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LINEAR SERIES WITH CUSPS AND n-FOLD POINTS

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ABSTRACT. A linear series $(V, \mathscr{L})$ on a curve X has an n-fold point along a divisor D of degree n if $\dim(V \cap H^0(X, \mathscr{L}(-D))) \geqslant \dim(V) - 1$. The linear series has a cusp of order e at a point P if $\dim(V \cap H^0(X, \mathscr{L}(-(e+1)P))) \geqslant \dim(V) - 1$. Linear series with cusps and n-fold points are shown to exist if certain inequalities are satisfied. The dimensions of the families of linear series with cusps are determined for general curves.

Introduction. Throughout this paper we work over the complex numbers **C**.

Let X be a smooth projective curve of genus g. A g_d^r on X is a linear series of degree d and dimension r on X, i.e., a pair $(V, \mathscr{L})$ consisting of a line bundle $\mathscr{L}$ of degree d on X and an $(r+1)$-dimensional subspace V of $H^0(X, \mathscr{L})$.

When $\rho(g, r, d) = g - (r+1)(g + r - d) \geqslant 0$, then X has g_d^r's and the family of g_d^r's on X forms a projective scheme $G_d^r(X)$ of dimension $\geqslant \rho(g, r, d)$. Furthermore, if X is general in moduli, then $G_d^r(X)$ is smooth of dimension $\rho(g, r, d)$ **[ACGH]**.

DEFINITION. If D is an effective divisor of degree $n \geqslant 2$ on X, we say that a g_d^r $(V, \mathscr{L})$ has an *n-fold point along D* if $\dim(V \cap H^0(X, \mathscr{L}(-D))) \geqslant r$. We say that a g_d^r has a *cusp of order e at a point P* if it has an $(e+1)$-fold point along the divisor $(e+1)P$.

The aim of this paper is to prove the following:

(i) If $g \geqslant n$, $\rho(g, r, d) - (n-1)r + n \geqslant 0$, and $\rho(g, r-1, d-n) \geqslant 0$, then there exists a g_d^r on X with an n-fold point along some divisor D of degree n.

(ii) If $\rho(g, r, d) - (n-1)r \geqslant 0$, $\rho(g, r-1, d-n) \geqslant 0$, and D is any divisor of degree n on X, then there exists a g_d^r on X with an n-fold point along D.

(iii) If $\rho(g, r, d) - er + 1 \geqslant 0$ and $\rho(g, r-1, d-e-1) \geqslant 0$, then there exists a g_d^r on X with a cusp of order e at some point P.

(iv) If X is general in moduli, then the family of g_d^r's on X with a cusp of order e has dimension $\rho(g, r, d) - er + 1$ if it is nonempty.

Assertion (iv) has been proved independently by Marc Coppens **[C]**.

It is easy to see that the hypotheses $\rho(g, r-1, d-n) \geqslant 0$ in assertions (i) and (ii), and $\rho(g, r-1, d-e-1) \geqslant 0$ in assertion (iii) are necessary if X is general in moduli.

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Our method of proof uses the theory of limit linear series as developed in [**EH3**] which generalizes the notion of linear series on smooth curves to curves of compact type, i.e., curves which are a union of complete smooth curves which meet at ordinary double points and are such that the dual graph is a tree.

We say that a sequence $a = (a_0, \ldots, a_r)$ is of *type* (r, d) if $0 \leqslant a_0 < \cdots < a_r \leqslant d$. We say that a g^r_d $(V, \mathscr{L})$ *has vanishing sequence* a *at* $P \in X$ if $\{\operatorname{ord}_P s \mid s \in V\} = \{a_i \mid i = 0, 1, \ldots, r\}$ where $\operatorname{ord}_P s$ denotes the order of vanishing of s as a section of $H^0(X, \mathscr{L})$. We say that $(V, \mathscr{L})$ *satisfies the vanishing condition* b *at* P if it has vanishing sequence a at P and $b_i \leqslant a_i$ for $i = 0, 1, \ldots, r$.

DEFINITION. *A limit* g^r_d (*or limit linear series*) on a curve X of compact type is a collection of g^r_d's $(V_i, \mathscr{L}_i)$ on each of the components X_i of X satisfying the *compactability conditions*: If X_i and X_j meet at P, then there are sequences a^i and a^j of type (r, d) such that $a^i_k + a^j_{r-k} = d$ for $k = 0, 1, \ldots, r$, and $(V_i, \mathscr{L}_i)$ and $(V_j, \mathscr{L}_j)$ have vanishing sequences a^i and a^j, respectively, at P.

A fundamental result of [**EH3**] is that if a family $f\colon X \to B$ of curves of compact type is sufficiently nice, then there is a quasi-projective B-scheme $G^r_d(X/B)$ whose fiber over a point $q \in B$ is a scheme parametrizing limit g^r_d's on $X_q = f^{-1}(q)$. Furthermore, every component of $G^r_d(X/B)$ must have dimension $\geqslant \rho(g, r, d) + \dim B$. We prove the existence of g^r_d's with a desired property (such as having a cusp of order e) by considering the subscheme $H \subset G^r_d(X/B)$ parametrizing such g^r_d's. We find a lower bound N on the dimension of each component of H. We then show that on some singular curve in the family there exists a limit g^r_d with the desired property which varies in a family of dimension $N - \dim B$. The component of H containing this limit g^r_d must now extend over B.

In §1 we consider subschemes of $G^r_d(X/B)$ for suitable X and B which parametrize g^r_d's with cusps of order e and g^r_d's with n-fold points, and we give lower bounds for the dimensions of the components of these subschemes.

In §2 we show the existence of limit g^r_d's with cusps and n-fold points on a singular curve which vary in a family of the "expected" dimension. The results of §1 are then used to show the existence of desired g^r_d's on smooth curves.

In §3 we determine the dimension of the family of g^r_d's with cusps on a general smooth curve by finding an upper bound for the dimension of the family of limit g^r_d's with cusps on a singular curve.

We will use the following three notations. If $f\colon X \to Y$ is a morphism and $q \in Y$, then X_q will denote the fiber of f over q. If V is a vector space, then $\operatorname{Gr}_k(V)$ will denote the Grassmannian of k-planes in V. If D and E are divisors on a smooth curve, $D \sim E$ will denote D is linearly equivalent to E.

I would like to thank Ziv Ran for his explanations of the theory of limit linear series and their application to the topics in this paper.

1. Families of linear series. The following is Theorem 3.3 of [**EH3**].

THEOREM 1.1. *Let* $\pi\colon X \to B$, $p_1, \ldots, p_s\colon B \to X$ *be an s-pointed (relative) genus g curve such that*: *B is irreducible*; π *is flat and proper*; *the fibers of* π *are curves of compact type*; *the images of the* p_i *are disjoint and in the smooth locus of* π; *there* *exists a relatively ample divisor D on X whose support is disjoint from all the sections $p_i(B)$; and the components of the singular locus of π map isomorphically onto their images in B.*

Let $a^1, \dots, a^s$ be sequences of type (r, d). Then there exists a scheme

$$G = G_d^r\big(X/B, (p_1, a^1), \dots, (p_s, a^s)\big),$$

quasi-projective over B, compatible with base extension, whose points over any $q \in B$ correspond to limit g_d^r's on X_q satisfying vanishing conditions $a^1, \dots, a^s$ at $p_1, \dots, p_s$, respectively. Further, every component of G has dimension $\geqslant \rho(g, r, d) - \sum_{0 \leqslant i \leqslant r; 1 \leqslant j \leqslant s}(a_i^j - i) + \dim B$.

