fields, with Galois group G . Then the order $\mathcal{A}$ of O_L in KG is a Hopf algebra and O_L is a tame $\mathcal{A}$ -extension if and only if for each prime p of O_K dividing lO_K ,*

- (a) l does not divide $v_p(w)$, or
- (b) l divides $v_p(w)$ and there is some c in K so that $v_p(c^l w - 1) \geq l v_p(l)$ or $v_p(c^l w - 1) \equiv 1 \pmod{l}$.

If (a) holds for all primes p of O_K dividing lO_K , then $\mathcal{A} = (O_K G)^*$. If (b) holds for some p dividing lO_K , then $\mathcal{A} = H_{\mathcal{B}}$ where for each prime p of O_K dividing lO_K , choosing $c \in K$ so that $v_p(c^l w - 1) \geq le$ or $\equiv 1 \pmod{l}$, we have

If $v_p(c^l w - 1) \geq le$, then $v_p(\mathcal{B}) = 0$,

If $v_p(c^l w - 1) = k$, $0 < k < le$, then $v_p(\mathcal{B}) = (k-1)(l-1)/l$.

We may also give a complete classification of Galois extensions inside O_L , globalizing Theorem 14.1:

THEOREM 17.5. *Let $L = K[z]$, $z^l = w \in K$ be a Kummer extension of prime order l of number fields. The set $\text{Gal}(L/K)$ of Galois extensions of O_K contained in O_L is as follows:*

- (a) *if for some prime p of O_K , $l \nmid v_p(w)$, then $\text{Gal}(L/K)$ is empty.*
- (b) *if for all primes p of O_K , $l \mid v_p(w)$, then the set $\text{Gal}(L/K)$ is in 1-1 lattice-inverting correspondence with the ideals of O_K which are $(l-1)$ th powers and which contain $(lO_K)(\text{tr}(O_L))^{-1}$.*

PROOF. An order $S \subseteq O_L$ over O_K in L is a Galois $H_{\mathcal{B}}$ -extension if and only if S_p is a Galois $H_{\mathcal{B}_p}$ -extension for each prime p of O_K . If l does not divide $v_p(w)$ for some prime p , then there are no Galois extension of $O_{K,p}$ contained in L by Corollary 8.2, hence (a) holds.

Suppose that l divides $v_p(w)$ for all primes p of O_K . If p is a prime which does not divide lO_K , then $O_{L,p}$ itself is a Galois $H_{1,p}$ -extension of $O_{K,p}$ and is unique. If p divides lO_K we have a chain $S_0 \subset S_1 \subset \dots \subset S_h \subseteq O_{L,p}$ where S_k is a Galois $H_{p^{k(l-1)}}$ -extension of $O_{K,p}$. Set $L = K[z]$, $z^l = 1 + up^k$ where $k \geq le$ or u is a unit in $O_{K,p}$ and l does not divide k . If $k \geq le$ then $h = e$ and $\text{tr}(O_{L,p}) = O_{K,p}$. If $k < le$, $k = ql + r$, $0 < r < l$, then $h = q$.

Thus the set of Galois extensions of O_K contained in L is in 1-1 correspondence with ideals ℓ so that at p not dividing lO_K , $\ell_p = (1)$ and at p dividing lO_K , $\ell_p = p^{k(l-1)}$ for $0 \leq k \leq h$. Since $S_h \subseteq O_{L,p}$, $\text{tr}(S_k) \subseteq \text{tr}(O_{L,p})$ for all $k \leq h$. But $\text{tr}(S_h) = p^{(e-h)(l-1)}\phi(S_h)$ where ϕ generates the space of integrals of $H_{p^{h(l-1)}}$; since S_h is a Galois $H_{p^{h(l-1)}}$ -extension, $\phi(S_h) = O_{K,p}$. Thus

$$\text{tr}(S_h) = p^{(e-h)(l-1)} \subseteq \text{tr}(O_{L,p})$$

and so

$$(lO_K)(\text{tr}(O_{L,v}))^{-1} \subseteq \mathfrak{p}^{h(l-1)} \subseteq \mathfrak{p}^{k(l-1)}$$

for all k , $0 \leq k \leq h$. That completes the proof.

