COMPARISON BETWEEN ANALYTIC CAPACITY AND THE BUFFON NEEDLE PROBABILITY

TAKAFUMI MURAI

ABSTRACT. We show that analytic capacity and the Buffon needle probability are not comparable.

1. Introduction. For a compact set E in the complex plane $\mathbb{C}$, we denote by $H^\infty(E^c)$ the Banach space of bounded analytic functions outside E with supremum norm $\| \cdot \|_{H^\infty(E^c)}$. The analytic capacity of E is defined by

$$\gamma(E) = \sup \{ |u'(\infty)|; u \in H^\infty(E^c), \|u\|_{H^\infty(E^c)} \leq 1 \},$$

where $u'(\infty) = \lim_{z \rightarrow \infty} z(u(z) - u(\infty))$ [3, p. 6]. We are concerned with estimating $\gamma(E)$. Let $D(z, r)$ denote the open disk of center z and of radius r . For $\varepsilon > 0$, we put $|E|_{(\varepsilon)} = 2 \inf \sum_{k=1}^\infty r_k$, where the infimum is taken over all coverings $\{D(z_k, r_k)\}_{k=1}^\infty$ of E with radii less than ε . The generalized length of E is defined by $|E| = \lim_{\varepsilon \rightarrow 0} |E|_{(\varepsilon)}$; the limit exists since $|E|_{(\varepsilon)}$ is decreasing with respect to ε . It is well known that

$$(1) \quad \gamma(E) \leq \text{Const} |E| \quad [3, \text{p. 48}].$$

Let $\mathfrak{P}^\theta$ ($-\pi/2 < \theta \leq \pi/2$) denote the straight line defined by the equation $x \sin \theta - y \cos \theta = 0$ and let $\mathfrak{L}(r, \theta)$ ($r > 0, -\pi < \theta \leq \pi$) denote the straight line defined by the equation $x \cos \theta + y \sin \theta = r$. The Buffon length of E is defined by

$$\text{Bu}(E) = \iint_{\{(r, \theta); \mathfrak{L}(r, \theta) \cap E \neq \emptyset\}} dr d\theta.$$

We easily see that $\text{Bu}(E) = \int_{-\pi/2}^{\pi/2} |E^\theta| d\theta$, where E^θ is the projection of E to $\mathfrak{P}^\theta$. If E is contained in $D(0, 1)$, $\text{Bu}(E)/2\pi$ is called the Buffon needle probability; this gives probability (measured by $dr|_{(0,1)} d\theta/2\pi$) of straight lines $\mathfrak{L}(r, \theta)$ intersecting with E .

Suppose that the boundary ∂E of E consists of a finite number of rectifiable Jordan curves. We put

$$\text{Cr}(E) = \iint_{r>0, |\theta| \leq \pi} n(r, \theta) dr d\theta,$$

where $n(r, \theta)$ is the cardinal number of $\mathfrak{L}(r, \theta) \cap \partial E$. Then we have evidently $\text{Bu}(E) \leq \text{Cr}(E)$. Crofton's formula [8, p. 13] shows that $\text{Cr}(E) = 2|\partial E|$. Since

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The set E_n which will be constructed later has a form like a crank with many pedals (see Figure 1). Our theorem shows that the inequality " $\gamma(\cdot) \leq \text{Const Bu}(\cdot)$ " is not valid. To prove our theorem, we remark

PROPOSITION. *Let E be a compact set on a rectifiable graph $\Gamma = \{x + iA(x); \alpha \leq x \leq \beta\}$. Then $\gamma(E) \geq \text{Const}|E^0|^{3/2}/|\Gamma|^{1/2}$.*

We do not know whether these exponents $3/2$ and $1/2$ are sharp or not. To prove our proposition, we shall use an estimate of the norm of the Calderón-Cauchy operator [7].

2. Proof of Proposition. Let L^p ($1 \leq p \leq \infty$) denote the L^p space on the real line $\mathbf{R}$. Its norm is denoted by $\|\cdot\|_p$. For a real-valued function $b \in L^\infty$, we define a kernel by

$$C[b](x, y) = 1/\{(x - y) + i(B(x) - B(y))\},$$

where $B(x) = \int_0^x b(t) dt$. The singular integral operator $C[b]$ defined by the above kernel is called the Calderón-Cauchy operator. We denote by $\|C[b]\|_{2,2}$ the norm of $C[b]$ as an operator from L^2 to itself. We use an auxiliary operator $C^*[b]$ defined by

$$C^*[b]f(x) = \sup_{\epsilon > 0} |C_\epsilon[b]f(x)|,$$

where

$$C_\epsilon[b]f(x) = \int_{|x-y|>\epsilon} C[b](x, y)f(y) dy.$$

The following fact is already known.

LEMMA 1 [7]. $\|C[b]\|_{2,2} \leq \text{Const}(1 + \sqrt{\|b\|_\infty})$.

In the same manner as Lemma 2 in [1, p. 139], the separation theorem yields

LEMMA 2. *Let T be an operator defined by a kernel $T(x, y)$ satisfying $T(x, y) = -T(y, x)$ ($x, y \in \mathbf{R}$) and $\sup_{x,y \in \mathbf{R}} |T(x, y)|(1 + |x - y|) < \infty$. Then, for any compact set $F \subset \mathbf{R}$, there exists a nonnegative function $f_0 \in L^\infty$ such that*

$$\|Tf_0\|_\infty \leq 1, \quad \|f_0\|_\infty \leq 1/\|T\|_w, \quad \|f_0\|_1 \geq |F|/10\|T\|_w, \quad \text{supp}(f_0) \subset F,$$

where $\text{supp}(f_0)$ is the support of f_0 and $\|T\|_w$ is the weak-(1,1) norm of T , that is,

$$\|T\|_w = \sup\{|x \in \mathbf{R}; |Tf(x)| > 1/\|f\|_1; f \in L^1\}.$$

LEMMA 3. $\|C^*[b]\|_w \leq \text{Const}(1 + \sqrt{\|b\|_\infty})$.