Remarks on the proof of Theorem 1.1. If the generic fiber of π: $X \to B$ is smooth, we have the following situation around a point $q \in B$ if we replace B by a sufficiently small neighborhood of q.

If Y is an irreducible component of X_q, let $\mathrm{Pic}^Y(X/B)$ be the relative Picard scheme of invertible sheaves whose degree on Y is d and whose degree on each of the other components of X_q is 0. Let $\tilde{\mathscr{L}}_Y$ be the universal Poincaré line bundle. By replacing D with a multiple of itself we may assume that it meets each component of X_q with high degree. Let D_Y be the union of the components of D that meet Y. Let π_1 and π_2 be the projections of $X \times \mathrm{Pic}^Y(X/B)$ to X and $\mathrm{Pic}^Y(X/B)$, respectively. Let G_Y denote the Grassmannian of $(r + 1)$ planes in $\pi_{2*}\tilde{\mathscr{L}}_Y(\pi_1^*D_Y)$, and let V_Y be the universal subbundle on G_Y. There is a morphism α such that the following diagram commutes, where β and γ are the natural morphisms,

$$G_d^r(X/B) \xrightarrow{\alpha} G_Y \xrightarrow{\gamma} \mathrm{Pic}^Y(X/B) \to B.$$

$$\beta$$

If $z \in G_d^r(X/B)$, then $V_{Y\alpha(z)}$ corresponds to an $(r + 1)$-dimensional subspace of $H^0(X_{\beta(z)}, \tilde{\mathscr{L}}_{Y|X_{\beta(z)}})$, and $\tilde{\mathscr{L}}_{Y|X_{\beta(z)}}$ has degree d on one component $Y_{\beta(z)}$ of $X_{\beta(z)}$ and degree zero on each of the other components. Thus $\alpha(z)$ determines a g_d^r on $Y_{\beta(z)}$. This is the same g_d^r on $Y_{\beta(z)}$ as the one in the limit g_d^r corresponding to z. The component $Y_{\beta(z)}$ specializes to a union of components containing Y in X_q.

Let E be a relative divisor of degree n on X whose support is disjoint from the singular locus of π and the support of D. Suppose all the components of E meet Y. The subscheme $H \subset G_d^r(X/B)$ of limit g_d^r's with n-fold points along E is the inverse image of the points in G_Y where the vector bundle map $V_Y \to \gamma^*\pi_{2*}\pi_1^*\mathcal{O}_E$ has rank $\leqslant 1$. Thus $\operatorname{codim}(H, G_d^r(X/B)) \leqslant r(n - 1)$ and H is closed in $G_d^r(X/B)$.

COROLLARY 1.2. *Let T be an irreducible curve containing a point* 0, *and let π: $X \to T$ be a flat family of genus g curves of compact type such that all fibers over $T \setminus \{0\}$ are nonsingular. Let $B = X \setminus \{$singular points of $X_0\}$. Then there exists a closed subscheme $H \subset G_d^r(X \times_T B/B)$ such that the fiber over any point $q \in B$ corresponds to limit g_d^r's on $X_{\pi(q)}$ with a cusp of order e at q. Furthermore, every component of H has dimension $\geqslant \rho(g, r, d) - er + 2$.*

PROOF. For any relatively ample divisor E on X contained in B, let $B_E = B \setminus E$. Let Δ be the divisor of the diagonal morphism $B_E \to X \times_T B_E$. By Theorem 1.1 and the remarks on its proof, there exists a closed subscheme $H_E \subset G^r_d(X \times_T B_E/B_E)$ parametrizing limit linear series with cusps of order e along Δ, and

$$\operatorname{codim}(H_E, G^r_d(X \times_T B_E/B_E)) \leqslant er.$$

The subschemes H_E patch together to form the desired subscheme

$$H \subset G^r_d(X \times_T B/B). \quad \square$$

We will call the subscheme H in Corollary 1.2 the *subscheme of g^r_d's with cusps of order e*.

COROLLARY 1.3. *Let T be an irreducible curve containing a point* 0, *and let π: $X \to T$ be a flat family of genus g curves of compact type such that all fibers over $T \setminus \{0\}$ are nonsingular. Let Y be an open smooth connected subset of X_0, and let $Z = X_0 \setminus Y$. Let $B = (X \setminus Z)^n$ where the product is fibered over T. Then there exists a subscheme $H \subset G^r_d(X \times_T B/B)$ such that the fiber over any point $(P_1, \dots, P_n) \in B$ corresponds to limit g^r_d's on $X_{\pi(P_1)}$ with an n-fold point along $P_1 + \cdots + P_n$.*

Furthermore, every component of H has dimension $\geqslant \rho(g, r, d) - (n-1)r + n + 1$.

PROOF. For any effective relative divisor E on X whose support does not include the singular points of X_0, let $B_E = (X \setminus (Z \cup E))^n$. For each $i = 1, \dots, n$, let $\Delta_i = \{(P, (P_1, \dots, P_n)) \in X \times_T B_E \mid P_i = P\}$, and let $D = \sum \Delta_i$. As in the proof of the previous corollary, Theorem 1.1 and the remark on its proof imply the existence of subschemes H_E which patch together to form the desired subscheme H, and $\operatorname{codim}(H, G^r_d(X \times_T B/B)) \leqslant (n-1)r$. $\square$

We will call the subscheme H of Corollary 1.3 the *subscheme of g^r_d's with n-fold points*.

2. Existence of linear series with cusps and n-fold points. The following is Theorem 4.5 of **[EH3]**.

THEOREM 2.1. *Let C be a curve of compact type, let $p_1, \dots, p_s$ be smooth points of C, and let $a^1, \dots, a^s$ be sequences of type (r, d). Every component of the family $G^r_d(C, (p_1, a^1), \dots, (p_n, a^s))$ has dimension $\geqslant \rho(g, r, d) - \sum_{0 \leqslant i \leqslant r;\, 1 \leqslant j \leqslant s}(a^j_i - i)$, and equality holds if each component of C is a general curve of its genus, and the singular points of C and $p_1, \dots, p_s$ are general points on the components in which they lie.*

DEFINITION. We say a curve C of compact type and smooth points $p_1, \dots, p_s$ are *general for d* if for any closed connected subcurve $X \subset C$ and points $Q_1, \dots, Q_t \in (\{p_1, \dots, p_s\} \cap X) \cup \{$singular points of C which are smooth points of $X\}$ and any sequences $b^1, \dots, b^t$ of type (r, e) with $e \leqslant d$, then every component $G^r_e(X, (Q_1, b^1), \dots, (Q_t, b^t))$ has dimension

$$\rho(g_X, r, e) - \sum_{j=1}^{t} \sum_{i=1}^{r} (b^j_i - i)$$

where the g_X is the genus of X.

We have C general for d if each component of C is a general curve of its genus, and the singular points of C and $p_1, \dots, p_s$ are general in the components in which they lie. This is because there are only finitely many sequences b of type (r, e) with $e \leqslant d$.

We also have that if C and $p_1, \dots, p_s$ are general for e and $d \leqslant e$, then a general member of $G_d^r(C, (p_1, a^1), \dots, (p_s, a^s))$ has vanishing sequences a^i at each p_i.