Note that (b) holds if and only if the Kummer order $\tilde{O}_L$ of O_L is a Galois H_1 -extension of O_K . Then $\tilde{O}_L$ corresponds to the unit ideal, and is contained in all other Galois extensions inside O_L . This observation allows determination of an upper bound on the number of Galois extensions of rank l of O_K :

COROLLARY 17.6. *Let K be a number field containing ζ , a primitive l th root of unity, l prime, and suppose $lO_K = (\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g})^{l-1}$ is the factorization of $lO_K = (1 - \zeta)^{l-1}O_K$ into a product of prime ideals. Then the number of Galois extensions S of O_K of rank l such that $S \otimes_{O_K} K$ is a Galois extension of K with group G , cyclic of order l , is at most*

$$|U(O_K)/U(O_K)^l| \cdot |Cl_l(O_K)| \cdot \prod_{i=1}^g (e_i + 1).$$

PROOF. The first two factors represent the number of Galois H_1 -extensions of O_K . This follows from the exact sequence

$$1 \rightarrow \text{NB}(O_K, H_1) \rightarrow \text{Gal}(O_K, H_1) \rightarrow \text{Prim Pic}(H_1) \rightarrow 1$$

given by the Picard invariant map [26], where $\text{Gal}(O_K, H_1)$ is the group of Galois H_1 -extensions, $\text{NB}(O_K, H_1)$ is the subgroup of Galois H_1 -extensions with normal basis, and $\text{Prim Pic}(H_1)$ is the subgroup of primitive elements of $\text{Pic}(H_1)$, the group of rank one projective H_1 -modules. Now $\text{NB}(O_K, H_1) \cong U(O_K)/U(O_K)^l$ by [14], and $\text{Prim Pic}(H_1) \cong Cl_l(O_K)$ the l -torsion subgroup of the class group of O_K , essentially by [7, Example 4.16]. The third factor is the number of ideals containing lO_K which are $(l-1)$ th powers: this factor is an upper bound for the number of Galois extensions of O_K contained in any Galois extension of K with group G of order l , by Theorem 17.5.

COROLLARY 17.7. *Let $K = Q(\sqrt{p})$, p a prime $\equiv 3 \pmod{4}$. Then K has at most 12 Galois extensions of rank 2.*

For $U(O_R)/U(O_K)^2$ has order 4, $\text{Pic}(O_K)$ has odd order, $2O_K$ is the square of a prime ideal, and for any Galois extension S of rank 2, $S \otimes K$ is a Galois extension of K with group G of order 2.

In fact, by genus theory, $\text{Gal}(O_K, H_2) = \text{Gal}(O_K, O_K G)$ has order 2, so the bound of 12 is not best possible: we suspect the correct number is 8.

Corollary 17.7 may be used to show the existence of many Azumaya O_K -algebras which are not crossed products for large p : see [31].

In Corollary 17.6 the hypothesis that $S \otimes K$ be a KG -Galois extension of K is necessary. For example, if $l = 3$, there exist nonnormal cubic field extensions of K and any such is a Galois H -extension for some rank 3 Hopf K -algebra $H \neq KG$: see Theorem 4.6 of [27].

REMARK 17.8. It is a straightforward matter to classify the set $\mathcal{T}(O_L/O_K)$ of tame extensions of O_K contained in O_L :

$$\mathcal{T}(O_L/O_K) = \prod_{\mathfrak{p}}' \mathcal{T}(O_{L,\mathfrak{p}}/O_{K,\mathfrak{p}})$$

where $\prod'$ mean the elements of the direct product over all primes p of O_K such that at all but a finite number of primes p , $S_p = O_{L,p}$. Here $\mathcal{T}(O_{L,\mathfrak{p}}/O_{K,\mathfrak{p}})$ is the union of the Galois extensions of $O_{K,p}$ contained in $O_{L,p}$, described in Theorem 14.1, and the non-Galois, tame H_1 -extensions, which are described in Theorem 8.1.