PROOF. Let $\omega(C[b])$ denote the minimum of all M satisfying the following three inequalities:

$$|C[b](x, y)| \leq M/|x - y|.$$

$$|C[b](x, y) - C[b](x', y)| \leq M|x - x'|^{1/2}/|x - y|^{3/2} \quad (|x - x'| \leq |x - y|/2).$$

$$|C[b](x, y) - C[b](x, y')| \leq M|y - y'|^{1/2}/|x - y|^{3/2} \quad (|y - y'| \leq |x - y|/2).$$

We put $\|C[b]\|_{CZ} = \|C[b]\|_{2,2} + \omega(C[b])$. It is known that $\|C^*[b]\|_w \leq \text{Const}\|C[b]\|_{CZ}$ [5, Chapter 4]. Since $\omega(C[b]) \leq \text{Const}(1 + \sqrt{\|b\|_\infty})$, Lemma 1 gives the required inequality.

LEMMA 4. For a compact set $F \subset \mathbf{R}$, there exists a nonnegative function $f_F \in L^\infty$ such that

$$\begin{aligned} \|C^*[b]f_F\|_\infty &\leq 1, \quad \|f_F\|_\infty \leq 1/(1 + \sqrt{\|b\|_\infty}), \\ \|f_F\|_1 &\geq \text{Const}|F|/(1 + \sqrt{\|b\|_\infty}), \quad \text{supp}(f_F) \subset F. \end{aligned}$$

PROOF. Lemma 3 shows that $\|C_\varepsilon[b]\|_w \leq \text{Const}(1 + \sqrt{\|b\|_\infty})$ ($\varepsilon > 0$). Hence Lemma 2 shows that, to each $\varepsilon > 0$, there corresponds a nonnegative function $f_\varepsilon \in L^\infty$ satisfying

$$\begin{aligned} \|C_\varepsilon[b]f_\varepsilon\|_\infty &\leq 1, \quad \|f_\varepsilon\|_\infty \leq 1/(1 + \sqrt{\|b\|_\infty}), \\ \|f_\varepsilon\|_1 &\geq \text{Const}|F|/(1 + \sqrt{\|b\|_\infty}) \quad \text{and} \quad \text{supp}(f_\varepsilon) \subset F. \end{aligned}$$

Cotlar's inequality [5, p. 56] gives that

$$\begin{aligned} |C_\eta[b]f_\varepsilon(x)| &\leq \text{Const}\{m(C_\varepsilon[b]f_\varepsilon)(x) + \|C[b]\|_{CZ}m f_\varepsilon(x)\} \\ &\quad (x \in \mathbf{R}, 0 < \varepsilon < \eta), \end{aligned}$$

where m is the noncentered maximal operator [5, p. 9]. Hence

$$\|C_\eta[b]f_\varepsilon\|_\infty \leq \text{Const}\{\|C_\varepsilon[b]f_\varepsilon\|_\infty + \|C[b]\|_{CZ}\|f_\varepsilon\|_\infty\} \leq \text{Const}.$$

Let f_F^0 be a weak* cluster point of $(f_\varepsilon)_{\varepsilon>0}$ in the space of measures. Then we have $\|C_\eta[b]f_F^0\|_\infty \leq \text{Const}$, which gives $\|C^*[b]f_F^0\|_\infty \leq \text{Const}$. Multiplying f_F^0 by a suitable absolute constant, we obtain the required function f_F . Q.E.D.

LEMMA 5. Let F, f_F be the same as in Lemma 4 and let

$$\hat{f}_F(z) = \int_{-\infty}^{\infty} \frac{f_F(y)}{\{z - (y + iB(y))\}} dy \quad (z \notin \hat{F} = \{x + iB(x); x \in F\}).$$

Then $\|\hat{f}_F\|_{H^\infty(\hat{F}^c)} \leq \text{Const}$ and $|\hat{f}_F'(\infty)| \geq \text{Const}|F|/(1 + \sqrt{\|b\|_\infty})$.

PROOF. By Lemma 4, we have

$$|\hat{f}_F'(\infty)| = \|f_F\|_1 \geq \text{Const}|F|/(1 + \sqrt{\|b\|_\infty}).$$

We show $\|\hat{f}_F\|_{H^\infty(\hat{F}^c)} \leq \text{Const}$. For any $z \notin \hat{F}$, we have

$$\hat{f}_F(z) = \int_{-\infty}^{\infty} \frac{f_F(y)}{(\text{Re } z - y) + i\{B(\text{Re } z) - B(y) + (\text{Im } z - B(\text{Re } z))\}} dy$$

as long as $\text{Im } z \neq B(\text{Re } z)$. Hence it is sufficient to show that

$$\sup_{x, s \in \mathbf{R}, s \neq 0} \left| \int_{-\infty}^{\infty} \frac{f_F(y)}{(x - y) + i(B(x) - B(y) + s)} dy \right| \leq \text{Const}.$$