LEMMA 2.2. *Let E be an elliptic curve, and let P and Q be distinct points of E. Suppose $a = (a_0, \dots, a_r)$ and $b = (b_0, \dots, b_r)$ are sequences of type (r, d) such that for some k we have $a_k + b_{r-k} \leqslant d$ and $a_i + b_{r-i} \leqslant d - 1$ for $i \neq k$. Then*

$$G_d^r(E, (P, a), (Q, b)) \neq \varnothing.$$

PROOF. Note that if $a_k + b_{r-k} = d$, then $k = 0$ or $a_k - a_{k-1} \geqslant (d - b_{r-k}) - (d - 1 - b_{r-k+1}) \geqslant 2$.

Let $i_0 < \cdots < i_s$ be the elements of $\{i \mid a_i - a_{i-1} \geqslant 2 \text{ or } i = 0\}$. Let $j_s = r + 1 - i_s$, and let $j_n = i_{n+1} - i_n$ for $n < s$. Let $\mathscr{L} = \mathscr{O}(a_k P + (d - a_k)Q)$, and for each $n = 0, \dots, s$, let

$$V_n = H^0\Big(E, \mathscr{L}\big(-a_{i_n}P - (d - a_{i_n} - j_n)Q\big)\Big).$$

If $n > m$, then $\max\{\operatorname{ord}_P(f) \mid f \in V_m\} < \min\{\operatorname{ord}_P(f) \mid f \in V_n\}$ and $\min\{\operatorname{ord}_Q(f) \mid f \in V_m\} > \max\{\operatorname{ord}_Q(f) \mid f \in V_n\}$, because $a_{i_n} \geqslant a_{i_m} + j_m + 1$. Let $V = \bigoplus_{n=0}^{s} V_n$. Then $\dim V = \sum \dim V_n = \sum j_n = r + 1$. Each V_n satisfies the vanishing condition $(a_{i_n}, \dots, a_{i_{n+1}-1})$ at P for $n < s$, and V_s satisfies $(a_{i_s}, \dots, a_r)$ at P. Thus V satisfies the vanishing condition a at P. If $k = i_n$, then V_n has an element f such that $\operatorname{ord}_P(f) = a_k$ and $\operatorname{ord}_Q(f) = d - a_k$. Thus each V_n satisfies the vanishing condition $(b_{r-i_{n+1}+1}, \dots, b_{r-i_n})$ at Q for $n < s$, and V_s satisfies $(b_0, \dots, b_{r-i_s})$ at Q. So we have $(V, \mathscr{L}) \in G_d^r(E, (P, a), (Q, b))$. □

LEMMA 2.3. *Let X be a curve of compact type consisting of a chain of elliptic curves $E_1, \dots, E_g$. Let P be a point in E_1, and let Q be a point in E_g such that X, P, and Q are general for d. Suppose $a = (a_0, \dots, a_r)$ and $b = (b_0, \dots, b_r)$ are sequences of type (r, d) such that $a_r - a_0 \leqslant r + 1$, $b_r - b_0 \leqslant r + 1$, and $\rho(g, r, d) - \sum(a_i - i) - \sum(b_i - i) \geqslant 0$. Then $G_d^r(X, (P, a), (Q, b)) \neq \varnothing$.*

PROOF. We use induction on g.

Suppose $g = 1$. By replacing the b_i's with larger values, if necessary, we may assume that $\rho(g, r, d) - \sum(a_i - i) - \sum(b_i - i) = 0$. By replacing d with $d - a_0 - b_0$, a_i with $a_i - a_0$, and b_i with $b_i - b_0$ for $i = 0, \dots, r$, we may assume $a_0 = b_0 = 0$. Now $\sum(a_i - i) + \sum(b_i - i) \leqslant 2r$, so $\rho(g, r, d) = 1 - (r + 1)(1 + r - d) = \sum(a_i - i) + \sum(b_i - i)$ implies that $d - r - 1 = -1, 0$ or 1. If $d - r - 1 = -1$, then $r = 0$ and $a_0 = b_0 = 0$, so $(H^0(C, \mathscr{O}), \mathscr{O})$ is a suitable g_d^r. If $d - r - 1 = 0$, then $d = r + 1$, $a_i = b_i = i$ for $i < r$, and $\{a_r, b_r\} = \{r, r + 1\}$; so Lemma 2.2 applies. If $d - r - 1 = 1$, then $d = r + 2$, and $r - \min\{i \mid a_i - i > 0\} = \min\{i \mid b_i - i > 0\}$; so again Lemma 2.2 applies.

Now suppose $g > 1$. We construct $(c_0, \ldots, c_r)$ in the following manner. If $a_r - a_0 = r$, then let $c_0 = a_0$, and let $c_i = a_i + 1$ for $i \geqslant 1$. If $a_r - a_0 = r + 1$, let $k = \min\{i \mid a_i - a_0 = i + 1\}$. In this case we set $c_k = a_k$ and $c_i = a_i + 1$ for $i \neq k$. Note that $c_r \leqslant d$, because $a_r = d$, $r \geqslant 1$, and $a_r - a_0 \leqslant r + 1$ would imply that $\Sigma(a_i - i) \geqslant (r+1)(d-r-1) + 1 > \rho(g, r, d)$ since $g \geqslant 2$.

There is a g_d^r L_1 on E_1 with vanishing sequences a at P and $(d - c_r, \ldots, d - c_0)$ at $R = E_1 \cap E_2$ by Lemma 2.2. We always have $c_r - c_0 \leqslant r + 1$ and $\Sigma c_i - \Sigma a_i = r$. Thus

$$\begin{aligned}\rho(g-1, r, d) - \sum (c_i - i) - \sum (b_i - i) \\ &= \rho(g, r, d) + r - \sum (c_i - i) - \sum (b_i - i) \\ &= \rho(g, r, d) - \sum (a_i - i) - \sum (b_i - i) \geqslant 0,\end{aligned}$$

so the induction hypothesis implies that there exists a limit g_d^r L_2 on $E_2 \cup \cdots \cup E_g$ with vanishing sequences c at R and b at Q. Now L_1 and L_2 determine a point in $G_d^r(X, (P, a), (Q, b))$. □

LEMMA 2.4. *Let X and P be as in Lemma 2.3. Suppose $a = (a_0, \ldots, a_r)$ is such that $a_r - a_1 \leqslant r$,*

$$\rho(g, r, d) - \sum_{i=0}^{r} (a_i - i) \geqslant 0,$$

and

$$\rho(g, r-1, d-a_1) - \sum_{i=0}^{r-1} (a_{i+1} - a_1 - i) \geqslant 0.$$

Then $G_d^r(X, (P, a)) \neq \varnothing$.