18. A cubic example. By way of illustrating the trace condition of Theorem 17.3, we consider $K = Q(\xi)$, $\xi = (-1 + \sqrt{-3})/2$, a cube root of unity. H. Wada [25] has determined relative integral bases for the rings of integers O_L of $L = K[z]$, $z^3 = w$.

Write $w = st^2$, where s, t are cube-free elements of O_K , with s, t both $\not\equiv -1 \pmod{\sqrt{-3}}$. Then Wada considers three cases.

(i) If $s \not\equiv t \pmod{3}$, then $1, z, z^2/t$ is an O_K -basis of O_L .

In this case $\text{tr}(O_L) = 3O_K$; since the only prime ideal $\mathfrak{p}$ of O_K containing $l = 3$ is $\mathfrak{p} = \sqrt{-3} O_K$, $v_{\mathfrak{p}}(\text{tr}(O_L)) = 2$. So O_L is a tame $(O_K G)^*$ -extension of O_K .

(ii) If $s \equiv t \pmod{3\sqrt{-3}}$, then s and t are relatively prime to 3, for otherwise $\sqrt{-3}$ divides t or s , hence both, and $3\sqrt{-3} = \sqrt{-3}^3$ would divide w , contrary to the assumption that w is cube-free. In this case, O_L has an O_K -basis consisting of $1, (1-z)/\sqrt{-3}$, and $((s+z+z^2)/t)/3$. Then $\text{tr}(O_L)$ is generated by $3, -3/\sqrt{-3} = \sqrt{-3}$ and s , so $\text{tr}(O_L) = O_K$. Thus O_L is a tame $O_K G$ -extension of O_K (that is, tame in the classical sense [11]).

(iii) If $s \equiv t \pmod{3}$, $s \not\equiv t \pmod{3\sqrt{-3}}$, then Wada shows that O_L has an O_K -basis $1, z, ((1+z+z^2)/t)/\sqrt{-3}$ and $\text{tr}(O_L) = \sqrt{-3} O_K$. Thus $v_{\mathfrak{p}}(\text{tr}(O_L)) = 1$ is not divisible by $3-1=2$, so by Theorem 6.1 the order $\mathcal{A}$ of O_L in KG is not a Hopf algebra.

This last fact can be seen directly:

Locally at (3), hence globally, the only Hopf algebras of order 3 contained in KG are H_{-3} and H_1 by Corollary 7.2. Since

$$\alpha = \frac{1}{\sqrt{-3}} \text{tr} = \frac{1}{\sqrt{-3}} (1 + \sigma + \sigma^2)$$

has $\alpha O_L = O_L$, the order $\mathcal{A}$ of O_L in KG contains α but not

$$-\alpha/\sqrt{-3} = (1 + \sigma + \sigma^2)/3.$$

Thus $\mathcal{A}$ lies properly between $H_{-3} = O_K G$ and $H_1 = (O_K G)^*$, and so is not a Hopf algebra.

REFERENCES

1. A.-M. Bergé, *Arithmétique d'une extension Galoisienne a groupe d'inertie cyclique*, Ann. Inst. Fourier (Grenoble) **28** (1978), 17–44.
2. F. Bertrandias, *Decomposition du Galois-module des entiers d'une extension cyclique de degre premier d'un corps de nombres ou d'un corps local*, Ann. Inst. Fourier (Grenoble) **29** (1979), 33–48.
3. F. Bertrandias and M. J. Ferton, *Sur l'anneau des entiers d'une extension cyclique de degre premier d'un corps local*, C. R. Acad. Sci. Paris A **274** (1972), 1330–1333.