Let $s_0 = 2(1 + \|b\|_\infty)$. We have, for any $x, s \in \mathbf{R}, s \neq 0$,

$$\begin{aligned}
 & \left| \int_{-\infty}^{\infty} \frac{f_F(y)}{(x-y) + i(B(x) - B(y) + s)} dy \right| \\
 & \leq \left| \int_{|x-y| \leq |s|/s_0} \frac{f_F(y)}{(x-y) + i(B(x) - B(y) + s)} dy \right| + \left| \int_{|s|/s_0 < |x-y| \leq |s|} \frac{f_F(y)}{(x-y) + i(B(x) - B(y) + s)} dy \right| + \left| \int_{|x-y| > |s|} \frac{f_F(y)}{(x-y) + i(B(x) - B(y) + s)} dy \right| \\
 & \leq \int_{|x-y| \leq |s|/s_0} \frac{f_F(y)}{|s| - |B(x) - B(y)|} dy + \int_{|s|/s_0 < |x-y| \leq |s|} \frac{f_F(y)}{|x-y|} dy \\
 & \quad + \left| \int_{|x-y| > |s|} \frac{s f_F(y)}{\{(x-y) + i(B(x) - B(y) + s)\} \{(x-y) + i(B(x) - B(y))\}} dy \right| \\
 & \quad + |C_{|s|}[b] f_F(x)| \\
 & \leq \text{Const} \|f_F\|_\infty + \|f_F\|_\infty \log s_0 + \|f_F\|_\infty |s| \int_{|x-y| > |s|} \frac{f_F(y)}{(x-y)^2} dy + C^*[b] f_F(x) \\
 & \leq \text{Const}(1 + \log s_0) \|f_F\|_\infty + \|C^*[b] f_F\|_\infty \leq \text{Const}. \quad \text{Q.E.D.}
 \end{aligned}$$

LEMMA 6. Let E and Γ be the same as in the theorem. Then there exists a graph $\Gamma^* = \{x + iB(x); \alpha \leq x \leq \beta\}$, $B(x) = \int_\alpha^x b(t) dt + A(\alpha)$; such that $\|b\|_\infty \leq 4|\Gamma|/|E^0|$ and $|\alpha \leq x \leq \beta; A(x) \neq B(x)| \leq |E^0|/2$.

PROOF. Let $t_0 = 2|\Gamma|/|E^0|$. Then

$$t_0 \geq \frac{|\Gamma|}{\beta - \alpha} \geq \frac{1}{\beta - \alpha} \int_\alpha^\beta |a(t)| dt \quad (a = A').$$

The Calderón-Zygmund decomposition [5, p. 12] shows that there exists a sequence $(I_k)_{k=1}^\infty$ of mutually nonoverlapping intervals in $[\alpha, \beta]$ such that

$$|a(x)| \leq t_0 \quad \text{a.e. in } [\alpha, \beta] - \bigcup_{k=1}^\infty I_k,$$

$$t_0 \leq \frac{1}{|I_k|} \int_{I_k} |a(t)| dt \leq 2t_0 \quad (k \geq 1).$$

We put

$$b(x) = \begin{cases} a(x) & \left(x \in [\alpha, \beta] - \bigcup_{k=1}^\infty I_k \right), \\ \frac{1}{|I_k|} \int_{I_k} a(t) dt & (x \in I_k, k \geq 1). \end{cases}$$

Then $\|b\|_\infty \leq 2t_0 = 4|\Gamma|/|E^0|$. We have, with $B(x) = \int_\alpha^x b(t) dt + A(\alpha)$,

$$\begin{aligned}
 |\alpha \leq x \leq \beta; A(x) \neq B(x)| & \leq \sum_{k=1}^\infty |I_k| \leq \frac{1}{t_0} \sum_{k=1}^\infty \int_{I_k} |a(t)| dt \\
 & \leq |\Gamma|/t_0 = |E^0|/2.
 \end{aligned}$$

Thus $\Gamma^* = \{x + iB(x); \alpha \leq x \leq \beta\}$ satisfies the required two inequalities. Q.E.D.

We now prove our proposition. Let Γ^* be the graph given in Lemma 6. We can choose a compact set F in $E^0 \cap \{\alpha \leq x \leq \beta; A(x) = B(x)\}$ so that $|F| \geq |E^0|/4$. Let $\hat{f}_F$ be the function in Lemma 5. Since $\hat{F} \subset E$, we have $\hat{f}_F \in H^\infty(E^c)$, $\|\hat{f}_F\|_{H^\infty(E^c)} \leq \text{Const}$. Thus

$$\begin{aligned} \gamma(E) &\geq |\hat{f}_F(\infty)| \geq \text{Const}|F|/(1 + \sqrt{\|b\|_\infty}) \\ &\geq \text{Const}|F|/\left\{1 + \sqrt{4|\Gamma|/|E^0|}\right\} \geq \text{Const}|E^0|^{3/2}/|\Gamma|^{1/2}. \end{aligned}$$

This completes the proof of our proposition.