PROOF. We use induction on g. If $g = 1$, then $(H^0(X, \mathcal{O}((r+1)P)), \mathcal{O}(dP)) \in G_d^r(X, (P, a))$ provided that $a_{r-1} \leqslant d - 2$. This holds, because otherwise $a_r - a_1 \leqslant r$ and $\rho(g, r-1, d-a_1) - \Sigma_{i=0}^{r-1}(a_{i+1} - a_1 - i) \geqslant 0$ would imply

$$\begin{aligned}0 &\leqslant r(d - a_1 - (r-1)) - (r-1) - \sum_{i=0}^{r-1} (a_{i+1} - i) + ra_1 \\ &\leqslant r(d-r) + 1 - (r(d-r) + 2) = -1.\end{aligned}$$

Suppose $g \geqslant 2$. If $a_r - a_0 \leqslant r + 1$, then we are done by Lemma 2.3. Assume $a_0 \leqslant a_r - (r+2)$, and note that this implies $a_0 \leqslant a_1 - 2$. We construct a sequence $c = (c_0, \ldots, c_r)$ of type (r, d) in the following manner. Let $c_0 = a_0 + 1$. If $a_r - a_1 = r - 1$, then let $c_1 = a_1$ and let $c_i = a_i + 1$ for $i \geqslant 1$. If $a_r - a_1 = r$, let $k = \min\{i \mid a_i - a_1 = i\}$, and let $c_k = a_k$ and $c_i = a_i + 1$ for $i \neq k$. The conditions $a_r - a_1 \leqslant r - 1$ and $\rho(g, r-1, d-a_1) - \Sigma_{i=0}^{r-1}(a_{i+1} - a_1 - i) \geqslant 0$ imply $c_r \leqslant d$ by an argument similar to one in the proof of Lemma 2.3.

We always have $c_r - c_1 \leqslant r$,

$$\sum_{i=0}^{r} c_i - \sum_{i=0}^{r} a_i = r \quad \text{and} \quad \sum_{i=1}^{r} c_i - \sum_{i=1}^{r} a_i = r - 1.$$

Thus

$$\rho(g-1,r,d)-\sum_{i=0}^{r}(c_i-i)=\rho(g,r,d)-\sum_{i=0}^{r}(a_i-i)\geqslant 0,$$

and

$$\rho(g-1,r-1,d-c_1)-\sum_{i=0}^{r-1}(c_{i+1}-c_1-i)$$
$$=\rho(g,r-1,d-a_1)-\sum_{i=0}^{r-1}(a_{i+1}-a_1-i)\geqslant 0.$$

Hence, the induction hypothesis implies that there exists a limit g_d^r on $E_2\cup\cdots\cup E_g$ with vanishing sequence c at $R=E_1\cap E_2$. The lemma now follows, because there exists a g_d^r on E_1 with vanishing sequences a at P and $(d-c_r,\dots,d-c_0)$ at R by Lemma 2.2. □

THEOREM 2.5. *Let C be a smooth curve of genus g. Let e and r be integers $\geqslant 1$. Suppose $\rho(g,r,d)-er+1\geqslant 0$ and $\rho(g,r-1,d-e-1)\geqslant 0$. Then there exists a g_d^r on C with a cusp of order e.*

PROOF. If $g=0$ or 1, the existence of a cusp of order e follows from

$$\rho(g,r-1,d-e-1)\geqslant 0 \quad\text{and}\quad h^0(\mathscr{L}(2p))>h^0(\mathscr{L})$$

if $\deg\mathscr{L}\geqslant 0$.

Assume $g\geqslant 2$. Let C_0 be a curve of compact type consisting of a chain of elliptic curves $E_1,\dots,E_g$, and let C_0 be general for d. Since C_0 is stable in the sense of Mumford and Deligne, there exists a flat proper family of curves $\pi\colon X\to T$ such that T is a smooth connected curve, the fibers of π are nonsingular except over a point $0\in T$ where $X_0\cong C_0$, and there is a point $q\in T$ where $X_q\cong C$. Let $B=X\setminus\{\text{singular points of } X_0\}$, and let $H\subset G_d^r(X\times_T B/B)$ be the subscheme of g_d^r's with cusps of order e. Corollary 1.2 says that every component of H has dimension $\geqslant\rho(g,r,d)-er+2$. We have $H\setminus H_0$ is projective over $T\setminus\{0\}$, because it is a closed subset of $G_d^r(X\times_T B/B)\setminus G_d^r(X_0\times B_0/B_0)$ which is projective over $T\setminus\{0\}$. Thus the theorem will follow when we show that there exists a component of H_0 with dimension $\rho(g,r,d)-er+1$ because then this component must extend over T.

Let P be a point in E_1 such that $P\neq Q=E_1\cap E_2$. If $(V_1,\mathscr{L}_1)$ is a g_d^r on E_1 with a cusp of order e at P and satisfying $b=(d-e-r-1,\dots,d-e-3,d-e-1,d)$ at Q, then we must have $\mathscr{L}_1\cong\mathscr{O}(dQ)$ and

$$V_1=H^0(E_1,\mathscr{L}_1(-dQ))\oplus H^0(E_1,\mathscr{L}_1(-(e+1)P-(d-e-r-1)Q)).$$

This linear series $(V_1,\mathscr{L}_1)$ will have vanishing sequence b and Q if and only if $(e+1)P\sim(e+1)Q$, and there are only finitely many of such points P.

Let $a=(d-b_r,\dots,d-b_0)=(0,e+1,e+3,\dots,e+r+1)$. Now

$$\rho(g-1,r,d)-\sum_{i=0}^{r}(a_i-i)=\rho(g,r,d)+r-(e+1)r+1$$
$$=\rho(g,r,d)-er+1\geqslant 0,$$

and

$$\rho(g-1, r-1, d-(e+1)) - \sum_{i=0}^{r-1} (a_{i+1} - a_1 - i)$$
$$= \rho(g, r-1, d-(e+1)) + r - 1 - (r-1)$$
$$= \rho(g, r-1, d-(e+1)) \geq 0.$$

Thus Lemma 2.4 implies that there exists a nonempty irreducible family F of limit g_d^r's on $E_2 \cup \cdots \cup E_g$ with vanishing sequence a at Q. Further, this family F has dimension $\rho(g-1, r, d) - \Sigma(a_i - i) = \rho(g, r, d) - er + 1$, because C_0 is general for d. Now F and $(V_1, \mathscr{L}_1)$ as described above for some point P with $P \sim (e+1)Q$ in E_1 form a component of H_0 with dimension $\rho(g, r, d) - er + 1$. □

LEMMA 2.6. *Let $(V, \mathscr{L})$ be a g_d^r on a smooth curve satisfying conditions $a = (a_0, \ldots, a_r)$ and $b = (b_0, \ldots, b_r)$ at points P and Q, respectively. Then there exists a basis $\{s_0, \ldots, s_r\}$ of V and a permutation σ of $\{0, \ldots, r\}$ such that* $\operatorname{ord}_P(s_i) \geq a_i$ *and* $\operatorname{ord}_Q(s_i) \geq b_{\sigma(i)}$ *for $i = 0, \ldots, r$.*

PROOF. We may assume that a and b are the vanishing sequences for $(V, \mathscr{L})$ at P and Q, respectively. We can choose $t_i \in V$ successively so that $\operatorname{ord}_P(t_i) = a_{r-i}$ and $\operatorname{ord}_Q t_i \neq \operatorname{ord}_Q t_j$ for $j < i$. This is so, because if $\operatorname{ord}_P(t) = a_{r-i}$, then $\operatorname{ord}_P(t - \alpha t_j) = a_{r-i}$ for any $\alpha \in \mathbf{C}$ and $j < i$. So, if $\operatorname{ord}_Q(t_j) = \operatorname{ord}_Q(t)$, then there exists $\alpha \in \mathbf{C}$ so that $\operatorname{ord}_Q(t - \alpha t_j) > \operatorname{ord}_Q(t_j)$. Now let $s_i = t_{r-i}$ for $i = 0, \ldots, r$. □

LEMMA 2.7. *Let E be an elliptic curve containing points P and Q such that E, P, and Q are general for d. Then*:

(i) *$mP \nsim mQ$ for $0 < m \leq d$; and*

(ii) *If a and b are sequences of type (r, d) such that*

$$\rho(1, r, d) - \Sigma(a_i - i) - \Sigma(b_i - i) = 0,$$

then $(V, \mathscr{L}) \in G_d^r(E, (P, a), (Q, b))$ implies that $\mathscr{L} \cong \mathscr{O}(mP + (d-m)Q)$ with $0 \leq m \leq d$.