4. F. Bertrandias, J.-P. Bertrandias and M. J. Ferton, *Sur l'anneau des entiers d'une extension cyclique de degre premier d'un corps local*, C. R. Acad. Sci. Paris A **274** (1972), 1388–1391.
5. N. Bourbaki, *Algèbre commutative*, Chapitre II, Hermann, Paris, 1961.
6. S. U. Chase, D. K. Harrison and A. Rosenberg, *Galois theory and Galois cohomology of commutative rings*, Mem. Amer. Math. Soc. No. 52, 1965, pp. 15–33.
7. S. U. Chase and M. E. Sweedler, *Hopf algebras and Galois theory*, Lecture Notes in Math., vol. 97 Springer, 1969.
8. L. N. Childs, *Representing classes in the Brauer group of quadratic number rings as smash products*, Pacific J. Math. (to appear).
9. L. N. Childs and S. Hurley, *Tameness and local normal bases for objects of finite Hopf algebras*, Trans. Amer. Math. Soc. **298** (1986), 763–778.
10. F. DeMeyer and E. Ingraham, *Separable Algebras Over Commutative Rings*, Lecture Notes in Math., vol. 181, Springer, 1971.
11. A. Frohlich, *Local fields*, Algebraic Number Theory, (J. W. S. Cassels and A. Frohlich, eds.), Thompson, Washington, D. C., 1967.
12. ———, *The module structure of Kummer extensions over Dedekind domains*, J. Reine Angew. Math. **209** (1962), 39–53.
13. S. Hurley, *Tame and Galois Hopf algebras with normal bases*, Thesis, SUNY at Albany, 1984.
14. ———, *Galois objects with normal bases for free Hopf algebras of prime degree*, J. Algebra (to appear).
15. H. Jacobinski, *Über die Hauptordnung eines Körpers als Gruppenmodul*, J. Reine Angew. Math. **213** (1964), 151–164.
16. R. Larson and M. Sweedler, *An associative orthogonal bilinear form for Hopf algebra*, Amer. J. Math. **91** (1969), 75–94.
17. H. W. Leopoldt, *Über die Hauptordnung der ganzen Elementen eines abelschen Zahlkörpers*, J. Reine Angew. Math. **201** (1959), 119–149.
18. T. Ligon, *Galois-Theorie in monoidalen Kategorien*, Algebra Ber. **35** (1978).
19. B. Pareigis, *When Hopf algebras are Frobenius algebras*, J. Algebra **18** (1971), 588–596.
20. P. Ribenboim, *Algebraic numbers*, Wiley-Interscience, New York, 1972.
21. J.-P. Serre, *Corps locaux*, Hermann, Paris, 1962.
22. M. E. Sweedler, *Hopf algebras*, Benjamin, New York, 1969.
23. J. Tate and F. Oort, *Group schemes of prime order*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 1–21.
24. M. Taylor, *Relative Galois module structure of rings of integers and elliptic functions. II*, Ann. of Math. **121** (1985), 519–535.
25. H. Wada, *On cubic Galois extensions of $Q(\sqrt{-3})$* , Proc. Japan Acad. **46** (1970), 397–400.
26. L. Childs and A. Magid, *The Picard invariant of a principal homogeneous space*, J. Pure Appl. Algebra **4** (1974), 273–286.
27. C. Greither and B. Pareigis, *Hopf Galois theory for separable field extensions*, J. Algebra **106** (1987), 239–258.
28. M. Taylor, *Hopf structure and the Kummer theory of formal groups*, J. Reine Angew. Math. **375 / 376** (1987), 1–11.
29. G. Bergman, *Everybody knows what a Hopf algebra is*, Contemp. Math., vol. 43, Amer. Math. Soc., Providence, R. I., 1985, pp. 25–48.
30. R. G. Larson, *Group rings over Dedekind domains*, J. Algebra **5** (1967), 358–361.
31. L. N. Childs, *Azumaya algebras which are not smash products*, Rocky Mountain J. Math. (to appear).

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