3. Construction of (E_n) . In this section we construct a sequence $(E_n)_{n=2}^\infty$ of compact sets in $D(0, 2)$ so that

$$(2) \quad \gamma(E_n) \geq \text{Const}/\sqrt{n},$$

$$(3) \quad \text{Bu}(E_n) \leq \text{Const}(\log n)^2/n.$$

If such a sequence has been constructed, the assertion of our theorem is evidently deduced. Let $I_0 = [0, 1]$ and $U = I_0 \times [-1, 1]$. For two sets G, G' in U and a positive integer ν , we define

$$\sigma(G, G') = \left(\frac{1}{2}G + \frac{i}{2}\right) \cup \left(\frac{1}{2}G' + \frac{1-i}{2}\right), \quad \tau(\nu, G) = \bigcup_{\mu=0}^{\nu-1} \left(\frac{1}{\nu}G + \frac{\mu}{\nu}\right),$$

where $\zeta G + \zeta' = \{\zeta z + \zeta'; z \in G\}$ ($\zeta, \zeta' \in \mathbb{C}$). Given $n \geq 2$, we shall define two finite increasing sequences $(\nu'_k)_{k=1}^n, (\nu''_k)_{k=1}^n$ of positive integers later. To these two sequences, we associate $n+1$ sets $G_0, G_1, \dots, G_n$ as follows:

$$G_0 = \sigma(I_0, I_0), \quad G_k = \sigma(\tau(\nu'_k, G_{k-1}), \tau(\nu''_k, G_{k-1})) \quad (1 \leq k \leq n).$$

We put $E_n = G_n$ (see Figure 1).

Here is a lemma necessary for the proof of (2).

LEMMA 7. *There exists a graph $\Gamma_n = \{x + iA_n(x); 0 \leq x \leq 1\}$ such that $|\Gamma_n| \leq \text{Const } n$ and $|\Gamma_n \cap E_n^0| \geq \frac{1}{2}$.*

PROOF. For two sets G, G' in U , we put

$$\tilde{\sigma}(G, G') = \sigma(G, G') \cup \left(\frac{i}{2}I_0\right) \cup \left(iI_0 + \frac{1-i}{2}\right) \cup \left(-\frac{i}{2}I_0 + 1\right).$$

Then

$$|\tilde{\sigma}(G, G')| \leq |\sigma(G, G')| + 2 \leq \frac{1}{2}(|G| + |G'|) + 2.$$

For a positive integer ν and a set G in U , we have $|\tau(\nu, G)| \leq |G|$. We now define $n+1$ arcwise connected sets $\Lambda_0, \Lambda_1, \dots, \Lambda_n$ by

$$\Lambda_0 = \tilde{\sigma}(I_0, I_0), \quad \Lambda_k = \tilde{\sigma}(\tau(\nu'_k, \Lambda_{k-1}), \tau(\nu''_k, \Lambda_{k-1})) \quad (1 \leq k \leq n).$$

Then we have $\Lambda_n \supset E_n$ and

$$\begin{aligned} |\Lambda_n| &\leq \frac{1}{2}(|\tau(\nu'_n, \Lambda_{n-1})| + |\tau(\nu''_n, \Lambda_{n-1})|) + 2 \\ &\leq |\Lambda_{n-1}| + 2 \leq \dots \leq |\Lambda_0| + 2n = 2n + 3. \end{aligned}$$

We see that $\Lambda_n \cap \mathfrak{L}(r, 0)$ is a singleton except for $r > 0$ such that $\mathfrak{L}(r, 0)$ passes through one of the endpoints of E_n . Hence we can define a graph Γ_n satisfying the required two inequalities. Q.E.D.

We now show (2). Let Γ_n be the graph given in Lemma 7. We put $\tilde{E}_n = E_n \cap \Gamma_n$. Then our proposition shows that

$$\begin{aligned} \gamma(E_n) &\geq \gamma(\tilde{E}_n) \geq \text{Const} |\tilde{E}_n^0|^{3/2} / |\Gamma_n|^{1/2} \\ &\geq \text{Const} / |\Gamma_n|^{1/2} \geq \text{Const} / \sqrt{n}. \end{aligned}$$

Thus we have (2).

4. Proof of (3). In this section we define $(\nu'_k)_{k=1}^n, (\nu''_k)_{k=1}^n$ such that (3) holds. Geometric observation immediately yields

LEMMA 8. For $0 \leq k \leq n$, G_k is a union of $\prod_{j=0}^k (\nu'_j + \nu''_j)$ segments in U ($\nu'_0 = \nu''_0 = 1$).

LEMMA 9. Let X be a union of at most ν intervals in an interval $I = [s, s + r]$ on $\mathbf{R}$ and let Y be a set in I of the form $Y = \bigcup_{j=0}^{\mu''-1} (Y' + (jr/\mu''))$ for some $\mu'' \geq 1$ and $Y' \subset [s, s + (r/\mu'')]$. Then

$$|X \cup Y| \leq |X| + |Y| - |X| |Y| / r + 3r\nu/\mu''.$$

PROOF. We can write $X = \bigcup_{j=1}^{\mu'} J_j$ with mutually disjoint intervals $\{J_j\}_{j=1}^{\mu'}$, $\mu' \leq \nu$. Then, for each J_j , the number of k 's satisfying $(s + (kr/\mu''), s + ((k+1)r/\mu'')) \subset J_j$ is larger than $(\mu''|J_j|/r) - 3$. Hence

$$\begin{aligned} |X \cap Y| &= \sum_{j=1}^{\mu'} |J_j \cap Y| \geq \sum_{j=1}^{\mu'} |Y'| \left\{ \frac{\mu''|J_j|}{r} - 3 \right\} \\ &= \frac{\mu''|Y'|}{r} \sum_{j=1}^{\mu'} |J_j| - 3\mu'|Y'| \geq \frac{|X||Y|}{r} - \frac{3r\nu}{\mu''}, \end{aligned}$$

which gives the required inequality. Q.E.D.

For $z \in \mathbf{C}$ and $\xi \in \mathbf{R}$, we write $x(z, \xi) = \text{Re } z + \xi \text{Im } z$. The straight line passing through z and $x(z, \xi)$ is perpendicular to $\mathfrak{P}^{\arctan \xi}$. For a set $G \subset \mathbf{C}$, we write $G(\xi) = \{x(z, \xi); z \in G\}$. Then $(sG + \xi)(\xi) = sG(\xi) + x(\xi, \xi)$ ($s \in \mathbf{R}, \xi \in \mathbf{C}$).