PROOF. Suppose assertion (i) is false. Let $\mathscr{L} = \mathscr{O}(mP)$. Then

$$(H^0(\mathscr{L}(-mP)) \oplus H^0(\mathscr{L}(-mQ)), \mathscr{L}) \in G_m^1(E, (P, (0, m)), (Q, (0, m)))$$

contradicts E, P, and Q are general for d, because

$$\rho(1, 1, m) - 2(m-1) = 2(m-1) - 1 - 2(m-1) = -1.$$

Assertion (ii) will hold if there exists an $s \in V$ such that $\operatorname{ord}_P(s) + \operatorname{ord}_Q(s) = d$. By Lemma 2.6 we may choose a basis $s_0, \ldots, s_r$ such that $\operatorname{ord}_P(s_i) = a_i$ and $\operatorname{ord}_Q(s_i) = b_{\sigma(i)}$ for some permutation σ of $\{0, \ldots, r\}$. Then

$$0 = \rho(1, r, d) - \sum (a_i - i) - \sum (b_i - i)$$
$$= (r+1)(d-r) - r + (r+1)r - \sum (a_i + b_{\sigma(i)})$$
$$= (r+1)(d-1) + 1 - \sum (a_i + b_{\sigma(i)}).$$

So $a_k + b_{\sigma(k)} = d$ for some $0 \leq k \leq r$. □

LEMMA 2.8. *Let n and r be $\geqslant 2$. Suppose $a = (a_0, \dots, a_r)$ is a sequence of type (r, d) such that $a_{r-1} - a_0 \leqslant r$, $a_r = d$, and $\rho(n, r-1, d-n) - \sum_{i=0}^{r-1}(a_i - i) = 0$. Then there exists a smooth curve C of genus n containing a point P with the following property. Let $B = (C \setminus \{P\})^n$, and let $H \subset G^r(C \times B/B)$ be the subscheme of g_d^r's with n-fold points. Let $\bar{p}$: $B \to C \times B$ be the morphism which sends Q to (P, Q). Then $A = H \cap G_d^r(C \times B/B, (\bar{p}, a))$ contains an isolated point, and the g_d^r on C corresponding to this point has vanishing sequence a at P.*

PROOF. There is a smooth connected curve T containing a point 0, a flat proper family of curves f: $X \to T$ and a T-morphism g: $T \to X$ such that: X_0 is a curve of compact type consisting of a chain of curves $Y_1, \dots, Y_{n-1}$ where Y_1 is genus 2 and Y_i is elliptic for $i \geqslant 2$;

$g(0) = P_n$ is a smooth point of Y_{n-1} such that X_0 and P_n are general for d;

X_q is nonsingular for $q \in T \setminus \{0\}$; and

$G_{d-n}^{r-1}(X_q, (g(q), (a_0, \dots, a_{r-1})))$ is finite for all $q \in T$.

Let $a' = (a_0, \dots, a_{r-1})$, and let $b = (b_0, \dots, b_{r-1}) = (1, 2, 4, 5, \dots, r+1)$. Let $P_i = Y_{i-1} \cap Y_i$ for $i = 2, \dots, n-1$. There exists a limit linear series $L_2 \in G_{d-n}^{r-1}(Y_2 \cup \cdots \cup Y_{n-1}, (P_n, a')(P_1, b))$ by Lemma 2.3. If K is a canonical divisor on Y_1, then $L_1 = (H^0(Y_1, \mathcal{O}(K + (r-1)P_2)), \mathcal{O}(K + (d-n-2)P_2))$ is a linear series in $G_{d-n}^{r-1}(Y_1, (P_2, (d - b_{r-1}, \dots, d - b_0)))$. Thus L_1 and L_2 determine a limit linear series L in $G_{d-n}^{r-1}(X_0, (P_n, a'))$. Let Z be a component of $G_{d-n}^{r-1}(X/T, (g, a'))$ which contains L. Since X_0 and P_n are general for d, Z_0 is finite. Hence, Theorem 1.1 implies that $\dim Z = 1$, and Z extends over T. If we replace T by a suitable base extension, we may assume that there exists a T-morphism ϕ: $T \to Z$. Hence, for each irreducible component Y_i of X_0, we have morphisms $T \overset{\phi}{\to} Z \overset{\psi}{\to} \mathrm{Pic}_{d-n}^{Y_i}(X/T)$ where for each $q \neq 0$ in T $\psi(\phi(q))$ corresponds to the line bundle of the linear series corresponding to $\phi(q)$, and $\psi(\phi(0))$ corresponds to a line bundle on X_0 whose restriction to Y_i is the line bundle on Y_i in the limit linear series L. Let $\mathcal{K}$ on X be the line bundle associated to the map $\psi \circ \phi$. Let $\mathcal{M} = \mathcal{O}_X(dg(T)) \otimes \mathcal{K}^{-1}$, and let $\mathcal{M}_q = \mathcal{M}|_{X_q}$ for each $q \in T$.

CLAIM. There is an open subset $U \subset T \setminus \{0\}$ such that if $q \in U$ and $s \in H^0(X_q, \mathcal{M}_q)$ is nonzero, then s does not vanish on $g(q)$.

PROOF OF CLAIM. Let η denote the generic point of T. It is sufficient to show that if $s \in H^0(X_\eta, \mathcal{M}_\eta)$ is nonzero, then s does not vanish on $g(\eta)$. The section s extends to a section in $H^0(X, \mathcal{M}(E))$ where the support of E maps to a finite set of T, and all components of the support of the divisor D of relative degree n associated to s map onto T. The claim will follow when we show that D induces a divisor of degree $\geqslant 2$ on $Y_1 \setminus \{P_2\}$, and that D meets each $Y_i \setminus \{P_i, P_{i+1}\}$.

For each Y_i, we have line bundles $\mathcal{K}_i$ and $\mathcal{K}_i'$ on Y_i induced by considering the maps $T \setminus \{0\} \to \mathrm{Pic}_{d-n}^{Y_i}(X/T)$ and $T \setminus \{0\} \to P_d^{Y_i}(X/T)$ that come from $\mathcal{K}$ and $\mathcal{K}(D)$, respectively. Since $\mathcal{K}(D)|_{X_\eta} = \mathcal{O}(dg(\eta))$, it is easy to see that $\mathcal{K}_i' \cong \mathcal{O}_{Y_i}(dP_{i+1})$ for each i. The line bundles $\mathcal{K}_i$ are the same as the line bundles of the limit linear series L on X_0.

We have $\mathcal{K}_1' \cong \mathcal{O}_{Y_1}(dP_2)$ and $\mathcal{K}_1' \cong \mathcal{K}_1(mP_2 + D_1)$ where D_1 is the intersection of D with $Y_1 \setminus \{P_2\}$ and $m + \deg D_1 = d$. Now $\mathcal{K}_1 \cong \mathcal{O}(K + (d-n-2)P_2)$, so $\mathcal{O}(K + D_1) \cong \mathcal{O}((d - m + 2)P_2)$. If $D_1 = 0$, then $K \sim 2P_2$. This is impossible, because X_0 is general for d implies that $G_2^1(Y_1, (P_2, (0, 2))) = \varnothing$. If $\deg D_1 = 1$ then $H^0(\mathcal{O}(K)) = H^0(\mathcal{O}(K + D_1)) = H^0(\mathcal{O}(3P_2))$ implies that $D_1 = P_2$ and $K \sim 2P_2$ which is impossible. Thus $\deg D_1 \geqslant 2$.

For each $i \geqslant 2$ we have $\mathcal{K}_i \cong \mathcal{O}(m_i P_i + (d - m_i)Q)$ with $0 \leqslant m_i \leqslant d$ by Lemma 2.7. Also, $\mathcal{K}_i' \cong \mathcal{O}(dP_{i+1}) \cong \mathcal{K}_i(s_i P_i + t_i P_{i+1} + D_i)$ where D_i is the intersection of D with $Y_i \setminus \{P_i, P_{i+1}\}$, $s_i \geqslant 1$ and $t_i \geqslant 0$. Now deg $D_i \geqslant 1$, because Lemma 2.7 implies $s_i P_i \sim (d - t_i)P_{i+1}$. Hence the claim holds.

Let $C = X_q$ for some $q \in U$. Let $P = g(q)$, and let $\mathcal{L} = \mathcal{K}|_C$. Note that $h^0(C, \mathcal{O}(dP) \otimes \mathcal{L}^{-1}) \leqslant 1$, because $|\mathcal{O}(dP) \otimes \mathcal{L}^{-1}| = |\mathcal{M}_q|$ contains a divisor with P in its support if $\dim|\mathcal{O}(dP) \otimes \mathcal{L}^{-1}| \geqslant 1$ **[FL]**. Thus there is a unique divisor D on C such that $\mathcal{L}(D) \cong \mathcal{O}(dP)$, and P is not in the support of D. We have a natural map

$$A = H \cap G_d^r(C \times B/B, (\bar{p}, a)) \to B \xrightarrow{\delta} \mathrm{Pic}_{d-n}(C)$$

where $\delta(P_1, \ldots, P_n)$ corresponds to $\mathcal{O}(dP - \Sigma P_i)$. The image of A in $\mathrm{Pic}_{d-n}(C)$ is finite, because it corresponds to line bundles of linear series in $G_{d-n}^{r-1}(C, (P, a'))$ which is finite. Let $V \subset H^0(C, \mathcal{L})$ be the subspace with vanishing sequence a' at P, and let $s \in H^0(C, \mathcal{L}(D))$ be such that $\mathrm{ord}_P(s) = d$. Then $(V + s, \mathcal{L}(D))$ lies in A, and the fiber over $\mathrm{Pic}_{d-n}(C)$ containing it is finite. Hence $(V + S, \mathcal{L}(D))$ is an isolated point of A. □

THEOREM 2.9. *Let C be a nonsingular curve of genus $g \geqslant n \geqslant 2$, and let $r \geqslant 2$. If $\rho(g, r, d) - (n-1)r + n \geqslant 0$ and $\rho(g, r-1, d-n) \geqslant 0$, then there exists a g_d^r on C with an n-fold point.*

PROOF. If $g = n$, then $\rho(g, r-1, d-n) \geqslant 0$ implies that there exists a $g_{d-n}^{r-1}(V, \mathcal{L})$ on C. Choose $P \in C$ so that $\mathcal{L} \not\cong \mathcal{O}((d-n)P)$. There exists a divisor D of degree n such that $\mathcal{L}(D) \cong \mathcal{O}(dP)$. Choose $s \in H^0(C, \mathcal{L}(D))$ such that $\mathrm{ord}_P s = d$. Now $(V + s, \mathcal{L}(D))$ is the desired linear series.

Suppose $g > n$. We can choose a curve of compact type C_0 consisting of a chain of curves $Y_0, \ldots, Y_{g-n}$ where Y_0 and $Q = Y_0 \cap Y_1$ are as C and P are in Lemma 2.8, $Y_1, \ldots, Y_{g-n}$ are elliptic, and $Y_1 \cup \cdots \cup Y_{g-n}$ and Q are general for d. Let $B_0 = Y_0 \setminus \{Q\}$, and let $H_0 \subset G_d^r(C_0 \times B_0/B_0)$ be the subscheme of g_d^r's with n-fold points. An argument similar to one found in the proof of Theorem 2.5 shows that the theorem will hold if there exists a component of H_0 of dimension $\rho(g, r, d) - n(r-1) + n$.

We can choose a sequence $a = (a_0, \ldots, a_r)$ so that $a_r = d$, $a_{r-1} - a_0 \leqslant r$, and $\rho(n, r-1, d-n) = \Sigma_{i=0}^{r-1}(a_i - i)$. Now Lemma 2.8 applies to Y_0, Q, and a, so there exists an isolated L in the space of g_d^r's on Y_0 with an n-fold point along a divisor whose support does not contain Q and with vanishing sequence a at Q.

Let $b = (b_0, \ldots, b_r) = (d - a_r, \ldots, d - a_0)$. Then

$$\sum_{i=0}^{r}(b_i - i) = \sum_{i=0}^{r}(d - a_{r-i} - i) = \sum_{i=0}^{r}(d - a_i - r + i)$$
$$= r(d-r) - \sum_{i=0}^{r-1}(a_i - i)$$
$$= r(d-r) - [n - r(n + (r-1) - (d-n))]$$
$$= -n + r(2n-1),$$

and

$$\sum_{i=0}^{r-1}(b_{i+1} - i) = \sum_{i=0}^{r}(b_i - i + 1) - b_0 - 1$$
$$= r + \sum_{i=0}^{r}(b_i - i) = -n + 2rn.$$

So

$$\rho(g-n, r, d) - \sum_{i=0}^{r}(b_i - i)$$
$$= g - n - (r+1)(g - n + r - d) + n - r(2n-1)$$
$$= g - (r+1)(g - n + r - d) - 2rn + r$$
$$= g - (r+1)(g + r - d) + (r+n)n - 2rn + r$$
$$= \rho(g, r, d) - r(n-1) + n \geqslant 0,$$

and

$$\rho(g-n, r-1, d-b_1) - \sum_{i=0}^{r-1}(b_{i+1} - b_1 - i)$$
$$= g - n - r(g - n + r - 1 - d + b_1) + rb_1 + n - 2rn$$
$$= g - r(g + n + r - 1 - d) = \rho(g, r-1, d-n) \geqslant 0.$$

So Lemma 2.4 implies that there exists a family F of limit g_d^r's on $Y_1 \cup \cdots \cup Y_{g-n}$ with vanishing sequence b at Q. This family has dimension $\rho(g-n, r, d) - \Sigma(b_i - i) = \rho(g, r, d) - r(n-1) + n$. Now L and F determine the desired component of H_0. □

THEOREM 2.10. *Let C be a curve of genus g, and let D be a divisor of degree $n \geqslant 2$ on C. If $r \geqslant 2$, $\rho(g, r, d) - r(n-1) \geqslant 0$, and $\rho(g, r-1, d-n) \geqslant 0$, then there exists a g_d^r on C with an n-fold point along D.*

PROOF. If $g = 0$ and $h^0(C, \mathscr{L}) \geqslant 1$, then $h^0(C, \mathscr{L}(D)) > h^0(C, \mathscr{L})$, so the theorem holds in this case.