LEMMA 10. Let G be a set in U with at most ν components. For three nonnegative integers ν', ν'', m with $\nu'' \geq \nu' \geq m + 1$, we put

$$H = H(\nu', \nu'', m, G) = (\tfrac{1}{2}\tau(\nu', G) + i/2) \cup (\tfrac{1}{2}\tau(\nu'', G) + (1-i)/2 + m).$$

Let $m \leq \xi \leq m + 1$. Then

$$|H(\xi)| \leq \{1 - \xi_m |G(\xi)|\} |G(\xi)| + \text{Const}\{(m+1)/\nu' + \nu\nu'/\nu''\}$$

as long as $|G(\xi)| \leq 1$, where $\xi_m = \min\{\xi - m, m + 1 - \xi\}$.

PROOF. Suppose that $m \leq \xi \leq m + \frac{1}{2}$. We put

$$X_j = \frac{1}{2\nu'}G + \frac{j}{2\nu'} + \frac{i}{2} \quad (0 \leq j \leq \nu' - 1),$$

$$Y_j = \frac{1}{2\nu''}G + \frac{j}{2\nu''} + \frac{1-i}{2} + m \quad (j = 0, \pm 1, \pm 2, \dots).$$

Then $|X_j(\xi)| = |G(\xi)|/2\nu'$, $|Y_j(\xi)| = |G(\xi)|/2\nu''$. Let $\mu' = \iota(2\nu'\xi_m)$ and $\mu'' = \iota(2\nu''\xi_m)$, where $\iota(x)$ is the integral part of x . If $\mu' \leq 2m + 2$, then, by $2\nu'|X_j(\xi)| = 2\nu''|Y_j(\xi)| = |G(\xi)| \leq 1$, we have

$$\begin{aligned} |H(\xi)| &= \left| \left(\frac{1}{2}\tau(\nu', G) + \frac{i}{2} \right)(\xi) \cup \left(\frac{1}{2}\tau(\nu'', G) + \frac{1-i}{2} + m \right)(\xi) \right| \\ &= \left| \left\{ \bigcup_{j=0}^{\nu'-1} X_j(\xi) \right\} \cup \left\{ \bigcup_{j=0}^{\nu''-1} Y_j(\xi) \right\} \right| \\ &\leq \sum_{j=0}^{\nu'-1} |X_j(\xi)| + \sum_{j=0}^{\nu''-1} |Y_j(\xi)| = |G(\xi)| \\ &\leq |G(\xi)| + \left\{ \frac{\mu' + 1}{2\nu'} - \xi_m \right\} \leq |G(\xi)| + \left\{ \frac{\mu' + 1}{2\nu'} - \xi_m |G(\xi)|^2 \right\} \\ &\leq \left\{ 1 - \xi_m |G(\xi)| \right\} |G(\xi)| + \frac{2m + 3}{2\nu'}, \end{aligned}$$

which gives the required inequality. Suppose that $\mu' \geq 2m + 3$. We apply Lemma 9 to

$$X = \bigcup_{j=\nu'-\mu'+m+1}^{\nu'-m-2} X_j(\xi), \quad Y = \left\{ \bigcup_{j=-\infty}^{\infty} Y_j(\xi) \right\} \cap I,$$

$$I = [x((1-i)/2 + m, \xi), x(\mu''/2\nu'' + (1-i)/2 + m, \xi)];$$

geometric observation shows that $X \subset I$ and $Y = \bigcup_{j=0}^{\mu''-1} (Y' + (j|I|/\mu''))$ with

$$Y' = Y \cap [x((1-i)/2 + m, \xi), x((1-i)/2 + m, \xi) + |I|/\mu'']$$

(see Figure 2). Since $\mu' \leq \nu'$, $|I| = \mu''/2\nu''$ and X is a union of at most $\nu\mu'$ intervals, Lemma 9 gives that

$$\begin{aligned} |X \cup Y| &\leq |X| + |Y| - |X||Y|/|I| + 3\nu\mu'|I|/\mu'' \\ &\leq |Y| + |X|\{1 - |Y|/|I|\} + 2\nu\nu'/\nu''. \end{aligned}$$

Since $|G(\xi)| \leq 1$, $|X| \leq (\mu' - 2m - 2)|G(\xi)|/2\nu'$,

$$\frac{\mu' - 2m - 2}{2\nu'} \frac{|G(\xi)|}{|I|} \leq \frac{\xi_m - 1/2\nu'}{|I|} \leq \frac{\mu'' + 1/2\nu'' - 1/2\nu'}{|I|} \leq \frac{\mu''}{2\nu''|I|} = 1$$

and

$$|Y| = \left| \left\{ \bigcup_{j=-m-1}^{\mu''+m} Y_j(\xi) \right\} \cap I \right| \leq \frac{\mu'' + 2m + 2}{2\nu''} |G(\xi)|,$$