Suppose $g \geqslant 1$. As in the proof of Theorem 2.5, there exists a smooth connected curve T containing a point 0 and a flat proper family of curves $\pi\colon X \to T$ such that X_q is nonsingular for $q \neq 0$, $X_q \cong C$ for some $q \in T$, X_0 is a curve of compact type consisting of a chain of g elliptic curves $Y_1, \ldots, Y_g$, and X_0 and P are general for d where $P \in Y_1$. Let $B = X \setminus (Y_2 \cup \cdots \cup Y_n)$. Let $H \subset G^r_d(X \times_T B/B)$ be the subscheme of g^r_d's with n-fold points. The fiber of H over $(P, P, \ldots, P) \in B$ is $G = G^r_d(X_0, (P, (0, n, n+1, \ldots, n+r-1)))$, and has dimension $\rho(g, r, d) - (n-1)r$, since it is nonempty by Lemma 2.4. The component of H containing G has dimension $\geqslant \rho(g, r, d) - (n-1)r + n + 1$ by Corollary 1.3 so it must extend over an open subset of B. The theorem now follows, because H is projective over $T \setminus \{0\}$. □

Linear series with a cusp on a general curve. In this section we show that if X is a smooth genus g curve which is general in moduli, then every component of the subscheme $H \subset G^r_d(X \times X/X)$ of g^r_d's with cusps of order e has dimension $\rho(g, r, d) - er + 1$.

The following combinatorial fact is Lemma 1.4 of **[EH1]**.

LEMMA 3.1. *If $a_0 < \cdots < a_r$ and $b_0 < \cdots < b_r$, and if for some permutation f of $\{0, \ldots, r\}$ we have $a_i \leqslant b_{f(i)}$ for $i = 0, \ldots, r$, then in fact $a_i \leqslant b_i$ for $i = 0, \ldots, r$. Further, if for some i we have $a_i = b_i$, then $f(i) = i$ so that $a_i = b_{f(i)}$ as well.*

LEMMA 3.2. *Let $\mathscr{L}$ be a line bundle of degree d on a smooth curve C which contains points P and Q. Let σ be a permutation of $\{0, \ldots, r\}$, and let $n = \#\{i \mid \sigma(i) > r - i\}$. Let $a = (a_0, \ldots, a_r)$ and $b = (b_0, \ldots, b_r)$ be sequences of type (r, d). Then the rational map*

$$\Phi\colon \prod_{i=0}^{r} \mathbf{P}\big(H^0\big(C, \mathscr{L}(-a_i P - b_{\sigma(i)} Q)\big)\big) \to \mathrm{Gr}_{r+1}\big(H^0(C, \mathscr{L})\big)$$

which sends $(s_0, \ldots, s_r)$ to the $(r+1)$-dimensional subspace spanned by $s_0, \ldots, s_r$ has all its fibers of dimension $\geqslant n$ wherever it is a morphism.

PROOF. For ease of notation we will let X_i denote $\mathbf{P}(H^0(C, \mathscr{L}(-a_i P - b_{\sigma(i)} Q)))$ for $i = 0, \ldots, r$ and we will let G_k denote $\mathrm{Gr}_k(h^0(C, \mathscr{L}))$ for $k = 1, \ldots, r+1$.

We use induction on n. There is nothing to prove if $n = 0$.

Suppose $n \geqslant 1$. Let $k = \max\{i \mid \sigma(i) > r - i\}$. Note that $k \geqslant 1$. We have the following factorization of Φ.

$$\prod_{i=0}^{r} X_i \xrightarrow{\alpha} G_k \times \prod_{i=k}^{r} X_i \xrightarrow{\beta} G_{k+1} \times \prod_{i=k+1}^{r} X_i \xrightarrow{\gamma} G_{r+1}.$$

The rational maps α, β, and γ are defined in the obvious manner. Let S be the open subset of $\prod_{i=0}^{r} X_i$ consisting of points $(s_0, \ldots, s_r)$ such that $s_0, \ldots, s_r$ span an $(r+1)$-dimensional subspace of $H^0(C, \mathscr{L})$. Let T be a quasi-projective dense subset of $\alpha(S)$. The lemma will follow when we show that a general fiber of $\alpha|_S$ has dimension $\geqslant n - 1$, and a general fiber of $\beta|_T$ has dimension $\geqslant 1$.

Let $(c_0, \ldots, c_{k-1})$ be the sequence of type $(k-1, d)$ such that for each $i = 0, \ldots, k-1$ $c_i = b_{\sigma(j)}$ for some $j \leqslant k-1$. Note that if $j > k$, then $\sigma(j) \leqslant r - j < r - k$. It follows that $i \leqslant k$ implies $\sigma(i) \geqslant r - k$. Thus

$$c_i = \begin{cases} b_{i+(r-k)} & \text{if } i + r - k < \sigma(k), \\ b_{i+(r-k)+1} & \text{if } i + r - k \geqslant \sigma(k). \end{cases}$$

Let f be the permutation of $\{0,\ldots,k-1\}$ defined by $c_{f(i)} = b_{\sigma(i)}$. If $\sigma(i) > r-i$, then $f(i) > r-i-(r-k)-1 = k-1-i$, because $\sigma(i) \leqslant f(i)+(r-k)+1$. Hence $\#\{i \mid f(i) \leqslant k-1-i\} \geqslant n-1$, and the induction hypothesis implies that a general fiber of α has dimension $\geqslant n-1$.

Suppose $(\operatorname{span}(s_0,\ldots,s_{k-1}), (s_k,\ldots,s_r)) \in T$. There exists $j<k$ so that $\sigma(j) = r-k < \sigma(k)$. For each $\lambda \in \mathbf{C}$, let $V_\lambda = \operatorname{span}\{t_0,\ldots,t_{k-1}\}$ where

$$t_i = \begin{cases} s_i & \text{if } i \neq j, \\ s_j + \lambda s_k & \text{if } i = j. \end{cases}$$

Since $j<k$ and $\sigma(j) < \sigma(k)$, we have $\operatorname{ord}_P(t_j) \geqslant a_j$ and $\operatorname{ord}_Q(t_j) \geqslant b_{\sigma(j)}$. Hence $(V_\lambda, (s_k,\ldots,s_r)) \in \alpha(S)$ for all $\lambda \in \mathbf{C}$. It is clear that $V_\lambda \neq V_\mu$ for $\lambda \neq \mu$ and $\beta(V_\lambda, (s_k,\ldots,s_r)) = \beta(V_\mu, (s_k,\ldots,s_r))$. Therefore a general fiber of β has dimension $\geqslant 1$. □

LEMMA 3.3. *Let E be an elliptic curve containing a point P, and $a = (a_0,\ldots,a_r)$ be a sequence of type (r,d). Let $H \subset G^r_d(E \times E/E) = G^r_d(E) \times E$ be the subscheme of g^r_d's on E with a cusp of order e. Let $\hat{H} = H \cap G^r_d(E,(P,a)) \times (E \setminus \{P\})$ be the subscheme of g^r_d's on E satisfying vanishing condition a at P and having a cusp of order e at a point distinct from P. Then* $\dim \hat{H} \leqslant \rho(1,r,d) - \Sigma(a_i - i) - er + 1$.

PROOF. Let $b = (0, e+1, \ldots, e+r)$ and let $Q \neq P$ be a point in E. Let $H_{Q,\mathscr{L}}$ denote the fiber of the morphism $G^r_d(E,(P,a),(Q,b)) \to \operatorname{Pic}_d(E)$ over the point corresponding to the line bundle $\mathscr{L}$. For each permutation σ of $\{0,\ldots,r\}$, let S_σ denote the open subset of $\prod_{i=0}^r \mathbf{P}(H^0(E,\mathscr{L}(-a_iP - b_{\sigma(i)}Q)))$ of points $(s_0,\ldots,s_r)$ such that $\dim \operatorname{span}(s_0,\ldots,s_r) = r+1$. Lemma 2.6 implies that $H_{Q,\mathscr{L}}$ is covered by the images of morphisms $\Phi_{Q,\mathscr{L}}$: $S_\sigma \to \operatorname{Gr}_{r+1}(H^0(E,\mathscr{L}))$, and Lemma 3.2 says that the general fiber of Φ_σ has dimension $\geqslant \#\{i \mid \sigma(i) > r-i\}$. If $S_\sigma \neq \varnothing$, Lemma 3.1 implies $a_i \leqslant d - b_{r-i}$ for $i = 0,\ldots,r$ and that $b_{r-i} = b_{\sigma(i)}$ if $a_i = d - b_{r-i}$. In particular, we have $a_i + b_{r-i} = d$ implies $\mathscr{L} \cong \mathscr{O}(a_iP + b_{r-i}Q)$. Note that $a_{r-2} + b_2 = d$ implies that $a_{r-1} + b_1 = d$, because $a_{r-1} > a_{r-2}$ and $b_2 - b_1 = 1$. Thus $a_{r-2} + b_2 < d$, because otherwise we would have $P \sim Q$. It follows that $a_{r-i} + b_i < d$ for $i \geqslant 2$, because $a_{r-2} - a_{r-i} \geqslant i-2$ and $b_{r-i} - b_2 = i-2$ for $i \geqslant 2$. Note that if $a_r + b_0 = d$ and $a_{r-1} + b_1 = d_1$ then $(e+1)P \sim (e+1)Q$.

Let $N = \rho(1,r,d) - er + 1$. We have the following if $S_\sigma \neq \varnothing$:

$$\begin{aligned} \dim S_\sigma &\leqslant \sum_{i=0}^{r} \left(d - a_i - b_{\sigma(i)} - 1\right) + 2 + \#\{i \mid \sigma(i) > r-i\} \\ &= \sum_{i=0}^{r} \left(d - r - 1 - (a_i - i) - (b_{r-i} + i - r)\right) + 2 + \#\{i \mid \sigma(i) > r-i\} \\ &= (r+1)(d-r) - r - \Sigma(a_i - i) - er + 1 + \#\{i \mid \sigma(i) > r-i\} \\ &= N + \#\{i \mid \sigma(i) > r-i\}. \end{aligned}$$

Thus $\dim H_{Q,\mathscr{L}} \leqslant N$. If $(e+1)Q \nsim (e+1)P$ and $S_\sigma \neq \varnothing$, then $\dim S_\sigma \leqslant N - 1 + \#\{i \mid \sigma(i) > r-i\}$, so $\dim H_{Q,\mathscr{L}} \leqslant N-1$. If $\mathscr{L}$ is not isomorphic to $\mathscr{O}(dP)$ or $\mathscr{O}((d-e-1)P + (e+1)Q)$, then we must have $\dim H_{Q,\mathscr{L}} \leqslant N-2$.

Let Z be a component of $\hat{H}$, and let α: $\hat{H} \to \mathrm{Pic}_d(E)$ and β: $\hat{H} \to E \setminus \{P\}$ be the morphisms which are defined in the obvious manner. We have three cases to consider.

Case 1. Suppose $\alpha|_Z$ and $\beta|_Z$ are constant. Then $Z = H_{Q,\mathscr{L}}$ for some Q and $\mathscr{L}$, so $\dim Z = \dim H_{Q,\mathscr{L}} \leqslant N$.

Case 2. Suppose $\alpha|_Z$ is constant, but $\beta|_Z$ is not constant. Then there is a $Q \in \beta(Z)$ such that $(e+1)Q \nsim (e+1)P$. Hence for some L we have $\dim(Z) \leqslant \dim H_{Q,\mathscr{L}} + 1 \leqslant N$.

Case 3. Suppose $\alpha|_Z$ is not constant. Then there exists an $\mathscr{L}$ corresponding to a point in $\alpha(Z)$ such that $\mathscr{L} \not\equiv \mathscr{O}(dP)$ and $\mathscr{L} \not\equiv \mathscr{O}((d-e-1)P + (e+1)Q)$. Thus for some $Q \in \beta(Z)$ we have $\dim(Z) \leqslant \dim H_{Q,\mathscr{L}} + 2 \leqslant N$. □

THEOREM 3.4. *Let X be a smooth curve of genus g, and let $H_X \subset G_d^r(X \times X/X)$ be the subscheme of g_d^r's with cusps of order e. If X is general in moduli, then every component of H_X has dimension $\rho(g,r,d) - er + 1$.*

PROOF. By Corollary 1.2, every component of H_X has dimension $\geqslant \rho(g,r,d) - er + 1$, so it remains to show an upper bound for $\dim H_X$ if X is general in moduli.

Let T be a smooth affine curve containing a point 0, and let π: $X \to T$ be a flat proper family of genus g curves such that X_q is smooth for $q \neq 0$ and X_0 is a curve of compact type which is general for d, consists only of rational and elliptic curves, and is such that every elliptic subcurve meets the rest of X_0 at most one point. Let $B = X \setminus \{\text{singular points of } X_0\}$, and let Δ: $B \to X \times_T B$ be the diagonal morphism. Let

$$H = G_d^r(X \times_T B/B, (\Delta, (0, e+1, \ldots, e+r))).$$

Then for $q \neq 0$, we have $H_{X_q} = H_q$. It follows from Proposition 2.5 and Theorem 2.6 of **[EH3]** that if we replace π: $X \to T$ by what we obtain after blowing up the nodes of X_0 sufficiently often, making finite base change of T, and resolving the resulting singularities of X we may assume that every component of H which does not map to a point in T meets X_0. Since our new X_0 is obtained by inserting chains of rational curves at the nodes of the old X_0, it will consist of only rational and elliptic curves and each elliptic curve will meet the rest of X_0 at most one point.

It is sufficient to show that $\dim H_0 \leqslant \rho(g,r,d) - er + 1$. Theorem 2.3 of **[EH2]** shows that the codimension of $G_d^r(X_0, (Q, (0, e+1, \ldots, e+r)))$ in $G_d^r(X_0)$ is $\geqslant er$ if Q is a smooth point lying in one of the rational components of X_0. Lemma 3.3 shows that any component of H_0 which corresponds to limit g_d^r's with a cusp of order e on an elliptic subcurve has codimension $\geqslant er$ in $G_d^r(X_0 \times B_0/B_0)$. Thus $\dim H_0 \leqslant \rho(g,r,d) - er + 1$ as desired. □

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