we have

$$\begin{aligned} |X \cup Y| &\leq |Y| + \left(\frac{\mu' - 2m - 2}{2\nu'} |G(\xi)| \right) \left(1 - \frac{|Y|}{|I|} \right) + \frac{2\nu\nu'}{\nu''} \\ &= \frac{\mu' - 2m - 2}{2\nu'} |G(\xi)| + |Y| \left(1 - \frac{\mu' - 2m - 2}{2\nu'} \frac{|G(\xi)|}{|I|} \right) + \frac{2\nu\nu'}{\nu''} \\ &\leq \frac{\mu' - 2m - 2}{2\nu'} |G(\xi)| \\ &\quad + \left(\frac{\mu'' + 2m + 2}{2\nu''} |G(\xi)| \right) \left(1 - \frac{\mu' - 2m - 2}{2\nu'} \frac{|G(\xi)|}{|I|} \right) + \frac{2\nu\nu'}{\nu''} \\ &\leq \frac{\mu' - 2m - 2}{2\nu'} |G(\xi)| \\ &\quad + \frac{\mu''}{2\nu''} |G(\xi)| \left(1 - \frac{\mu' - 2m - 2}{2\nu'} \frac{|G(\xi)|}{|I|} \right) + \left\{ \frac{m+1}{\nu''} + \frac{2\nu\nu'}{\nu''} \right\} \\ &= \left\{ \frac{\mu' - 2m - 2}{2\nu'} + \frac{\mu''}{2\nu''} \right\} |G(\xi)| - \frac{\mu' - 2m - 2}{2\nu'} |G(\xi)|^2 + \left\{ \frac{m+1}{\nu''} + \frac{2\nu\nu'}{\nu''} \right\} \\ &\leq \left\{ \frac{\mu' - 2m - 2}{2\nu'} + \frac{\mu''}{2\nu''} \right\} |G(\xi)| - \xi_m |G(\xi)|^2 + \left\{ \frac{2m+3}{2\nu'} + \frac{m+1}{\nu''} + \frac{2\nu\nu'}{\nu''} \right\}. \end{aligned}$$

FIGURE 2

Thus

$$\begin{aligned}
 |H(\xi)| &= \left| \left\{ \bigcup_{j=0}^{\nu'-1} X_j(\xi) \right\} \cup \left\{ \bigcup_{j=0}^{\nu''-1} Y_j(\xi) \right\} \right| \\
 &\leq \left| \left\{ \bigcup_{j=0}^{\nu'-1} X_j(\xi) - X \right\} \cup \left[\left\{ \bigcup_{j=0}^m + \bigcup_{j=\mu''-m-1}^{\nu''-1} \right\} Y_j(\xi) \right] \cup (X \cup Y) \right| \\
 &\leq \left\{ \sum_{j=0}^{\nu'-\mu'+m} + \sum_{j=\nu'-m-1}^{\nu'-1} \right\} |X_j(\xi)| + \left\{ \sum_{j=0}^m + \sum_{j=\mu''-m-1}^{\nu''-1} \right\} |Y_j(\xi)| + |X \cup Y| \\
 &= \left\{ \frac{\nu' - \mu' + 2m + 2}{2\nu'} + \frac{\nu'' - \mu''}{2\nu''} \right\} |G(\xi)| + |X \cup Y| + \frac{2m + 2}{2\nu''} |G(\xi)| \\
 &\leq \{1 - \xi_m |G(\xi)|\} |G(\xi)| + \text{Const} \left\{ \frac{m+1}{\nu'} + \frac{\nu\nu'}{\nu''} \right\}.
 \end{aligned}$$

In the case where $m + \frac{1}{2} < \xi \leq m + 1$, we discuss $H(\xi)$ on an interval

$$\left[x \left(-\frac{\iota(2\nu''\xi_m)}{2\nu''} + \frac{1-i}{2} + m + \frac{1}{2}, \xi \right), x \left(\frac{1-i}{2} + m + \frac{1}{2}, \xi \right) \right].$$

Then, in the same manner as above, we obtain the required inequality. Q.E.D.

LEMMA 11. $|G_k(\xi)| \leq |G_{k-1}(\xi)| \leq 1$ ($\xi \in \mathbf{R}$, $1 \leq k \leq n$).

PROOF. We easily see that $|G_0(\xi)| \leq 1$. We have

$$\begin{aligned}
 |G_k(\xi)| &= |\sigma(\tau(\nu'_k, G_{k-1}), \tau(\nu''_k, G_{k-1}))(\xi)| \\
 &= \left| \left(\frac{1}{2} \tau(\nu'_k, G_{k-1})(\xi) + \frac{\xi}{2} \right) \cup \left(\frac{1}{2} \tau(\nu''_k, G_{k-1})(\xi) + \frac{1-\xi}{2} \right) \right| \\
 &\leq \frac{1}{2} \{ |\tau(\nu'_k, G_{k-1})(\xi)| + |\tau(\nu''_k, G_{k-1})(\xi)| \} \\
 &= \frac{1}{2} \left\{ \left| \bigcup_{j=0}^{\nu'_k-1} \left(\frac{1}{\nu'_k} G_{k-1}(\xi) + \frac{j}{\nu'_k} \right) \right| + \left| \bigcup_{j=0}^{\nu''_k-1} \left(\frac{1}{\nu''_k} G_{k-1}(\xi) + \frac{j}{\nu''_k} \right) \right| \right\} \\
 &\leq |G_{k-1}(\xi)|. \quad \text{Q.E.D.}
 \end{aligned}$$

We now define $(\nu'_k)_{k=1}^n$, $(\nu''_k)_{k=1}^n$ such that (3) holds. We see that $|E_n^\theta| = |E_n(\tan \theta)| \cos \theta$ ($-\pi/2 < \theta \leq \pi/2$). Hence we have

$$\begin{aligned}
 \text{Bu}(E_n) &= \int_{-\pi/2}^{\pi/2} |E_n^\theta| d\theta = \int_{-\infty}^{\infty} \frac{|E_n(\xi)|}{(1+\xi^2)^{3/2}} d\xi \\
 &= \sum_{m=-n}^n \int_m^{m+1} + \int_{-\infty}^{-n} + \int_{n+1}^{\infty} \left(= \sum_{m=-n}^n \delta_m + \delta_{-n-1} + \delta_{n+1}, \text{ say} \right).
 \end{aligned}$$

We first estimate δ_0 . Recall that G_{k-1} is a union of $\prod_{j=0}^{k-1} (v'_j + v''_j)$ segments in U . Applying Lemma 10 to $G = G_{k-1}$, $v' = v'_k$, $v'' = v''_k$, $m = 0$, we have, for any $0 \leq \xi < 1$,

$$\begin{aligned} |G_k(\xi)| &= |\sigma(\tau(v'_k, G_{k-1}), \tau(v''_k, G_{k-1}))(\xi)| = |H(v'_k, v''_k, 0, G_{k-1})(\xi)| \\ &\leq \{1 - \xi_0 |G_{k-1}(\xi)|\} |G_{k-1}(\xi)| + \text{Const} \left[\frac{1}{v'_k} + \left\{ \prod_{j=0}^{k-1} (v'_j + v''_j) \right\} \frac{v'_k}{v''_k} \right]. \end{aligned}$$

Since this estimate is valid for any $1 \leq k \leq n$, we obtain

$$\begin{aligned} (4) \quad |E_n(\xi)| &= |G_n(\xi)| \\ &\leq \prod_{k=1}^n \{1 - \xi_0 |G_{k-1}(\xi)|\} |G_0(\xi)| + C_0 \sum_{k=1}^n \left[\frac{1}{v'_k} + \left\{ \prod_{j=0}^{k-1} (v'_j + v''_j) \right\} \frac{v'_k}{v''_k} \right] \\ &\leq \{1 - \xi_0 |E_n(\xi)|\}^n + C_0 \rho_n, \end{aligned}$$

by Lemma 11, where C_0 is an absolute constant and

$$(5) \quad \rho_n = \sum_{k=1}^n \left[\frac{n}{v'_k} + \left\{ \prod_{j=0}^{k-1} (v'_j + v''_j) \right\} \frac{v'_k}{v''_k} \right].$$

We define ξ'_0 by $1/\xi'_0 = 1/\xi_0 + C_0 n \rho_n$. If $|E_n(\xi)| > (1/n\xi'_0) \log(10 + n\xi'_0)$, then (4) shows that

$$\begin{aligned} |E_n(\xi)| &\leq \left\{ 1 - \xi_0 \frac{1}{n\xi'_0} \log(10 + n\xi'_0) \right\}^n + C_0 \rho_n \\ &\leq \left\{ 1 - \frac{1}{n} \log(n\xi'_0) \right\}^n + C_0 \rho_n \\ &\leq \frac{1}{n\xi'_0} + C_0 \rho_n \leq \frac{2}{n\xi'_0} \leq \frac{1}{n\xi'_0} \log(10 + n\xi'_0) < |E_n(\xi)|, \end{aligned}$$

which is a contradiction. Thus (4) yields that $|E_n(\xi)| \leq (1/n\xi'_0) \log(10 + n\xi'_0)$ ($0 \leq \xi \leq 1$). We have

$$\begin{aligned} (6) \quad \delta_0 &\leq \int_0^1 |E_n(\xi)| d\xi \leq \frac{2}{n} + \int_{1/n}^{1-1/n} \frac{1}{n\xi'_0} \log(10 + n\xi'_0) d\xi \\ &\leq \frac{2}{n} + 2 \int_{1/n}^{1/2} \left(\frac{1}{n\xi} + C_0 \rho_n \right) \log(10 + n\xi) d\xi \\ &\leq \text{Const} \left\{ \frac{(\log n)^2}{n} + \rho_n \log n \right\}. \end{aligned}$$

We next estimate δ_m ($1 \leq m \leq n$). We have, for any $2 \leq k \leq n$, $m \leq \xi \leq m+1$,

$$|G_k(\xi)| \leq \frac{1}{2} \left\{ |\tau(v'_k, G_{k-1})(\xi)| + |\tau(v''_k, G_{k-1})(\xi)| \right\}.$$

To estimate $|\tau(\nu'_k, G_{k-1})(\xi)|$, we put

$$X_j^* = \frac{1}{\nu'_k} \left(\frac{1}{2} X^* + \frac{i}{2} \right) + \frac{j}{\nu'_k}, \quad Y_j^* = \frac{1}{\nu'_k} \left(\frac{1}{2} Y^* + \frac{1-i}{2} \right) + \frac{j}{\nu'_k} \\ (0 \leq j \leq \nu'_k - 1),$$

where $X^* = \tau(\nu'_{k-1}, G_{k-2})$ and $Y^* = \tau(\nu''_{k-1}, G_{k-2})$. Lemmas 8, 10 and 11 show that

$$\begin{aligned} |\tau(\nu'_k, G_{k-1})(\xi)| &= \left| \bigcup_{j=0}^{\nu'_k-1} \{X_j^*(\xi) \cup Y_j^*(\xi)\} \right| \\ &\leq \sum_{j=0}^{m-1} |Y_j^*(\xi)| + \sum_{j=\nu'_k-m}^{\nu'_k-1} |X_j^*(\xi)| + \sum_{j=0}^{\nu'_k-m-1} |(X_j^* \cup Y_{j+m}^*)(\xi)| \\ &= \frac{m}{2\nu'_k} |Y^*(\xi)| + \frac{m}{2\nu'_k} |X^*(\xi)| \\ &\quad + \frac{\nu'_k - m}{\nu'_k} \left| \left(\left(\frac{1}{2} X^* + \frac{i}{2} \right) \cup \left(\frac{1}{2} Y^* + \frac{1-i}{2} + m \right) \right) (\xi) \right| \\ &\leq \frac{m}{\nu'_k} |G_{k-2}(\xi)| + \frac{\nu'_k - m}{\nu'_k} |H(\nu'_{k-1}, \nu''_{k-1}, m, G_{k-2})(\xi)| \\ &\leq \frac{m}{\nu'_k} + \frac{\nu'_k - m}{\nu'_k} \left[\{1 - \xi_m |G_{k-2}(\xi)|\} |G_{k-2}(\xi)| \right. \\ &\quad \left. + \text{Const} \left\{ \frac{m+1}{\nu'_{k-1}} + \left(\prod_{j=0}^{k-2} (\nu'_j + \nu''_j) \right) \frac{\nu'_{k-1}}{\nu''_{k-1}} \right\} \right] \\ &\leq \{1 - \xi_m |E_n(\xi)|\} |G_{k-2}(\xi)| + \text{Const} \left[\frac{m+1}{\nu'_{k-1}} + \left\{ \prod_{j=0}^{k-2} (\nu'_j + \nu''_j) \right\} \frac{\nu'_{k-1}}{\nu''_{k-1}} \right]. \end{aligned}$$

These inequalities are valid with ν'_k replaced by ν''_k , and hence

$$\begin{aligned} |G_k(\xi)| &\leq \{1 - \xi_n |E_n(\xi)|\} |G_{k-2}(\xi)| \\ &\quad + \text{Const} \left[\frac{m+1}{\nu'_{k-1}} + \left\{ \prod_{j=0}^{k-2} (\nu'_j + \nu''_j) \right\} \frac{\nu'_{k-1}}{\nu''_{k-1}} \right]. \end{aligned}$$

Using this inequality for $k = n, n-2, \dots, n-2(\iota(n/2)+1)$, we have, with an absolute constant C'_0 ,

$$\begin{aligned} |E_n(\xi)| &= |G_n(\xi)| \leq \{1 - \xi_m |E_n(\xi)|\}^{\iota(n/2)} |E_{n-2\iota(n/2)}(\xi)| \\ &\quad + \text{Const} \sum_{l=0}^{\iota(n/2)} \left[\frac{m+1}{\nu'_{n-2l-1}} + \left\{ \prod_{j=0}^{n-2l-2} (\nu'_j + \nu''_j) \right\} \frac{\nu'_{n-2l-1}}{\nu''_{n-2l-1}} \right] \\ &\leq \{1 - \xi_m |E_n(\xi)|\}^{\iota(n/2)} + C'_0 \rho_n. \end{aligned}$$

In the same manner as in the estimate of δ_0 , this yields that $|E_n(\xi)| \leq (3/n\xi'_m)\log(10 + n\xi'_m)$ ($m \leq \xi \leq m+1$), where ξ'_m is defined by $1/\xi'_m = 1/\xi_m + C'_0 n \rho_n$. Thus

$$\begin{aligned}
 (7) \quad \delta_m &\leq \frac{1}{m^3} \int_m^{m+1} |E_n(\xi)| d\xi \\
 &\leq \frac{2}{m^3 n} + \frac{1}{m^3} \int_{m+1/n}^{m+1-1/n} \frac{3}{n\xi'_m} \log(10 + n\xi'_m) d\xi \\
 &\leq \text{Const} \left\{ \frac{(\log n)^2}{n} + \rho_n \log n \right\} \frac{1}{m^3} \quad (1 \leq m \leq n).
 \end{aligned}$$

By (6) and (7), we have

$$\begin{aligned}
 \sum_{m=0}^{n+1} \delta_m &\leq \sum_{m=0}^n \delta_m + \int_{n+1}^{\infty} \frac{d\xi}{\xi^3} \\
 &\leq \text{Const} \left\{ \frac{(\log n)^2}{n} + \rho_n \log n \right\} \left\{ 1 + \sum_{m=1}^n \frac{1}{m^3} \right\} + \frac{1}{2(n+1)} \\
 &\leq \text{Const} \left\{ \frac{(\log n)^2}{n} + \rho_n \log n \right\}.
 \end{aligned}$$

In the same manner, we have $\sum_{m=1}^{n+1} \delta_{-m} \leq \text{Const}\{(\log n)^2/n + \rho_n \log n\}$. Consequently we have

$$(8) \quad \text{Bu}(E_n) = \sum_{m=-n-1}^{n+1} \delta_m \leq \text{Const} \left\{ \frac{(\log n)^2}{n} + \rho_n \log n \right\}.$$

Now we put

$$\begin{aligned}
 v'_k &= n^3 \quad (1 \leq k \leq n), \\
 v''_k &= n^2 \left\{ \prod_{j=0}^{k-1} (v'_j + v''_j) \right\} v'_k \quad (1 \leq k \leq n),
 \end{aligned}$$

where $v'_0 = v''_0 = 1$. Then, by (5), we have $\rho_n \leq 2/n$. By (8), we have $\text{Bu}(E_n) \leq \text{Const}(\log n)^2/n$. This completes the proof of (3). As easily seen, (2) and (3) yield the required equality in our theorem. This completes the proof of our theorem.

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DEPARTMENT OF MATHEMATICS, COLLEGE OF GENERAL EDUCATION, NAGOYA UNIVERSITY, CHIKUSA -
KU, NAGOYA, 464, JAPAN